The Annals of Mathematical Statistics
1971, Vol. 42, No. 2, 662–679

ADMISSIBILITY OF THE USUAL CONFIDENCE SETS FOR A CLASS OF BIVARIATE POPULATIONS

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For a bivariate population, with a probability density $p(|x-\theta|)$ where $p(t)$ is strictly decreasing for $t \geqq 0$, when a single observation is taken, the usual confidence sets are circles of constant radius centered at the observed value. Previous results of Kiefer imply that these circles have the minimax property of minimizing the maximum expected Lebesgue measure of the confidence sets for a given lower confidence level. It is now shown here that subject to mild conditions on $p(t)$, the confidence circles are also essentially unique in having the minimax property. The result generalizes the result proved previously for the bivariate normal case.

1. Introduction. For an m-variate normal population with the identity matrix as covariance matrix, the usual confidence sets for estimating the population mean are m-dimensional spheres of fixed volume centered at the observed sample mean. They have the minimax property that amongst the confidence procedures with a given lower confidence level, they minimize the maximum expected Lebesgue measure of the confidence sets.

Investigating a conjecture of Stein (1962), it was shown in previous papers that in the univariate and bivariate cases the usual confidence sets are essentially unique in having the minimax property (1969), and that for $m \geqq 3$, the usual sets are not essentially unique and are in fact inadmissible (1967).

The uniqueness in the univariate case was later shown to hold generally for the invariant confidence sets based on n observations for any population with a probability density function $f(x-\theta)$ subject to two mild conditions (1970).

In the following we now show that the property of essential uniqueness in the bivariate case holds also for any population with a density function $p(|x-\theta|]$ where the function $p(t)$ is strictly decreasing for $t \geqq 0$ and satisfies certain other conditions, and only one observation is taken from the population. The restriction to the single observation case is necessary as the proof depends upon the monotonicity of the function $p(t)$.

The minimax property however holds without any restriction on the density function p. In fact from the results proved by Kiefer (1957) it follows that the minimax property holds for the invariant confidence sets for an m-dimensional random vector with a density function $f(|x-\theta|)$ where θ is the m-dimensional translation parameter, without any restrictions on f.

2. Preliminaries. $X = (X_1, X_2)$ denotes a random vector which assumes values $x = (x_1, x_2)$ in the 2-dimensional Euclidean sample space R; X has a probability density $p(|x-\theta|)$ where $\theta = (\theta_1, \theta_2)$ is a point in 2-dimensional parameter space

Received October 1, 1969; revised October 12, 1970.

Ω; for any vector $u = (u_1, u_2)$, $|u|$ denotes the absolute magnitude $= (u_1^2 + u_2^2)^{\frac{1}{2}}$; the function $p(t)$ strictly decreases as t increases for $t \geqq 0$. On R, Ω and the product space $R \times \Omega$ is defined the Lebesgue measure, all sets considered being Lebesgue measurable. Confidence procedures including randomized ones are defined as in Section 2 of the previous paper (1969). We obtain the class of all such procedures by taking as the decision space

$$\mathscr{D} = \{\phi(x, \theta);\ \phi(x, \theta) \text{ jointly measurable in } x \text{ and } \theta;\quad 0 \leqq \phi(x, \theta) \leqq 1\}. \tag{1}$$

For any confidence procedure, $\phi(x, \theta)$ is the probability that the point θ is included in the confidence set selected, when x is the observed value. Equivalence of procedures is defined by

DEFINITION 2.1. Two procedures ϕ_1 and ϕ_2 are equivalent if $\phi_1(x, \theta) = \phi_2(x, \theta)$ for almost all $(x, \theta) \in R \times \Omega$.

For a more detailed discussion of randomized confidence procedures, definition of equivalent procedures, and restrictions on the geometrical form of the sets which may be imposed as an alternative to the definition of equivalence procedures, we refer to Sections 2, 3 and 4 of [2].

For given confidence level $(1-\alpha)$, the smallest confidence sets are given by

$$|x-\theta| \leqq h \tag{2}$$

where h is fixed by $\int_{[|x-\theta|<h]} p(|x-\theta|)\, dx = 1-\alpha$.

Here we have written dx for short for $dx_1 dx_2$. Similarly for any vector u we shall write du for $du_1 du_2$.

If $\phi_0(x, \theta)$ denotes the procedure consisting of the confidence sets defined by (2), then since for each x, $\phi(x, \theta)$ gives the probability that the point θ is included in the confidence sets, we have

$$\begin{aligned} \phi_0(x, \theta) &= 1 \quad \text{if } |x-\theta| \leqq h, \\ &= 0 \quad \text{if } |x-\theta| > h. \end{aligned} \tag{3}$$

3. Further conditions on $p(t)$. We assume that besides being strictly decreasing, $p(t)$ satisfies the following further conditions:

CONDITION 1. The frequency function $p(|x-\theta|)$, has finite second moment, i.e.

$$\int_R |x-\theta|^2 p(|x-\theta|)\, dx = 2\pi \int_0^\infty t^3 p(t)\, dt < \infty. \tag{4}$$

CONDITION 2. There exists a positive function $\beta(\tau)$ such that

$$\lim_{\tau\to\infty} \tau^2 \int_{|u|>\tau/\beta} p(|u|)\, du = 0, \tag{5}$$

and

$$\frac{\beta}{[\log \tau]^{\frac{1}{2}}} \to \infty \qquad \text{as } \tau \to \infty. \tag{6}$$

Here we write β for short for $\beta(\tau)$.

CONDITION 3. If Δ denotes a positive or negative increment in the value of t,

(7) $$\liminf_{\Delta\to 0} \frac{1}{\Delta}\left[1-\frac{p(h+\Delta)}{p(h)}\right] > 0.$$

We note that Condition 1 implies that $\lim_{\tau\to\infty} \tau^2 \int_{|u|>\tau} p(|u|)\,du = 0$. Condition 2 involves a slightly stronger assumption.

If a β satisfying (5) and (6) exists, by replacing it, if necessary, by a more slowly varying function, we can always secure that

(8) $$\beta/\tau \to 0 \qquad \text{as} \quad \tau \to \infty.$$

Let 2γ be the inferior limit in Condition 3. Then the condition implies that there exists a positive number $\delta_0 < h$, such that for any δ, $0 < \delta \leqq \delta_0 < h$

(9) $$p(h+\delta) \leqq p(h)(1-\gamma\delta) \qquad \text{and,}$$

$$p(h-\delta) \geqq p(h)(1+\gamma\delta).$$

Condition 3 thus implies that $p(t)$ decreases at a certain minimum rate in the neighbourhood of the point $t = h$.

4. Bayes risk. We assume a prior distribution on Ω given by

(10) $$\xi_\tau(\theta) = \frac{1}{2\pi\tau^2}\exp\left(-\frac{|\theta|^2}{2\tau^2}\right).$$

We define a loss function for the procedure ϕ by

(11) $$L_\phi(x, \theta) = bv\phi(x,\cdot)-\phi(x, \theta),$$

where,

(12) $$v\phi(x,\cdot) = \int_\Omega \phi(x, \theta)\,d\theta,$$

and $$b = p(h).$$

For the prior distribution in (10), let $\phi_\tau(x, \theta)$ be the Bayes procedure; let $L_\tau(x, \theta)$ denote the loss function of the Bayes procedure and $L_0(x, \theta)$ that of the procedure ϕ_0 in (3). Let E_τ denote expectation with respect to the prior distribution (4). We now prove the following:

LEMMA 4.1. *If the function $p(t)$ satisfies Conditions* 1, 2, *and* 3 *given in Section* 4, *then as $\tau \to \infty$*

(13) $$E_\tau L_0(x, \theta)-E_\tau L_\tau(x, \theta) = O(1/\tau^2).$$

PROOF. Let δ be a given arbitrarily small positive number, such that

(14) $$\delta \leqq \delta_0, \qquad \text{where} \quad \delta_0 \text{ is as in (9).}$$

We put

(15) $$p(h+\delta) = b(1-\eta), \qquad 0 < \eta < 1,$$

(16) $$p(h-\delta) = b(1+\eta'), \qquad 0 < \eta',$$

and

(17) $$\varepsilon = \min(\eta/3, \eta'/3, \delta).$$

Since $\eta < 1$,

(18) $$\varepsilon < \tfrac{1}{3}.$$

The Bayes risk is given by,

(19) $$E_\tau L_\tau(x, \theta) = \frac{1}{(2\pi)\tau^2}\int_\Omega \exp\left(-\frac{|\theta|^2}{2\tau^2}\right) d\theta \int_R [bv_\tau(x) - \phi_\tau(x, \theta)]p(|x-\theta|)\, dx$$

where

(20) $$v_\tau(x) = v\phi_\tau(x, \cdot) = \textstyle\int_\Omega \phi_\tau(x, \theta)\, d\theta.$$

We now partition, the domain of integration $R\times\Omega$ of the integral in (19), into subsets, D_1, D_2, D_3 respectively defined by

$$\{|x-\theta| > \tau/\beta\}, \{|x|/\tau > \varepsilon\beta, |x-\theta| \leqq \tau/\beta\}$$

and $\{|x|/\tau \leqq \varepsilon\beta, |x-\theta| \leqq \tau/\beta\}$. Let I_1, I_2 and I_3 respectively denote the values of the integral on D_1, D_2, D_3.

We now take, τ_0 so that for $\tau \geqq \tau_0$ each of the following conditions is satisfied viz.

(21 i) $$\textstyle\int_{|u|>\tau/\beta} p(|u|)\, du \leqq \delta/\tau^2,$$

(21 ii) $$\frac{1}{2\pi\tau^2}\int_{|u|/\tau > (\varepsilon\beta - 1/\beta)} \exp\left(-\frac{|u|^2}{2\tau^2}\right) du \leqq \delta/\tau^2 \qquad \text{by (6) and (7)},$$

(21 iii) $$\frac{m_0 + 2\delta_0}{2\tau^2} \leqq \varepsilon,$$

where $m_0 = \int_{|u|<\infty} |u|^2 p(|u|)\, du$ which is finite by Condition 1 and

(21 iv) $$\beta h_1/\tau \leqq 1 \qquad \text{where} \quad h_1 = h+\delta_0.$$

Then using that $0 \leqq \phi\,(x, \theta) \leqq 1$ and that $\int_R p(|x-\theta|)\, dx = 1$, we have

(22) $$I_1 \geqq -\delta/\tau^2 \qquad \text{by (21 i)}$$

and $$I_2 \geqq -\frac{1}{2\pi\tau^2}\int_{|\theta|/\tau > (\varepsilon\beta - 1/\beta)} \exp\left(-\frac{|\theta|^2}{2\tau^2}\right) d\theta \geqq -\delta/\tau^2 \qquad \text{by (21 ii)}$$

In the integral for I_3, by interchanging the order of integration with respect to x and θ, putting $u = x-\theta$, and writing $(x\cdot u)$ for the inner product x and u and substituting $\int\phi_\tau(x, \theta)\,d\theta$ for $v_\tau(x)$, we get

$$(23)\qquad I_3 = \frac{1}{2\pi\tau^2}\int_{|x|/\tau \leqq \beta\varepsilon} \exp\left(-\frac{|x|^2}{2\tau^2}\right)$$
$$\cdot\, dx \int_{|x-\theta|\leqq\tau/\beta}\left\{bK_\tau(x) - p(|u|)\exp\left[\frac{(x\cdot u)}{\tau^2}-\frac{|u|^2}{2\tau^2}\right]\right\}\phi_\tau(x, \theta)\,d\theta$$

where

$$(24)\qquad K_\tau(x) = \int_{|u|\leqq\tau/\beta} p(|u|)\exp\left[\frac{(x\cdot u)}{\tau^2}-\frac{|u|^2}{2\tau^2}\right]du,$$
$$\geqq \int_{|u|\leqq\tau/\beta} p(|u|)\left[1+\frac{(x\cdot u)}{\tau^2}-\frac{|u|^2}{2\tau^2}\right]du.$$

The integral of the term $(x\cdot u)$ vanishes by symmetry and the integral of $|u|^2 \leqq$ the second moment of the distribution $= m_0$ by (21iii). Hence using (21i), we get from (24)

$$(25)\qquad K_\tau(x) \geqq 1-\frac{\delta}{\tau^2}-\frac{m_0}{2\tau^2} \geqq 1-\frac{m_1}{2\tau^2}.$$

where we put $m_1 = m_0+2\delta_0$.

Hence we have from (23),

$$(26)\qquad I_3 \geqq \frac{1}{2\pi\tau^2}\int_{|x|/\tau\leqq\beta\varepsilon}\exp\left(-\frac{|x|^2}{2\tau^2}\right)$$
$$\cdot\, dx\int_{|x-\theta|\leqq\tau/\beta}\left\{b\left(1-\frac{m_1}{2\tau^2}\right)-p(|u|)\exp\left[\frac{(x\cdot u)}{\tau^2}\right]\right\}\phi_\tau(x, \theta)\,d\theta.$$

Let B_x denote the subset of Ω, defined by $\theta \in B_x$, if, and only if,

$$(27)\qquad p(|u|)\exp\left[\frac{(x\cdot u)}{\tau^2}\right] > b\left(1-\frac{m_1}{2\tau^2}\right) \qquad \text{and,}$$
$$|u| \leqq \tau/\beta.$$

We shall show that the set B_x is completely contained in the circle in Ω, defined by $|u| \leqq h+\delta$, and completely contains the circle defined by $|u| \leqq h-\delta$.

For $(x, \theta)\in D_3$, $|(x\cdot u)|/\tau^2 \leqq \varepsilon$. Hence

$$(28)\qquad \exp\left[\frac{(x\cdot u)}{\tau^2}\right] > 1-\varepsilon$$

and
$$\exp\left[\frac{(x\cdot u)}{\tau^2}\right] < 1+\frac{\varepsilon}{1-\varepsilon}$$
$$< 1+2\varepsilon \qquad \text{since} \quad \varepsilon < \tfrac{1}{3} \text{ by (18).}$$

Also

(29)
$$1-\frac{m_1}{2\tau^2} \geqq 1-\varepsilon \qquad \text{by (21iii).}$$

If $|u| > h+\delta$, $p(|u|) < b(1-3\varepsilon)$ by (15) and (17), and hence by (28),

(30)
$$p(|u|)\exp\left[\frac{(x\cdot u)}{\tau^2}\right] < b(1-3\varepsilon)(1+2\varepsilon) < b(1-\varepsilon)$$

and if $|u| \leqq h-\delta$, $p(|u|) \geqq b(1+3\varepsilon)$ by (16) and (17),
and hence by (28),

(31)
$$p(|u|)\exp\left[\frac{(x\cdot u)}{\tau^2}\right] \geqq b(1+3\varepsilon)(1-\varepsilon)$$
$$> b(1-\varepsilon).$$

(29), (30) and (31) combined with the definition of B_x in (27) show that the boundary of the set B_x lies wholly within the area bounded by two concentric circles with centre x and radii $h-\rho$ and $h+\rho$ where

(32)
$$\rho \leqq \delta.$$

Let $f(x)$ denote the inner integral in the right-hand side of (26). Clearly $f(x)$ is minimized by putting $\phi_\tau(x, \theta) = 1$ for $\theta \in B_x$, and $\phi_\tau(x, \theta) = 0$ for $\theta \notin B_x$, and taking the integral over the whole set B_x. We also substitute in the inner integral by

$$\exp\left[\frac{(x\cdot u)}{\tau^2}\right] \leqq 1+\frac{(x\cdot u)}{\tau^2}+\frac{|x|^2|u|^2}{2\tau^4}[1+\varepsilon+\cdots]$$
$$\leqq 1+\frac{(x\cdot u)}{\tau^2}+\frac{|x|^2|u|^2}{\tau^4}$$
$$\leqq 1+\frac{(x\cdot u)}{\tau^2}+\frac{{h_1}^2|x|^2}{\tau^4}$$

where $h_1 = h+\delta_0 \geqq h+\delta \geqq |u|$ by (32) and (14). We then obtain from (26).

$$f(x) \geqq \int_{B_x}[b-p(|u|)]\,du-\int_{B_x}\left[\frac{bm_1}{2\tau^2}+\frac{{h_1}^2|x|^2}{\tau^4}\right]du$$

(33)
$$-\int_{B_x}\frac{(x\cdot u)}{\tau^2}p(|u|)\,du$$
$$= t_1+t_2+t_3 \quad \text{say.}$$

Now since $p(|u|) > b$ for $|u| < h$, $p(|u|) < b$ for $|u| > h$, t_1 is minimized when B_x coincides with the circle $|u| \leqq h$. Hence putting $v_0 = \pi h^2$, and using (2), we get,

$$t_1 \geqq bv_0-(1-\alpha). \tag{34}$$

Next, since the set B_x is wholly contained in circle of radius $h_1 = h+\delta_0$,

$$t_2 \geqq -\pi h_1{}^2\left[\frac{bm_1}{2\tau^2}+\frac{h_1{}^2|x|^2}{\tau^4}\right]. \tag{35}$$

Let C_x denote the circle in Ω, defined by $|u| \leqq h$. In the integral for t_3, we partition the set B_x by putting

$$B_x = C_x+(B_x-B_x\cdot C_x)-(C_x-C_x\cdot B_x). \tag{36}$$

By symmetry the integral of $(x\cdot u)$ on C_x vanishes. Hence using that in the domain of integration $|u| \leqq h_1$, we get

$$t_3 \geqq -\int_{(B_x-B_x\cdot C_x)}\frac{h_1|x|}{\tau^2}p(|u|)\,du-\int_{(C_x-C_x\cdot B_x)}\frac{h_1|x|}{\tau^2}p(|u|)\,du. \tag{37}$$

To obtain a lower bound for t_3, we now prove that ρ in (32) is in fact $0(1/\tau^2)$.

Since in (27),

$$\exp\left[\frac{(x\cdot u)}{\tau^2}\right] \leqq 1+\frac{|x|\,|u|}{\tau^2}\,[1+\varepsilon+\varepsilon^2\cdots]$$

$$\leqq 1+2\frac{|x|\cdot h_1}{\tau^2},$$

therefore for $\theta \in B_x$

$$\begin{aligned} p(|u|) &> b\left(1-\frac{m_1}{2\tau^2}\right)\left(1+\frac{2h_1|x|}{\tau^2}\right)^{-1} \\ &> b\left(1-\frac{m_1}{2\tau^2}\right)\left(1-\frac{2h_1|x|}{\tau^2}\right) \\ &> b\left(1-\frac{m_1}{2\tau^2}-\frac{2h_1|x|}{\tau^2}\right). \end{aligned} \tag{38}$$

Let $u = h+\Delta$. Then since $\Delta \leqq \delta$ by (32), we have by Condition 3, and (9)

$$p(|u|) \leqq b(1-\gamma\Delta). \tag{39}$$

By (38) and (39),

$$\Delta < \frac{1}{\gamma}\left(\frac{m_1}{2\tau^2}+\frac{2h_1|x|}{\tau^2}\right). \tag{40}$$

Again in (27)

$$\exp\left[\frac{(x\cdot u)}{\tau^2}\right] \geqq 1-\frac{|x|h_1}{\tau^2}.$$

Hence for $\theta \notin B_x$

$$\begin{aligned} p(|u|) &\leqq b\left(1-\frac{m_1}{2\tau^2}\right)\left(1-\frac{|x|h_1}{\tau^2}\right)^{-1} && \text{by (27)} \\ &\leqq b\left(1+\frac{|x|h_1}{\tau^2}\frac{1}{1-\varepsilon}\right) \\ &\leqq b\left(1+\frac{2\,|x|\,h_1}{\tau^2}\right) && \text{by (18).} \end{aligned}$$

If $u = h-\Delta$, then by Condition 3, $p(|u|) \geqq b(1+\gamma\Delta)$.

Hence for $\theta \in B_x$,

$$\Delta \leqq \frac{2}{\gamma}\frac{h_1|x|}{\tau^2}. \tag{41}$$

Combining (40) and (41), we get in (32),

$$\rho \leqq \frac{1}{\gamma}\left(\frac{m_1}{2\tau^2}+\frac{2h_1|x|}{\tau^2}\right). \tag{42}$$

Thus in (37), the sets $(B_x-B_x\cdot C_x)$ and $(C_x-C_x\cdot B_x)$ are both contained within the region enclosed by concentric circles with center x and radii $(h+\rho)$ and $(h-\rho)$ whose area is $4\pi h\rho$. Also throughout this region,

$$\begin{aligned} p(|u|) &\leqq p(h-\delta_0) \\ &= b_0 \quad \text{say.} \end{aligned} \tag{43}$$

Note that by the strictly decreasing property of $p(t)$, b_0 must be finite as $h-\delta_0 > 0$ by (9) and hence $p(h-\delta_0) < p((h-\delta_0)/2) < \infty$. We thus obtain in (37), using (42),

$$\begin{aligned} t_3 &\geqq -\frac{4\pi hh_1b_0}{\gamma}\left[\frac{m_1|x|}{2\tau^4}+\frac{2h_1|x|^2}{\tau^4}\right] \\ &\geqq -\frac{4\pi hh_1b_0}{\gamma}\left[\frac{m_1|x|}{2\tau^3}+\frac{2h_1|x|^2}{\tau^4}\right] && \text{assuming that} \quad \tau_0 > 1. \end{aligned} \tag{44}$$

Combining (34), (35) and (44) with (33) we obtain the lower bound for $f(x)$. We substitute this lower bound for the inner integral in (26). As the lower bound is everywhere negative, the integral with respect to x can be taken over the whole space R. Hence integrating out with respect to x and using

$$\frac{1}{2\pi\tau^2}\int_R \exp\left(-\frac{|x|^2}{2\tau^2}\right)\frac{|x|}{\tau}\,dx = \frac{(2\pi)^{\frac{1}{2}}}{2}$$

$$\frac{1}{2\pi\tau^2}\int_R \exp\left(-\frac{|x|^2}{2\tau^2}\right)\frac{|x|^2}{\tau^2}\,dx = 2,$$

we get

$$I_3 \geqq bv_0-(1-\alpha)-\pi h_1{}^2\left(\frac{bm_1}{2\tau^2}+\frac{2h_1{}^2}{\tau^2}\right) \tag{45}$$
$$-\frac{4\pi hh_1 b_0}{\gamma}\left[\frac{m_1}{2\tau^2}\cdot\frac{(2\pi)^{\frac{1}{2}}}{2}+\frac{4h_1}{\tau^2}\right].$$

We now combine (45) with (22) to obtain the lower bound for $E_\tau L_\tau(x, \theta)$. It is also easily verified from the definition of ϕ_0 in (3) that

$$E_\tau L_0(x, \theta) = bv_0-(1-\alpha). \tag{46}$$

Substituting (46), we obtain from (22) and (45) that

$$E_\tau L_0(x, \theta)-E_\tau L_\tau(x, \theta) \leqq A/\tau^2 \qquad \text{for all} \quad \tau \geqq \tau_0 \tag{47}$$

where A is a constant independent of τ and δ. This completes the proof of Lemma 4.1.

5. Main Theorem. The loss function for the procedure $\phi_0(x, \theta)$ is denoted by $L_0(x\ \theta)$. Similarly we shall denote the loss function for the procedure $\phi_1(x, \theta)$ by $L_1(x, \theta)$. Using the result in (47), we now state and prove the following:

THEOREM 5.1. *$\phi_0(x, \theta)$ being the usual confidence procedure defined by* (3), *if any other procedure $\phi_1(x, \theta)$ exists such that,*

$$E_\theta L_1(x, \theta) \leqq E_\theta L_0(x, \theta) \qquad \text{for all} \quad \theta \in \Omega, \tag{48}$$

then ϕ_1 is equivalent to ϕ_0 i.e. $\phi_1(x, \theta) = \phi_0(x, \theta)$ for almost all $(x, \theta) \in (R \times \Omega)$.

PROOF. We define functions $U_1(x)$, $U_0(x)$ on R by

$$U_1(x) = \int_\Omega [b-p(|x-\theta|)]\phi_1(x, \theta)\, d\theta, \qquad \text{and} \tag{49}$$
$$U_0(x) = \int_\Omega [b-p(|x-\theta|)]\phi_0(x, \theta)\, d\theta.$$

Using the definition of ϕ_0 in (3), we get,

$$U_1(x)-U_0(x) = \int_{|x-\theta|\leqq h}[p(|x-\theta|)-b][1-\phi_1(x, \theta)]\, d\theta \tag{50}$$
$$+\int_{|x-\theta|>h}[b-p(|x-\theta|)]\phi_1(x, \theta)\, d\theta.$$

Hence as $p(t)$ is strictly decreasing and $p(h) = b$

$$U_1(x)-U_0(x) \geqq 0. \tag{51}$$

Suppose ϕ_1 is not equivalent to ϕ_0. Then the strict inequality in (51) holds on a non-null subset of R. Hence there exist positive constants k and a such that

$$\int_{|x|\leqq a}[U_1(x)-U_0(x)]\, dx = k;\ k > 0. \tag{52}$$

Let T_a denote the subset of R defined by $\{x\colon |x| \leqq a\}$, and let $T_a{}^c$ be its complementary set.

We now write down the expressions for $E_\tau L_1(x, \theta)$ and $E_\tau L_0(x, \theta)$ as in (19).

Then combining the two expressions we obtain the integral for $E_\tau L_1(x, \theta) - E_\tau L_0(x, \theta)$. We interchange the order of integration with respect to x and θ and partition the integral on R into integrals on T_a and $T_a{}^c$. We thus obtain, writing $G_\tau(x)$ for the integrand on R,

$$(53) \qquad E_\tau L_1(x, \theta) - E_\tau L_0(x, \theta) = \frac{1}{2\pi\tau^2}\int_{T_a} G_\tau(x)\,dx + \frac{1}{2\pi\tau^2}\int_{T_a{}^c} G_\tau(x)\,dx$$

$$= T_1 + T_2 \quad \text{say.}$$

The integrals corresponding to T_1 and T_2 have been dealt with in the previous paper ((1969) (44) to (54)). A similar argument applies here and it is unnecessary to repeat the detailed argument. We show that the integrand $G_\tau(x)$ is bounded in absolute magnitude, uniformly in τ by a function $G(x)$ which is integrable on T_a. Also as $\tau \to \infty$, $G_\tau(x) \to [U_1(x) - U_0(x)]$. Hence by the Dominated Convergence Theorem, for sufficiently large τ.

$$(54) \qquad (2\pi\tau^2)T_1 \geqq k - \delta \qquad \text{by (52).}$$

Similarly $(-T_2)$ gives the improvement in risk of the procedure ϕ_1 over that of ϕ_0. This improvement must be less than that of the Bayes procedure. Hence

$$(55) \qquad T_2 \geqq -A/\tau^2 \qquad \text{by (47).}$$

Since the left-hand side of (53) must be non-positive by (48), we have $k \leqq 2\pi A + \delta$, and so, as δ in (14) can be taken arbitrarily small,

$$(56) \qquad k \leqq 2\pi A.$$

Since (56) holds however large a in (52) may be, letting $a \to \infty$, we obtain

$$(57) \qquad \int_R [U_1(x) - U_0(x)]\,dx = \text{a finite number} \quad M \quad \text{say,}$$

and since $M \geqq k$ in (52), $M > 0$.

[Explanatory note: As the following argument is rather long we shall give its brief outline. We consider the improvement in the expected risk of the procedure ϕ_1 over that of ϕ_0 viz. $E_\tau L_1(x, \theta) - E_\tau L_0(x, \theta)$, whose value is given by the right-hand side of (53). It is shown that the worsening of the expected risk of ϕ_1 over that of ϕ_0 on the set T_a can be made arbitrarily close to M by taking a sufficiently large. This worsening has to be offset by the improvement in risk on the complementary set $T_a{}^c$. But it is shown that the latter, for any fixed a can be made arbitrarily small by making τ sufficiently large. Hence M must be $= 0$. The theorem follows from this.]

Now for each $x \in R$, we partition Ω into subsets E_x, F_x, G_x and H_x defined by

$$\begin{aligned} &\theta \in E_x, && \text{if and only if } |x-\theta| \leqq h-\delta_0 \\ &\theta \in F_x, && \text{if and only if } h-\delta_0 < |x-\theta| \leqq h, \\ &\theta \in G_x, && \text{if and only if } h < |x-\theta| \leqq h+\delta_0 \\ &\theta \in H_x, && \text{if and only if } h+\delta_0 < |x-\theta|. \end{aligned} \tag{58}$$

In (58), δ_0 is the number defined in (9). Then substituting for $U_1(x)$ and $U_0(x)$ by (49) and for $\phi_0(x, \theta)$ by (3), we obtain from (57),

$$\begin{aligned} &\textstyle\int_R dx \int_{E_x} [p(|u|)-b][1-\phi_1(x, \theta)]\, d\theta + \int_R dx \int_{F_x} [p(|u|)-b][1-\phi_1(x, \theta)]\, d\theta \\ &\quad \textstyle + \int_R dx \int_{G_x} [b-p(|u|)]\phi_1(x, \theta)\, d\theta + \int_R dx \int_{H_x} [b-p(|u|)]\phi_1(x, \theta)\, d\theta = M. \end{aligned} \tag{59}$$

The integrands of each of the integrals in the left-hand side of (59), being non-negative, each of these integrals must converge and we have (writing ϕ_1 for short for $\phi_1(x, \theta)$),

$$\textstyle\int_R dx \int_{E_x} [p(|u|)-b] \cdot (1-\phi_1)\, d\theta = M_1 \geqq 0, \tag{60 i}$$

$$\textstyle\int_R dx \int_{F_x} [p(|u|)-b] \cdot (1-\phi_1)\, d\theta = M_2 \geqq 0. \tag{60 ii}$$

$$\textstyle\int_R dx \int_{G_x} [b-p(|u|)]\phi_1\, d\theta = M_3 \geqq 0, \tag{60 iii}$$

$$\textstyle\int_R dx \int_{H_x} [b-p(|u|)]\phi_1 = M_4 \geqq 0, \tag{60 iv}$$

where $M_1+M_2+M_3+M_4 = M$.

We now put

$$\begin{aligned} v_E(x) &= \textstyle\int_{E_x} (1-\phi_1)\, d\theta, \\ v_F(x) &= \textstyle\int_{F_x} (1-\phi_1)\, d\theta, \\ v_G(x) &= \textstyle\int_{G_x} \phi_1\, d\theta, \qquad \text{and} \\ v_H(x) &= \textstyle\int_{H_x} \phi_1\, d\theta. \end{aligned} \tag{61}$$

We next prove

LEMMA 5.1. *The convergence of the integrals in* (60) *implies that*

$$\textstyle\int_R v_E(x)\, dx < \infty, \tag{62 i}$$

$$\textstyle\int_R dx \int E_x\, p(|u|)(1-\phi_1)\, d\theta < \infty, \tag{62 ii}$$

$$\textstyle\int_R v_H(x)\, dx < \infty \tag{62 iii}$$

$$\textstyle\int_R v_F^2(x)\, dx < \infty \qquad \textit{and} \tag{62 iv}$$

$$\textstyle\int_R v_G^2(x)\, dx < \infty. \tag{62 v}$$

PROOF. (62i) follows from (60i) as for $\theta \in E_x$, $p(|u|)-b \geqq p(h-\delta_0)-b$. (62i) with (60i) then yields (62ii). (62iii) follows from (60iv) as for $\theta \in H_x$, $b-p(|u|) \geqq b-p(h+\delta_0)$. Next in (62iv), let $v_F(x)$ = area enclosed between two concentric circles with center x and radii h and h_2 where $h_2 = h_2(x) < h$.

Then

$$v_F(x) = \pi(h^2 - h_2^2). \tag{63}$$

As $p(|u|)$ is a decreasing function of $|u|$, the inner integral in the left-hand side of (60ii) is minimized for given $v_F(x)$, by putting $\phi_1 = 0$ for $h_2 \leqq |u| \leqq h$, and $\phi_1 = 1$ otherwise. Also since in this region $h - |u| < \delta_0$ by the definition of $v_F(x)$ in (58), we have by (9),

$$p(|u|) \geqq b[1 + \gamma(h - |u|)]. \tag{64}$$

From (60ii), using (64), we get,

$$\begin{aligned}
\textstyle\int_{F_x} [p(|u|) - b](1 - \phi_1)\, d\theta &\geqq 2\pi\gamma b \textstyle\int_{h_2}^{h} \rho(h-\rho)\, d\rho \\
&= \frac{\pi\gamma b}{3}(h - h_2)^2(h + 2h_2) \\
&= \frac{\gamma b}{3\pi}\frac{(h+2h_2)}{(h+h_2)^2} v_F^2(x) \qquad \text{by (63)} \\
&\geqq \frac{\gamma b}{3\pi}\frac{3(h-\delta_0)}{4h^2} v_F^2(x) \\
&\qquad \text{as } \; h \geqq h_2 > (h - \delta_0) \quad \text{by (58).}
\end{aligned} \tag{65}$$

Substituting the right-hand side of (65) in the left-hand side of (60ii) we obtain (62iv).

Lastly, let $v_G(x) =$ the area enclosed between two concentric circles with center x and radii h and $h_3 = h_3(x) \geqq h.$, so that

$$v_G(x) = \pi(h_3^2 - h^2). \tag{66}$$

For given $v_G(x)$, the inner integral in the left-hand side of (60iii) is minimized by putting $\phi_1 = 1$, for $h \leqq |u| \leqq h_3$ and $\phi_1 = 0$ otherwise. As by (58), $h_3 = h + \delta \leqq h + \delta_0$, we have by (9),

$$p(|u|) \leqq b[1 - \gamma(|u| - h)]. \tag{67}$$

Hence

$$\begin{aligned}
\textstyle\int_{G_x} [b - p(|u|)]\phi_1\, d\theta &\geqq 2\pi\gamma b \textstyle\int_{h}^{h_3} \rho(\rho - h)\, d\rho \\
&= \frac{\pi\gamma b}{3}(h_3 - h)^2(2h_3 + h) \\
&= \frac{\gamma b}{3\pi}\frac{(h+2h_3)}{(h+h_3)^2} v_G^2(x) \qquad \text{by (66)} \\
&\geqq \frac{\gamma b}{3\pi}\frac{3h}{4(h+\delta_0)^2} v_G^2(x) \qquad \text{as } \; h \leqq h_3 \leqq h + \delta_0
\end{aligned} \tag{68}$$

On substituting (68) in the left-hand side of (60iii), (62v) is proved. This completes the proof of Lemma 5.1.

6. Main Theorem continued. We now resume the proof of the Main Theorem 5.1. Using (57), (62iv) and (62v), we fix the number a, which determines the set $T_a = \{x\,|x| \leqq a$, so that each of the following inequalities hold viz.,

(69 i) $$\int_{T_a} [U_1(x) - U_0(x)]\,dx \geqq M - \delta,$$

(69 ii) $$\int_{T_a{}^c} v_F{}^2(x)\,dx \leqq \delta^2,$$ and

(69 iii) $$\int_{T_a{}^c} v_G{}^2(x)\,dx \leqq \delta^2,$$

where δ is the fixed positive number in (14).

We now keep a fixed, and show that by making τ sufficiently large, the improvement in risk of the procedure ϕ_1 over that of ϕ_0, on the set $T_a{}^c$ can be made arbitrarily small.

In the right-hand side of (53), as observed in the remarks following (53), as $\tau \to \infty$, $(2\pi\tau^2)T_1 \to$ the left-hand side of (69i) $\geqq M - \delta$.

Hence we can take τ_0 sufficiently large so that,

(70) $$T_1 > \frac{1}{2\pi\tau^2}(M - 2\delta) \qquad \text{for all} \quad \tau \geqq \tau_0.$$

Next in the integral for T_2, in the right-hand side of (53), we partition the domain of integration $T_a{}^c X\Omega$ as follows:

(71) $$D_1' = \left\{(x, \theta);\ x \in T_a{}^c;\ |x - \theta| > \frac{\tau}{\beta}\right\},$$
$$D_2' = \left\{(x, \theta);\ x \in T_a{}^c;\ \frac{|x|}{\tau} > \varepsilon\beta;\ |x - \theta| \leqq \frac{\tau}{\beta}\right\}, \quad \text{and}$$
$$D_3' = \left\{(x, \theta);\ x \in T_a{}^c;\ \frac{|x|}{\tau} \leqq \varepsilon\beta,\ |x - \theta| \leqq \frac{\tau}{\beta}\right\}.$$

The partitioning is similar to that into the sets D_1, D_2, D_3 described below equation (20), except that the values of x are restricted to the set $T_a{}^c$. Let J_1, J_2, J_3 be the components of T_2 arising from integration on D_1', D_2' and D_3' respectively.

Now on D_1', interchanging the order of integration with respect to x and θ, using that $0 \leqq \phi_1 \leqq 1$, and writing $v_1(x)$ for $v\phi_1(x, \cdot)$, we have,

(72) $$\begin{aligned} J_1 &= \frac{1}{2\pi\tau^2}\int_\Omega \exp\left[-\frac{|\theta|^2}{2\tau^2}\right] \\ &\quad \cdot d\theta \int_{|x|>a,\ |u|>\tau/\beta} \{[bv_1(x) - \phi_1] - [bv_0 - \phi_0]\}\, p(|x - \theta|)\,dx \\ &\geqq -\frac{1}{2\pi\tau^2}\int_\Omega \exp\left(-\frac{|\theta|^2}{2\tau^2}\right) d\theta \int_{|u|>\tau/\beta} [1 + bv_0]\, p(|x - \theta|)\,dx \\ &\geqq -(1 + bv_0)\cdot\frac{\delta}{\tau^2} \qquad \text{by (21i).} \end{aligned}$$

Similarly,

$$
\begin{aligned}
J_2 &= \frac{1}{2\pi\tau^2}\int_{|\theta|/\tau \geqq (\varepsilon\beta - 1/\beta)} \exp\left(-\frac{|\theta|^2}{2\tau^2}\right) \\
&\quad \cdot d\theta \int_{|x|/\tau > \varepsilon\beta,\ |x|>a,\ |u|>\tau/\beta} \{[bv_1(x)-\phi_1]-[bv_0-\phi_0]\} p(|x-\theta|)\, d\theta \qquad (73)\\
&\geqq -(1+bv_0)\cdot\frac{1}{2\pi\tau^2}\int_{|\theta|/\tau > (\varepsilon\beta - 1/\beta)} \exp\left(-\frac{|\theta|^2}{2\tau^2}\right) d\theta \\
&\geqq -\frac{\delta}{\tau^2}(1+bv_0) \qquad \text{by (21ii).}
\end{aligned}
$$

Next in the expression for J_3, we interchange the order of integration with respect to x and θ, and obtain

$$
\begin{aligned}
J_3 &= \frac{1}{2\pi\tau^2}\int_{T_{a'}{}^c} \exp\left[-\frac{|x|^2}{2\tau^2}\right] \\
&\quad \cdot dx\left\{\left\{bv_1(x)K_\tau(x) - \int_{|x-\theta|\leqq\tau/\beta} p(|x-\theta|)\exp\left[\frac{(x\cdot u)}{\tau^2}-\frac{|u|^2}{2\tau^2}\right]\phi_1\, d\theta\right\}\right. \qquad (74)\\
&\quad \left. -\left\{bv_0K_\tau(x) - \int_{|x-\theta|\leqq\tau/\beta} p(|x-\theta|)\exp\left[\frac{(x\cdot u)}{\tau^2}-\frac{|u|^2}{2\tau^2}\right]\phi_0\, d\theta\right\}\right\}
\end{aligned}
$$

where K_τ is as defined in (24) and $T_{a'}{}^c$ denotes the set $\{x;\, a < |x| \leqq \tau\beta\varepsilon\}$.

Now,

$$
\begin{aligned}
(75)\qquad v_1(x) &= \textstyle\int_\Omega \phi_1\, d\theta \qquad \geqq \int_{|x-\theta|\leqq\tau/\beta}\phi_1\, d\theta \qquad \text{and}\\
v_0 &= \textstyle\int_{|x-\theta|\leqq h}\phi_0\, d\theta = \int_{|x-\theta|\leqq\tau/\beta}\phi_0\, d\theta \qquad \text{by (3).}
\end{aligned}
$$

Substituting by (75), we have from (74)

$$
\begin{aligned}
(76)\qquad J_3 \geqq \frac{1}{2\pi\tau^2}\int_{T_{a'}{}^c}\exp\left[-\frac{|x|^2}{2\tau^2}\right]dx\int_{|u|\leqq\tau/\beta}\left\{bK_\tau(x) - \exp\left[\frac{(x\cdot u)}{\tau^2}-\frac{|u|^2}{2\tau^2}\right]\right.\\
\left.\cdot\, p(|x-\theta|)\right\}(\phi_1-\phi_0)\, d\theta.
\end{aligned}
$$

Now in the inner integral in the right-hand side of (76), we partition the domain of integration $\{\theta;\, |x-\theta| \leqq \tau/\beta\}$ into the subsets of Ω, E_x, F_x, G_x and $H_x' = H_x \cap \{\theta;\, |x-\theta| \leqq \tau/\beta\}$ where E_x, F_x, G_x and H_x are the sets defined by (58). Let J_4, J_5, J_6 and J_7 denote the components arising from integrations on E_x ,F_x, G_x and H_x'. We assume here that $\tau/\beta > h+\delta_0$, so that E_x, F_x and G_x are wholely contained in the set $|u| \leqq \tau/\beta$. Then

$$
(77)\qquad J_3 \geqq J_4+J_5+J_6+J_7.
$$

Now from (76), substituting for ϕ_0 by (3)

$$(78)\qquad J_4 = \frac{1}{2\pi\tau^2}\int_{T_{a}'^{c}} \exp\left(-\frac{|x|^2}{2\tau^2}\right) \cdot dx \int_{E_x} \left\{ bK_\tau(x) - p(|u|)\exp\left[\frac{(x\cdot u)}{\tau^2} - \frac{|u|^2}{2\tau^2}\right]\right\}(\phi_1 - 1)\, d\theta.$$

From (24),

$$(79)\qquad K_\tau(x)\exp\left(-\frac{|x|^2}{2\tau^2}\right) = \int_{|u|\leqq \tau/\beta} p(|u|)\exp\left(-\frac{|x-u|^2}{2\tau^2}\right) du \leqq 1$$

and

$$\exp\left(-\frac{|x|^2}{2\tau^2}\right)\exp\left[\frac{(x\cdot u)}{\tau^2} - \frac{|u|^2}{2\tau^2}\right] = \exp\left[-\frac{|x-u|^2}{2\tau^2}\right] \leqq 1.$$

Hence in the right-hand side of (78), the integrand is bounded in absolute magnitude, uniformly in τ in the domain of integration by

$$g(x, \theta) = [b + p(|u|)](1 - \phi_1),$$

and by (62i) and (62ii), $g(x, \theta)$ has a finite integral on $T_a'^{c} \times \Omega$. Hence by the Dominated Convergence Theorem, the limit of $(2\pi\tau^2)\cdot J_4$ as $\tau \to \infty$ can be taken under the integral sign. But as $\tau \to \infty$, the integrand $\to [b - p(|u|)](\phi_1 - 1) \geqq 0$ for $\theta \in E_x$. Hence $\lim_{\tau\to\infty} (2\pi\tau^2 J_4) \geqq 0$.

Therefore we can take τ_0 sufficiently large, so that

$$(80)\qquad J_4 \geqq -\delta \qquad \text{for all} \quad \tau \geqq \tau_0.$$

Similarly,

$$(81)\qquad J_7 = \frac{1}{2\pi\tau^2}\int_{T_{a}'^{c}} \exp\left(-\frac{|x|^2}{2\tau^2}\right) \cdot dx \int_{H_x'} \left\{ bK_\tau(x) - p(|u|)\exp\left[\frac{(x\cdot u)}{\tau^2} - \frac{|u|^2}{2\tau^2}\right]\right\}\phi_1\, d\theta.$$

Using (79) and that $p(|u|) \leqq b$, the integrand is seen to be bounded in absolute magnitude, uniformly in τ by $g_1(x, \theta) \doteq 2b\phi_1$ and the integral of $g_1(x, \theta)$ on $T_a'^{c} \times H_x'$ is finite by (62iii). Hence by the Dominated Convergence Theorem, the limit of $(2\pi\tau^2)\cdot J_7$ can be taken under the integral sign. Also as $\tau \to \infty$, the integrand $\to [b - p(|u|)]\phi_1 \geqq 0$ for $\theta \in H_x'$. Thus $\lim_{\tau\to\infty} (2\pi\tau^2)\cdot J_7 \geqq 0$.

Hence we can take τ_0 sufficiently large so that

$$(82)\qquad J_7 \geqq -\delta \qquad \text{for all} \quad \tau \geqq \tau_0$$

We next have

$$(83)\qquad J_5 = \frac{1}{2\pi\tau^2}\int_{T_a'^c} \exp\left(-\frac{|x|^2}{2\tau^2}\right) \cdot dx \int_{F_x} \left\{p(|u|)\exp\left[\frac{(x\cdot u)}{\tau^2}-\frac{|u|^2}{2\tau^2}\right]-bK_\tau(x)\right\}(1-\phi_1)\,du$$

In the inner integral in the right-hand side of (83), $p(|u|) \geqq b$,

$$(84)\qquad \exp\left[\frac{(x\cdot u)}{\tau^2}-\frac{|u|^2}{2\tau^2}\right] \geqq 1-\frac{|x|\cdot|u|}{\tau^2}-\frac{|u|^2}{2\tau^2}. \geqq 1-\frac{h|x|}{\tau^2}-\frac{h^2}{2\tau^2}.$$

Also since for $x \in T_a'^c$, $|x| \leqq \varepsilon\beta\tau$ by (74),

$$K_\tau(x) = \int_{|u|\leqq\tau/\beta} p(|u|)\exp\left[\frac{(x\cdot u)}{\tau^2}-\frac{|u|^2}{2\tau^2}\right]du \qquad \text{by (24)}$$

$$(85)\qquad \leqq \int_{|u|\leqq\tau/\beta} p(|u|)\exp\left(\frac{|x|\cdot|u|}{\tau^2}\right)du$$

$$\leqq \int_{|u|\leqq\tau/\beta} p(|u|)\left[1+\frac{2|x|\,|u|}{\tau^2}\right]du, \quad \text{as} \quad \frac{|x|\,|u|}{\tau^2} \leqq \varepsilon < \tfrac{1}{3} \qquad \text{by (18)}$$

$$\leqq \int_{|u|<\infty} p(|u|)\left[1+\frac{2h|x|}{\tau^2}\right]du = 1+\frac{2h|x|}{\tau^2}.$$

Denote the inner integral on the right-hand side of (83) by $f_5(x)$. Then substituting by (84) and (85), and noting the definition of $v_F(x)$ in (61), we have,

$$(86)\qquad f_5(x) \geqq -b\left(\frac{h^2}{2\tau^2}+\frac{3h|x|}{\tau^2}\right)v_F(x).$$

We now use the following relations which have been proved in Lemma 6.2 of the previous paper [2] viz. that the relation (69ii) implies that,

$$(87)\qquad \frac{1}{2\pi\tau^2}\int_{T_a^c}\exp\left(-\frac{|x|^2}{2\tau^2}\right)v_F(x)\,dx \leqq \frac{\delta}{\tau(2\pi)^{\frac{1}{2}}},$$

(Inequality (93) of [2])

and

$$(88)\qquad \frac{1}{2\pi\tau^2}\int_{T_a^c}\exp\left(-\frac{|x|^2}{2\tau^2}\right)\cdot\frac{|x|}{\tau}\cdot v_F(x) \leqq \frac{2}{(2\pi)^{\frac{1}{2}}}\frac{\delta}{\tau}.$$

(Inequality (99) of [2])

Since the set $T_a'^c$ which occurs in the right-hand side of (83) is a subset of T_a^c, (87) and (88) continue to hold when $T_a'^c$ is substituted for T_a^c.

NOTE 5.1. In Lemma 6.2 of [2], the exponential term is $\exp(-|x|^2/(2g\tau^2)$ but it is easily verified that the proof of that lemma remains valid for any positive value of g and in particular for $g = 1$.

Substituting by (86) in (83), and using (87) and (88), we obtain that,

$$(89) \qquad J_5 \geqq -b\left(\frac{h^2}{2(2\pi)^{\frac{1}{2}}}\frac{\delta}{\tau^3}+\frac{6h}{(2\pi)^{\frac{1}{2}}}\frac{\delta}{\tau^3}\right)$$
$$\geqq -\frac{b}{(2\pi)^{\frac{1}{2}}}\left(\frac{h^2}{2}+6h\right)\frac{\delta}{\tau^2} \qquad \text{for all } \tau \geqq \tau_0$$

where we assume that $\tau \geqq \tau_0 \geqq 1$.

Lastly,

$$(90) \qquad J_6 = \frac{1}{2\pi\tau^2}\int_{T_a'^c} \exp\left(-\frac{|x|^2}{2\tau^2}\right)$$
$$\cdot\, dx \int_{G_x} \left\{bK_\tau(x) - p(|u|)\exp\left[\frac{(x\cdot u)}{\tau^2}-\frac{|u|^2}{2\tau^2}\right]\right\}\phi_1\, d\theta.$$

Here we put,

$$K_\tau(x) \geqq 1-\frac{m_1}{2\tau^2} \qquad \text{by (25)},$$
$$(91) \qquad p(|u|) \leqq b \qquad \text{by (58)},$$
$$\exp\left[\frac{(x\cdot u)}{\tau^2}-\frac{|u|^2}{2\tau^2}\right] \leqq \exp\left(\frac{|x|\cdot|u|}{\tau^2}\right) \leqq 1+2\frac{|x|\cdot|u|}{\tau^2}$$
$$\leqq 1+\frac{2h_1|x|}{\tau^2}$$

as $(|x|\cdot|u|)/(\tau^2) \leqq \varepsilon < \frac{1}{3}$ for $(x, \theta)\varepsilon D_3'$, by (71) and (18), and for $\theta \in G_x$, $|u| \leqq h+\delta_0 = h_1$ by (58).

Because of (69iii) the relations (87), (88), also hold when $v_G(x)$ is substituted in them for $v_F(x)$. Hence substituting by (91) in (90), and using (87) and (88) with $v_F(x)$ replaced by $v_G(x)$, we get

$$(92) \qquad J_7 \geqq -\left(\frac{bm_1}{2\tau^2}\frac{\delta}{\tau(2\pi)^{\frac{1}{2}}}+\frac{4b}{(2\pi)^{\frac{1}{2}}}\frac{h_1\delta}{\tau^3}\right) \qquad \text{for all } \tau \geqq \tau_0,$$
$$\geqq -\frac{b}{(2\pi)^{\frac{1}{2}}}\left(\frac{m_1}{2}+4h_1\right)\frac{\delta}{\tau^2} \qquad \text{for all } \tau \geqq \tau_0,$$

as $\tau > 1$, by (89).

Now adding up the lower bound for T_1 and J_1, J_2 and J_4 to J_7 in (70), (72), (73), (80), (82), (89) and (92), we obtain that

$$E_\tau L_1(x, \theta) - E_\tau L_0(x, \theta) \geqq \frac{M - B\delta}{2} \qquad \text{for all } \tau \geqq \tau_0 \tag{93}$$

where B is a fixed constant independent of δ and τ. Since by (48), the left-hand side of (93) $\leqq 0$, and δ in (14) is arbitrarily small, (93) implies that

$$M = 0. \tag{94}$$

But by (57)

$$M \geqq k > 0. \tag{95}$$

Thus the assumption that ϕ_1 is not equivalent to ϕ_0 leads to a contradiction. Therefore ϕ_1 must be equivalent to ϕ_0, as was to be proved.

Acknowledgment. I am deeply grateful to the referees for their valuable comments. I have also to thank Mr. V. J. A. Mudaliar for doing the typing work.

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