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SOME MULTIVARIATE COMPARISON PROCEDURES BASED ON RANKS

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1. Introduction. There are the p-variate treatment populations π_i with the cdf $F_i(\mathbf{x})$, $i = 1, \cdots, c$ and a p-variate control π_0 with $F_0(\mathbf{x})$ where we assume that $F_0(\mathbf{x}) = F(\mathbf{x})$, $F_i(\mathbf{x}) = F(\mathbf{x} - \boldsymbol{\theta}_i)$, $\boldsymbol{\theta}_i' = (\boldsymbol{\theta}_i^{(1)}, \cdots, \boldsymbol{\theta}_i^{(p)})$, $i = 1, \cdots, c$ and $F(\mathbf{x})$ is continuous, but unknown otherwise. Now set $\boldsymbol{\Delta}_i' = (\Delta_i^{(1)}, \cdots, \Delta_i^{(l)})$, $\Delta_i^{(h)} = \mathbf{a}_h'\boldsymbol{\theta}_i$, $i = 1, \cdots, c$, $h = 1, \cdots, l$ for the l given constant vectors $\mathbf{a}_h' = (a_h^{(1)}, \cdots, a_h^{(p)})$ where the matrix $\mathbf{A} = (\mathbf{a}_1, \cdots, \mathbf{a}_l)$ has rank l, $l \leqq p$. Then the criterion for the goodness of the treatment is defined as follows:

(i) the control π_0 is best if $\boldsymbol{\Delta}_i \leqq \mathbf{0}$ for $i = 1, \cdots, c$;

(ii) π_i is better than the control π_0 if $\boldsymbol{\Delta}_i \geqq \mathbf{0}$ and $\boldsymbol{\Delta}_i \neq \mathbf{0}$ hold;

(iii) π_i is not better at the $\mathbf{h}$th component than the control π_0, $\mathbf{h} = (h_1, \cdots, h_t)$, $1 \leqq h_1 < \cdots < h_t \leqq l$, $1 \leqq t \leqq l$, if $\Delta_i^{(h_\alpha)} < 0$ and $\Delta_i^{(h_\beta)} \geqq 0$ hold for $\alpha = 1, \cdots, t$ and $\beta \not\in \{1, \cdots, t\}$ where generally $\mathbf{x} \leqq \mathbf{y}$ means that each component of $\mathbf{x}$ is not larger than the corresponding component of $\mathbf{y}$ and $\mathbf{0}$ means the zero vector.

If $l = p$, and $\mathbf{A} = \mathbf{I}$, the identity matrix, then (i)–(iii) reduce to comparing the p components of $\boldsymbol{\theta}_i$ with the p vector zero, $i = 1, \cdots, c$. Then a multivariate comparison problem to separate the better treatments than π_0 from the not better ones may be introduced. Some similar problems have been respectively dealt with by Krishnaiah-Rizvi [2] under the assumption of the normal populations, and by Tamura [4], [5] under the nonparametric circumstances. This paper attempts some generalization for them. A formulation for the above problem and some lemmas are given in Section 2. The procedures based on (a) the randomized normal score statistics, (b) the statistics of Wilcoxon type, and (c) the classical sample means will be respectively proposed in Section 3 and their properties will be also investigated in this section.

2. Some lemmas. Though the following Lemma 1 is elementary, it plays an important part for our formulation.

LEMMA 1. *Let the cdf of the random vector* $\mathbf{X}$ *of p-variates be* $F(\mathbf{x} - \boldsymbol{\theta})$ *with the pdf* $f(\mathbf{x} - \boldsymbol{\theta})$ *and covariance matrix* $\boldsymbol{\Sigma}$ *where* $\boldsymbol{\theta}' = (\theta^{(1)}, \cdots, \theta^{(p)})$. *Then the pdf of the random vector* $\mathbf{Y}' = (Y^{(1)}, \cdots, Y^{(l)})$, $Y^{(h)} = \mathbf{a}_h'\mathbf{X}$ *for* $h = 1, \cdots, l$, *is given by the form* $g(\mathbf{y} - \boldsymbol{\Delta})$ *where* $\boldsymbol{\Delta} = (\Delta^{(1)}, \cdots, \Delta^{(l)})$, $\Delta^{(h)} = \mathbf{a}_h'\boldsymbol{\theta}$ *with the covariance matrix* $\mathbf{A}'\boldsymbol{\Sigma}\mathbf{A}$.

PROOF. Without loss of generality assume that $[A_1] = [a_i^{(h)}]$, $i, h = 1, \cdots, l$, and $|A_1| \neq 0$. Then by the transformation

$$\begin{bmatrix} \mathbf{Y} \\ \mathbf{Z} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \\ \mathbf{0} & \mathbf{I}_{p-l} \end{bmatrix} \mathbf{X}$$

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where $\mathbf{Z}' = (Z^{(l+1)}, \cdots, Z^{(p)})$, $\mathbf{0}$ is a $(p - l) \times l$ matrix of zeros, and $\mathbf{I}_{p-l}$ is the identity matrix of order $(p - l)$. The pdf of $(\mathbf{Y}', \mathbf{Z})$ is given by

$$(\mathrm{mod}\ |A_1|)^{-1} f(\mathbf{b}_{11}'(\mathbf{y} - \mathbf{\Delta}) + \mathbf{b}_{12}'(\mathbf{z} - \mathbf{\theta}^*), \cdots, \mathbf{b}_{p1}'(\mathbf{y} - \mathbf{\Delta}) + \mathbf{b}_{p2}'(\mathbf{z} - \mathbf{\theta}^*))$$

where $\mathbf{b}_{hk}$'s are some constant vectors depending on only $a_i^{(h)}$'s and $\mathbf{\theta}^{*\prime} = (\theta^{(l+1)}, \cdots, \theta^{(p)})$. After integrating out $\mathbf{z}$, the pdf of $\mathbf{Y}$ may be given by the form $g(\mathbf{y} - \mathbf{\Delta})$. That the covariance matrix becomes $\mathbf{A}'\mathbf{\Sigma}\mathbf{A}$ is trivial.

We may obtain a formulation of our problem from this lemma. Let the random sample of size n_j from the l-variate population π_j with the cdf $G_j(\mathbf{y}) = G(\mathbf{y} - \mathbf{\Delta}_j)$, $j = 0, 1, \cdots, c$, be $\{\mathbf{Y}_{j1}, \cdots, \mathbf{Y}_{jn_j}\}$, $\mathbf{Y}_{j\alpha} = (Y_{j\alpha}^{(1)}, \cdots, Y_{j\alpha}^{(l)})$, $\alpha = 1, \cdots, n_j$, where π_0 is the control with $\mathbf{\Delta}_0 = \mathbf{0}$. Assume that $G(\mathbf{y})$ is continuous and has the covariance matrix $\mathbf{A}'\mathbf{\Sigma}\mathbf{A}$ but unknown otherwise. Moreover let $G^{(h)}(y)$ and $G^{(h,k)}(y, z)$ be the marginal cdf of the hth and (h, k)th components of $G(\mathbf{y})$. When the criterion for the goodness of π_i is given by the same one in Section 1, our problem is to find the procedures how to separate the treatments better than the control π_0 from the not better ones under the restriction

$$\Pr\,[\pi_0 \text{ is selected as best when all } \mathbf{\Delta}_i = \mathbf{0}] \geqq 1 - \alpha. \tag{1}$$

It is convenient to define the following decisions,

D^0: the control π_0 is best

$D_{j_1(\mathbf{h}_1)\cdots j_s(\mathbf{h}_s)}^{i_1\cdots i_r}$: π_{i_β} for $\beta = 1, \cdots, r$, $1 \leqq r \leqq c$, are better than π_0, and π_{j_γ} for $\gamma = 1, \cdots, s$ are not better than π_0 at the $\mathbf{h}_\gamma$ component where $(i_1 < \cdots < i_r$, $j_1 < \cdots < j_s)$, $r + s = c$, is a permutation from $(1, \cdots, c)$ and $\mathbf{h}_\gamma = (h_\gamma^{(1)}, \cdots, h_\gamma^{(t_\gamma)})$, $t_\gamma = 1, \cdots, l$. We here notice that the notations i_β, j_γ and $\mathbf{h}_j$ will be used in the same sense above in later statements although we shall omit their explanation without confusion. Thus our purpose is to select one of the above decisions by a process satisfying the restriction (1).

Now we define some statistics which will be used in the selection procedures.

Definition 1. The randomized normal score statistics.

$$T_j^{(h)} = n_j^{-1} \textstyle\sum_{\alpha=1}^{n_j} V(R(Y_{j\alpha}^{(h)})), \qquad h = 1, \cdots, l, \qquad j = 0, 1, \cdots, c; \tag{2}$$

where $R(Y_{j\alpha}^{(h)})$ is the rank of $Y_{j\alpha}^{(h)}$ among the combined sample of the hth component $\{Y_{j\alpha}^{(h)}, \alpha = 1, \cdots, n_j, j = 0, 1, \cdots, c\}$ and $V(1) < \cdots < V(N)$, $N = \sum_{j=0}^{c} n_j$, are the order statistics from the standard normal population with the cdf $\Phi(x)$. We also define some random vectors.

$$\begin{aligned} \mathbf{T}^{(h)\prime} &= ((n_0 n_i/(n_0 + n_i))^{\frac{1}{2}}(T_i^{(h)} - T_0^{(h)}), \\ &\qquad\qquad i = 1, \cdots, c), \qquad h = 1, \cdots, l. \\ \mathbf{T}_i' &= ((n_0 n_i/(n_0 + n_i))^{\frac{1}{2}}(T_i^{(h)} - T_0^{(h)}), \\ &\qquad\qquad h = 1, \cdots, l), \qquad i = 1, \cdots, c. \end{aligned} \tag{3}$$

Definition 2. The statistics of the Wilcoxon type.

$$W_j^{(h)} = (Nn_j)^{-1} \textstyle\sum_{\alpha=1}^{n_j} R(Y_{j\alpha}^{(h)}), \qquad h = 1, \cdots, l, \qquad j = 0, 1, \cdots, c. \tag{4}$$

$$\mathbf{W}^{(h)\prime} = ((12\, n_0 n_i/(n_0 + n_i))^{\frac{1}{2}}(W_i^{(h)} - W_0^{(h)}),\ i = 1, \cdots, c),$$

(5) $$h = 1, \cdots, l.$$

$$\mathbf{W}_i' = ((12\, n_0 n_i/(n_0 + n_i))^{\frac{1}{2}}(W_i^{(h)} - W_0^{(h)}),\ h = 1, \cdots, l),$$

$$i = 1, \cdots, c.$$

DEFINITION 3. The sample means.

$$\bar{\mathbf{Y}}^{(h)\prime} = ((n_0 n_i/(n_0 + n_i))^{\frac{1}{2}}(\bar{Y}_i^{(h)} - \bar{Y}_0^{(h)})/(\mathbf{a}_h'\mathbf{S}\mathbf{a}_h)^{\frac{1}{2}},\ i = 1, \cdots, c),$$

(6) $$h = 1, \cdots, l.$$

$$\bar{\mathbf{Y}}_i' = ((n_0 n_i/(n_0 + n_i))^{\frac{1}{2}}(\bar{Y}_i^{(h)} - \bar{Y}_0^{(h)})/(\mathbf{a}_h'\mathbf{S}\mathbf{a}_h)^{\frac{1}{2}},\ h = 1, \cdots, l),$$

$$i = 1, \cdots, c.$$

where $\bar{Y}_j^{(h)} = n_j^{-1} \sum_{\alpha=1}^{nj} Y_{j\alpha}^{(h)}, j = 0, 1, \cdots, c, h = 1, \cdots, l$ and $\mathbf{S}$ is a consistent estimate of $\mathbf{\Sigma}$.

LEMMA 2. *Under the hypothesis that all $\mathbf{\Delta}_i = \mathbf{0}$, the exact distribution of $\mathbf{T}^{(h)}$ is the c-variate normal $N(\mathbf{0}, \mathbf{\Lambda})$ and the statistics $\mathbf{W}^{(h)}$ and $\bar{\mathbf{Y}}^{(h)}$ are asymptotically distributed according to $N(\mathbf{0}, \mathbf{\Lambda})$ where $\mathbf{\Lambda} = [\rho_{ij}]$, $\rho_{ij} = (\rho_i \rho_j/(\rho_i + \rho_0)(\rho_j + \rho_0))^{\frac{1}{2}}$ (or 1) for $i \neq j$ (or $i = j$) and $\rho_j = n_j/N, j = 0, 1, \cdots, c$.*

LEMMA 3. *Under all $\mathbf{\Delta}_i = \mathbf{\delta}_i/N^{\frac{1}{2}}$, the asymptotic distribution of $[\mathbf{T}^{(1)}, \cdots, \mathbf{T}^{(l)}]$, $[\mathbf{W}^{(1)}, \cdots, \mathbf{W}^{(l)}]$ and $[\bar{\mathbf{Y}}^{(1)}, \cdots, \bar{\mathbf{Y}}^{(l)}]$ are respectively the normal $N(\boldsymbol{\mu}, \mathbf{\Lambda} \otimes \mathbf{\Gamma})$, $N(\boldsymbol{\lambda}, \mathbf{\Lambda} \otimes \mathbf{\Pi})$ and $N(\boldsymbol{\nu}, \mathbf{\Lambda} \otimes \mathbf{\Omega})$ where $\boldsymbol{\mu} = [\mu_i^{(h)}]$, $\boldsymbol{\lambda} = [\lambda_i^{(h)}]$, $\boldsymbol{\nu} = [\nu_i^{(h)}]$, $i = 1, \cdots, c$, $h = 1, \cdots, l$,*

$$(7)\quad \mu_i^{(h)} = (\rho_0 \rho_i/(\rho_0 + \rho_i))^{\frac{1}{2}} \delta_i^{(h)} \int_{-\infty}^{\infty} \frac{d}{dy} \Phi^{-1}(G^{(h)}(y))\, dG^{(h)}(y)$$

$$(8)\quad \lambda_i^{(h)} = (12\rho_0\rho_i/(\rho_0 + \rho_i))^{\frac{1}{2}} \delta_i^{(h)} \int_{-\infty}^{\infty} g^{(h)}(y)\, dg^{(h)}(y), \qquad g^{(h)}(x) = \frac{d}{dx} G^{(h)}(x)$$

$$(9)\quad \nu_i^{(h)} = (\rho_0\rho_i/(\rho_0 + \rho_i))^{\frac{1}{2}} \delta_i^{(h)}/(\mathbf{a}_h \mathbf{\Sigma} \mathbf{a}_h)^{\frac{1}{2}}$$

and $\mathbf{\Gamma} = [\gamma_{hk}]$, $\mathbf{\Pi} = [\pi_{hk}]$, $\mathbf{\Omega} = [\omega_{hk}]$, $h, k = 1, \cdots, l$,

$$(10)\quad \gamma_{hk} = \textstyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi^{-1}(G^{(h)}(x))\Phi^{-1}(G^{(k)}(y))dG^{(h,k)}(x, y) \quad (or\ 1)$$

$$for\ h \neq k\ (or\ h = k),$$

$$(11)\quad \pi_{hk} = 12 \textstyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G^{(h)}(x)G^{(k)}(y)dG^{(h,k)}(x, y) - 3 \quad (or\ 1)$$

$$for\ \ h \neq k\ (or\ h = k),$$

$$(12)\quad \omega_{hk} = \mathbf{a}_h'\mathbf{\Sigma}\mathbf{a}_k/(\mathbf{a}_h'\mathbf{\Sigma}\mathbf{a}_h)^{\frac{1}{2}}(\mathbf{a}_k'\mathbf{\Sigma}\mathbf{a}_k)^{\frac{1}{2}} \quad (or\ 1) \qquad for\ h \neq k\ (or\ h = k).$$

The proof of Lemma 2 or 3 immediately follows from the results of Bell-Doksum [1] and Tamura [3].

3. Selection procedures and their properties. In this section, we shall propose three selection procedures and investigate their properties.

Randomized normal score procedure M:

Accept D^0 if $\mathbf{T}_i \leqq \mathbf{z}_\alpha$ for $i = 1, \cdots, c$ and accept $D_{j_1(h_1)\cdots j_s(h_s)}^{i_1\cdots i_r}$ if $\mathbf{T}_{i_\beta} > \mathbf{z}_\alpha$ for $\beta = 1, \cdots, r$ and

$$\begin{aligned} T_{j_\gamma}^{(h_\gamma(\delta))} &\leqq z_\alpha \quad \text{for } \delta = 1, \cdots, t_\gamma; \quad \gamma = 1, \cdots, s \\ &> z_\alpha \quad \text{for } \delta \not\in \{1, \cdots, t_\gamma\}; \quad \gamma = 1, \cdots, s, \end{aligned}$$

where $\mathbf{z}_\alpha$ is the constant vector whose components are equal to z_α determined by the following

$$\int_{-\infty}^{z_\alpha} \cdots \int_{-\infty}^{z_\alpha} n(\mathbf{0}, \mathbf{\Lambda})\, d\mathbf{y} = 1 - \alpha/l. \tag{13}$$

Procedure W of the Wilcoxon type or Procedure U by the classical sample means is obtained by using the statistics $\mathbf{W}_i$ or $\mathbf{Y}_i$ instead of $\mathbf{T}_i$ in the Procedure M.

THEOREM 1. *Procedure M satisfies the restriction* (1) *and also Procedure W and U asymptotically satisfy the restriction* (1).

PROOF. We first prove this result for the Procedure M. That is,

$$\begin{aligned} &\Pr\,[D^0 \text{ is accepted when all } \mathbf{\Delta}_i = \mathbf{0}] \\ &= \Pr\,[\mathbf{T}_i \leqq \mathbf{z}_\alpha \quad \text{for} \quad i = 1, \cdots, c \mid \text{all } \mathbf{\Delta}_i = \mathbf{0}] \\ &= \Pr\,[\mathbf{T}^{(h)} \leqq \mathbf{z}_\alpha \quad \text{for} \quad h = 1, \cdots, l \mid \text{all } \mathbf{\Delta}_i = \mathbf{0}] \\ &\geqq 1 - \textstyle\sum_{h=1}^{l} \Pr\,[\mathbf{T}^{(h)} \nleqq \mathbf{z}_\alpha \mid \text{all } \mathbf{\Delta}_i = \mathbf{0}] \end{aligned}$$

where the last line has been derived from Bonferroni's inequality and $\nleqq$ means the negation of $\leqq$.

Then the last line of the above is transformed by using (13) to

$$\textstyle\sum_{h=1}^{l} \Pr\,[\mathbf{T}^{(h)} \leqq \mathbf{z}_\alpha \mid \text{all } \mathbf{\Delta}_i = \mathbf{0}] - (l - 1) = 1 - \alpha.$$

The result for Procedure W or U is similarly derived from Lemma 2.

THEOREM 2. *Each procedure asymptotically satisfies the following*:

$$\Pr\,[D^0 \text{ is accepted when all } \mathbf{\Delta}_i = \mathbf{\delta}_i/N^{\frac{1}{2}},\ \mathbf{\delta}_i \leqq \mathbf{0}] \geqq 1 - \alpha. \tag{14}$$

PROOF. By the same considerations as the proof of Theorem 1,
$\Pr\,[D^0 \text{ is accepted} \mid \text{all } \mathbf{\Delta}_i = \mathbf{\delta}_i/N^{\frac{1}{2}}, \mathbf{\delta}_i \leqq \mathbf{0}]$

$$\geqq 1 - \textstyle\sum_{h=1}^{l} \Pr\,[\mathbf{T}^{(h)} \nleqq \mathbf{z}_\alpha \mid \mathbf{\Delta}_i = \mathbf{\delta}_i/N^{\frac{1}{2}},\ \mathbf{\delta}_i \leqq \mathbf{0}].$$

Using Lemma 3, the right hand is asymptotically equal to

$$\begin{aligned} &\textstyle\sum_{h=1}^{l} \int_{-\infty}^{z_\alpha} \cdots \int_{-\infty}^{z_\alpha} n(\mathbf{\mu}^{(h)}, \mathbf{\Lambda})\, d\mathbf{x} - (l - 1) \\ &\qquad = \textstyle\sum_{h=1}^{l} \int_{-\infty}^{z_\alpha - \mu_1^{(h)}} \cdots \int_{-\infty}^{z_\alpha - \mu_c^{(h)}} n(\mathbf{0}, \mathbf{\Lambda})\, d\mathbf{x} - (l - 1) \end{aligned}$$

where $\mathbf{\mu}^{(h)} = (\mu_1^{(h)}, \cdots, \mu_c^{(h)})$ and $\mathbf{\mu}^{(h)} \leqq \mathbf{0}$ since $\mathbf{\delta}_i \leqq \mathbf{0}$. Thus the last expression is larger than $(1 - \alpha)$ from $\mathbf{\mu}^{(h)} \leqq \mathbf{0}$ and (13). We also get the same result for the other procedures.

Secondly let the probability that the decision $D_{j_1(h_1)\cdots j_s(h_s)}^{i_1\cdots i_r}$ is accepted under the assumption $\mathbf{\Delta}_i = \boldsymbol{\delta}_i/N^{\frac{1}{2}}$, $i = 1, \cdots, c$ in the procedure S be denoted by $P^{i_1 \cdots i_r}_{j_1(h_1) \cdots j_s(h_s)}(S)$. We here notice that this probability is an analogous notion to the power under the Pitman alternatives in testing hypothesis.

THEOREM 3.

$$P^{i_1 \cdots i_r}_{j_1(h_1) \cdots j_s(h_s)}(M) \sim \int R_M \, n(\mathbf{0}, \mathbf{\Lambda} \otimes \mathbf{\Gamma}) \, d\mathbf{v} \tag{15}$$

where R_M is the domain determined by the following,

$$\begin{aligned} \mathbf{v}_{i_\beta} &> \mathbf{z}_\alpha - \boldsymbol{\mu}_{i_\beta} \quad \text{for} \quad \beta = 1, \cdots, r \quad \text{and} \quad \boldsymbol{\mu}_i = (\mu_i^{(1)}, \cdots, \mu_i^{(l)}), \\ v_{j_\gamma}^{(h_\gamma(\delta))} &\leqq z_\alpha - \mu_{j_\gamma}^{(h_\gamma(\delta))} \quad \text{for} \quad \delta = 1, \cdots, t_\gamma; \qquad \gamma = 1, \cdots, s \\ &> z_\alpha - \mu_{j_\gamma}^{(h_\gamma(\delta))} \quad \text{for} \quad \delta \not\in \{1, \cdots, t_\gamma\}; \qquad \gamma = 1, \cdots, s \end{aligned}$$

$$P^{i_1 \cdots i_r}_{j_1(h_1) \cdots j_s(h_s)}(W) \sim \int_{R_W} n(\mathbf{0}, \mathbf{\Lambda} \otimes \mathbf{\Pi}) \, d\mathbf{v} \tag{16}$$

$$P^{i_1 \cdots i_r}_{j_1(h_1) \cdots j_s(h_s)}(U) \sim \int_{R_U} n(\mathbf{0}, \mathbf{\Lambda} \otimes \mathbf{\Omega}) \, d\mathbf{v} \tag{17}$$

where R_W or R_U express the domain obtained by setting λ's or ν's instead of μ's in R_M.

PROOF.

$$\begin{aligned} P^{i_1 \cdots i_r}_{j_1(h_1) \cdots j_s(h_s)}(M) &= \Pr[\mathbf{T}_{i_\beta} > \mathbf{z}_\alpha \quad \text{for} \quad \beta = 1, \cdots, r \quad \text{and} \\ &(n_0 n_{j_\gamma}/(n_0 + n_{j_\gamma}))^{\frac{1}{2}}(T_{j_\gamma}^{(h_\gamma(\delta))} - T_0^{(h_\gamma(\delta))}) \leqq z_\alpha \, (\text{or} > z_\alpha) \quad \text{for } \delta = 1, \cdots, t_\gamma \\ &(\text{or } \delta \not\in \{1, \cdots, t_\gamma\}); \gamma = 1, \cdots, s \mid \mathbf{\Delta}_i = \boldsymbol{\delta}_i/N^{\frac{1}{2}}, i = 1, \cdots, c] \end{aligned}$$

Using Lemma 3, we get

$$\text{right hand side} \sim \int_R n(\boldsymbol{\mu}, \mathbf{\Lambda} \otimes \mathbf{\Gamma}) \, d\mathbf{y}$$

where the domain R is determined by the following: $\mathbf{y}_{i_\beta} > \mathbf{z}_\alpha$ for $\beta = 1, \cdots, r$ and $y_{j_\gamma}^{(h_\gamma(\delta))} \leqq z_\alpha$ (or $> z_\alpha$) for $\delta = 1, \cdots, t_\gamma$ (or $\delta \not\in \{1, \cdots, t_\gamma\}$), $\gamma = 1, \cdots, s$. Thus we may obtain the expression (15). The other (16) and (17) are similarly proved.

COROLLARY. *If the marginal cdf $G^{(h,k)}(x, y)$ is the bivariate normal distribution with zero mean vector and covariance matrix* $\begin{pmatrix} \mathbf{a}_h'\mathbf{\Sigma}\mathbf{a}_h & \mathbf{a}_h'\mathbf{\Sigma}\mathbf{a}_k \\ \mathbf{a}_k'\mathbf{\Sigma}\mathbf{a}_h & \mathbf{a}_k'\mathbf{\Sigma}\mathbf{a}_k \end{pmatrix}$, *the Procedure M is asymptotically equivalent to the Procedure U.*

PROOF. Since we may get $G^{(h)}(x) = \Phi(x/(\mathbf{a}_h'\mathbf{\Sigma}\mathbf{a}_h))$ and

$$\begin{aligned} \mu_i^{(h)} &= (\rho_0\rho_i/\rho_0 + \rho_i)^{\frac{1}{2}}\delta_i^{(h)}/(\mathbf{a}_h'\mathbf{\Sigma}\mathbf{a}_h)^{\frac{1}{2}} \\ \gamma_{hk} &= \mathbf{a}_h'\mathbf{\Sigma}\mathbf{a}_k/(\mathbf{a}_h'\mathbf{\Sigma}\mathbf{a}_h)^{\frac{1}{2}}(\mathbf{a}_k'\mathbf{\Sigma}\mathbf{a}_k)^{\frac{1}{2}} \text{ (or 1)} \qquad \text{for} \quad h \neq k \text{ (or } h = k), \end{aligned} \tag{18}$$

it holds that $\mu_i^{(h)} = \nu_i^{(h)}$ and $\Gamma = \Omega$ and these show the result.

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