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INADMISSIBILITY OF THE BEST INVARIANT TEST IN THREE OR MORE DIMENSIONS

BY STEPHEN PORTNOY[1] AND CHARLES STEIN[2]

Harvard University

Consider the problem where one observes (X, Y) with X an arbitrary random quantity and Y a p-dimensional random vector, and where there are two hypotheses, H_i, under which $(X, Y-\eta)$ is distributed according to P_i $(i = 0, 1)$ for some unknown point $\eta \in R^P$. Lehmann and Stein [1] show that if $p = 1$, then under certain reasonable conditions, the best invariant test of H_0 versus H_1 is admissible. Here we present an example showing that the analogous result does not hold if $p \geqq 3$. The example was suggested by certain problems in the recovery of interblock information in the randomized designs considered in [2]. Although the present example is somewhat artificial and not directly applicable to the above problems, it is not unlikely that similar methods might also work there.

Let η be an unknown point in R^P and let $\varepsilon_i = (-1)^i$ for $i = 0, 1$. Suppose $X = (W, V)$ and Y are distributed as follows under H_i: W is normally distributed with mean ε_i and variance 1, V is independent of W and uniformly distributed over the surface of the unit sphere in R^P, and $Y = \eta+\varepsilon_i V$. The problem is invariant under translation: $(W, V, Y) \to (W, V, Y+c)$ for $c \in R^P$. An invariant test is one depending only on (W, V) and the best invariant test of given size accepts H_0 if and only if $W > K$. We shall take $K = 0$ for ease of computation. We want to show that for sufficiently small $a > 0$ and sufficiently large b, we have for all η

$$P_{0,\eta}\left\{W+\frac{aV'Y}{b+||Y||^2} > 0\right\} > P_0\{W > 0\}, \tag{1}$$

$$P_{1,\eta}\left\{W+\frac{aV'Y}{b+||Y||^2} > 0\right\} < P_1\{W > 0\}, \tag{2}$$

where V and Y are column vectors, V' denotes the transpose of V and $||Y||^2 = Y'Y$. Because of the symmetry of the problem under interchange of the two hypotheses, we need only prove (1).

We use the identity

$$\frac{1}{A+B} = \frac{1}{A}-\frac{B}{A^2}+\frac{B^2}{A^2(A+B)} \tag{3}$$

with

$$A = b+||\eta||^2, \qquad B = 2\eta' V+1, \tag{4}$$

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and note that (under H_0)

$$\|Y\|^2 = \|\eta+V\|^2 = \|\eta\|^2+2\eta'V+1 \tag{5}$$

(since $\|V\|^2 = 1$). Thus,

$$\begin{aligned}\frac{V'Y}{b+\|Y\|^2} &= \frac{1+\eta'V}{b+\|\eta\|^2}\left\{1-\frac{2\eta'V+1}{b+\|\eta\|^2}+\frac{(2\eta'V+1)^2}{(b+\|\eta\|^2)(b+\|\eta+V\|^2)}\right\} \\ &= \frac{1}{b+\|\eta\|^2}\left\{1+\eta'V-\frac{2(\eta'V)^2}{b+\|\eta\|^2}+O\left(\frac{1}{b^{\frac{1}{2}}}\right)\right\}\end{aligned} \tag{6}$$

where the remainder is uniform in both η and V; this follows from the fact that $\|V\| = 1$ and the results

$$\left|\frac{\eta'V}{b+\|\eta\|^2}\right| \leqq \frac{\|\eta\|}{b+\|\eta\|^2} = O\left(\frac{1}{b^{\frac{1}{2}}}\right), \tag{7}$$

$$\begin{aligned}\frac{(1+2V'\eta)^2}{b+\|\eta+V\|^2} &= \frac{(2V'V+2V'\eta-1)^2}{b+\|\eta+V\|^2} \leqq \frac{8[V'(V+\eta)]^2+2}{b+\|\eta+V\|^2} \\ &\leqq \frac{8\|V+\eta\|^2+2}{b+\|\eta+V\|^2} = O(1).\end{aligned} \tag{8}$$

Therefore

$$P_{0,\eta}\left\{W+\frac{aV'Y}{b+\|Y\|^2} > 0\right\} = E\Phi\left(1+\frac{aV'Y}{b+\|Y\|^2}\right)$$

$$\begin{aligned}&= E\Phi\left(1+\frac{a}{b+\|\eta\|^2}\left[1+\eta'V-\frac{2(\eta'V)^2}{b+\|\eta\|^2}+O\left(\frac{1}{b^{\frac{1}{2}}}\right)\right]\right) \\ &= \Phi(1)+\frac{a\Phi'(1)}{b+\|\eta\|^2}E\left[1+\eta'V-\frac{2(\eta'V)^2}{b+\|\eta\|^2}+O\left(\frac{1}{b^{\frac{1}{2}}}\right)\right] \\ &\quad+\frac{a^2}{2(b+\|\eta\|^2)^2}E\Phi''(U)\left[1+\eta'V-\frac{2(\eta'V)^2}{b+\|\eta\|^2}+O\left(\frac{1}{b^{\frac{1}{2}}}\right)\right]^2\end{aligned} \tag{9}$$

where U is a value between 1 and $(1+aV'Y/(b+\|Y\|^2)$. But

$$E(\eta'V) = 0 \quad\text{and}\quad E(\eta'V)^2 = \frac{1}{p}\|\eta\|^2, \tag{10}$$

and since $\Phi''(x)$ is uniformly bounded, the final expectation in (9) is bounded by

$$KE\left[3+\|\eta\|+O\left(\frac{1}{b^{\frac{1}{2}}}\right)\right]^2 \leqq (1+\|\eta\|^2)O(1) \tag{11}$$

(again uniformly in η and V). Therefore,

$$(12)\quad P_{0,\eta}\left\{W+\frac{aV'Y}{b+\|Y\|^2}>0\right\}=\Phi(1)+\frac{a\Phi'(1)}{b+\|\eta\|^2}\left[1-\frac{(2/p)\|\eta\|^2}{b+\|\eta\|^2}+O\left(\frac{1}{b^{\frac{1}{2}}}\right)\right.$$
$$\left.+\frac{a(1+\|\eta\|^2)}{b+\|\eta\|^2}O(1)\right]>\Phi(1)=P_{0,\eta}\{W>0\}$$

for all $\eta\in R^p$ if $p\geqq 3$ provided $a>0$ is sufficiently small and b sufficiently large.

Some comments about the inessentiality of special features of this example should be made.

(1) We have not really used the fact that $W\sim N(\varepsilon_i, 1)$, but only that (i) the distribution of W under H_0 is the same as the distribution of $-W$ under H_1, (ii) the condition $W>0$ is equivalent to rejection of the likelihood ratio test (based on W alone) for some critical level, and (iii) W has a density under H_0 which is positive at zero and has a uniformly bounded first derivative.

(2) We have strongly used the condition that the distribution of V is spherically symmetric (to obtain (10)), but the condition $\|V\|^2=1$ could be circumvented with somewhat greater care (both in the definition of the improved test and in the calculations).

(3) The assumption of a singular distribution for Y, that is $Y=\eta+\varepsilon_i V$, is also inessential since η can be replaced by $\eta+Z$ (where Z has an arbitrary distribution independent of (V, W) and the same under H_0 and H_1). Using conditional distributions (given Z), the above proof shows that (12) holds for the conditional probability given Z and hence, for the unconditional probability.

REFERENCES

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