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A NECESSARY AND SUFFICIENT CONDITION THAT FOR REGULAR MULTIPLE DECISION PROBLEMS OF TYPE I EVERY UNBIASED PROCEDURE HAS MINIMAX RISK[1]

BY WILHELMINE STEFANSKY

University of California, Berkeley

1. Introduction and summary. We shall use the notation of [1], Sections 2 and 3. Briefly, multiple decision problems (m.d.p.'s) of Type 1 have the following formulation. On the basis of the outcome of a random variable (rv) X, which takes on values in the space $\mathscr{X}$, one element of the finite decision space $\mathscr{D} = \{d_0, d_1, \cdots, d_n\}$ is to be selected. The rv X has a density p_θ w.r.t. some σ-finite measure μ for some $\theta \in \Omega$. The parameter space Ω is partitioned into $(m+1)$ disjoint subsets $\Omega = \Omega_0 \cup \cdots \cup \Omega_m$ and $\Omega_1 \cup \cdots \cup \Omega_m$ is denoted by Ω'. The loss function $L: \Omega' x\mathscr{D} \to [0, \infty)$ with $L(\theta, d) = w_{ij} \geqq 0$ for $d = d_j$ and $\theta \in \Omega_i$ $(i = 1, \cdots, m; j = 0, \cdots, n)$ is constant over each Ω_i and is not defined for $\theta \in \Omega_0$. Type 1 m.d.p.'s satisfy the following assumptions: (i) $\Omega \subset R^s$, (ii) $[\Omega_1] \cap \cdots \cap [\Omega_m] = \Omega_0' \neq \emptyset$, where [ ] denotes closure w.r.t. the usual topology, (iii) for every test function φ, $E_\theta[\varphi(X)]$ is continuous in θ. A m.d.p. of Type 1 will be called *regular*, if it satisfies the additional assumption (iv) for every test function φ, $E_{\theta_0}[\varphi(X)] = 0$ for all $\theta_0 \in \Omega_0'$ implies that $E_\theta[\varphi(X)] = 0$ for all $\theta \in \Omega'$.

A procedure $\delta = \delta(\varphi_0, \cdots, \varphi_n)$ is *unbiased* if and only if for $i = 1, \cdots, m$ the following inequalities hold

$$\sum_{j=0}^n w_{hj} E_\theta[\varphi_j(X)] \geqq \sum_{j=0}^n w_{ij} E_\theta[\varphi_j(X)], \qquad \theta \in \Omega_i; \quad h = 1, \cdots, m. \tag{1}$$

Let $W(M)$ denote the class of all unbiased (minimax risk) procedures for a m.d.p. of Type 1. Procedures with the same risk function will be identified. Let $w_{\cdot j}$ denote the point $(w_{1j}, \cdots, w_{mj})$ of R^m and let S denote the convex hull of $w_{\cdot j}$ $(j = 0, \cdots, n)$. Finally, let E denote the set of points $e \in S$ with $e_1 = \cdots = e_m$.

In [1] Theorem 4.1 it was proved that for problems of Type 1, a *sufficient* condition for $W \subset M$ is that there exists a point $e \in E$ that is both a minimax and a maximin point of S. This note establishes the following result.

THEOREM. *For regular problems of Type* 1, *necessary and sufficient for* $W \subset M$ *is that one of the following conditions holds*: (i) $E = \emptyset$, (ii) *E consists of exactly one point e and this is a minimax point of S.*

The following non-trivial example shows that for a regular m.d.p. of Type 1 the sufficient condition of Theorem 4.1 [1] is not necessary. Take $m = 3$, $n = 2$, $w_{\cdot 0} = (3, -3, 0)$, $w_{\cdot 1} = (-3, 3, 0)$, $w_{\cdot 2} = (6, -4, 6)$.

Then $e = (0, 0, 0) = \frac{1}{2}w_{\cdot 0} + \frac{1}{2}w_{\cdot 1}$ is the unique convex representation with all coordinates equally large. Moreover, e is a minimax point, so that condition

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(ii) of the Theorem is satisfied, but e is not a maximin point, since the point $3w_{\cdot 1}/5+2w_{\cdot 2}/5 = (3/5, 1/5, 12/5)$ has all coordinates strictly larger than zero.

2. Proof of the theorem.

Necessity. First we shall show that $W \subset M$ implies that every point of E is a minimax point of S. If $E = \emptyset$, there is nothing to prove. Therefore, suppose that e is an arbitrary point of E with some convex representation $\sum_{j=0}^{n} p_j w_{\cdot j}$. The constant procedure $\delta^* = \delta^*(\varphi_0{}^*, \cdots, \varphi_n{}^*)$ with $\varphi_j{}^*(x) = p_j$ $(j = 0, \cdots, n)$ is unbiased, because equality holds in (1). By assumption $W \subset M$, so that δ^* also has minimax risk. Since for all $\theta_0 \in \Omega_0'$, $E_{\theta_0}[\varphi_j{}^*(X)] = p_j$, it follows from Theorem 3.1 (ii) [1] that e is a minimax point of S. Next we need to show that if every point of E is a minimax point of S and $E \neq \emptyset$, then condition (ii) of the Theorem holds. Suppose e and e' both belong to E. The common value of the coordinates of one point must be strictly smaller than the common value of the coordinates of the other point, so that it cannot be the case that both e and e' are minimax points of S.

Sufficiency. If $E = \emptyset$, then Corollary 4.1 [1] implies that $W = \emptyset$, so that $W \subset M$ holds trivially. Therefore, we may assume that E consists of exactly one point e and that this is a minimax point of S. Let $J = \{j_0, \cdots, j_r\}$ denote the collection of indices j such that there is a convex representation $\sum_{k=0}^{n} p_k w_{\cdot k}$ of e with $p_j > 0$. We need an auxiliary m.d.p. of Type 1, all of whose symbols will be provided with a star. This auxiliary problem is obtained from the original one by omitting all decisions d_j from $\mathscr{D}$ for which $j \notin J$. Formally we have $n^* = r$, $d_h{}^* = d_{j_h}$, $w_{ih}^* = w_{ij_h}(i = 1, \cdots, m;\, h = 0, \cdots, r)$. Let S^* denote the convex hull in R^m of the points $w_{\cdot h}^*$ $(h = 0, \cdots, r)$. Let $W^*(M^*)$ denote the class of all unbiased (minimax risk) procedures for the auxiliary problem. In part (a) we show that e is a minimax point of S^*, and in (b) that e is also a maximin point of S^*. Then it follows from Theorem 4.1 [1] that $W^* \subset M^*$. But every procedure for the auxiliary problem corresponds in a natural way to a procedure for the original problem: if $\delta^* = \delta^*(\varphi_0{}^*, \cdots, \varphi_r{}^*)$ is a procedure for the auxiliary problem, then the corresponding procedure $\delta = i(\delta^*) = \delta(\varphi_0, \cdots, \varphi_n)$ is defined by $\varphi_j = 0$ if $j \notin J$ and $\varphi_j = \varphi_h{}^*$ if $j = j_h \in J$. Then, of course, $W^* \subset M^*$ implies $i(W^*) \subset i(M^*)$. In (c) we show that $i(M^*) \subset M$, and in (d) that $i(W^*) = W$, which completes the proof. Part (d) is the only part of the proof in which assumption (iv) is used.

(a) *e is a minimax point of S^*.* This follows straightforwardly from the facts that $e \in S^* \subset S$ and e is a minimax point of S.

(b) *e is a maximin point of S^*.* We shall show that if e is not a maximin point of S^*, then there exists a point $s^* \in S^*$ all of whose coordinates are strictly smaller than the common value $\bar{e}$ of the coordinates of e, so that e cannot be a minimax point of S^*. Since this contradicts (a), it follows that e is a maximin point of S^*. The point s^* can be constructed as follows. From the definition of J it is seen that there exists a convex representation $\sum_{h=0}^{r} p_h w_{\cdot h}^*$ of e such that $p_h > 0$ $(h = 0, \cdots, r)$; for example, one can take an average of the various convex representations. In particular, $\varepsilon = \min(p_h) > 0$. If e is not a maximin point of S^*, then there exists a point $s' = \sum_{h=0}^{r} p_h' w_{\cdot h} \varepsilon S^*$ whose smallest coordinate is larger than $\bar{e}$. Let $s^* = (1+\varepsilon)e - \varepsilon s' = \sum_{h=0}^{r} [(1+\varepsilon)p_h - \varepsilon p_h']w_{\cdot h}^* = \sum_{h=0}^{r} p_h^* w_{\cdot h}^*$. The point s^* belongs to S^* because $\sum_{h=0}^{r} p_h^* = 1$ and $p_h^* = (1+\varepsilon)p_h - \varepsilon p_h' > p_h - \varepsilon \geqq 0$ by definition of ε. Since $s^* = e + \varepsilon(e - s')$ and every coordinate of $e - s'$ is less than zero, every coordinate of s^* is strictly less than $\bar{e}$.

(c) $i(M^*) \subset M$. From Theorem 3.1 [1] it follows that the minimax risk for both the original and the auxiliary problem is equal to $\bar{e}$. If $\delta^* = \delta^*(\varphi_0^*, \cdots, \varphi_r^*)$ has minimax risk for the auxiliary problem, then for all θ, $R(\theta, i(\delta^*)) = R^*(\theta, \delta^*) \leqq \bar{e}$, so that $i(\delta^*)$ has minimax risk for the original problem.

(d) $i(W^*) = W$. If $\delta^* \in W^*$, then it is easy to verify that (1) holds for $i(\delta^*)$, so that $i(W^*) \subset W$. If $\delta = \delta(\varphi_0, \cdots, \varphi_n) \in W$ then by Corollary 4.1 [1], for all $\theta_0 \in \Omega_0'$, $\sum_{j=0}^{n} w_{\cdot j} E_{\theta_0}[\varphi_j(X)] = e$, which implies that for all $\theta_0 \in \Omega_0'$, $E_{\theta_0}[\varphi_j(X)] = 0$ for all $j \notin J$. Then by assumption (iv), $E_\theta[\varphi_j(X)] = 0$ for all $j \notin J$ and for all $\theta \in \Omega'$. Since procedures with the same risk function are identified, we may assume that $\varphi_j(x) = 0$ for all $j \notin J$, so that $\delta^* = \delta^*(\varphi_0^*, \cdots, \varphi_r^*)$ with $\varphi_h^*(x) = \varphi_{j_h}(x)$ for $j_h \in J$ is a procedure for the auxiliary problem. It is easily verified that $\delta^* \in W^*$ and that $\delta = i(\delta^*)$ so that $W \subset i(W^*)$. $\square$

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REFERENCE

[1] SCHAAFSMA, WILLEM. (1969). Minimax risk and unbiasedness for multiple decision problems of Type 1. *Ann. Math. Statist.* **40** 1684–1720.