THE DISTRIBUTION OF HOTELLING'S GENERALISED T_0^2

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1. Introduction and summary. The T_0^2-statistic was introduced by Hotelling [5], [6] as a measure of multivariate dispersion in connection with the problem of testing the accuracy of bombsights. A similar statistic was considered earlier by Lawley [10] as a generalisation of the F-test for testing significance in multivariate analyses of variance. In general, if the $m \times m$ matrices S_1 and S_2 are independently distributed on n_1 and n_2 degrees of freedom respectively, estimating the same covariance matrix, then T_0^2 is defined by

$$T = T_0^2/n_2 = \operatorname{tr} S_1 S_2^{-1}.$$

For instance, in a one-way classification analysis of variance S_1 and S_2 may be the "between classes" and "within classes" matrices of sums of squares and products. The matrix S_1 may be singular, i.e., $n_1 < m$, but S_2 is assumed non-singular. Assuming that one is sampling from a normal population, S_2 has the Wishart distribution and S_1 the (possibly) non-central Wishart distribution if $n_1 \geqq m$. If $n_1 < m$, the distribution of T may be obtained from that for $n_1 \geqq m$ by a simple substitution (see Section 4).

Hotelling derived the distribution of T when $n_1 = 1$ (in which case T is the generalisation of "Student's" t [4]) and for $m = 2$, [6]. In Section 4, the distribution of T will be derived for arbitrary m, n_1 and n_2 in the non-central case. More precisely, if Ω is the matrix of non-centrality parameters, the probability density function of T has the series expansion

$$(1)\quad [\Gamma_m(\tfrac{1}{2}(n_1+n_2))/\Gamma(\tfrac{1}{2}mn_1)\Gamma_m(\tfrac{1}{2}n_2)]e^{\operatorname{tr}(-\Omega)}T^{\frac{1}{2}mn_1-1} \cdot \sum_{k=0}^{\infty}[(-T)^k/(\tfrac{1}{2}mn_1)_k k!]\sum_{\kappa}(\tfrac{1}{2}(n_1+n_2))_\kappa L_\kappa^\gamma(\Omega),$$

$$\gamma = \tfrac{1}{2}(n_1 - m - 1), \qquad |T| < 1.$$

The functions $L_\kappa^\gamma(\Omega)$ are polynomials in the elements of Ω and are extensions of the classical Laguerre polynomials, to which they reduce when $m = 1$. They will be defined and studied in Section 3. The constants and coefficients occurring in the series are defined in Section 2. If $\Omega = 0$, the density function is

$$(2)\quad [\Gamma_m(\tfrac{1}{2}(n_1+n_2))/\Gamma(\tfrac{1}{2}mn_1)\Gamma_m(\tfrac{1}{2}n_2)]T^{\frac{1}{2}mn_1-1} \cdot \sum_{k=0}^{\infty}[(-T)^k/(\tfrac{1}{2}mn_1)_k k!]\sum_{\kappa}(\tfrac{1}{2}n_1)_\kappa(\tfrac{1}{2}(n_1+n_2))_\kappa C_\kappa(I),$$

where $C_\kappa(I)$ is the *zonal polynomial* evaluated at the identity matrix (see Section 2).

Both series (1) and (2) converge only for $|T| < 1$. In the case $m = 1$, the series in (2) reduces to the binomial series for $(1+T)^{-\frac{1}{2}(n_1+n_2)}$. Unfortunately,

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this is not very useful since one is usually interested in the upper tail of the distribution. However, it is hoped that the series may be simplified or be the basis for further studies of the distributions.

In Section 5, the moments of T are given, again in terms of the generalised Laguerre polynomials in the non-central case. As in the case of the F-distribution, only moments of sufficiently low order exist.

2. Notation and preliminary results. In the following sections, use will be made of zonal polynomials and hypergeometric functions of matrix argument. For convenience, those results required here are listed below. Definitions, proofs, etc., may be found in the papers [7], [8] and [9] of A. T. James and [1] of Constantine.

Let S be a complex symmetric $m \times m$ matrix. Corresponding to each partition $\kappa = (k_1, k_2, \cdots, k_m)$ of the integer k into not more than m parts, there is a *zonal polynomial* $C_\kappa(S)$, see [7], [8]. $C_\kappa(S)$ is a homogeneous symmetric polynomial in the elements of S, and hence in the eigenvalues of S. An explicit formula for C_κ is not known, but they can be relatively easily calculated and have been tabulated by James [9] up to order 6. The value of the zonal polynomial at the identity is known [1] to be

$$(3) \qquad C_\kappa(I_m) = [2^{2k}k! \textstyle\prod_{i<j}(p_i - p_j)/p_1!\, p_2! \cdots p_m!](\frac{1}{2}m)_\kappa,$$

where $p_i = 2k_i + m - i$, and

$$(4) \quad (a)_\kappa = \textstyle\prod_{i=1}^m (a - \frac{1}{2}(i-1))_{k_i}, \qquad (a)_n = a(a+1)\cdots(a+n-1).$$

The generalised "binomial" type coefficient $(a)_\kappa$ will occur frequently in the sequel. The fundamental property of the zonal polynomials, indeed almost the defining relation, is the average over the orthogonal group, $O(m)$, given by

$$(5) \qquad \int_{O(m)} C_\kappa(AH'BH)\, d(H) = C_\kappa(A)C_\kappa(B)/C_\kappa(I),$$

where the measure $d(H)$ on $O(m)$ is normalised to make the volume of $O(m)$ unity. (5) was proved by James [7].

For functions of matrix argument, there is the *Laplace transform* $g(Z) = \int_{R>0} e^{\mathrm{tr}-RZ} f(R)\, dR$, the integral being taken over the space of positive definite matrices, and the corresponding inverse transform

$$f(R) = [2^{\frac{1}{2}m(m-1)}/(2\pi i)^{\frac{1}{2}m(m+1)}] \int_{R(Z)>X_0>0} e^{\mathrm{tr}RZ} g(Z)\, dZ,$$

where $Z = X + iY$, $R(Z) = X$ is fixed and $> K_0$ for some $X_0 > 0$, and the integration is over all real symmetric matrices Y. For a discussion of the Laplace transform including conditions on $f(R)$ and $g(Z)$ for the integrals to converge absolutely, the reader is referred to Herz [3].

The Laplace transform of the zonal polynomial is [1]

$$(6) \quad \int_{R>0} e^{\mathrm{tr}-RZ}(\det R)^{a-\frac{1}{2}(m+1)} C_\kappa(R)\, dR = \Gamma_m(a, \kappa)(\det Z)^{-a} C_\kappa(Z^{-1}),$$

where

$$(7) \qquad \Gamma_m(a, \kappa) = \pi^{\frac{1}{4}m(m-1)} \textstyle\prod_{i=1}^m \Gamma(a + k_i - \frac{1}{2}(i-1)),$$

and the integral converges for $R(a) + k_m > \frac{1}{2}(m - 1)$. If a is such that the gamma functions are defined then the binomial type coefficient (4) is

$$(8) \qquad (a)_\kappa = \Gamma_m(a, \kappa)/\Gamma_m(a), \qquad \Gamma_m(a) = \Gamma_m(a, 0).$$

The inverse form of (6) is then

$$(9) \quad [2^{\frac{1}{2}m(m-1)}/(2\pi i)^{\frac{1}{2}m(m+1)}] \int_{R(Z)>0} e^{\mathrm{tr}RZ}(\det Z)^{-a}C_\kappa(Z^{-1})\, dZ = [1/\Gamma_m(a, \kappa)](\det R)^{a-\frac{1}{2}(m+1)}C_\kappa(R).$$

An extension of (6) that will be useful later is the integral

$$(10) \quad \int_{R>0} e^{\mathrm{tr}-RZ}(\det R)^{a-\frac{1}{2}(m+1)}C_\kappa(R^{-1})\, dR = [(-1)^k\Gamma_m(a)/(-a + \tfrac{1}{2}(m+1))_\kappa](\det Z)^{-a}C_\kappa(Z),$$

valid for $R(a) > k_1 + \frac{1}{2}(m - 1)$. (10) may be proved in a manner similar to (6) or, alternatively, by noting that

$$(\det R)^n C_\kappa(R^{-1}) = [C_\kappa(I)/C_\kappa^*(I)]C_{\kappa^*}(R)$$

where n is any integer $\geqq k_1$ and $\kappa^* = (n - k_m, \cdots, n - k_1)$.

The zonal polynomials may be used as a basis for symmetric functions, and a number of important functions have simple expansions as series of zonal polynomials. In particular, the exponential function has the expansion

$$(11) \qquad e^{\mathrm{tr}R} = \sum_{k=0}^{\infty} \sum_\kappa [C_\kappa(R)/k!],$$

and the "binomial" function has the expansion

$$(12) \qquad \det(I - R)^{-a} = \sum_{k=0}^{\infty} \sum_\kappa [(a)_\kappa/k!]C_\kappa(R).$$

The generalised "Bessel" function (Herz [3]) occurring in the non-central Wishart distribution has the expansion [1]

$$(13) \quad A_\gamma(R) = [1/\Gamma_m(\gamma + \tfrac{1}{2}(m+1))]\sum_{k=0}^{\infty} \sum_\kappa [C_\kappa(-R)/(\gamma + \tfrac{1}{2}(m+1))_\kappa k!].$$

3. The generalised Laguerre polynomials. The distribution of T, derived in the next section, will be expressed as a series of *generalised Laguerre polynomials*. They are polynomials in the elements of the $m \times m$ matrix S and reduce to the classical polynomials when $m = 1$. Many of the results for the classical polynomials generalise to the case of matrix variables, and some of these results will be derived here. The reader is referred to Chapter 10 of "Higher Transcendental Functions" by Erdelyi et al. for the case $m = 1$, especially Section 12. Our definition parallels that of C. S. Herz [3] and the polynomials here are normalised a little differently from those in [2].

For each homogeneous, symmetric polynomial $\sigma(R)$ in the $m \times m$ matrix R, Herz defines the function $L_\sigma^\gamma(S)$ by

$$r^{\mathrm{tr}-S}L_\sigma^\gamma(S) = \int_{R>0} e^{\mathrm{tr}-R}(\det R)^\gamma \sigma(R)A_\gamma(RS)\, dR,$$

where $\gamma > -1$, and the Bessel function A_γ is given by (13). He showed that $L_\sigma^\gamma(S)$ is a polynomial of the same degree as σ, and if σ ranges over a basis for the homogeneous symmetric polynomials, then the L_σ^γ form a complete set of polynomials in the L^2-space of functions $f(S)$ on $S > 0$ with respect to the weight function $w(S) = e^{\text{tr}-S}(\det S)^\gamma$, $S > 0$. Furthermore, if the degree $\sigma_1 \neq$ degree σ_2, then L_{σ_1} and L_{σ_2} are orthogonal on $S > 0$ (with respect to the weight function $w(S)$).

Here, we shall take the zonal polynomials as a basis for symmetric functions, and define

$$(14) \qquad e^{\text{tr}-S}L_\kappa^\gamma(S) = \int_{R>0} e^{\text{tr}-R}(\det R)^\gamma C_\kappa(R)A_\gamma(RS)\,dR.$$

Now the Bessel function has the integral definition

$$(15) \qquad A_\gamma(R) = [2^{\frac{1}{2}m(m-1)}/(2\pi i)^{\frac{1}{2}m(m+1)}] \int_{R(Z)>0} e^{\text{tr}Z}e^{\text{tr}-RZ^{-1}}(\det Z)^{-\gamma-p}\,dZ$$

where, throughout this section,

$$(16) \qquad p = \tfrac{1}{2}(m+1).$$

(15) may be proved by expanding exp (tr $- RZ^{-1}$) in zonal polynomials and integrating term-by-term using (9). Substituting (15) in (14) and reversing the order of integration,

$$(17) \quad L_\kappa^\gamma(S) = \Gamma_m(\gamma+p,\kappa)[2^{\frac{1}{2}m(m-1)}/(2\pi_i)^{\frac{1}{2}m(m+1)}] \cdot \int_{R(Z)>0} e^{\text{tr}Z}(\det Z)^{-\gamma-p}C_\kappa(I - SZ^{-1})\,dZ.$$

This last expression is seen to be the Laplace inverse of

$$(18) \quad \int_{S>0} e^{\text{tr}-SZ}(\det S)^\gamma L_\kappa^\gamma(S)\,dS = \Gamma_m(\gamma+p,\kappa)(\det Z)^{-\gamma-p}C_\kappa(I - Z^{-1}).$$

Equation (17) allows the calculation of the Laguerre polynomials. Expanding $C_\kappa(I - SZ^{-1})$,

$$(19) \qquad C_\kappa(I - SZ^{-1})/C_\kappa(I) = \sum_{n=0}^{k}\sum_\nu (-1)^n a_{\kappa,\nu}C_\nu(SZ^{-1})/C_\nu(I),$$

and performing the integration in (17) using (9),

$$(20) \quad L_\kappa^\gamma(S) = (\gamma+p)_\kappa C_\kappa(I)\sum_{n=0}^{k}(-1)^n[a_{\kappa,\nu}/(\gamma+p)_\nu][C_\nu(S)/C_\nu(I)].$$

An explicit formula for the $a_{\kappa,\nu}$ is not known, but they may be readily calculated from (19). They are tabulated up to order $k = 4$ in an appendix at the end of this paper. (20) shows that, in general, $L_\kappa^\gamma(S)$ is a polynomial of degree k in S, unless S is singular when the degree may be less than k. When $m = 1$, $S = s$, $\kappa = (k)$, $L_k^\gamma(s)$ is readily seen to be identical with the classical Laguerre polynomial as defined in [2], except for the constant $k!$. The polynomial in [2] is $(1/k!)L_k^\gamma(s)$ in our notation.

From (20), the value of L_κ^γ at the origin $S = 0$ is seen to be

$$(21) \qquad L_\kappa^\gamma(0) = (\gamma+p)_\kappa C_\kappa(I).$$

Next, we consider a generating function for the Laguerre polynomials. If $m = 1$, there is the well known result

$$(1 - Z)^{-\gamma-1}\exp[-sZ/(1-Z)] = \sum_{k=0}^{\infty}[L_k^\gamma(s)Z^k/k!], \qquad |Z| < 1.$$

For arbitrary m this generalises in the following manner:

THEOREM 1. *The generating function for the Laguerre polynomials is*

$$\det(I - Z)^{-\gamma-p} \int_{O(m)} \exp(\operatorname{tr} - SH'Z(I - Z)^{-1}H)\, d(H)$$

$$(22) \qquad = \det(I - Z)^{-\gamma-p} \textstyle\sum_{k=0}^{\infty} \sum_{\kappa} [C_\kappa(-S) C_\kappa(Z(I - Z)^{-1})/k!\, C_\kappa(I)].$$

$$= \textstyle\sum_{k=0}^{\infty} \sum_{\kappa} [L_\kappa^{\gamma}(S) C_\kappa(Z)/k!\, C_\kappa(I)], \qquad \|Z\| < 1.$$

PROOF. Multiply the series on each side of (22) by $\exp(\operatorname{tr} - SW)(\det S)^{\gamma}$ and integrate term-by-term over $S > 0$. Using (18), the right hand side becomes

$$\Gamma_m(\gamma + p)(\det W)^{-\gamma-p} \textstyle\sum_{k=0}^{\infty} \sum_{\kappa} (\gamma + p)_\kappa C_\kappa(Z) C_\kappa(I - W^{-1})/k!\, C_\kappa(I)$$

$$= \Gamma_m(\gamma + p)(\det W)^{-\gamma-p} \int_{O(m)} \det(I - ZH'(I - W^{-1})H)^{-\gamma-p}\, d(H)$$

$$= \Gamma_m(\gamma + p)(\det W)^{-\gamma-p} \det(I - Z)^{-\gamma-p}$$

$$\cdot \int_{O(m)} \det(I + Z(I - Z)^{-1}H'W^{-1}H)^{-\gamma-p}\, d(H).$$

Using (6), the left hand side becomes

$$\Gamma_m(\gamma + p)(\det W)^{-\gamma-p} \det(I - Z)^{-\gamma-p} \textstyle\sum_{k=0}^{\infty} \sum_{\kappa} (\gamma + p)_\kappa$$

$$\cdot C_\kappa(Z(I - Z)^{-1}) C_\kappa(-W^{-1})/k!\, C_\kappa(I)$$

which equals the previous expression. The theorem then follows by the uniqueness of Laplace transforms. In this proof, the expansion (12) for $\det(I - R)^{-\gamma-p}$ together with the averaging operation (5) have been used. Q.E.D.

The generating function can be used to show that $L_\kappa^{\gamma}(S)$ and $L_\nu^{\gamma}(S)$ are orthogonal unless $\kappa = \nu$. This strengthens Herz's result since he showed only that polynomials of different degree are orthogonal.

THEOREM 2. *$L_\kappa^{\gamma}(S)$ and $L_\nu^{\gamma}(S)$ are orthogonal on $S > 0$ with respect to the weight function*

$$W(S) = e^{\operatorname{tr} - S}(\det S)^{\gamma},$$

unless $\kappa = \nu$.

PROOF. Multiply both sides of the generating function (22) by $e^{\operatorname{tr} - S}(\det S)^{\gamma}\, C_\nu(S)$, where ν is a partition of any integer n, and integrate over $S > 0$. The left hand side becomes

$$\det(I - Z)^{-\gamma-p} \int_{O(m)} \int_{S>0} \exp(\operatorname{tr} - S(I + H'Z(I - Z)^{-1}H))\, d(H)$$

$$\cdot (\det S)^{\gamma} C_\nu(S)\, dS = \Gamma_m(\gamma + p, \nu) C_\nu(I - Z)$$

by (6). The right hand side becomes

$$\textstyle\sum_{k=0}^{\infty} \sum_{\kappa} [C_\kappa(Z)/C_\kappa(I)k!] \int_{S>0} e^{\operatorname{tr} - S}(\det S)^{\gamma} C_\nu(S) L_\kappa^{\gamma}(S)\, dS.$$

Now $C_\nu(I - Z) = (-1)^n C_\nu(Z) +$ terms of lower degree, so that comparing coefficients of $C_\mu(Z)$ on both sides, we have

$$\int e^{\operatorname{tr} - S}(\det S)^{\gamma} C_\nu(S) L_\kappa^{\gamma}(S)\, dS = 0 \qquad \text{for } k \geqq n,$$

unless $\kappa = \nu$. Hence $L_\kappa^\gamma(S)$ is orthogonal to all Laguerre polynomials of lower degree, and since, from (20), $L_\kappa^\gamma(S) = (-1)^k C_\kappa(S)$ + terms of lower degree, it is also orthogonal to all Laguerre polynomials $L_\nu^\gamma(S)$ of the *same* degree unless $\kappa = \nu$.

Comparing coefficients of $C_\nu(Z)$ gives the L^2-norm of L_ν^γ as

$$\|L_\nu^\gamma\|^2 = n!\, C_\nu(I)\Gamma_m(\gamma + p, \nu).$$

Q.E.D.

Finally, for examining the convergence of series, we need some estimates for the Laguerre polynomials. The ones derived below are rather crude and could be considerably improved, but they serve our purposes. In the case $m = 1$, one can proceed as follows. First obtain an estimate for some special value of γ and then apply the formula ([2], p. 192)

$$(23)\qquad (1/k!)L_k^\beta(s) = \textstyle\sum_{m=0}^{k} [(\beta - \gamma)_m/m!\,(k - m)!]\, L_{k-m}^\gamma(s).$$

We shall generalise this procedure.

THEOREM 3.

$$(24)\qquad |L_\kappa^\beta(S)| \leqq (\beta + p)_\kappa C_\kappa(I) e^{\operatorname{tr} S}.$$

PROOF. We first obtain an estimate for $\gamma = -\frac{1}{2}$. The Bessel function $A_{-\frac{1}{2}}(R)$ has the integral representation ([3], [9])

$$A_{-\frac{1}{2}}(R) = [1/\Gamma_m(\tfrac{1}{2}m)] \textstyle\int_{O(m)} \exp\,(\operatorname{tr}\, 2iR^{\frac{1}{2}}H)\, d(H),$$

and hence $|A_{-\frac{1}{2}}(R)| \leqq 1/\Gamma_m(\frac{1}{2}m)$. Substituting this estimate in (14), with $\gamma = -\frac{1}{2}$,

$$(25)\qquad \begin{aligned} L_\kappa^{-\frac{1}{2}}(S) &\leqq [1/\Gamma_m(\tfrac{1}{2}m)]e^{\operatorname{tr} S} \textstyle\int_{R>0} e^{\operatorname{tr} -R}(\det R)^{\frac{1}{2}m-\frac{1}{2}(m+1)} C_\kappa(R)\, dR \\ &= (\tfrac{1}{2}m)_\kappa C_\kappa(I) e^{\operatorname{tr} S}. \end{aligned}$$

The next step is to generalise (23). The result is

$$(26)\qquad L_\kappa^\beta(S)/k!\, C_\kappa(I) = \textstyle\sum_{t=0}^{k} \sum_\tau \sum_\nu [(\beta - \gamma)_\tau/t!\, n!] g_{\tau\nu}^\kappa [L_\nu^\gamma(S)/C_\nu(I)],$$

where the summation is over all partitions τ of t and ν of n such that $t + n = k$, and $g_{\tau\nu}^\kappa$ is the coefficient of $C_\kappa(S)$ in $C_\tau(S)C_\nu(S)$, i.e.

$$(27)\qquad C_\tau(S)C_\nu(S) = \textstyle\sum_\kappa g_{\tau\nu}^\kappa C_\kappa(S).$$

To prove (26), multiply both sides of the generating function (22) by $\det\,(I - Z)^{-(\beta-\gamma)}$. The left hand side then becomes the generating function for $L_\kappa^\beta(S)$,

$$(28)\qquad \textstyle\sum_{k=0}^{\infty} \sum_\kappa [L_\kappa^\beta(S) C_\kappa(Z)/k!\, C_\kappa(I)].$$

The right hand side is

$$\det\,(I - Z)^{-(\beta-\gamma)} \textstyle\sum_{k=0}^{\infty} \sum_\kappa [L_\nu^\gamma(S) C_\nu(Z)/n!\, C_\nu(I)],$$

and we require the coefficient of $C_\kappa(Z)$ in the expansion of this product. Expanding det $(I - Z)^{-(\beta-\gamma)}$ in zonal polynomials as in (12), the term of degree i is

$$\textstyle\sum_\tau [(\beta - \gamma)_\tau/t!]C_\tau(Z) \sum_\nu [L_\nu^\gamma(S)C_\nu(Z)/n!\, C_\nu(Z)], \qquad t + n = k.$$

Hence, multiplying the zonal polynomials according to (27) and comparing coefficients of $C_\kappa(Z)$ with (28) gives the required result.

We now apply (26) with $\gamma = -\frac{1}{2}$. Substituting the estimate (25) for $L_\nu^{-\frac{1}{2}}(S)$, one has

$$| L_\kappa^\beta(S)| \leqq k!\, C_\kappa(I)e^{\operatorname{tr} S}\textstyle\sum_{t=0}^k \sum_\tau \sum_\nu [(\beta + \frac{1}{2})_\tau(\frac{1}{2}m)_\nu/t!\, n!]g_{\tau\nu}^\kappa$$

and this sum is seen to be the coefficient of $C_\kappa(Z)$ in the expansion of det $(I - Z)^{-(\beta+\frac{1}{2})}$ det $(I - Z)^{-\frac{1}{2}m}$, i.e., of det $(I - Z)^{-(\beta+p)}$. Hence the sum reduces to $(\beta + p)_\kappa/k!$. Q.E.D.

4. The distribution of T. In Theorem 4 below, the distribution of $T = \operatorname{tr} S_1S_2^{-1}$ will be derived assuming that S_2 has the central Wishart distribution on n_2 df and S_1 the non-central Wishart distribution on n_1 df, $n_1, n_2 \geqq m$. However, T is well-defined even if $n_1 < m$. If $n_1, n_2 \geqq m$, it is known [1] that the density function of the eigenvalues $w_1, w_2, \cdots, w_m$ of $W = S_1S_2^{-1}$ is

$$\begin{aligned}&[\pi^{\frac{1}{2}m^2}\Gamma_m(\tfrac{1}{2}(n_1 + n_2))/\Gamma_m(\tfrac{1}{2}n_1)\Gamma_m(\tfrac{1}{2}n_2)\Gamma_m(\tfrac{1}{2}m)]e^{\operatorname{tr}-\Omega}(\textstyle\prod w_i)^{\frac{1}{2}(n_1-m-1)}\\ &\cdot\textstyle\prod (1 + w_i)^{-\frac{1}{2}(n_1+n_2)}\prod_{i<j} (w_i - w_j)\sum_{k=0}^\infty \sum_\kappa \\ &\cdot[(\tfrac{1}{2}(n_1+n_2))_\kappa/(\tfrac{1}{2}n_1)_\kappa]C_\kappa(\Omega)C_\kappa(W(I+W)^{-1})/k!\, C_\kappa(I),\end{aligned} \tag{29}$$

where Ω is the matrix of non-centrality parameters. If $n_1 < m$, $S_1S_2^{-1}$ has only n_1 non-zero roots and these have the density function obtained from (29) by making the substitutions

$$n_1 \to m, \qquad n_2 \to n_1 + n_2 - m, \qquad m \to n_1 . \tag{30}$$

Now $T = \sum w_i$, and hence its distribution for $n_1 < m$ can be obtained from its distribution for $n_1 \geqq m$ by making the substitutions (30).

THEOREM 4. *Let the $m\times m$ matrices S_1 and S_2 be independently distributed, S_2 with the Wishart distribution on n_2* df *and S_1 with the non-central Wishart distribution on n_1* df *and matrix of non-centrality parameters Ω, the population covariance in each case being Σ. Then the density function of $T = \operatorname{tr} S_1S_2^{-1}$ is given by*

$$\begin{aligned}&[\Gamma_m(\tfrac{1}{2}(n_1 + n_2))/\Gamma_m(\tfrac{1}{2}n_2)\Gamma(\tfrac{1}{2}mn_1)]e^{\operatorname{tr}-\Omega}T^{\frac{1}{2}mn_1-1}\\ &\cdot \textstyle\sum_{k=0}^\infty [(-T)^k/(\frac{1}{2}mn_1)_k k!]\sum_\kappa (\frac{1}{2}(n_1 + n_2))_\kappa L_\kappa^\gamma(\Omega),\end{aligned} \tag{31}$$

where $\gamma = \frac{1}{2}(n_1 - m - 1)$, and $L_\kappa^\gamma(\Omega)$ is the generalised Laguerre polynomial defined in Section 3. If $\Omega = 0$, the null density of T is

$$\begin{aligned}&[\Gamma_m(\tfrac{1}{2}(n_1 + n_2))/\Gamma_m(\tfrac{1}{2}n_2)\Gamma(\tfrac{1}{2}mn_1)]T^{\frac{1}{2}mn_1-1}\textstyle\sum_{k=0}^\infty [(-T)^k/(\frac{1}{2}mn_1)_k k!]\\ &\cdot \textstyle\sum_\kappa (\frac{1}{2}n_1)_\kappa(\frac{1}{2}(n_1 + n_2))_\kappa C_\kappa(I),\end{aligned} \tag{32}$$

both series being convergent for $|T| < 1$.

PROOF. T is clearly independent of Σ, so that the joint distribution of S_1 and S_2 may be taken to be

$$(33)\qquad [e^{\operatorname{tr}-\Omega}/\Gamma_m(\tfrac{1}{2}n_2)]e^{\operatorname{tr}-S_1}e^{\operatorname{tr}-S_2}(\det S_1)^{\frac{1}{2}(n_1-m-1)}(\det S_2)^{\frac{1}{2}(n_2-m-1)}A_\gamma(-\Omega S_1),$$

where $\gamma = \frac{1}{2}(n_1 - m - 1)$ and A_γ is the Bessel function [1], [3]. The distribution of T will be defined by inverting the Laplace transform

$$g(t) = E[\exp(\operatorname{tr} - tS_1S_2^{-1})]$$

of its density function. Multiplying (33) by $\exp(\operatorname{tr} - tS_1S_2^{-1})$ and integrating over $S_1 > 0$, we have

$$\begin{aligned} g(t) &= [e^{\operatorname{tr}-\Omega}/\Gamma_m(\tfrac{1}{2}n_2)] \int_{S_2>0} e^{\operatorname{tr}-S_2}(\det S_2)^{\frac{1}{2}(n^2-m-1)} \det(I + tS_2^{-1})^{-\frac{1}{2}n_1} \\ (34)\qquad &\quad\cdot\exp(\operatorname{tr} \Omega(I + tS_2^{-1})^{-1})\, dS_2 \\ &= [e^{\operatorname{tr}-\Omega}/\Gamma_m(\tfrac{1}{2}n_2)]t^{-\frac{1}{2}mn_1} \int_{S_2>0} e^{\operatorname{tr}-S_2}(\det S_2)^{\frac{1}{2}(n_1+n_2-m-1)} \\ &\quad\cdot\det(I + t^{-1}S_2)^{-\frac{1}{2}n_1} \exp(\operatorname{tr} \Omega t^{-1}S_2(I + t^{-1}S_2)^{-1})\, dS_2 . \end{aligned}$$

It appears very difficult to carry out the integration with respect to S_2 in this expression and obtain $g(t)$ explicitly. However, it is possible to perform the Laplace inversion

$$(35)\qquad f(T) = (1/2\pi i) \int_{c-i\infty}^{c+i\infty} e^{tT}g(t)\, dt$$

first, and then integrate over S_2.

First, we note that the distribution of T is clearly a symmetric function of Ω. Hence, making the transformation $\Omega \to H'\Omega H$ in (34) and integrating over $H \varepsilon O(m)$ using Theorem 1,

$$\begin{aligned} &\det(I + t^{-1}S_2)^{-\frac{1}{2}n_1} \int_{O(m)} \exp(\operatorname{tr} H'\Omega H t^{-1}S_2(I + t^{-1}S_2)^{-1})\, d(H) \\ &\qquad = \textstyle\sum_{k=0}^{\infty} \sum_\kappa [L_\kappa^\gamma(\Omega)C_\kappa(-t^{-1}S_2)/k!\, C_\kappa(I)], \qquad \gamma = \tfrac{1}{2}(n_1 - m - 1). \end{aligned}$$

Therefore

$$\begin{aligned} (36)\qquad g(t) &= [e^{\operatorname{tr}-\Omega}t^{-\frac{1}{2}mn_1}/\Gamma_m(\tfrac{1}{2}n_2)] \int_{S_2>0} e^{\operatorname{tr}-S_2}(\det S_2)^{\frac{1}{2}(n_1+n_2-m-1)} \\ &\qquad\cdot \textstyle\sum_{k=0}^{\infty} \sum_\kappa [L_\kappa^\gamma(\Omega)C_\kappa(-t^{-1}S_2)/k!\, C_\kappa(I)]\, dS_2 . \end{aligned}$$

Applying the estimate (24) for $L_\kappa^\gamma(\Omega)$, we see that the series in (36) is dominated termwise by the series

$$e^{\operatorname{tr}\Omega} \textstyle\sum_{k=0}^{\infty} \sum_\kappa [(\tfrac{1}{2}n_1)_\kappa C_\kappa(-t^{-1}S_2)/k!] = e^{\operatorname{tr}\Omega} \det(I + t^{-1}S_2)^{-\frac{1}{2}n_1}.$$

Hence, for S_2 fixed, $R(t) = c$ sufficiently large, the series in (36) can be integrated term-by-term with respect to t, the same being true for $\det(I + t^{-1}S_2)^{-\frac{1}{2}n_1}$. Since

$$(1/2\pi i) \int_{c-i\infty}^{c+i\infty} e^{tT}t^{-\frac{1}{2}mn_1-k}\, dt = T^{\frac{1}{2}mn_1+k-1}/\Gamma(\tfrac{1}{2}mn_1 + k),$$

we have

$$\begin{aligned} (37)\quad f(T) &= [e^{\operatorname{tr}-\Omega}/\Gamma_m(\tfrac{1}{2}n_2)\Gamma(\tfrac{1}{2}mn_1)]T^{\frac{1}{2}mn_1-1} \int_{S_2>0} e^{\operatorname{tr}-S_2}(\det S_2)^{\frac{1}{2}(n_1+n_2-m-1)} \\ &\qquad\cdot \textstyle\sum_{k=0}^{\infty}[(-T)^k/(\tfrac{1}{2}mn_1)_k k!]\sum_\kappa[L_\kappa^\gamma(\Omega)C_\kappa(S_2)/C_\kappa(I)]\, dS_2 . \end{aligned}$$

Again applying the estimates for $L_\kappa^\gamma(\Omega)$ in (37), we see that the series is dominated termwise by the series $e^{\mathrm{tr}\,\Omega}\sum_{k=0}^{\infty}[1/(\frac{1}{2}mn_1)_k k!]\sum_\kappa(\frac{1}{2}n_1)_\kappa C_\kappa(TS_2)$ and, since $(\frac{1}{2}n_1)_\kappa/(\frac{1}{2}mn_1)_k \leqq 1$ for all m, this series is dominated termwise by the series

$$e^{\mathrm{tr}\,\Omega}\sum_{k=0}^{\infty}\sum_\kappa[C_\kappa(-TS_2)/k!] = e^{\mathrm{tr}\,\Omega}e^{\mathrm{tr}-TS_2}.$$

Hence, the series in (37) may be integrated term-by-term for $|T| < 1$ to give the required result.

Finally, (32) follows from (31) by putting $\Omega = 0$ and substituting the value (21) of $L_\kappa^\gamma(0)$. Q.E.D.

COROLLARY. *The distribution of* T *for* $n_1 < m$ *is obtained from* (31) (*and from* (32) *in the null case* $\Omega = 0$) *by making the substitutions* (30).

We now consider the special case $m = 1$, in which case the series (32) and (31) should reduce respectively to the density functions for the familiar variance ratio F, and non-central F. If $m = 1$, only the partitions $\kappa = (k)$ into one part occur, so that

$$(a)_\kappa = a_k, \qquad C_{(k)}(I) = 1, \qquad \Omega = w.$$

The series in (32) then becomes

$$\sum_{k=0}^{\infty}[(\tfrac{1}{2}(n_1 + n_2))_k(-T)^k/k] = (1 + T)^{-\frac{1}{2}(n_1+n_2)},$$

and hence $F = (n_1/n_2)T$ has the variance ratio distribution on n_1 and n_2 df as required. The density function (31) becomes

$$[\Gamma(\tfrac{1}{2}(n_1 + n_2))/\Gamma(\tfrac{1}{2}n_1)\Gamma(\tfrac{1}{2}n_2)]e^{-w}T^{\frac{1}{2}n_1-1}\sum_{k=0}^{\infty}[(\tfrac{1}{2}(n_1 + n_2))_k/(\tfrac{1}{2}n_1)_k k!]$$
$$\cdot L_k^\gamma(w)(-T)^k, \qquad \gamma = \tfrac{1}{2}n_1 - 1,$$

and according to [2], p. 215, formula 27, the series is

$$(1 + T)^{-\frac{1}{2}(n_1+n_2)}\,{}_1F_1[\tfrac{1}{2}(n_1 + n_2); \tfrac{1}{2}n_1; wT/(1 + T)].$$

Hence $(n_1/n_2)T$ has the non-central F distribution. Similarly, if $n_1 = 1$, the series reduce to the density functions for Hotelling's generalisation of "Student's" t.

So far, it has proved impossible to simplify the series or to extend them beyond $|T| < 1$. In the case $m = 2$, Hotelling [6] derived the null distribution of T in the form

$$[\Gamma(n_1 + n_2 - 1)/\Gamma(n_1)\Gamma(n_2 - 1)](\tfrac{1}{2}T)^{n_1-1}(1 + \tfrac{1}{2}T)^{-(n_1+n_2)}$$
$${}_2F_1(1, \tfrac{1}{2}(n_1 + n_2); \tfrac{1}{2}(n_1 + 1); r^2)$$

where $r = \frac{1}{2}T/(1 + \frac{1}{2}T)$, and ${}_2F_1$ is the Gaussian hypergeometric function. This and other considerations suggest expressing the distributions as series in the variable $(T/m)/(1 + T/m)$, but no progress has been made in this direction. Such series would probably converge for all values of T and would allow tabulation of the distribution functions if they converged reasonably fast.

5. The moments of T. To calculate the moments, we commence with the joint distribution of S_1 and S_2 given by (33), and note that [6]

$$T^k = (\operatorname{tr} S_1S_2^{-1})^k = \sum_\kappa C_\kappa(S_1S_2^{-1}).$$

Therefore,

$$E(T^k) = [e^{\operatorname{tr}-\Omega}/\Gamma_m(\tfrac{1}{2}n_2)] \int_{S_1>0}\int_{S_2>0} e^{\operatorname{tr}-S_1}e^{\operatorname{tr}-S_2}(\det S_1)^{\frac{1}{2}(n_1-m-1)}$$
$$\cdot(\det S_2)^{\frac{1}{2}(n_2-m-1)}A_\gamma(-\Omega S_1) \sum_\kappa C_\kappa(S_1S_2^{-1})\, dS_1\, dS_2, \qquad \gamma = \tfrac{1}{2}(n_1 - m - 1).$$

The integration with respect to S_2 can be carried out for $n_2 > 2k + m - 1$ using (10), and then the integration with respect to S_1 using (14). The result is

$$(38)\quad E(T^k) = (-1)^k \sum_\kappa [L_\kappa^\gamma(-\Omega)/(\tfrac{1}{2}(m+1-n_2))_\kappa], \quad \gamma = \tfrac{1}{2}(n_1 - m - 1),$$

if $n_2 > 2k + m - 1$, the moments not existing otherwise. If $\Omega = 0$, $L_\kappa^\gamma(0) = (\frac{1}{2}n_1)_\kappa C_\kappa(I)$, and (38) becomes

$$(39)\qquad E(T^k) = (-1)^k \sum_\kappa [(\tfrac{1}{2}n_1)_\kappa C_\kappa(I)/(\tfrac{1}{2}(m+1-n_2))_\kappa].$$

The first few moments are, from (39),

$$E(T) = n_1m/(n_2 - m - 1),$$

$$E(T^2) = n_1m(mn_1n_2 - 2mn_1 - m^2n_1 + 2n_1 + 2n_2 - 2)/$$
$$(n_2 - m)(n_2 - m - 1)(n_2 - m - 3),$$

whence

$$\operatorname{Var}(T) = 2n_1m(n_2 - 1)(n_1 + n_2 - m - 1)/(n_2 - m - 1)^2(n_2 - m)(n_2 - m - 3).$$

APPENDIX

The values of $a_{\kappa,\tau}$ up to order $k = 4$ in the expression

$$C_\kappa(I + A)/C_\kappa(I) = \sum_{t=0}^k \sum_\tau a_{\kappa,\tau}C_\tau(A)/C_\tau(I).$$

Entries not shown in the tables are zero.

$k = 1$

κ	τ: (0)	(1)
(1)	1	1

$k = 2$

κ	τ: (0)	(1)	(2)	(1^2)
(2)	1	2	1	
(1^2)	1	2		1

$k = 3$

	τ						
κ	(0)	(1)	(2)	(1^2)	(3)	(21)	(1^3)
(3)	1	3	3		1		
(21)	1	3	4/3	5/3		1	
(1^2)	1	3		3			1

$k = 4$

	τ											
κ	(0)	(1)	(2)	(1^2)	(3)	(21)	(1^3)	(4)	(31)	(2^2)	(21^2)	(1^4)
(4)	1	4	6		4			1				
(31)	1	4	11/3	7/3	6/5	14/5			1			
(2^2)	1	4	8/3	10/3		4				1		
(21^2)	1	4	5/3	13/3		5/2	3/2				1	
(1^4)	1	4		6			4					1

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