DECISION PROCEDURES FOR FINITE DECISION PROBLEMS UNDER COMPLETE IGNORANCE[1]

By F. V. Atkinson, J. D. Church and B. Harris

University of Toronto, University of Nebraska, and Mathematics Research Center, U. S. Army

1. Introduction. This paper deals with a class of statistical decision problems considered as games against Nature. We are concerned with problems in which the number of states of Nature is finite, the set of decisions available to the statistician is finite, and the statistician is strictly without information as to which state of Nature will occur. Our purpose is to describe what we would consider to be a satisfactory method for selecting a decision in such problems. A list of properties, which we believe should characterize such a method is given. Modifying a method due to J. W. Milnor, we exhibit a procedure which has the required properties.

Related formulations of the problem have been treated by H. Chernoff [2] and J. W. Milnor [3].

The following is a more explicit characterization of the class of decision problems we will deal with.

We are given a probability space (S, P) where $S = \{s_1, s_2, \cdots, s_n\}$ is a set of states of Nature and P is a probability measure defined on all the subsets of S. A vector $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ is associated with S, where ξ is a point in Ξ, an $n - 1$ dimensional proability simplex, such that

$$P(\{s_j\}) = \xi_j \geqq 0 \qquad \text{for } 1 \leqq j \leqq n$$

$$P(\{s_i\} \cup \{s_j\}) = \xi_i + \xi_j \qquad \text{for } i \neq j$$

$$P(S) = \sum_{j=1}^{n} \xi_j = 1$$

We call ξ an *a priori* probability distribution on the elements of S. It is assumed that knowledge concerning ξ can be completely specified by a statement of the form $\xi \, \varepsilon \, \Xi_0$ where $\Xi_0 \subset \Xi$. If $\Xi_0 = \Xi$, we say we are in complete ignorance of ξ.

We are also given a set $D = \{d_1, d_2, \cdots, d_m\}$ of decisions exactly one of which must be selected prior to learning which state of Nature has occurred. We may regard the d_i as strategies in our game against Nature and will call the d_i pure strategies.

Also given is a real-valued function u defined on $D \times S$. $u(d_i, s_j)$ is a real number measuring the loss we incur when we choose d_i and state s_j occurs.

Then $P = (S, D, u)$ is a decision problem, which can be regarded as a finite game against Nature. The problem P can be identified with the loss matrix A,

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where

$$A = \begin{bmatrix} u(d_1, s_1) & u(d_1, s_2) & \cdots & u(d_1, s_n) \\ u(d_2, s_1) & u(d_2, s_2) & \cdots & u(d_2, s_n) \\ \vdots & \vdots & & \vdots \\ u(d_m, s_1) & u(d_m, s_2) & \cdots & u(d_m, s_n) \end{bmatrix}$$

The row vector of the matrix A corresponding to pure strategy d_i is called the loss vector associated with d_i.

In selecting a strategy, our sole motivation is to minimize our loss with respect to the utility function u. The purpose of this paper is to discuss methods for selecting a strategy in cases where we are in complete ignorance of $\tilde{\xi}$.

2. Characterization of strategies. Let Φ be an $m - 1$ dimensional probability simplex over the elements of D. Instead of selecting a pure strategy directly we may select an element $\phi \varepsilon \Phi$, $\phi = (p_1, p_2, \cdots, p_m)$, and make our choice of $d \varepsilon D$ on the basis of the outcome of a random experiment constructed so that d_i will be selected with probability p_i. Such a method for selecting a pure strategy is called a mixed strategy. We will identify mixed strategies with their associated vectors in Φ and we will say that $\phi = (p_1, p_2, \cdots, p_m)$ assigns probabilities $p_1, \cdots, p_m$ to $d_1, \cdots, d_m$ respectively. Clearly, Φ contains all pure strategies.

We can extend the utility function to the set Φ of mixed strategies by defining for all $\phi \varepsilon \Phi$, and $1 \leqq j \leqq n$,

$$u(\phi, s_j) = \sum_{i=1}^{m} p_i u(d_i, s_j), \qquad \text{where } \phi = (p_1, \cdots, p_m).$$

The vector $(u(\phi, s_1), u(\phi, s_2), \cdots, u(\phi, s_n))$ is called the loss vector attained by the strategy ϕ. Strategies ϕ_1 and ϕ_2 are said to be u-equivalent if $u(\phi_1, s_j) = u(\phi_2, s_j)$ for all $1 \leqq j \leqq n$.

A strategy ϕ_1 is said to be dominated by strategy ϕ_2 if $u(\phi_2, s_j) \leqq u(\phi_1, s_j)$ for all $1 \leqq j \leqq n$ and $u(\phi_2, s_j) < u(\phi_1, s_j)$ for at least one $1 \leqq j \leqq n$. A strategy is said to be admissible, if it is not dominated. A strategy φ_0 is said to be essential, if it is admissible, and if, for every pair $\varphi_1, \varphi_2 \varepsilon \Phi$, with φ_1 not u-equivalent to φ_0 and every λ, $0 < \lambda < 1$, we have $u(\varphi_0, s_j) \neq \lambda u(\varphi_1, s_j) + (1 - \lambda)u(\varphi_2, s_j)$ for at least one index j, $1 \leqq j \leqq n$.

3. Desirable properties for decision procedures. For any decision problem $P = (S, D, u)$, a decision procedure is a method of dividing the set of strategies Φ into two disjoint subsets; the set K, of strategies which are, in some sense, optimal, and the complement of K. It is assumed that all elements of K are equivalent with respect to the optimality criterion used to construct K, so that when employing a given decision procedure we may arbitrarily select a strategy ϕ from the resulting set K, provided K is non-null. Below we give a list of properties that should characterize a satisfactory decision procedure.

In order to simplify notation and language, we will identify a decision problem P with its associated loss matrix A and we will regard a decision procedure as a method for specifying the set $Q(A)$ of optimal loss vectors. The set K of optimal strategies will be the set of all strategies which attain loss vectors in $Q(A)$.

Let $C(A)$ denote the convex hull of the row vectors of the matrix A and let E_A denote the submatrix of row vectors of A corresponding to essential strategies. Note that an essential strategy is an admissible strategy, which attains a loss vector, which is an extreme point of $C(A)$.

We believe that a satisfactory decision procedure is characterized by the 8 properties listed below.

Property 1. For every A, the set $Q(A)$ is a non-null subset of $C(A)$.

Property 2. If A' can be obtained from A by permutations of the columns and rows of A then $Q(A')$ can be obtained from $Q(A)$ by applying the permutation on the columns of A to the coordinates of all vectors in $Q(A)$.

Property 3. If $\tilde{x} \in Q(A)$ and $\tilde{\epsilon}$ is a non-negative vector and $\tilde{\epsilon} \neq 0$ then $\tilde{x} - \tilde{\epsilon} \not\in C(A)$. (Every strategy which attains a loss vector in $Q(A)$ is admissible.)

Property 4. $Q(A)$ is a convex set.

Property 5. If

$$A_1 = \lambda A_0 + \begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & & \vdots \\ c_1 & c_2 & & c_n \end{bmatrix}$$

where $\lambda > 0$ and the c_i are any constants then $Q(A_1) = \{\lambda\tilde{x} + \tilde{c}: \tilde{x} \in Q(A_0)\}$ where $\tilde{c} = (c_1, c_2, \cdots, c_n)$.

Property 6. If $C(A_1^T) = C(A_2^T)$ and A_1 can be obtained from A_2 by deleting j columns from A_2 then $Q(A_1)$ can be obtained from $Q(A_2)$ by deleting the corresponding j coordinates from every vector in $Q(A_2)$.

Property 7. If for A_1 and A_2, $C(E_{A_1}) = C(E_{A_2})$ then $Q(A_1) = Q(A_2)$.

Property 8. If $\{A_j\}_{j=1}^{\infty}$ converges to A_0, and $\tilde{x}_j \in Q(A_j)$ for each $j \geqq 1$, then every limit point of $\{\tilde{x}_j\}_{j=1}^{\infty}$ is contained in $Q(A_0)$.

Properties 1 through 4 appear basic to the problem and are not controversial. They agree with properties given by both J. W. Milnor [3] and H. Chernoff [2].

Property 5 has two motivations. One is the relevance of regret theorem established by H. Chernoff [2]. If $\lambda = 1$ and $c_j = -\min_{1 \leqq i \leqq m} u(d_i, s_j)$ for all $1 \leqq j \leqq n$ where $A_0 = (u(d_i, s_j))$, then

$$A_1 = A_0 + \begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ \vdots & & & \vdots \\ c_1 & c_2 & \cdots & c_n \end{bmatrix}$$

is the regret matrix for A_0. The other motivation is an invariance property of the utility function. If $c_j = \mu$ for all $1 \leqq j \leqq n$ and $A_0 = (u(d_i, s_j))$ we have $A_1 = (\lambda u(d_i, s_j) + \mu)$.

Property 6 includes the column duplication properties required by J. W. Milnor and H. Chernoff as a special case. The assignment, by Nature, of an *a priori* distribution $\tilde{\xi}$ to the states of Nature is analogous to the selection of a mixed strategy by the statistician. Nature has a corresponding pay off vector $(u(d_1, \tilde{\xi}), u(d_2, \tilde{\xi}), \cdots, u(d_m, \tilde{\xi}))^T$ where $u(d_i, \tilde{\xi}) = \sum_{j=1}^{n} \xi_j u(d_i, s_j)$. If, in problems A_1 and A_2, the sets of all such vectors are identical, then, under complete ignorance of $\tilde{\xi}$, the problem of selecting a strategy is essentially the same in A_1 and A_2.

Property 7 requires that the set of optimal loss vectors be the same for problems which have the same set of loss vectors corresponding to essential strategies. Thus Property 7 insures that the set of optimal loss vectors depend only on those loss vectors in $C(A)$, which uniquely minimize the expected loss for some $\tilde{\xi} \varepsilon \Xi$.

Property 8 is given by J. W. Milnor [3]. An example (given below in the proof of Theorem 1), showing the failure of the minimax regret criterion to satisfy Property 8, is evidence of the need for it.

It will be shown that two general classes of decision procedures contain no procedures which possess all of Properties 1 through 8.

Consider the class **D** of procedures, which select a non null subset $\Xi_1 \subset \Xi$, where Ξ_1 depends only on n, the number of states of Nature, and specify as optimal all loss vectors $\tilde{x} \varepsilon C(A)$ for which there is a $\tilde{\xi}_0 \varepsilon \Xi_1$ such $\tilde{\xi}_0 \cdot \tilde{x}^T \leqq \tilde{\xi}_1 \cdot \tilde{y}^T$ for all $\tilde{y} \varepsilon C(A)$.

The procedures in **D** which satisfy Property 2 are those which, for all $n \geqq 1$ select Ξ_1 so that if $\tilde{\xi} \varepsilon \Xi_1$ then $\tilde{\xi}' \varepsilon \Xi_1$ where $\tilde{\xi}$ consists of any permutation of the coordinates of $\tilde{\xi}$.

The procedures in **D** which satisfy Properties 3 and 4 are those which select Ξ_1 as a one point set. If $\tilde{\xi}_1, \tilde{\xi}_2 \varepsilon \Xi_1$ with $\tilde{\xi}_1 \neq \tilde{\xi}_2$ then we can find an A for which $C(A)$ has extreme points $\tilde{x}_1$, $\tilde{x}_2$, and $\tilde{x}_3$ such that, for $i = 1, 2$, $\tilde{\xi}_i \cdot \tilde{x}_i^T \leqq \tilde{\xi}_i \cdot \tilde{y}^T$ for all $\tilde{y} \varepsilon C(A)$ but the strategy corresponding to $\tilde{x}_3$ dominates the strategy corresponding to $\frac{1}{2}\tilde{x}_1 + \frac{1}{2}\tilde{x}_2$. Property 4 requires that $\frac{1}{2}\tilde{x}_1 + \frac{1}{2}\tilde{x}_2 \varepsilon Q(A)$ while Property 3 requires that $\frac{1}{2}\tilde{x}_1 + \frac{1}{2}\tilde{x}_2 \not\varepsilon Q(A)$.

Then the only member of **D** consistent with Properties 2, 3 and 4 is the procedure which selects $\Xi_1 = \{\tilde{\xi} = (1/n, \cdots, 1/n)\}$ for all $n \geqq 1$ (the procedure based on the Principle of Insufficient Reason), but this procedure is not consistent with Property 6.

Consider the class of decision procedures $\{\Delta_p : 1 \leqq p \leqq \infty\}$ defined below. Let $\tilde{v} = (v_1, v_2, \cdots, v_n)$ where $v_j = \min_{1 \leqq i \leqq m} u(d_i, s_j)$ (the v_j are the column minimums for the loss matrix A). For $1 \leqq p < \infty$, define

$$\|\tilde{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{p^{-1}},$$

and

$$\|\tilde{x}\|_\infty = \sup_{1 \leqq i \leqq n} |x_i|.$$

Denote by $B(A)$ the subset of $C(A)$ consisting of loss vectors corresponding to admissible strategies. Let Δ_p be the decision procedure which specifies $Q(A)$ as all $\tilde{x} \varepsilon B(A)$ for which $\|\tilde{v} - \tilde{x}\|_p \leqq \|\tilde{v} - \tilde{y}\|_p$ for all $\tilde{y} \varepsilon C(A)$. Note that Δ_1 coincides with the decision procedure based on the Principle of Insufficient Reason and Δ_∞ is the restriction of the minimax regret procedure to admissible strategies. Then, we have the following theorem.

THEOREM 1. *For* $1 \leqq p < \infty$, Δ_p *satisfies all properties except Property* 6, *and* Δ_∞ *satisfies all properties except Property* 8.

PROOF. Properties 2, 3, 5, and 7 are trivially satisfied. Property 1 is satisfied since $\|\tilde{v} - \tilde{x}\|_p$ is continuous on E_n and since $B(A)$ is compact, we have $\min_{\tilde{x}\varepsilon B(A)} \|\tilde{v} - \tilde{x}\|_p$ is attained in $B(A)$.

To see that $Q(A)$ is convex, note that the set $S_{p,\alpha} = \{\tilde{y}: \|\tilde{y} - \tilde{v}\|p \leqq \alpha\}$ is a closed, convex set in E_n and $Q(A)$ is given by $S_{p,\alpha^*} \cap C(A)$, where α^* is the least $\alpha \geqq 0$ such that $S_{p,\alpha} \cap C(A)$ is non-null. The conclusion follows on noting that the intersection of convex sets is convex. Hence Property 4 is satisfied.

To see that Property 6 is not satisfied for $1 \leqq p < \infty$, consider the games given by

$$A_1 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 2 \end{pmatrix}$$

Then, for any p, $1 < p < \infty$, it can easily be seen that $Q(A) = \{(3\lambda_1)2(1 - \lambda_1)\}$ where λ_1 minimizes $3^p\lambda + 2^p(1 - \lambda)^p$ on $[0, 1]$ and $Q(A_2) = \{(3\lambda_2), 2(1 - \lambda_2), 2(1 - \lambda_2)\}$, where λ_2 minimizes $3^p\lambda^p + 2^{p+1}(1 - \lambda)^p$ on $[0, 1]$. It then follows that $\lambda_1 \neq \lambda_2$ and $Q(A_1)$ can not be obtained from $Q(A_2)$ upon deleting the third coordinate. For $p = 1$, row 1 is the optimal loss vector in A_2 and row 1 is not the optimal loss vector in A_1. For $p = \infty$, Property 6 is easily seen to be satisfied by using well-known properties of the minimax regret procedure (D. Blackwell and M. A. Girshick [1]).

We now examine Property 8. For $1 \leqq p < \infty$ and $\lim_{j\to\infty} A_j = A_0$, let $\tilde{v}_j$ and $\tilde{v}_0$ be the vectors of column minimums for A_j and A_0 respectively. Then $\lim_{j\to\infty} \tilde{v}_j = \tilde{v}_0$. Now let $\{\tilde{x}_{j_m}\}$ be a convergent subsequence with $\tilde{x}_{j_m} \varepsilon Q(A_{j_m})$ for all j_m, and let $\tilde{x}_0 = \lim_{j_m\to\infty} \tilde{x}_{j_m}$. Then, $\tilde{x}_0$ must be in $C(A_0)$. Assume $\tilde{x}_0 \not\varepsilon Q(A_0)$. Then, there is a $\tilde{y} \varepsilon B(A_0) \subset C(A_0)$ and a $\delta > 0$, with $\|\tilde{v}_0 - \tilde{y}\|_p < \|\tilde{v}_0 - \tilde{x}_0\|_p - \delta$. In addition, there is a sequence $\{\tilde{y}_j\}$, $\tilde{y}_j \varepsilon C(A_j)$ with $\|\tilde{y} - \tilde{y}_j\|_p < \delta$ and $\|\tilde{v}_j - \tilde{v}_0\|_p < \delta$ for all $j > j^*(\delta)$. Then for $j_m > j^*(\delta)$, we have

$$\|\tilde{x}_{j_m} - \tilde{v}_{j_m}\|_p \leqq \|\tilde{y}_{j_m} - \tilde{v}_{j_m}\|_p < 2\delta + \|\tilde{y} - \tilde{v}_0\|_p < \|\tilde{v}_0 - \tilde{x}_0\|_p - \delta$$

which contradicts $\lim_{j_m\to\infty} \tilde{x}_{j_m} = \tilde{x}_0$.

For $p = \infty$, the following example will show that Property 8 does not hold. Let

$$A_j = \begin{pmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 + j^{-1} & 1 + j^{-1} \end{pmatrix}$$

and

$$A_0 = \begin{pmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Then $Q(A_j) = (1, 1, 1)$ and $Q(A_0) = (0, 1, 1)$.

Now let $v_0 = \min_{1 \leqq i \leqq n} v_i$ and define $\tilde{v}' = (v_0, v_0, \cdots, v_0)$. Further let Δ_p' be the decision procedure which specifies as optimal all $\tilde{x} \varepsilon B(A)$ for which $\|\tilde{v}' - \tilde{x}\|_p \leqq \|\tilde{v}' - \tilde{y}\|_p$ for all $\tilde{y} \varepsilon C(A)$. Then for all p, $1 \leqq p < \infty$ the procedure Δ_p' is not consistent with either Property 5 or Property 6. In addition Δ_∞' is the restriction of the minimax procedure to admissible strategies and violates Properties 5 and 8. The proof is similar to that of Theorem 1 and will be omitted.

J. W. Milnor [3] initially gives a list of axioms for decision procedures, which he proves are inconsistent. Milnor then gives an alernate list of less restrictive axioms, which can be shown to be consistent. The requirements of Milnor's alternate list are included in the list of properties given here.

H. Chernoff [2] gives a list of postulates for decision procedures, which he shows to be inconsistent. He does not list a requirement similar to Property 8, but his Postulate 4 excludes the minimax regret procedure. H. Chernoff's Postulate 4 is stated below as Property 9.

Property 9. If A_2 can be obtained from A_1 by deleting rows from A_1, then

$$Q(A_1) \cap C(A_2) \subset Q(A_2).$$

Property 9 says that a non-optimal strategy in A_2 can not be optimal in A_1. It is our feeling that if the rows deleted change the set of the loss vectors corresponding to essential strategies, Property 9 should not be required. In particular, the inclusion of Property 9 would make the set of properties given here inconsistent.

J. W. Milnor [3] proves his alternate list of axioms are consistent by giving a family of decision procedures, each member of which satisfies his axioms. Modifying Milnor's procedures, we can produce a similar family, each member of which possesses Properties 1 through 8.

Consider the following procedure. Let $\{\epsilon_j\}_{j=1}^{\infty}$ be a monotone non-increasing positive sequence converging to zero. Let $d(\tilde{x}, \tilde{y}) = \max_{1 \leqq i \leqq n} |x_i - y_i|$. Then for the $m \times n$ loss matrix A denote the convex hull of A, $C(A)$ by Q_1. Further, we denote by $\tilde{v}_1$ the vector $(v_1^{(1)}, v_2^{(1)}, \cdots, v_n^{(1)})$, whose coordinates are defined by $v_i^{(1)} = \min_{\tilde{x} \varepsilon Q_1} x_i$. Let $z_1 = \min_{\tilde{x} \varepsilon Q_1} d(\tilde{v}_1, \tilde{x})$. Proceeding inductively, for all $h \geqq 1$, let $Q_{h+1} = \{\tilde{x} \varepsilon Q_h : d(\tilde{v}_h, \tilde{x}) \leqq z_h + \epsilon_h z_1\}$, where $\tilde{v}_h = (v_1^{(h)}, v_2^{(h)}, \cdots, v_n^{(h)})$ and $v_i^{(h)} = \min_{\tilde{x} \varepsilon Q_h} x_i$, and $z_h = \min_{\tilde{x} \varepsilon Q_h} d(\tilde{v}_h, \tilde{x})$.

The sets Q_h are closed convex subsets of $C(A)$ for every h and $Q_h \supset Q_{h+1}$ for all $h \geqq 1$. Let $Q(A) = \inf_h Q_h$.

Lemma 1. *$Q(A)$ satisfies Properties* 1 *and* 4.

Proof. By Cantor's theorem $Q(A)$ is a closed non-null subset of $C(A)$. Futher, $Q(A)$ is convex, since Q_h is clearly convex for every h.

LEMMA 2. *The set* $Q(A)$ *consists of a single point* $\tilde{s}$.

PROOF. Let $\tilde{v} = (v_1, v_2, \cdots, v_n)$ where $v_i = \min_{\tilde{x}\varepsilon Q(A)} x_i$. Then, it is easily seen that $\lim_{h\to\infty} v_i^{(h)} = v_i^{(0)} \leqq v_i$. Let $K_i = \{\tilde{x}\,\varepsilon\, E_n : x_i = v_i^{(0)}\}$. Then $K_i \cap Q(A) = \bigcap_{h=1}^{\infty} (K_i \cap Q_h)$ is non-empty and there is an $\tilde{x}\,\varepsilon\, Q(A)$ with $x_i = v_i^{(0)}$ and $v_i \leqq v_i^{(0)}$. Hence, $\lim_{h\to\infty} \tilde{v}_h = \tilde{v}$. Define $z = \min_{x\varepsilon Q(h)} d(\tilde{v}, \tilde{x})$. Then, for any $\epsilon > 0$ and all sufficiently large h, and every $\tilde{x}\,\varepsilon\, Q(A)$, $z \leqq d(\tilde{v}, \tilde{x}) \leqq d(\tilde{v}, \tilde{v}_h) + d(\tilde{v}, \tilde{x}) < \epsilon + z_h + z_1\epsilon_h$. But,

$$z_h = \min_{\tilde{x}\varepsilon Q_h} d(\tilde{v}_h, \tilde{x}) \leqq \min_{\tilde{x}\varepsilon Q(A)} d(\tilde{v}_h, \tilde{x}) \leqq \min_{\tilde{x}\varepsilon Q(A)} [d(\tilde{v}_h, \tilde{v}) + d(\tilde{v}, \tilde{x})] < \epsilon + \min_{\tilde{x}\varepsilon Q(A)} d(\tilde{v}, \tilde{x}) = z + \epsilon.$$

But $\epsilon_h \to 0$ and hence $d(\tilde{v}, \tilde{x}) = z$ for all $\tilde{x}\,\varepsilon\, Q(A)$.

Thus, $Q(A)$ lies on the surface of a sphere (hypercube) of radius z with center at $\tilde{v}$. However, since $Q(A)$ is convex, $Q(A)$ must lie entirely on one face of the hypercube. Hence, we have that for all $\tilde{x}\,\varepsilon\, Q(A)$, $x_i = v_i + z$ for some i, $1 \leqq i \leqq n$; which implies that $z = 0$, since $v_i = \min_{\tilde{x}\varepsilon Q(A)} x_i = v_i + z$. Therefore, $Q(A)$ is a single point, $\tilde{s}$, since for all $\tilde{x}\,\varepsilon\, Q(A)$, $d(\tilde{v}, \tilde{x}) = 0$.

LEMMA 3. $\tilde{s}$ *is attainable only by admissible strategies. Thus Property* 3 *is satisfied.*

PROOF. Assume the contrary. Then there exists an $\tilde{x}'\,\varepsilon\, C(A)$ with $x_i' \leqq s_i$ for all $1 \leqq i \leqq n$, and $x_j' < s_j$ for at least one j, $1 \leqq j \leqq n$, and $\tilde{x} \not\varepsilon\, Q(A)$. Since $\{Q_h\}$ is a non-increasing sequence of sets, there is an $h \geqq 1$ such that $\tilde{x}\,\varepsilon\, Q_h$, and $\tilde{x}' \not\varepsilon\, Q_{h+1}$. Also, we clearly have $v_i^{(h)} \leqq x_i$ for all $1 \leqq i \leqq n$. Since $\tilde{x}' \not\varepsilon\, Q_{h+1}$, $d(\tilde{v}_h, \tilde{x}') > z_h + z_1\epsilon_h$. This implies $d(\tilde{v}_h, s) > z_h + z_1\epsilon_h$ and $\tilde{s} \not\varepsilon\, Q_{h+1}$, contradicting $\tilde{s}\,\varepsilon\, Q(A)$. Thus we have shown that no strategy attaining the loss vector $\tilde{s}$ can be dominated.

Some auxiliary results will be used to show that Property 8 is satisfied. Since these results are general properties of convex polyhedra, they are sketched in the appendix as Theorems A and B. Hence, we now establish:

LEMMA 4. *If* $\{A_k\}_{k=1}^{\infty}$ *converges to* A, *then,* $\{\tilde{s}_k\}_{k=1}^{\infty}$ *converges* to $\tilde{s}$, *and hence Property 8 is satisfied.*

PROOF. For any set R, let $d(\tilde{x}, R) = \inf_{y\varepsilon R} d(\tilde{x}, \tilde{y})$. Let $\tilde{\gamma} = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ and define $T(\tilde{\gamma}) = \{\tilde{x}: x_i \leqq \gamma_i, i = 1, 2, \cdots, n\}$. Since $\{A_k\}_{k=1}^{\infty}$ converges to A, for any $\delta > 0$, there is a $k(\delta)$ sufficiently large such that $\max_{i,j} |a_{ij}^{(k)} - a_{ij}| < \delta$ for all $k \geqq k(\delta)$. Let $Q_1^{(k)} = C(A_k)$ and $Q_1 = C(A)$. Further, let

$$\tilde{v}_1^{(k)} = (v_{11}^{(k)}, v_{12}^{(k)}, \cdots, v_{1n}^{(k)}),$$

where $v_{1i}^{(k)} = \min_{\tilde{x}\varepsilon Q_i^{(k)}} \tilde{x}_i$, and let $\tilde{v}_1 = (v_{11}, v_{12}, \cdots, v_{1n})$, where $v_{1i} = \min_{\tilde{x}\varepsilon Q_1} \tilde{x}_i$. Let $z_1^{(k)} = d(\tilde{v}_1^{(k)}, Q_1^{(k)})$ and $z_1 = d(\tilde{v}_1, Q_1)$. Let $\tilde{u}$ be the vector $(1, 1, \cdots, 1)$. Then for $p \geqq 1$, define $Q_{p+1}^{(k)} = \{Q_p^{(k)} \cap T(\tilde{v}_p^{(k)} + (z_p^{(k)} + \epsilon_p z_1^{(k)})\tilde{u})\}$, where $\tilde{v}_p^{(k)} = (v_{p1}^{(k)}, v_{p2}^{(k)}, \cdots, v_{pn}^{(k)})$ and $v_{pi}^{(k)} = \min_{\tilde{x}\varepsilon Q_p^{(k)}} x_i$, $z_p^{(k)} = d(v_p^{(k)}, Q_p^{(k)})$; and let $Q_{p+1} = \{Q_p \cap T(\tilde{v}_p + (z_p + \epsilon_p z_1)\,\tilde{u}\}$, where $\tilde{v}_p = (v_{p1}, v_{p2}, \cdots, v_{pn})$ and $v_{pi} = \min_{\tilde{x}\varepsilon Q_p} x_i$, $z_p = d(\tilde{v}_p, Q_p)$. From Theorem B of the appendix, we have; for any closed, bounded convex set C, with a finite number of extreme points, and any $\tilde{\eta} = (\eta_1, \eta_2, \cdots, \eta_n)$, $\eta_i \geqq 0$, and any $\tilde{\gamma}$ with $T(\tilde{\gamma}) \cap C \neq \theta$, then for any

$\tilde{w} \varepsilon T(\tilde{\gamma} + \tilde{\eta}) \cap C$, there is a number $q(C)$ (depending only on C) and a $\tilde{t} \varepsilon T(\tilde{\gamma}) \cap C$ with $d(\tilde{t}, \tilde{w}) < q(C)[\max_i \eta_i]$. Let $q = q(C(A))$ and let $q' = 2 + 2q(2 + \epsilon_1)$. In addition, let $\delta_0 = 1$ and let δ_h be a sequence of real numbers satisfying

$$0 < \delta_h < \min\left[\frac{z_1 \epsilon_h}{2(q')^{h-1} 2(2 + \epsilon_1)}, \delta_{h-1}\right],$$

and let $k_h = k(\delta_h)$. Next, we define

$$Q'_{p+1,h} = \{Q_p \cap T(\tilde{v}_p + (z_p + \epsilon_p z_1 - 2(2 + \epsilon_p)\delta_h (q')^{p-1})\tilde{u})\}.$$

Assume $z_1 > 0$. Then we first show that for $k \geqq k_h$, $h \geqq p$,

$$\max\{\max_{x \varepsilon Q_p^{(k)}} d(\tilde{x}, Q_p), \max_{\tilde{x} \varepsilon Q_p} d(\tilde{x}, Q_p^{(k)})\} < (q')^{p-1} \delta_h. \tag{1}$$

Since, for $k \geqq k_h$, $\max_{i,j} |a_{ij}^{(k)} - a_{ij}| < \delta_h$, we have, trivially,

$$\max\{\max_{\tilde{x} \varepsilon Q_1^{(k)}} d(\tilde{x}, Q_1), \max_{\tilde{x} \varepsilon Q_1} d(\tilde{x}, Q_1^{(k)})\} < \delta_h. \tag{2}$$

Further, since $|v_{1i}^{(k)} - v_{1i}| < \delta_h$, $i = 1, 2, \cdots, n$, we have

$$d(\tilde{v}_1^{(k)}, v_1) < \delta_h. \tag{3}$$

In addition, since there is an $\tilde{x} \varepsilon Q_1^{(k)}$ with $d(\tilde{v}_1^{(k)}, \tilde{x}) = z_1^{(k)}$ and an $\tilde{x}' \varepsilon Q_1$ with $d(\tilde{x}, \tilde{x}') < \delta_h$, we have:

$$d(\tilde{v}_1, \tilde{x}') < d(\tilde{v}_1, \tilde{v}_1^{(k)}) + d(\tilde{v}_1^{(k)}, \tilde{x}) + d(\tilde{x}, \tilde{x}') < z_1^{(k)} + 2\delta_h$$

and therefore:

$$d(\tilde{v}_1, Q_1) = z_1 < z_1^{(k)} + 2\delta_h. \tag{4}$$

By symmetry,

$$z_1^{(k)} < z_1 + 2\delta_h. \tag{5}$$

Hence by (4) and (5), we have

$$|z_1 - z_1^{(k)}| < 2\delta_h. \tag{6}$$

In particular, setting $p = 1$, we have verified (1). Then, assume (1) holds for $p = 1, 2, \cdots, m - 1$. Then, trivially, we have for $h \geqq m - 1$, and $k \geqq k_h$,

$$d(\tilde{v}_{m-1}^{(k)}, \tilde{v}_{m-1}) < (q')^{m-2} \delta_h, \tag{7}$$

and similarly to (6), we can conclude,

$$|z_{m-1} - z_{m-1}^{(k)}| < 2(q')^{m-2} \delta_h. \tag{8}$$

Let $\tilde{y}$ be any element of $Q_m^{(k)}$. Then, from (8) and (6),

$$d(\tilde{v}_{m-1}^{(k)}, \tilde{y}) \leqq z_{m-1}^{(k)} + \epsilon_{m-1} z_1^{(k)} < z_{m-1} + 2(q')^{m-2} \delta_h + \epsilon_{m-1}(z_1 + 2\delta_h)$$

and

$$d(\tilde{v}_{m-1}^{(k)}, \tilde{y}) < z_{m-1} + z_1 \epsilon_{m-1} + 2(q')^{m-2} \delta_h (1 + \epsilon_{m-1}) \tag{9}$$

Since $Q_m^{(k)} \subset Q_{m-1}^{(k)}$, there is an $\tilde{x}' \varepsilon Q_{m-1}$ with $d(\tilde{y}, \tilde{x}') < (q')^{m-2}\delta_h$ and from (7) and (9),

$$d(\tilde{v}_{m-1}, \tilde{x}') < z_{m-1} + z_1\epsilon_{m-1} + 2(q')^{m-2}\delta_h(2 + \epsilon_{m-1})$$

and hence $\tilde{x}' \varepsilon T(\tilde{v}_{m-1} + [z_1\epsilon_{m-1} + 2(q')^{m-2}\delta_h(2 + \epsilon_{m-1})]\tilde{u}) \cap Q_{m-1}$. Then, by Theorem B, there is a $\tilde{y}' \varepsilon Q_m$ with

$$d(\tilde{x}', \tilde{y}') < q[2(q')^{m-2}\delta_h(2 + \epsilon_{m-1})]$$

and

$$d(\tilde{y}, \tilde{y}') < (q')^{m-2}\delta_h[1 + q(2 + \epsilon_{m-1})].$$

Hence

$$\max_{\tilde{y}\varepsilon Q_m^{(k)}} d(\tilde{y}, Q_m) < (q')^{m-2}\delta_h[1 + q(2 + \epsilon_{m-1})]$$

and

$$\max_{\tilde{y}\varepsilon Q_m^{(k)}} d(\tilde{y}, Q_m) < (q')^{m-1}\delta_h \tag{10}$$

for all $h \geqq m - 1, k \geqq k_h$.

Since for $h \geqq m - 1$,

$$\alpha_{m,h} = z_{m-1} + \epsilon_{m-1} z_1 - 2(2 + \epsilon_{m-1})\delta_h(q')^{m-2} > z_{m-1} + \epsilon_{m-1} z_1 - \frac{2(2 + \epsilon_{m-1})z_1\epsilon_h(q')^{m-2}}{2(q')^{h-1}2(2 + \epsilon_1)}$$

Inasmuch as $(2 + \epsilon_{m-1})/(2 + \epsilon_1) < 1$ and $q' > 1$, we have

$$\alpha_{m,h} > z_{m-1} + \epsilon_{m-1}z_1 - \tfrac{1}{2}(z_1\epsilon_h) > z_{m-1}$$

and therefore $Q'_{m,h}$ is non-null for $h \geqq m - 1$.

Now let $\tilde{x}$ be any element of Q_m. Applying Theorem B, there is an $\tilde{x}' \varepsilon Q'_{m,h}$, $h \geqq m - 1$ with

$$d(\tilde{x}, \tilde{x}') < 2q(2 + \epsilon_{m-1})\delta_h(q')^{m-2}. \tag{11}$$

In addition, since $\tilde{x}' \varepsilon Q'_{m,h} \subset Q_m \subset Q_{m-1}$, from (1) there is a $\tilde{y} \varepsilon Q_{m-1}^{(k)}$ with

$$d(\tilde{x}', \tilde{y}) < (q')^{m-2}\delta_h \tag{12}$$

and hence, by (11) and (12), we note that

$$d(\tilde{x}, \tilde{y}) < (q')^{m-2}\delta_h[1 + 2q(2 + \epsilon_{m-1})]. \tag{13}$$

However, then $\tilde{y} \varepsilon Q_m^{(k)}$ since

$$\begin{aligned} d(\tilde{v}_{m-1}^{(k)}, \tilde{y}) &\leqq d(\tilde{v}_{m-1}^{(k)}, \tilde{v}_{m-1}) + d(\tilde{v}_{m-1}, \tilde{x}') + d(\tilde{x}', \tilde{y}) \\ &< z_{m-1} + \epsilon_{m-1}z_1 - 2(q')^{m-2}\delta_h[1 + \epsilon_{m-1}] \\ &< z_{m-1}^{(k)} + \epsilon_{m-1}z_1^{(k)} - 2\delta_h\epsilon_{m-1}((q')^{m-2} - 1) \\ &< z_{m-1}^{(k)} + \epsilon_{m-1}z_1^{(k)}. \end{aligned}$$

Therefore, from (13),

$$\max_{\tilde{x}\varepsilon Q_m} d(\tilde{x}, Q_m^{(k)}) < (q')^{m-1}\delta_h \tag{14}$$

for $h \geqq m - 1$, and $k \geqq k_h$. Combining (10) and (14), we have established (1).

Then, since $\inf_p Q_p = \{\tilde{s}_0\}$, for any $\delta > 0$, there is an integer $p(\delta)$ such that the set $\{\tilde{x}: \|\tilde{x} - \tilde{s}_0\| < \delta/2\} = S(\tilde{s}_0, \delta/2)$ contains Q_p for all $p \geqq p(\delta)$.

In addition, there is a $p(\delta_h)$ with $(q')^{p-1}\delta_h < \delta/2$ for $p \geqq p(\delta_h)$ and $h \geqq p$, so that for $p \geqq \max[p(\delta), p(\delta_h)]$, $\tilde{s}_k \varepsilon S(\tilde{s}_0, \delta)$ for $k \geqq k(h)$ and therefore $d(\tilde{s}_0, \tilde{s}_h) < \delta$, establishing Lemma 4, whenever $z_1 > 0$.

If $z_1 = 0$, then $\tilde{s}_0 = \tilde{v}_1 \varepsilon Q_p$ for all $p \geqq 1$, and $\{\tilde{s}_0\} = Q_p$ for $p > 1$. Then, for any $\delta > 0$, there is a $k(\delta)$ such that for $k \geqq k(\delta)$, $\max_{ij}|a_{ij}^{(k)} - a_{ij}| < \delta$, $d(\tilde{s}_0, \tilde{v}_1^{(k)}) < \delta$, and $z_1^{(k)} < 2\delta$. In particular, $z_1^{(k)} + \epsilon_1 z_1^{(k)} < 2\delta(1 + \epsilon_1)$. Thus for $\tilde{y} \varepsilon Q_2^{(k)}$, $d(\tilde{s}_0, \tilde{y}) < d(\tilde{s}_0, \tilde{v}_1^{(k)}) + 2\delta(1 + \epsilon_1) = \delta + 2\delta(1 + \epsilon_1)$. Since $\tilde{s}_k \varepsilon Q_2^{(k)}$, the conclusion follows.

LEMMA 5. *Properties* 2, 5, *and* 7 *are satisfied.*

PROOF. The proof is trivial for Properties 2 and 5. Property 7 is easily verified by noting that $\tilde{v}_p$ and z_p depend only on loss vectors corresponding to admissible strategies.

LEMMA 6. *Property* 6 *is satisfied.*

PROOF. Let A_1 and A_2 be $m \times n - 1$ and $m \times n$ matrices respectively with $C(A_1^T) = C(A_2^T)$ and let the columns of A_1 be $\tilde{s}_1, \tilde{s}_2, \cdots, \tilde{s}_{n-1}$ and the columns of A_2 be $\tilde{s}_1, \tilde{s}_2, \cdots, \tilde{s}_n$. Then there exist constants $\lambda_k \geqq 0$, $\sum_{k=1}^{n-1} \lambda_k = 1$ such that for $x \varepsilon Q_1(A_2)$, $x_n = \sum_{k=1}^{n-1} \lambda_k x_k$. Then:

$$\begin{aligned} x_n - v_n &= \sum_{k=1}^{n-1} \lambda_k x_k - \min_{\tilde{y}\varepsilon Q_1(A_2)} y_n \\ &= \sum_{k=1}^{n-1} \lambda_k x_k - \min_{\tilde{y}\varepsilon Q_1(A_2)} \sum_{k=1}^{n-1} \lambda_k y_k \\ &\leqq \sum_{k=1}^{n-1} \lambda_k (x_k - \min_{\tilde{y}\varepsilon Q_1(A)} y_k) = \sum_{k=1}^{n-1} \lambda_k (x_k - v_k) \end{aligned}$$

Hence $x_n - v_n \leqq x_k - v_k$ for at least one $k < n$, and hence $d(\tilde{v}, \tilde{x}) = d(\tilde{v}', \tilde{x}')$' the primes indicating the omission of the nth column. Thus the decision procedure which depends only on d, is not affected.

Hence, we have established the following theorem.

THEOREM 2. *For any non-increasing positive sequence* $\{\epsilon_p\}_{p=1}^{\infty}$, *with* $\epsilon_p \to 0$, *there exists a decision procedure which satisfies Properties* 1 *through* 8.

APPENDIX—SOME PROPERTIES OF CONVEX POLYHEDRA

Let C be a closed, bounded convex set in E_n with a finite number of extreme points, let $\|\tilde{x}\|$ be the Euclidean norm of $\tilde{x}$, and let $\rho(\tilde{x}, \tilde{y})$ be the Euclidean distance between $\tilde{x}$ and $\tilde{y}$.

DEFINITION. A vector $\tilde{d}$ is said to leave C at the point $\tilde{v} \varepsilon C$; if for all $\eta > 0$, $\tilde{v} + \eta d \not\varepsilon C$. Then, letting $\rho(\tilde{x}, C) = \min_{\tilde{y}\varepsilon C} \rho(\tilde{x}, \tilde{y})$ we have the following theorem.

THEOREM A. *If $\tilde{d}$ leaves C at the point $\tilde{v}$, where $\|\tilde{d}\| = 1$, and*

$$f(\eta) = f(\eta, \tilde{v}, \tilde{d}) = \frac{\rho(\tilde{v} + \eta\tilde{d}, C)}{\eta} \quad \eta > 0$$

then

$$f(0) = \lim_{\eta\to 0+} f(\eta) > 0.$$

PROOF. Let $\tilde{p} \varepsilon C$ be the vector uniquely specified by

$$\rho(\tilde{v} + \eta\tilde{d}, C) = \rho(\tilde{v} + \eta\tilde{d}, \tilde{v} + \tilde{p}) \tag{A.1}$$

Then, for $0 \leqq \eta' < \eta$

$$0 \leqq \rho(\tilde{v} + \eta' d, C) \leqq (\eta'/\eta)\rho(\tilde{v} + \eta\tilde{d}, C)$$

and, hence it follows that $f(\eta)$ satisfies

(i) $0 \leqq f(\eta) \leqq 1$
(ii) $\lim_{\eta\to\infty} f(\eta) = 1$
(iii) $f(\eta)$ is continuous in η
(iv) $f(\eta)$ is monotone non-decreasing.

Then, write

$$\tilde{v} + \tilde{p}(\eta) = \tilde{y}(\eta) = \sum_{i\varepsilon K(\eta)} \lambda_i(\eta)\tilde{u}^{(i)}$$

where the $\tilde{u}^{(i)}$ are extreme points of C and $0 < \lambda_i \leqq 1$, $\sum_{i\varepsilon K(\eta)}\lambda_i = 1$. If for sufficiently small η, $K(\eta)$ is a one-element set, it follows that $\tilde{v}$ is an extreme point and $f(0) = f(\eta) = 1$. If $K(\eta)$ contains more than one element, it can be shown that for $0 < \eta < \eta^*$, $f(\eta)$ is constant, and positive, whenever $\tilde{d}$ leaves C at $\tilde{v} \varepsilon C$. Essentially we have shown that

$$0 < f(0) = \min_{\gamma_i} \|\tilde{d} - \sum_{i\varepsilon K} \gamma_i u^{(i)}\|, \ \sum_{i\varepsilon K} \gamma_i = 0 \tag{A.2}$$

and the set K is determined by $\tilde{v}$. Since there are a finite number of extreme points, there are only a finite number of possible sets K, and hence the minima (A.2) will have a positive lower bound independent of $\tilde{v}$.

For any vector $\tilde{z} = (z_1, \cdots, z_n)$, define $T_{\tilde{z}} = \{\tilde{x}: x_i \leqq z_i\}$. We now establish the following theorem.

THEOREM B. *For any $\tilde{\epsilon} = (\epsilon_1, \cdots, \epsilon_n)$ with $\epsilon_i \geqq 0$, $i = 1, 2, \cdots, n$ and any z such that $T_{\tilde{z}} \cap C \neq \theta$, and any $\tilde{w} \varepsilon T_{\tilde{z}+\tilde{\epsilon}} \cap C$, there is a $\tilde{v} \varepsilon T_{\tilde{z}} \cap C$ and a constant q depending only on C, such that $\|\tilde{w} - \tilde{v}\| \leqq q\|\tilde{\epsilon}\|$.*

PROOF. Assume the contrary, then there exist sequences $\tilde{z}^{(m)}$, $\tilde{\epsilon}^{(m)}$, $\tilde{w}^{(m)}$, $\tilde{v}^{(m)}$, such that

$$\|\tilde{w}^{(m)} - \tilde{v}^{(m)}\| \geqq m\|\tilde{\epsilon}^{(m)}\| > 0, \qquad m = 2, 3, \cdots \tag{A.3}$$

where $\tilde{v}^{(m)}$ is the unique best approximant to $\tilde{w}^{(m)}$ from the convex set $T_{\tilde{z}^{(m)}} \cap C$. Further the coordinates of $\tilde{w}^{(m)}$ and $\tilde{v}^{(m)}$ satisfy

$$w_i^{(m)} \leqq z_i^{(m)} + \epsilon_i^{(m)} \tag{A.4}$$

$$v_i^{(m)} \leqq z_i^{(m)} \tag{A.5}$$

We classify the values of i, $i = 1, 2, \cdots, n$, according to the following inequalities.

$$v_i^{(m)} = z_i^{(m)} < w_i^{(m)} \tag{A.6}$$

$$v_i^{(m)} = w_i^{(m)} \tag{A.7}$$

$$v_i^{(m)} < w_i^{(m)},\ v_i^{(m)} < z_i^{(m)} \tag{A.8}$$

$$w_i^{(m)} < v_i^{(m)} \tag{A.9}$$

and let I denote the set of indices i for which (A.6) or (A.7) hold and J the set for which (A.8) or (A.9) hold and note that neither I nor J can be empty.

Since there are only a finite number of partitions of the integers $\{1, 2, \cdots, n\}$ into the sets satisfying (A.6)–(A.9), we can choose a subsequence for which the partition is the same for all m. Hence, we shall suppose that the sequences satisfy this condition. Then, let

$$\tilde{d}^{(m)} = \frac{(\tilde{w}^{(m)} - \tilde{v}^{(m)})}{\|\tilde{w}^{(m)} - \tilde{v}^{(m)}\|}. \tag{A.10}$$

Clearly $\|\tilde{d}^{(m)}\| = 1$, and we can choose a convergent subsequence such that $\lim_{\alpha\to\infty}\tilde{d}^{(m_\alpha)} = \tilde{d}$, where $\|\tilde{d}\| = 1$.

Then, if $i \varepsilon I$, it follows that $d_i = 0$ and hence $d_i \neq 0$ for some $i \varepsilon J$. Hence, we have that $d_i \geqq 0$ if (A.8) holds and $d_i \leqq 0$ if (A.9) holds, with strict inequality for at least one i. Since $\tilde{v}^{(m_\alpha)}$ and $\tilde{w}^{(m_\alpha)} \varepsilon C$, there is an $\eta' > 0$ with $\tilde{v}^{(m_\alpha)} + \eta'\tilde{d}^{(m_\alpha)}$ in C. Thus, it can be shown that there exists an $\eta'' > 0$ with $\tilde{v}^{(m_\alpha)} + \eta''\tilde{d}$ closer to $\tilde{w}^{(m_\alpha)}$ than $\tilde{v}^{(m_\alpha)}$ and hence, $\tilde{v}^{(m_\alpha)} + \eta'' d$ is not in C. Hence, $\tilde{d}$ leaves C at the point $\tilde{v}^{(m_\alpha)}$, but is approximated at these points by the vectors $\tilde{d}^{(m_\alpha)}$ which do not leave C at $\tilde{v}^{(m_\alpha)}$. Then, from Theorem A, for sufficiently small $\eta > 0$,

$$f(\eta, \tilde{v}^{(m_\alpha)}, \tilde{d}) \leqq \frac{\|\tilde{d} - \tilde{d}^{(m_\alpha)}\|}{\eta} \to 0 \quad \text{as} \quad \alpha \to \infty,$$

which contradicts the conclusion of Theorem A, and establishes the existence of the constant q.

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