The Annals of Mathematical Statistics
1972, Vol. 43, No. 5, 1623-1633

STOCHASTIC ORDER RELATIONSHIPS BETWEEN GI/G/k SYSTEMS[1]

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Consider two GI/G/k queueing systems in which the second system has stochastically larger interarrival times and/or stochastically shorter service times than the first system. In this paper we find sufficient conditions under which this implies stochastically shorter waiting time, virtual waiting time, queue size, and imbedded queue size, for the second system as compared to the first.

0. Introduction. Consider two GI/G/k queueing systems with interarrival and service time distributions F, H and F', H', respectively. Assume that for all $x \in R$ we have $F(x) \geqq F'(x)$ and $H(x) \leqq H'(x)$, i.e., the primed system has stochastically larger interarrival times and smaller service times than the original system. In this paper we show that this implies a corresponding ordering of the waiting time and imbedded queue size, and in some, but not all cases, it implies an ordering of queue size and virtual waiting time.

For the one-server queue such a result has been derived by Daley and Moran (1968) for the waiting time distribution, by using a direct comparison of distribution functions. Stoyan and Stoyan (1969) analyze more general order relationships between random variables, and they obtain Daley and Moran's results as a special case. Our results generalize this to the many-server queue and to queueing variables other than waiting times. They are derived from a suitable construction of the basic probability space.

As pointed out by Daley and Moran, results of this type are useful in establishing upper and lower bounds for queueing variables in analytically unwieldy systems. To the extent that tractable and close upper and lower bounds approximating the F and H functions can be found, the bounds to the queueing variables can be made arbitrarily close. This follows from a recent result due to Kennedy (1970) and Whitt (1971).

1. Notation and standard construction. We assume throughout this paper that the interarrival times $\theta_1, \theta_2, \cdots$ and service times $\chi_1, \chi_2, \cdots$ are two independent sequences of identically distributed nonnegative independent random variables. The same assumption is to hold with respect to the primed system. We then assume

$$P(\theta_i \leqq x) = F(x) \geqq F'(x) = P(\theta_i' \leqq x)\,, \tag{1.1}$$

Received September 13, 1971; revised March 2, 1972.

[1] Research supported by the United States Department of Transportation, Federal Highway Administration, Bureau of Public Roads under Contract DOT-FH-11-7716 awarded to the Department of Statistics, The Johns Hopkins University.

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and

$$P(\chi_i \leqq x) = H(x) \leqq H'(x) = P(\chi_i' \leqq x)\,, \qquad x \in R,\ i \in I\,. \tag{1.2}$$

According to a well-known lemma (e.g., Lehmann (1959) page 73) there exists a probability space (Ω_1, A_1, P_1) and random variables $\bar{\theta}_1$ and $\bar{\theta}_1'$ defined on it such that $\bar{\theta}_1 =_d \theta_1$, $\bar{\theta}_1' =_d \theta_1'$, and $\bar{\theta}_1(\omega_1) \leqq \bar{\theta}_1'(\omega_1)$, $\omega_1 \in \Omega_1$ (with $\bar{\theta}_1 \equiv \bar{\theta}_1'$ in case $F \equiv F'$); similarly for each pair (θ_i, θ_i') and (χ_i, χ_i'), $i = 1, 2, \cdots$. If we take (Ω, A, P) to be the product of all these probability spaces and if we define on it random variables θ_i, θ_i' and χ_i, χ_i' in the obvious manner (each depending on one coordinate only), then all the distributional assumptions are satisfied. We have thus constructed a space (Ω, A, P) and random variables $\theta_1, \theta_2, \cdots; \chi_1, \chi_2, \cdots; \theta_1', \theta_2', \cdots; \chi_1', \chi_2', \cdots$ such that all the primed random variables are independent and all the unprimed random variables are independent. They satisfy (1.1), (1.2) and

$$\theta_i(\omega) \leqq \theta_i'\omega) \qquad \omega \in \Omega,\ i \in I \tag{1.3}$$

$$\chi_i(\omega) \geqq \chi_i'(\omega) \qquad \omega \in \Omega,\ i \in I\,. \tag{1.4}$$

This particular choice of a probability space, which supports both queueing systems, will be called *standard construction* and will be used frequently in the sequel.

2. Waiting time. Before proving the basic result of this section we need a lemma. Let x be an arbitrary k-vector and let Rx be the vector obtained by arranging the elements of x in ascending order. We say that $x \leqq y$ if $x_i \leqq y_i$ for all i. Then we get the following:

LEMMA 2.1. *$x \leqq y$ implies $Rx \leqq Ry$.*

PROOF. Since the relation $x \leqq y$ is invariant under identical permutations of the x- and y-components, we may assume that $x = Rx \leqq y$. Let $Ry = (y_{\pi(1)}, \cdots, y_{\pi(k)})$ for a suitable permutation π of the integers $1, \cdots, k$. If $x \nleqq Ry$, then there exists an i such that $x_i > y_{\pi(i)}$. We show that this leads to a contradiction: If $i = \pi(i)$, then $y_i \geqq x_i > y_{\pi(i)} = y_i$. If $i < \pi(i)$, then $x_i \leqq x_{\pi(i)} \leqq y_{\pi(i)} < x_i$. If $i > \pi(i)$, then there exists an $i' < i$ with $\pi(i') \geqq i$. Hence in this case $y_{\pi(i)} < x_i \leqq x_{\pi(i')} \leqq y_{\pi(i')} \leqq y_{\pi(i)}$. Thus we must have $x_i \leqq y_{\pi(i)}$ for all i, as was to be shown.

In this and all subsequent sections we assume that at time 0 all servers are free, and that $t = 0$ is the beginning of a new interarrival interval. It is, however, easy to see that many of the results and proofs are valid under more general conditions.

Let $\eta_n(\eta_n')$ be the waiting time (exclusive of service time) of the nth customer in the original (primed) system. Then we get the following:

THEOREM 2.2. *If $F(x) \geqq F'(x)$ and $H(x) \leqq H'(x)$ for $x \in R$, we have*

$$P(\eta_n \leqq x) \leqq P(\eta_n' \leqq x) \qquad \textit{for all } x \in R,\ n \in I\,. \tag{2.1}$$

PROOF. Let R^+ be the function which reorders the element of a k-vector in ascending order and replaces negative elements by zeros. It is an immediate consequence of Lemma 2.1 that if $x \leqq y$, then $R^+x \leqq R^+y$. Let $\boldsymbol{\eta}_n = (\eta_{n1}, \cdots, \eta_{nk})$ be the vector of ascendingly ordered times remaining, measured from the time of the nth arrival, until each of the various servers would first be available to serve customer n. Kiefer and Wolfowitz (1955) have derived the recursive relationship

$$\boldsymbol{\eta}_{n+1} = R^+(\boldsymbol{\eta}_n + \chi_n \mathbf{e}_1 - \theta_{n+1}\mathbf{1}) , \tag{2.2}$$

where

$$\mathbf{e}_1 = (1, 0, \cdots, 0)$$

and

$$\mathbf{1} = (1, \cdots, 1) .$$

We now assume that the interarrival and service time random variables are defined as in the standard construction outlined above. For each $\omega \in \Omega$ we have $\boldsymbol{\eta}_1(\omega) = 0 = \boldsymbol{\eta}_1'(\omega)$ because of the initial condition. Assuming that $\boldsymbol{\eta}_n(\omega) \geqq \boldsymbol{\eta}_n'(\omega)$ we get from (2.2), (1.3), (1.4) and the above lemma that

$$\begin{aligned} \boldsymbol{\eta}_{n+1}(\omega) &= R^+(\boldsymbol{\eta}_n(\omega) + \chi_n(\omega)\mathbf{e}_1 - \theta_{n+1}(\omega)\mathbf{1}) \\ &\geqq R^+(\boldsymbol{\eta}_n'(\omega) + \chi_n'(\omega)\mathbf{e}_1 - \theta_{n+1}'(\omega)\mathbf{1}) = \boldsymbol{\eta}_{n+1}'(\omega) . \end{aligned}$$

Hence by induction

$$\boldsymbol{\eta}_n(\omega) \geqq \boldsymbol{\eta}_n'(\omega) \tag{2.3}$$

for all n and all ω. Since $\eta_n(\omega) = \eta_{n1}(\omega)$ and $\eta_n'(\omega) = \eta_{n1}'(\omega)$ we have, in particular,

$$\eta_n(\omega) \geqq \eta_n'(\omega) , \qquad \text{for all} \quad \omega \quad \text{and all} \quad n . \tag{2.4}$$

This implies (2.1), since $\{\omega : \eta_n(\omega) \leqq x\} \subset \{\omega : \eta_n'(\omega) \leqq x\}$.

REMARK. Whenever limiting distributions ($n \to \infty$) of η_n, η_n' exist, they are similarly ordered. If the limiting distributions do not depend on initial conditions, the asymptotic inequalities are valid for any initial conditions. Similar remarks apply to limits of other variables to be discussed in this paper.

3. Imbedded queue size. Let ξ_n (ξ_n') be the number of customers in the original (primed) system (including customers being served), at the instant immediately preceding the arrival of the nth customer. We arrive at the following:

THEOREM 3.1. *If $F(x) = F'(x)$ and $H(x) \leqq H'(x)$, for $x \in R$, then,*

$$P(\xi_n \leqq x) \leqq P(\xi_n' \leqq x) \qquad \textit{for all} \quad x \in R,\ n \in I . \tag{3.1}$$

PROOF. Assume the standard construction. Let $\tau_n(\omega) = \sum_{m=1}^{n} \theta_m(\omega)$, $n = 1, 2, \cdots$ be the sequence of arrival epochs. Since $\theta_m(\omega) = \theta_m'(\omega)$, we have $\tau_n(\omega) = \tau_n'(\omega)$. Hence the instants of arrivals of customers are the same in the two systems. For the instants of departures we have

$$\tau_n(\omega) + \eta_n(\omega) + \chi_n(\omega) \geqq \tau_n'(\omega) + \eta_n'(\omega) + \chi_n'(\omega) \tag{3.2}$$

because of (1.4) and (2.4). For any time point the number of customers in the primed system is not greater than the number of customers in the unprimed systems. This then is true for the left-hand limits before arrivals. Hence, $\xi_n(\omega) \geqq \xi_n'(\omega)$, $\omega \in \Omega$, $n \in I$, which implies the desired result.

The preceding proof works primarily because of $\tau_n(\omega) = \tau_n'(\omega)$ in the standard construction. In the remaining case, $F(x) \geqq F'(x)$ and $H(x) = H'(x)$, $x \in R$, this relationship does not usually hold. But by using a "random time contraction", under which arrivals occur simultaneously in both systems, a similar proof can be set up to establish the following:

THEOREM 3.2. *If $F(x) \geqq F'(x)$ and $H(x) = H'(x)$ for $x \in R$, then*

$$P(\xi_n \leqq x) \leqq P(\xi_n' \leqq x), \qquad x \in R,\ n \in I. \tag{3.3}$$

PROOF. Again assume that we use the standard construction. Hence,

$$\theta_n(\omega) \leqq \theta_n'(\omega), \qquad \tau_n(\omega) \leqq \tau_n'(\omega), \qquad \chi_n(\omega) = \chi_n'(\omega), \tag{3.4}$$

for $\omega \in \Omega$, $n \in I$. For each fixed ω define the function

$$d(t, \omega) = \tau_n(\omega) + (t - \tau_n'(\omega)) \frac{\theta_{n+1}(\omega)}{\theta'_{n+1}(\omega)} \qquad \text{for } \tau_n'(\omega) \leqq t < \tau'_{n+1}(\omega). \tag{3.5}$$

Since $\theta'_{n+1}(\omega) = 0$ implies $\tau_n'(\omega) = \tau'_{n+1}(\omega)$, $d(t, \omega)$ is well defined for each $t \in R$, $\omega \in \Omega$. As a function of t, $d(t, \omega)$ is continuous and piecewise linear, with slopes between zero and one; hence it has the property

$$0 \leqq d(t_2, \omega) - d(t_1, \omega) \leqq t_2 - t_1 \qquad \text{for any } t_2 \geqq t_1,\ \omega \in \Omega. \tag{3.6}$$

Furthermore

$$d(\tau_n'(\omega), \omega) = \tau_n(\omega), \qquad \omega \in \Omega,\ n \in I. \tag{3.7}$$

Let $\xi(t, \omega)$ ($\xi'(t, \omega)$) be the number of customers in the original (primed) system at time t, including any arrival at time t. We define a pseudo-queue length process $\bar{\xi}(t, \omega)$ by

$$\bar{\xi}(t, \omega) = \xi'(d^{-1}(t, \omega), \omega), \tag{3.8}$$

where $d^{-1}(\cdot, \omega)$ is, for fixed ω, the left-continuous inverse of the function $t \to d(t, \omega)$. The points of increase of $\bar{\xi}$ are $\tau_1, \tau_2, \cdots$ just as for ξ. The points of decrease of $\bar{\xi}$ are of the form $d(\rho_n'(\omega), \omega)$, where

$$\rho_n' = \tau_n'(\omega) + \eta_n'(\omega) + \chi_n'(\omega). \tag{3.9}$$

From (36), (1.4), and (2.4) it follows easily that

$$\begin{aligned} d(\rho_n') &= d(\tau_n') + d(\rho_n') - d(\tau_n') \\ &\leqq d(\tau_n') + \rho_n' - \tau_n' \\ &= \tau_n + \eta_n' + \chi_n' \\ &\leqq \tau_n + \eta_n + \chi_n, \end{aligned}$$

where we have suppressed all ω. But $\tau_n + \eta_n + \chi_n$ are the points of decrease of ξ (departure epochs). Since departures in $\bar{\xi}$ are earlier than in ξ, we get,

$$\xi(t, \omega) \geqq \bar{\xi}(t, \omega), \qquad t \geqq 0,\ \omega \in \Omega. \tag{3.11}$$

Hence, in particular, this inequality holds for the left-hand limits at arrival epochs. This implies (3.3).

4. Virtual waiting time and queue size. Let $\eta(t)$ ($\eta'(t)$) be the time a customer would wait before being served if he joined the original (primed) queue at the instant t.

THEOREM 4.1. *If for all $x \in R$, $F(x) = F'(x)$ and $H(x) \leqq H'(x)$, then*

$$P(\eta(t) \leqq x) \leqq P(\eta'(t) \leqq x) \qquad \text{for all } x \in R,\ t \geqq 0 . \tag{4.1}$$

PROOF. Make the standard construction, so that

$$\theta_n(\omega) = \theta_n'(\omega)\,, \qquad \tau_n(\omega) = \tau_n'(\omega)\,, \qquad \chi_n(\omega) \geqq \chi_n'(\omega)\,, \tag{4.2}$$

for $\omega \in \Omega$, $n \in I$. The time which a customer would wait if he joined the queue at the instant t is equal to the least amount of time until a server will finish serving all those customers whom he must serve and who arrived before time t. That is, $\eta(t)$ is the minimum of the occupation times of the servers at time t. Since by assumption the servers are all free at $t = 0$, we have $\eta(0, \omega) = \eta'(0, \omega) = 0$. Furthermore, (2.4) and (4.2) imply that $\eta(\tau_n(\omega) - 0, \omega) \geqq \eta'(\tau_n(\omega) - 0, \omega)$, $n \in I$. As long as no arrivals occur, the occupation times are either constant at zero, or, if positive, decrease linearly with slope -1. Thus if the two virtual waiting time processes are ordered just after an arrival, that ordering remains in force at least until just before the next arrival. We now show that the ordering is not destroyed by an arrival. The server who has minimum occupation time at $\tau_n(\omega) + 0$ (immediately following arrival n) is either the same server as had minimum occupation time at $\tau_n(\omega) - 0$, or is the server who had next least occupation time then. Considering these two servers at these two epochs, in each system, we see that

$$\begin{aligned} \eta(\tau_n(\omega) + 0, \omega) &= \min(\eta_{n1}(\omega) + \chi_n(\omega), \eta_{n2}(\omega)) \\ \eta'(\tau_n(\omega) + 0, \omega) &= \min(\eta'_{n1}(\omega) + \chi'_n(\omega), \eta'_{n2}(\omega))\,. \end{aligned} \tag{4.3}$$

(2.3) and (4.2) guarantee that $\eta(\tau_n(\omega) + 0, \omega) \geqq \eta'(\tau_n(\omega) + 0, \omega)$, and this completes the proof that $\eta(\tau, \omega) \geqq \eta'(\tau, \omega)$, which in turn implies (4.1).

Let $\xi(t)$ ($\xi'(t)$) denote the number of customers either waiting or being served in the original (primed) system. Using the same technique we get easily the

THEOREM 4.2. *If, for $x \in R$, $F(x) = F'(x)$ and $H(x) \leqq H'(x)$, then*

$$P(\xi(t) \leqq x) \leqq P(\xi'(t) \leqq x)\,, \qquad x \in R,\ t \geqq 0 . \tag{4.4}$$

PROOF. Using the construction and the argument of the proof to Theorem 3.1, one shows that at any time t and for any $\omega \in \Omega$ the number of customers in the primed system is not greater than the number of customers in the unprimed system. This implies (4.4).

REMARK. A more formal proof of the above theorem would use the representation for queue size

$$\xi(t) = \textstyle\sum_{n=1}^{\infty} f(t - \tau_n, \eta_n + \chi_n)$$

where $f(u, v) = 1$ for $0 \leqq u \leqq v$, and $= 0$ otherwise. The essential point of the argument is the monotonicity of $f(u, v)$ in v for fixed u. Many other processes besides queue length are thus amenable to similar treatment; for instance, shot noise.

In the case $F(x) \geqq F'(x)$ and $H(x) = H'(x)$, $x \in R$, we have not obtained a complete answer to the question of stochastic ordering of virtual waiting time and queue size. In part this question is answered as follows. For finite t, the anticipated stochastic ordering for virtual time and queue size will not hold in general. As an example set $\theta = 2$ or 6 with probability $\frac{1}{2}$ each, $\theta' = 2$ or 8 with probability $\frac{1}{2}$ each, $\chi \equiv \chi' \equiv 3$. By direct enumeration of the possible outcomes it can be shown that

$$W(9.5, x) \leqq W'(9.5, x) \qquad \forall x \in R$$

$$W(10.5, x) \geqq W'(10.5, x) \qquad \forall x \in R$$

and also

$$Q(8.5, x) \leqq Q'(8.5, x) \qquad \forall x \in R$$

$$Q(9.5, x) \equiv Q'(9.5, x) \qquad \forall x \in R$$

$$Q(10.5, x) \geqq Q'(10.5, x) \qquad \forall x \in R\,.$$

All the inequalities are strict for some $x \in R$.

In Sections five and six we show that many of the anticipated order relationships hold for the M/G/k and GI/M/k systems. In the remainder of this section we get additional results for the one-server queue.

Denote by $W(\infty, x)$, $W_\infty(x)$ the weak limit of the distribution function of $\eta(t)$, η_n, as $t \to \infty$, $n \to \infty$, respectively. Similarly for the primed variables.

THEOREM 4.3. *Let the following conditions be satisfied for two queueing systems of the type* GI/G/1:

(i) $F(x) \geqq F'(x)$, $H(x) = H'(x)$, $x \in R$,
(ii) $\int x\,dH(x) < \int x\,dF(x) < \infty$,
(iii) *F, F' are not arithmetic distributions.*

Then $W(\infty, x)$ and $W'(\infty, x)$ exist as proper distributions and

$$W(\infty, x) \leqq W'(\infty, x)\,, \qquad x \in R\,. \tag{4.5}$$

PROOF. Takács (1963) shows that under the above conditions $W(\infty, x)$ and $W'(\infty, x)$ exist as proper distribution functions. He also derives an explicit formula:

$$W(\infty, x) = 1 - \frac{\alpha}{\beta}[1 - W_\infty * \hat{H}(x)]\,, \qquad x \geqq 0$$

$$= 0\,, \qquad x < 0$$

where

$$\hat{H}(x) = \frac{1}{\alpha}\int_0^x (1 - H(y))\,dy$$

and $\alpha = \int x\,dH(x)$, $\beta = \int x\,dF(x)$. Similarly for the primed system. Assumption (i) implies $\beta \leqq \beta'$. From Theorem 2.2 above we know that $W_\infty(x) \leqq W_\infty'(x)$,

$x \in R$. It is then easy to verify that

$$W_\infty * \hat{H}(x) \leqq W_\infty' * \hat{H}(x) . \tag{4.6}$$

This together with $\alpha/\beta \geqq \alpha/\beta'$ implies (4.5).

Denote by $Q(\infty, x)$ ($Q'(\infty, x)$) the weak limit of $\xi(t)$ ($\xi'(t)$).

THEOREM 4.4. *Under the assumptions of Theorem* 4.3, $Q(\infty, x)$ *and* $Q'(\infty, x)$ *exist as proper distributions and satisfy*

$$Q(\infty, x) \leqq Q'(\infty, x) , \qquad x \in R . \tag{4.7}$$

PROOF. Takács (1963) shows that proper limits exist and are given by

$$\begin{aligned} Q(\infty, x) &= 1 - \frac{\alpha}{\beta} \int_0^\infty F^{[x]*}(y)\, dW_\infty * H(y) , && x \geqq 0 \\ &= 0 , && x < 0 , \end{aligned} \tag{4.8}$$

where F^{r*} is the rth convolution of F with itself, and F^{0*} is the distribution function of a unit mass at 0. Similarly for the primed system. From Theorem 2.2 we get $W_\infty(y) \leqq W_\infty'(y)$ and hence $W_\infty * \hat{H}(y) \leqq W_\infty' * \hat{H}(y)$, $y \in R$. Using integration by parts this implies

$$\int_0^\infty F'^{[x]*}(y)\, dW_\infty' * \hat{H}(y) \leqq \int_0^\infty F'^{[x]*}(y)\, dW_\infty * \hat{H}(y) .$$

By assumption $F'(y) \leqq F(y)$, and hence $F'^{r*}(y) \leqq F^{r*}(y)$, thus

$$\int_0^\infty F'^{[x]*}(y)\, dW_\infty * \hat{H}(y) \leqq \int_0^\infty F^{[x]*}(y)\, dW_\infty * \hat{H}(y) .$$

Together with $\beta \leqq \beta'$ this implies

$$\beta \int_0^\infty F'^{[x]*}(y)\, dW_\infty' * \hat{H}(y) \leqq \beta' \int_0^\infty F^{[x]*}(y)\, dW_\infty * \hat{H}(y) , \qquad x \geqq 0 .$$

Using (4.8) this then leads to (4.7).

5. Additional results for the GI/M/k **system.** Whenever service times are exponentially distributed, additional information can be obtained about the limiting distribution of the virtual waiting time and the number of customers queueing.

Let $\xi_q(t) = (\xi(t) - k)^+$, the number of customers queueing but not being served at time t, and let $Q_q(r) = \lim_{t\to\infty} P(\xi_q(t) \leqq r)$, whenever the limit exists. Then we get

THEOREM 5.1. *Let the following conditions be satisfied for two queueing systems of the type* GI/M/k:

(i) $F(x) \geqq F'(x)$, $H(x) = H'(x) = 1 - e^{-\mu x}$, $x \geqq 0$,
(ii) $\mu^{-1} < k \int x\, dF(x)$, $\int x\, dF'(x) < \infty$,
(iii) F, F' *are not central arithmetic distributions.*

Then $Q_q(r)$, $Q_q'(r)$ *exist as proper distributions and*

$$Q_q(r) \leqq Q_q'(r) , \qquad r = 0, 1, 2, \cdots . \tag{5.1}$$

PROOF. It follows easily from Takács [(1962) Theorem 2, page 153] that the limits exist and are given by

$$Q_q(r) = 1 - \frac{1}{k\mu\beta} \sum_{j=r+k}^{\infty} P_j ,$$

where $\beta = \int x\, dF(x)$ and $P_j = \lim_{n\to\infty} P(\xi_n = j)$, $j = 0, 1, 2, \cdots$. Since $\beta \leqq \beta'$, the desired result follow directly from Theorem 3.2.

Before we state the result about limiting virtual waiting time distributions, we derive the following:

LEMMA 5.2. *Let $U_1, U_2, \cdots; U_1', U_2', \cdots$ be two sequences of independent nonnegative random variables with $G_{U_i}(x) \leqq G_{U_i'}(x)$, $x \in R$, $i \in I$. Let N, N' be integer-valued random variables, independent of the sequences and such that $G_N(x) \leqq G_{N'}(x)$, $x \in R$. Then*

$$P(\textstyle\sum_{i=1}^{N} U_i \leqq x) \leqq P(\sum_{i=1}^{N'} U_i' \leqq x) , \qquad x \in R . \tag{5.2}$$

PROOF. Assume that the random variables are jointly defined on a probability space $(\bar{\Omega}, \bar{A}, \bar{P})$ in such a manner that for $\omega \in \bar{\Omega}$, $N(\omega) \geqq N'(\omega)$ and $U_i(\omega) \geqq U_i'(\omega)$, $i = 1, 2, \cdots$. Such a space may be constructed by using the independence and stochastic ordering properties of the two sequences. Then it follows that for each $\omega \in \bar{\Omega}$

$$\textstyle\sum_{i=1}^{N} U_i \geqq \sum_{i=1}^{N'} U_i' ,$$

which implies (5.2).

THEOREM 5.3. *Under the conditions of Theorem 5.1 $W(\infty, x)$ and $W'(\infty, x)$ exist as proper distribution functions and the relationship*

$$W(\infty, x) \leqq W'(\infty, x) , \qquad x \in R , \tag{5.3}$$

holds.

PROOF. In the present queueing system the virtual waiting time is a random sum of independent, exponentially distributed random variables, each with parameter $k\mu$. The number of terms is given by $\xi_q(t)$, $\xi_q'(t)$, resp. Hence the result follows directly from Theorem 5.1 and the preceding lemma, letting $t \to \infty$.

6. Additional results for the M/G/k system. Assume in this section that all interarrival times are exponentially distributed. For such systems a suitable construction of a probability space, somewhat different from the one used above, will enable us to make a comparison of distribution functions by making a realization-by-realization comparison of two queueing systems with stochastically ordered interarrival times. It turns out that in such systems the anticipated ordering of queue size and virtual waiting time holds for any t.

The basic idea of the construction is this: If arrivals in the original queueing system are recorded with probability p, the recordings being independent of each other and of the (Poisson) arrival process, then the recorded arrivals form a new interarrival process with stochastically larger exponential interarrival times.

Thus we can construct the primed arrival process by thinning out the original process. For convenience, customers recorded are called type I customers, the others are type II customers. By a suitable rearrangement of the sequence of service times we are then able to get inequalities for each ω of the basic probability space.

To come to the specifics of the construction, let $\lambda(\lambda')$ be the arrival rates $(\lambda > \lambda')$ of the two systems to be compared and let $H(x)$ be the common distribution function of the service times. For $n = 1, 2, \cdots$ form probability spaces (Ω_n, A_n, P_n) and independent random variables $\bar{\theta}_n$, $\bar{\chi}_n$, $\bar{\nu}_n$ defined on them such that $P_n(\bar{\theta}_n \leqq x) = 1 - e^{-\lambda x}$, $P_n(\bar{\chi}_n \leqq x) = H(x)$, $P_n(\bar{\nu}_n = k) = p(1-p)^{k-1}$, $k = 1, 2, \cdots$, where $p = \lambda'/\lambda$. Then set $(\Omega, A, P) = (\Pi\Omega_n, \Pi A_n, \Pi P_n)$ and define on it random variables θ_n, χ_n, ν_n by

$$\theta_n(\omega) = \bar{\theta}_n(\omega_n)\,, \qquad \chi_n(\omega) = \bar{\chi}_n(\omega_n)\,, \qquad \nu_n(\omega) = \bar{\nu}_n(\omega_n)\,, \qquad n \in I\,.$$

Obviously $\theta_1, \theta_2, \cdots; \chi_1, \chi_2, \cdots$ are independent random variables. They will serve as interarrival and service times for the unprimed queueing system. Furthermore, define random variables $M_0 \equiv 0$, $M_n = \sum_{j=1}^{n} \nu_j$, $\theta_n' = \sum_{j=M_{n-1}+1}^{M_n} \theta_j$, $\chi_n' = \chi_{M_n}$. Then it is easy to see that $\theta_1', \theta_2', \cdots; \chi_1', \chi_2', \cdots$ are independent random variables with $P(\theta_n' \leqq x) = 1 - e^{-\lambda' x}$, $P(\chi_n' \leqq x) = H(x)$. They will serve as interarrival and service times of the primed system. This particular construction implies that for fixed ω the nth type I customer in the unprimed system arrives at the same time as the nth customer in the primed system and has the same service time.

Before we prove the main theorem of this section we need a lemma. For fixed t let $\eta_j(t)$ be the amount of time that the jth server would still remain busy, if the arrival process were to be shut off at the instant t. Let $\boldsymbol{\eta}(t) = (\eta_1(t), \cdots, \eta_k(t))$ and define $\boldsymbol{\eta}'(t)$ similarly.

LEMMA 6.1. *For some t_0 let $\boldsymbol{\eta}(t_0) \geqq \boldsymbol{\eta}'(t_0)$. Assume that after time t_0 customers arrive at times $t_1 < t_2 < \cdots < t_\nu$ in the unprimed system, with service times $s_1, s_2, \cdots, s_\nu$. Assume also that the first customer after t_0 in the primed system arrives at time t_ν, requiring s_ν time units of service. Then $\boldsymbol{\eta}(t) \geqq \boldsymbol{\eta}'(t)$ for $t_0 \leqq t < t_\nu$ and $R^+\boldsymbol{\eta}(t_\nu) \geqq R^+\boldsymbol{\eta}'(t_\nu)$.*

PROOF. Obviously $\boldsymbol{\eta}(t) \geqq \boldsymbol{\eta}'(t)$ for $t_0 \leqq t < t_\nu$, since $\eta(t)$ may have points of increase whereas $\eta'(t)$ will not. In particular $\boldsymbol{\eta}(t_\nu - 0) \geqq \boldsymbol{\eta}'(t_\nu - 0)$. $\boldsymbol{\eta}(t_\nu)\,[\boldsymbol{\eta}'(t_\nu)]$ is obtained from $\boldsymbol{\eta}(t_\nu - 0)\ [\boldsymbol{\eta}'(t_\nu - 0)]$ by adding s_ν to the minimal component of $\boldsymbol{\eta}(t_\nu - 0)\ [\boldsymbol{\eta}'(t_\nu - 0)]$. Let the minimal component have index $j\,[j']$. If $j = j'$, then $\boldsymbol{\eta}(t_\nu) \geqq \boldsymbol{\eta}'(t_\nu)$ and thus the result follows from Lemma 2.1. If $j \neq j'$, then it follows from the definition of j and j' that

$$\eta_{j'}'(t_\nu - 0) \leqq \eta_j'(t_\nu - 0) \leqq \eta_j(t_\nu - 0)$$

and

$$\eta_{j'}'(t_\nu - 0) \leqq \eta_j(t_\nu - 0) \leqq \eta_{j'}(t_\nu - 0)\,.$$

Hence if we interchange the jth and j'th component of $\boldsymbol{\eta}(t_\nu)$ to get $\bar{\boldsymbol{\eta}}(t_\nu)$, then

$\bar{\boldsymbol{\eta}}(t_\nu) \geqq \boldsymbol{\eta}'(t_\nu)$ and hence $R^+\bar{\boldsymbol{\eta}}(t_\nu) \geqq R^+\boldsymbol{\eta}'(t_\nu)$, again by Lemma 2.1. But since $R^+\mathbf{x}$ is invariant under interchanges of components of $\mathbf{x}$, we have $R^+\boldsymbol{\eta}(t) \geqq R^+\boldsymbol{\eta}'(t)$.

THEOREM 6.2. *In the system* M/G/k, *if* $F(x) \geqq F'(x)$ *and* $H(x) = H'(x)$ *for* $x \in R$, *then*

$$P(\eta(t) \leqq x) \leqq P(\eta'(t) \leqq x)\,, \qquad x \in R,\ t \geqq 0\,. \tag{6.2}$$

PROOF. Use the construction given above as the basic space with random sequences $\{\theta_n\}$, $\{\theta_n'\}$, $\{\chi_n\}$ and $\{\chi_n'\}$ defined on it. It follows from that construction that if we fix $\omega \in \Omega$, then we have the situation of Lemma 6.1 with $t_0 = 0$, $\nu = \nu_1(\omega)$, $t_i = \tau_i(\omega)$, and $s_i = \chi_i(\omega)$, $i = 1, \cdots, \nu_1(\omega)$. Also $\boldsymbol{\eta}(0) = \boldsymbol{\eta}'(0) = \mathbf{0}$, from our initial conditions. Hence the lemma applies and we have $\boldsymbol{\eta}(t, \omega) \geqq \boldsymbol{\eta}'(t, \omega)$ for $0 \leqq t < \tau_{\nu_1(\omega)}(\omega) = \tau_1'(\omega)$. Since $\eta(t)$ $[\eta'(t)]$ is the minimum component of $\boldsymbol{\eta}(t)$ $[\boldsymbol{\eta}'(t)]$, we get $\eta(t, \omega) \geqq \eta'(t, \omega)$ for $0 \leqq t < \tau_1'(\omega)$. At time $\tau_1'(\omega)$ we relabel the servers in the primed and unprimed systems, so that without loss of generality we may assume $\boldsymbol{\eta}(\tau_1'(\omega), \omega) = R^+\boldsymbol{\eta}(\tau_1'(\omega), \omega)$ and $\boldsymbol{\eta}'(\tau_1'(\omega), \omega) = R^+\boldsymbol{\eta}'(\tau_1'(\omega), \omega)$. Hence by Lemma 6.1 $\boldsymbol{\eta}(\tau_1'(\omega), \omega) \geqq \boldsymbol{\eta}'(\tau_1'(\omega), \omega)$. Taking this as the initial condition we can apply the previous argument to the t-interval $[\tau_1'(\omega), \tau_2'(\omega)]$, and, by induction on the subscript, to the entire positive real line. Hence for the minimum components of $\boldsymbol{\eta}(t)$, $\boldsymbol{\eta}'(t)$,

$$\eta(t, \omega) \geqq \eta'(t, \omega)\,, \qquad \omega \in \Omega,\ t \geqq 0\,, \tag{6.3}$$

which implies (6.2).

THEOREM 6.3. *If in the system* M/G/k $F(x) \geqq F'(x)$ *and* $H(x) = H'(x)$ *for* $x \in R$, *then*

$$P(\xi(t) \leqq x) \leqq P(\xi'(t) \leqq x)\,, \qquad x \in R,\ t > 0\,. \tag{6.4}$$

PROOF. Assume the construction given at the beginning of this section. Let $\{t_n'\}$ represent the sequence of type I arrival epochs. Since (6.3) holds for every value of t we have

$$\eta(t_n'(\omega) - 0, \omega) \geqq \eta'(t_n'(\omega) - 0, \omega)\,, \qquad n \in I\,. \tag{6.5}$$

In the unprimed system, the departure epoch of the nth type I arrival is

$$t_n'(\omega) + \eta(t_n'(\omega) - 0, \omega) + \chi_n'(\omega)\,;$$

whereas in the primed system it is

$$t_n'(\omega) + \eta'(t_n'(\omega) - 0, \omega) + \chi_n'(\omega)\,.$$

Therefore, using (6.5) the nth type I arrival departs sooner in the primed system than in the unprimed. Thus for every t, there will be at least as many type I customers in the unprimed queue, as there are customers in the primed queue. Furthermore, there may be type II customers in the unprimed system, but there can be none in the primed system. Hence $\xi(t, \omega) \geqq \xi'(t, \omega)$, $t \geqq 0$, $\omega \in \Omega$. This implies (6.4).

Acknowledgment. The authors are grateful to D. J. Daley for suggesting the idea of proof of the theorem of Section 2.

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