The Annals of Mathematical Statistics
1971, Vol. 42, No. 2, 552–559

EQUICONVERGENCE OF MARTINGALES

BY EDWARD S. BOYLAN
Rutgers, The State University

Let $(X, \mathscr{F}, P)$ be a probability space, $\mathscr{F}_n$, $n = 1, 2, \cdots$, a sequence of subfields of $\mathscr{F}$ increasing (or decreasing) to a limit subfield $\mathscr{F}_\infty$ and $f \in L_1(X)$. It is a (by now) classical result of martingale theory that $\lim_{n\to\infty} E(f \mid \mathscr{F}_n) = E(f \mid \mathscr{F}_\infty)$ almost everywhere. (See [2] for details.) In all such convergence proofs, however, there is little investigation as to the rate of convergence to the limit. One would expect that knowledge regarding how quickly the subfields $\mathscr{F}_n$ "approach" the limit field $\mathscr{F}_\infty$ should yield some information regarding how well $E(f \mid \mathscr{F}_n)$ approximates $E(f \mid \mathscr{F}_\infty)$. Such information, depending only on the $\mathscr{F}_n$, is, in some sense, independent of the particular L_1 function f. In this paper we first define a pseudometric, D, on the set of subfields of $\mathscr{F}$ and then show that if $D(\mathscr{F}_n, \mathscr{F}_\infty) \to 0$ then rates of convergence of $E(f \mid \mathscr{F}_n)$ to $E(f \mid \mathscr{F}_\infty)$ can be given which are, in a sense to be defined below, independent of f.

DEFINITION 1. If $F \in \mathscr{F}$ and $\mathscr{F}'$ is a subfield of $\mathscr{F}$, let

$$d(F, \mathscr{F}') = \inf_{F' \in \mathscr{F}'} P(F \triangle F'),$$

where $F \triangle F' = (F - F') \cup (F' - F)$, the symmetric difference of F and F'.

DEFINITION 2. If $\mathscr{F}_1$ and $\mathscr{F}_2$ are two subfields of $\mathscr{F}$, let

$$d(\mathscr{F}_1, \mathscr{F}_2) = \sup_{F_1 \in \mathscr{F}_1} d(F_1, \mathscr{F}_2),$$

and $D(\mathscr{F}_1, \mathscr{F}_2) = d(\mathscr{F}_1, \mathscr{F}_2) + d(\mathscr{F}_2, \mathscr{F}_1)$.

THEOREM 1. *D is a pseudometric on the set of subfields of $\mathscr{F}$.*

PROOF. Clearly (a) $D(\mathscr{F}_1, \mathscr{F}_2) \geqq 0$; (b) $D(\mathscr{F}_1, \mathscr{F}_2) = D(\mathscr{F}_2, \mathscr{F}_1)$. It remains to show

(c) $D(\mathscr{F}_1, \mathscr{F}_3) \leqq D(\mathscr{F}_1, \mathscr{F}_2) + D(\mathscr{F}_2, \mathscr{F}_3)$. By symmetry, however, it suffices to show

(c′) $d(\mathscr{F}_1, \mathscr{F}_3) \leqq d(\mathscr{F}_1, \mathscr{F}_2) + d(\mathscr{F}_2, \mathscr{F}_3)$.

Suppose $d(\mathscr{F}_1, \mathscr{F}_2) = a$ and $d(\mathscr{F}_2, \mathscr{F}_3) = b$. To show that $d(\mathscr{F}_1, \mathscr{F}_3) \leqq a + b$ it suffices to show that for every $\varepsilon > 0$ and $F_1 \in \mathscr{F}_1$ there exists an $F_3 \in \mathscr{F}_3$ such that $P(F_1 \triangle F_3) \leqq a + b + \varepsilon$. Let $F_1 \in \mathscr{F}_1$. Since $d(\mathscr{F}_1, \mathscr{F}_2) = a$, there exists an $F_2 \in \mathscr{F}_2$ such that $P(F_1 \triangle F_2) \leqq a + \varepsilon/2$. Since $d(\mathscr{F}_2, \mathscr{F}_3) = b$, there is an $F_3 \in \mathscr{F}_3$ such that $P(F_2 \triangle F_3) \leqq b + \varepsilon/2$. The inclusion $F_1 \triangle F_3 \subset (F_1 \triangle F_2) \cup (F_2 \triangle F_3)$ implies

$$P(F_1 \triangle F_3) \leqq P(F_1 \triangle F_2) + P(F_2 \triangle F_3) \leqq a + b + \varepsilon.$$

Received May 4, 1970.

It should be noted that d is not a pseudometric because it is not symmetric. In particular, if $\mathscr{F}_1 \subset \mathscr{F}_2$, then $d(\mathscr{F}_1, \mathscr{F}_2) = 0$ but $d(\mathscr{F}_2, \mathscr{F}_1)$ need not be zero. There is a significant relationship between D and conditional expectations, as the following theorem begins to show.

THEOREM 2. *$D(\mathscr{F}_1, \mathscr{F}_2) = 0$ if and only if $E(f \mid \mathscr{F}_1) = E(f \mid \mathscr{F}_2)$ almost everywhere for every $f \in L_1(X)$.*

PROOF. Suppose $D(\mathscr{F}_1, \mathscr{F}_2) = 0$. Let $f \in L_1(X)$ and real numbers a and b, $a < b$, be given. To prove the only if segment of the theorem, it suffices to show that

$$P(\{w: E(f \mid \mathscr{F}_1) < a < b < E(f \mid \mathscr{F}_2)\}) = 0.$$

(For simplicity of notation, the symbols $\{w: -\}$ will be omitted, e.g. the above equation will be written

$$P(E(f \mid \mathscr{F}_1) < a < b < E(f \mid \mathscr{F}_2)) = 0.$$

Let $A = (E(f \mid \mathscr{F}_1) < a)$ and $B = (E(f \mid \mathscr{F}_2) > b)$. Then $A \in \mathscr{F}_1$ and $B \in \mathscr{F}_2$. Since $D(\mathscr{F}_1, \mathscr{F}_2) = 0$, for any $\varepsilon > 0$ there exist sets $F_a \in \mathscr{F}_2$, $F_b \in \mathscr{F}_1$, such that $P(F_a \triangle A) \leqq \varepsilon$ and $P(F_b \triangle B) \leqq \varepsilon$. (We omit the dependence of F_a and F_b on ε.) We have

$$\begin{aligned} \int_{A\cap B} f\,dP &= \int_{A\cap F_b} f\,dP + \int_{A\cap(B-F_b)} f\,dP - \int_{A\cap(F_b-B)} f\,dP \\ &= \int_{A\cap F_b} E(f \mid \mathscr{F}_1)\,dP + \int_{A\cap(B-F_b)} f\,dP - \int_{A\cap(F_b-B)} f\,dP \\ &\leqq aP(A\cap F_b) + \int_{A\cap(B-F_b)} f\,dP - \int_{A\cap(F_b-B)} f\,dP. \end{aligned} \tag{1}$$

As $\varepsilon \to 0$, $P(A \cap F_b) \to P(A \cap B)$ while the remaining integrals all approach zero. Thus, we may conclude that

$$\int_{A\cap B} f\,dP \leqq aP(A\cap B). \tag{2}$$

In similar fashion, however, we have

$$\int_{A\cap B} f\,dP \geqq bP(A\cap B). \tag{3}$$

Since $a < b$, this is possible only when $P(A\cap B) = 0$.

Suppose $E(f \mid \mathscr{F}_1) = E(f \mid \mathscr{F}_2)$ almost everywhere for every $f \in L_1(X)$. Let $A \in \mathscr{F}_1$. Then $E(1_A \mid \mathscr{F}_2) = 1_A = E(1_A \mid \mathscr{F}_1)$ almost everywhere. Let $B = (E(1_A \mid \mathscr{F}_2) > 0)$. Then $B \in \mathscr{F}_2$ and

$$P(A \triangle B) = P(1_A \neq 1_B) \leqq P(1_A \neq E(1_A \mid \mathscr{F}_2)) = 0.$$

Since A was arbitrary, this implies $d(\mathscr{F}_1, \mathscr{F}_2) = 0$. The same logic shows $d(\mathscr{F}_2, \mathscr{F}_1) = 0$, and hence $D(\mathscr{F}_1, \mathscr{F}_2) = 0$.

COROLLARY 1. *If $D(\mathscr{F}_1, \mathscr{F}_2) = 0$ every set in $\mathscr{F}_1$ differs from a set in $\mathscr{F}_2$ by at most a set of measure zero.*

COROLLARY 2. *If $\mathscr{F}$ has no nonempty subsets of measure zero then D is a metric.*

Having defined D, we now return to the question raised in the first paragraph of the paper.

DEFINITION 3. Let f_{jn}, $j \in J$, J an indexing set, $n = 1, 2, \cdots$, be a collection of functions such that $\lim_{n\to\infty} f_{jn} = f_{j\infty}$ exists for every j. The functions f_{jn} are said to be equiconvergent in measure if for every $\varepsilon > 0$

$$\lim_{n\to\infty} P(|f_{jn} - f_{j\infty}| > \varepsilon) = 0$$

uniformly in j. The functions f_{jn} are said to be strongly equiconvergent if

$$\lim_{n\to\infty} \int |f_{jn} - f_{j\infty}|\, dP = 0$$

uniformly in j.

THEOREM 3. *Let $\mathscr{F}_n$, $n = 1, 2, \cdots, \infty$ be subfields of $\mathscr{F}$ with $\mathscr{F}_n$ increasing or decreasing to $\mathscr{F}_\infty$ and $\lim_{n\to\infty} D(\mathscr{F}_n, \mathscr{F}_\infty) = 0$. Then the functions $E(f \mid \mathscr{F}_n)$, $n = 1, 2, \cdots$, $\|f\|_\infty \leqq 1$, are strongly equiconvergent.*

PROOF. Strictly speaking, we have not put the functions $E(f \mid \mathscr{F}_n)$, $\|f\|_\infty \leqq 1$, in the form given by Definition 3. It should be clear, however, that the conclusion of the theorem is

$$\lim_{n\to\infty} \int |E(f \mid \mathscr{F}_n) - E(f \mid \mathscr{F}_\infty)|\, dP = 0$$

uniformly for all f such that $|f| \leqq 1$ almost everywhere.

Because all the functions in question are uniformly bounded, strong equiconvergence is equivalent to equiconvergence in measure, and this is what will be shown. The proof is similar to that of Theorem 2. For simplicity of notation, $E(f \mid \mathscr{F}_n)$ will henceforth be denoted simply by f_n. We wish to show that

$$\lim_{n\to\infty} P(|f_n - f_\infty| > \varepsilon) = 0 \tag{4}$$

uniformly in f, $|f| \leqq 1$. (Unless otherwise stated, all functions f appearing below will be assumed to be bounded in absolute value by 1.) Let $\varepsilon' > 0$ be given. To prove (4) it suffices to show there exists an integer $N(\varepsilon')$ such that

$$P(|f_n - f_\infty| > \varepsilon) \leqq \varepsilon', \qquad n \geqq N(\varepsilon'). \tag{5}$$

Since $|f| \leqq 1$ almost everywhere, $|f_n| \leqq 1$ almost everywhere. We can find a sequence of numbers a_i, $i = 0, 1, \cdots, k$, where k is a function of ε, with the following properties:

(a) $a_i = 0$ for some value i;

(b) $a_{i+1} - a_i \geqq \varepsilon/2$;

(c) $|a_i| \leqq 1$;

(d) $|a| \leqq 1$, $b - a > \varepsilon$ implies $a \leqq a_i$, $b > a_{i+1}$, for some i, $0 \leqq i \leqq k-1$.

(For example, if $\varepsilon = 1/m$, then $a_i = -1 + \frac{1}{2}i/m$, $0 \leqq i \leqq 4m$.) We now prove

$$P(f_\infty - f_n > \varepsilon) \leqq \varepsilon'/2, \qquad n \geqq N(\varepsilon') \tag{5'}$$

for some $N(\varepsilon')$ yet to be determined, uniformly in f. From the properties of the a_i, we have

$$P(f_\infty - f_n > \varepsilon) \leqq \sum_{i=0}^{k-1} P(f_n \leqq a_i, f_\infty > a_{i+1}).$$

To prove (5′) it thus suffices to show

$$P(f_n \leqq a_i, f_\infty > a_{i+1}) < \frac{\varepsilon'}{2k}, \qquad 0 \leqq i < k, n \geqq N(\varepsilon'), \tag{6}$$

uniformly in f.

Let $A = (f_n \leqq a_i)$, $B = (f_\infty > a_{i+1})$ and $D(\mathscr{F}_n, \mathscr{F}_\infty) = \delta$. There exist sets $F \in \mathscr{F}_n$, $G \in \mathscr{F}_\infty$ such that $P(F \triangle B) = P(G \triangle A) = \delta$. (Strictly speaking the value δ need not necessarily be attained, but we assume so for simplicity of notation. There are already more than enough ε's in use.) Then (assuming $a_i \geqq 0$)

$$\begin{aligned} \int_{A \cap B} f\,dP &= \int_{A \cap F} f\,dP + \int_{A \cap (B-F)} f\,dP - \int_{A \cap (F-B)} f\,dP \\ &\leqq \int_{A \cap F} f\,dP + 2\delta \leqq a_i P(A \cap F) + 2\delta \\ &\leqq a_i (P(A \cap B) + \delta) + 2\delta = a_i P(A \cap B) + \delta(2 + a_i). \end{aligned} \tag{7}$$

The next to last inequality is true because $(A \cap F) \subset (A \cap B) \cup (A \cap (F-B))$. Similarly,

$$\begin{aligned} \int_{A \cap B} f\,dP &= \int_{G \cap B} f\,dP + \int_{(A-G) \cap B} f\,dP - \int_{(G-A) \cap B} f\,dP \\ &\geqq \int_{G \cap B} f_\infty\,dP - 2\delta \geqq a_{i+1} P(G \cap B) - 2\delta \\ &\geqq a_{i+1}(P(A \cap B) - \delta) - 2\delta = a_{i+1} P(A \cap B) - \delta(2 + a_{i+1}). \end{aligned} \tag{8}$$

Combining (7) and (8), we have

$$a_{i+1} P(A \cap B) - \delta(2 + a_{i+1}) \leqq a_i P(A \cap B) + \delta(2 + a_i), \tag{9}$$

which implies

$$P(A \cap B) \leqq \frac{(4 + a_{i+1} + a_i)}{a_{i+1} - a_i} \leqq \frac{12\delta}{\varepsilon}. \tag{10}$$

If $a_i < a_{i+1} \leqq 0$, then similar reasoning shows

$$P(A \cap B) \leqq \frac{\delta(4 - a_{i+1} - a_i)}{a_{i+1} - a_i} \leqq \frac{12\delta}{\varepsilon}. \tag{11}$$

Therefore, if N is so large that $D(\mathscr{F}_n, \mathscr{F}_\infty) \leqq \varepsilon\varepsilon'/(24k)$, $n \geqq N$, we have

$$P(f_\infty - f_n > \varepsilon) \leqq \varepsilon'/2, \qquad n \geqq N.$$

An examination of the argument used to prove (5′) reveals that the roles played by $\mathscr{F}_n$ and $\mathscr{F}_\infty$ can be interchanged, i.e., it is also true that

$$P(f_n - f_\infty > \varepsilon) \leqq \varepsilon'/2, \qquad n \geqq N, \tag{5''}$$

where N is as defined above. (5′) and (5″), however, are equivalent to (5) and the proof is complete.

It should be noted that Theorem 3 is true in a somewhat broader context as well. In proving Theorem 3, use was made of the fact that all functions f were uniformly bounded. (The bound was 1, but this was not significant.) The set, H, of all functions f, $\|f\|_\infty \leqq 1$, is a uniformly integrable subset of $L_1(X)$. Theorem 3 remains true when $f \in H$, where H is any uniformly integrable subset of $L_1(X)$. The set $H = \{f:\|f\|_\infty \leqq 1\}$ was used in the statement of Theorem 3 for (relative) simplicity of representation. The modifications necessary to prove the more general result will be given in the proof of Theorem 4.

It should be noted that it is possible for $\mathscr{F}_n$ to increase or decrease to $\mathscr{F}_\infty$ without $D(\mathscr{F}_n, \mathscr{F}_\infty)$ approaching zero. For example let $X = \prod_{i=1}^\infty I_i$, $\mathscr{B} = \prod_{i=1}^\infty B_i$, $P = \prod_{i=1}^\infty L_i$, where $I_i = [0, 1)$, for all i, B_i the Borel field for all i, L_i standard Lebesgue measure for all i. If $\mathscr{F}_n = \prod_{i=1}^n B_i \times \prod_{i=n+1}^\infty \mathscr{B}_i'$, where $\mathscr{B}_i' = \{\phi, I_i\}$, then the $\mathscr{F}_n$ increase to $\mathscr{F}$, but $D(\mathscr{F}_{n+1}, \mathscr{F}_n) = \frac{1}{2}$, which implies that $D(\mathscr{F}_n, \mathscr{F})$ does not approach zero. To see that $D(\mathscr{F}_{n+1}, \mathscr{F}_n) = \frac{1}{2}$, consider two independent sets, A and B. The $P(A \triangle B) = P(A-B)+P(B-A) = P(A)+P(B)-2P(A \cap B) = P(A)(1-2P(B))+P(B)$. If $P(B) = \frac{1}{2}$, $P(A \triangle B) = P(B)$. Taking B a set in $\mathscr{F}_{n+1}$ independent of all sets in $\mathscr{F}_n$ with $P(B) = \frac{1}{2}$ (such a B clearly exists), we have $d(B, \mathscr{F}_n) = \frac{1}{2} = d(\mathscr{F}_{n+1}, \mathscr{F}_n) = D(\mathscr{F}_{n+1}, \mathscr{F}_n)$. In similar fashion, if $\mathscr{F}_{-n} = \prod_{i=1}^n \mathscr{B}_i' \times \prod_{i=n+1}^\infty B_i$, then the $\mathscr{F}_{-n}$ decrease to the tail field $\mathscr{F}_{-\infty}$, but $D(\mathscr{F}_{-n}, \mathscr{F}_{-\infty})$ does not approach zero.

A large class of cases where $D(\mathscr{F}_n, \mathscr{F}_\infty)$ does approach zero can be given, however. If $X = \bigcup_{n=1}^\infty A_n$, A_n disjoint, with $P(A_n) \to 0$, let $\mathscr{F}_n' = A_n \cap \mathscr{F}$, i.e., the measurable subsets of A_n. If, $\mathscr{F}_n = \bigcup_{k=1}^n \mathscr{F}_k'$ (or $\mathscr{F}_n = \bigcup_{i=1}^n A_i \cup \bigcup_{k=n+1}^\infty \mathscr{F}_k'$), then the $\mathscr{F}_n$ increase (decrease) to a limit field $\mathscr{F}_\infty$ and (clearly) $D(\mathscr{F}_n, \mathscr{F}_\infty)$ approaches zero.

THEOREM 4. *Let* $\mathscr{F}_n, n = 1, 2, \cdots,$ *be a sequence of subfields such that* $\lim_{m,n\to\infty} D(\mathscr{F}_m, \mathscr{F}_n) = 0$. *Then there exists a subfield,* $\mathscr{H}$, *such that*

$$\lim_{n\to\infty} D(\mathscr{F}_n, \mathscr{H}) = 0.$$

Moreover, for every $f \in L_1(X)$, $E(f \mid \mathscr{F}_n)$ *converges in measure to* $E(f \mid \mathscr{H})$.

PROOF. Let $f \in L_1(X)$. The proof of Theorem 3 shows that

$$\lim_{m,n\to\infty} P(|f_n - f_m| > \varepsilon) = 0$$

for any fixed $\varepsilon > 0$. Thus, the functions f_n are fundamental in measure. This implies (see [1]) that there exists an integrable function (which we denote) f_∞ such that f_n converges to f_∞ in measure. The function f_∞ is unique (up to sets of measure zero). Consider the mapping $T:L_1(X) \to L_1(X)$, $T(f) = f_\infty$. The mapping T clearly satisfies the following properties:

(a) $T:L_2(X) \to L_2(X)$;

(b) $T(k_1 f + k_2 g) = k_1 T(f) + k_2 T(g)$, k_1, k_2, constants, $f, g \in L_1(X)$;

(c) $f \geqq 0$ implies $Tf \geqq 0$

(d) $T(1) = 1$.

If it can be shown that

(e) $T(T(f)) = T(f)$,

then it follows that $T(f) = E(f \mid \mathscr{H})$ for some subfield $\mathscr{H}$. (See [2], page 123). In other words, to show that $T(f) = E(f \mid \mathscr{H})$ for some subfield $\mathscr{H}$, it suffices to show that if $g = \text{l.i.m.}_{n\to\infty} f_n$, then $g = \text{l.i.m.}_{n\to\infty} g_n$, where $\text{l.i.m.}_{n\to\infty}$ denotes convergence in measure (and $g_n = E(g \mid \mathscr{F}_n)$).

LEMMA 1. *Let $g = \text{l.i.m.}\ f_n$ for some $f \in L_1(X)$ and $F = (g \leqq a)$ for some real number a. For every $\varepsilon' > 0$ there exists an $N(\varepsilon')$ such that for every $n \geqq N(\varepsilon')$ there is an $F_n \in \mathscr{F}_n$, $P(F \bigtriangleup F_n) \leqq \varepsilon'$.*

PROOF. Let $\varepsilon' > 0$ be given. Let $\varepsilon > 0$ be so small that $P(a < g \leqq a+2\varepsilon) \leqq \varepsilon'/2$. Since $\lim_{n\to\infty} P(|f_n - g| > \varepsilon) = 0$, for n sufficiently large we have

$$P(|f_n - g| > \varepsilon) < \varepsilon'/2.$$

Let $F_n = (f_n \leqq a+\varepsilon)$. Then

$$F_n \bigtriangleup F = (f_n > a+\varepsilon, g \leqq a \quad \text{or} \quad f_n \leqq a+\varepsilon, g > a+2\varepsilon)$$
$$\cup\, (f_n \leqq a+\varepsilon, a < g \leqq a+2\varepsilon).$$

Thus $P(F_n \bigtriangleup F) \leqq P(|f_n - g| > \varepsilon) + P(a < g \leqq a+2\varepsilon) \leqq \varepsilon'/2 + \varepsilon'/2 = \varepsilon'$.

To show that T is idempotent, it suffices to show that for every $\varepsilon > 0$ and $\varepsilon' > 0$ there exists an $N(\varepsilon, \varepsilon')$ such that

$$P(|g - g_n| > \varepsilon) \leqq \varepsilon', \qquad n \geqq N(\varepsilon, \varepsilon'). \tag{12}$$

Choose M so large that $P(|g| > M) < \varepsilon'/4$. Let a_i, $i = 0, 1, \cdots, k$, be as in Theorem 3, except they span the interval $[-M, M]$ instead of $[-1, 1]$. (For example, (c) would be replaced by $|a_i| \leqq M$).

We now show that

$$\lim_{n\to\infty} P(g \leqq a_i, g_n \geqq a_{i+1}) = 0, \qquad 0 \leqq i < k. \tag{13}$$

Let $A = (g \leqq a_i)$, $B = (g_n \geqq a_{i+1})$. We have

$$\begin{aligned} a_i P(A \cap B) &\geqq \textstyle\int_{A\cap B} g\, dP = \int_{F_n \cap B} g\, dP + \int_{(A - F_n)\cap B} g\, dP - \int_{(F_n - A)\cap B} g\, dP \\ &\geqq \textstyle\int_{F_n \cap B} g_n\, dP - 2\delta' \\ &\geqq a_{i+1} P(F_n \cap B) - 2\delta' \geqq a_{i+1}(P(A\cap B) - \delta) - 2\delta', \end{aligned} \tag{14}$$

where F_n is the set (whose existence was proven in Lemma 1) with the property that $F_n \in \mathscr{F}_n$, $P(A \bigtriangleup F_n) = \delta$. $\delta' \to 0$ as $\delta \to 0$ because g is integrable, and thus $\int_F g\, dP \to 0$ as $P(F) \to 0$. We are also assuming that $a_i \geqq 0$. If $a_i < a_{i+1} \leqq 0$, then only minor modifications, as before, are necessary.

Rewriting (14), we have

$$P(A\cap B) \leqq \frac{a_{i+1}\delta + 2\delta'}{a_{i+1} - a_i} \leqq \frac{4(M\delta + \delta')}{\varepsilon}. \tag{15}$$

Pick δ so small that $M\delta+\delta' < \varepsilon\varepsilon'/16k$, and N so large that F_n exists for all $n \geqq N$. Then

$$P(g_n-g > \varepsilon) \leqq P(g_n-g > \varepsilon, |g| \leqq M)+P(|g| > M) \leqq \varepsilon'/4+\varepsilon'/4 = \varepsilon'/2, \qquad n \geqq N. \tag{16}$$

Similar logic, with g_n and g interchanged, shows that

$$P(g-g_n > \varepsilon) \leqq \varepsilon'/2, \qquad n \geqq N, \tag{17}$$

which implies (12). Therefore g_n converges to g in measure, $T(T(f)) = T(f)$, for every $f \in L_1(X)$, and there thus is a subfield, $\mathscr{H}$, such that $T(f) = E(f \mid \mathscr{H})$.

We have not completed the proof of the theorem, however, because we have not yet show that $\lim_{n\to\infty} D(\mathscr{F}_n, \mathscr{H}) = 0$. It follows from Theorem 3 that

$$\lim_{m,n\to\infty} P(|f_m-f_n| > \varepsilon) = 0$$

uniformly in f, $|f| \leqq 1$. This implies

$$\lim_{n\to\infty} P(|f_n-f_\infty| > \varepsilon) = 0 \tag{18}$$

uniformly in f, $|f| \leqq 1$. To show that $\lim_{n\to\infty} D(\mathscr{F}_n, \mathscr{H}) = 0$, it suffices to show that for every $\varepsilon' > 0$ there is an $N(\varepsilon')$ such that:

(a) if $H \in \mathscr{H}$ there exists an $F_n \in \mathscr{F}_n$, $P(F_n \triangle H) \leqq \varepsilon'$, $n \geqq N(\varepsilon')$;

(b) if $F_n \in \mathscr{F}_n$ there exists an $H \in \mathscr{H}$, $P(F_n \triangle H) \leqq \varepsilon'$, $n \geqq N(\varepsilon')$.

Each statement is proven in essentially the same manner. Let $\varepsilon' > 0$ be given. From (18) we know there exists an $N(\varepsilon')$ such that

$$P(|f_n-f_\infty| > \tfrac{1}{2}) \leqq \varepsilon', \qquad n \geqq N(\varepsilon'), \tag{19}$$

for all f, $|f| \leqq 1$. Let $H \in \mathscr{H}$ and $F_n = (E(1_H \mid \mathscr{F}_n) > \frac{1}{2})$. Then $H \triangle F_n \subset (|E(1_H \mid \mathscr{F}_n)-1_H| > \frac{1}{2})$. Thus $P(H \triangle F_n) \leqq \varepsilon'$, $n \geqq N(\varepsilon')$. Similarly, if $F_n \in \mathscr{F}_n$, let $H = (E(1_{F_n} \mid \mathscr{H}) > \frac{1}{2})$. Then $H \triangle F_n \subset (|1_{F_n}-E(1_{F_n} \mid \mathscr{H})| > \frac{1}{2})$ and

$$P(H \triangle F_n) \leqq \varepsilon', n \geqq N(\varepsilon').$$

This completes the proof of Theorem 4. It should be noted that pointwise convergence cannot be substituted for convergence in probability, as the following example (due to Burgess Davis) shows. Let $X = [0, 1)$, $\mathscr{F}$ be the Borel field and P be Lebesgue measure on X. Let A_n be a sequence of measurable subsets of $[\frac{1}{2}, 1)$ with $\limsup A_n = [\frac{1}{2}, 1)$, i.e. every $x \in [\frac{1}{2}, 1)$ belongs to an infinite number of A_n, and $P(A_n) \to 0$. Let $B_n = [0, \frac{1}{2}) \cup A_n$, $\mathscr{F}_n = \{\phi, B_n, X-B_n, X\}$ and $f = 1_{[0,\frac{1}{2})}$. Clearly $D(\mathscr{F}_n, \mathscr{F}_\infty) \to 0$, where $\mathscr{F}_\infty = \{\phi, [0, \frac{1}{2}), [\frac{1}{2}, 1), X\}$, but $E(f \mid \mathscr{F}_n)$ does not approach $E(f \mid \mathscr{F}_\infty)$ for any $x \in [\frac{1}{2}, 1)$, since $E(f \mid \mathscr{F}_\infty) = f = 0$ on $[\frac{1}{2}, 1)$, but $E(f \mid \mathscr{F}_n) > \frac{1}{2}$ for $x \in A_n$ when n is so large that $P(A_n) < \frac{1}{2}$. Thus

$$\limsup E(f \mid \mathscr{F}_n) \geqq \tfrac{1}{2} \quad \text{for} \quad x \in [\tfrac{1}{2}, 1).$$

Though the main point of defining D was in relation to finding sufficient conditions for equiconvergence of martingales, D may well be worthy of interest in its own right and, hopefully, will be of value in investigating other probabilistic questions.

REFERENCES

[1] HALMOS, P. (1950). *Measure Theory*. Van Nostrand, Princeton.
[2] NEVEAU, J. (1965). *Mathematical Foundations of the Calculus of Probability*. Holden-Day, San Francisco.