Electron. Commun. Probab. **18** (2013), no. 36, 1–8.
DOI: 10.1214/ECP.v18-2780
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

On Zhao-Woodroofe's condition for martingale approximation*

Jana Klicnarová† Dalibor Volný‡

Abstract

The Zhao-Woodroofe condition has been introduced in [19] and it is a necessary and sufficient condition for the existence of a martingale approximation of a causal stationary process. Here, a nonadapted version is given and the convergence of Cesaro averages is replaced by a convergence of a subsequence. The nonadapted version is of a different form than in other cases, e.g. of Wu-Woodroofe or Maxwell-Woodroofe conditions ([10], [18], [15]).

Keywords: stationary process; martingale approximation; nonadapted version.
AMS MSC 2010: 60G10.
Submitted to ECP on December 2, 2010, final version accepted on April 30, 2013.

1 Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and $T : \Omega \to \Omega$ a bijective, bimeasurable and measure preserving mapping. For a measurable function f the sequence $(f \circ T^i)$ is strictly stationary and every strictly stationary sequence of random variables can be represented in this way.

Denote

$$S_n(f) = \sum_{i=1}^{n} f \circ T^i.$$

By $(\mathcal{F}_i)$ we will denote an increasing filtration where $T^{-1}(\mathcal{F}_i) = \mathcal{F}_{i+1}$, P_i will denote the orthogonal projection onto $L^2(\mathcal{F}_i) \ominus L^2(\mathcal{F}_{i-1})$, i.e. $P_i f = E(f|\mathcal{F}_i) - E(f|\mathcal{F}_{i-1})$. If f is $\mathcal{F}_0$-measurable, we say that the process $(f \circ T^i)$ is adapted (to the filtration $(\mathcal{F}_i)$).

Definition 1.1. *We say that a function f (process $(f \circ T^i)$) has* ***a martingale approximation*** *with respect to a filtration $(\mathcal{F}_i)_{i=-\infty}^{+\infty}$ if there exists a martingale difference sequence $(m \circ T^i)_{i \in \mathbb{Z}}$ such that $P_0 m = m$ and $||S_n(f - m)||_2^2 = o(n)$.*

Because the CLT holds for stationary sequences of L^2 martingale differences, this condition guarantees a CLT for $(f \circ T^i)$ (if μ is ergodic, the limit law is normal and for μ non ergodic it can be mixture of normal laws). Since 1969, the date of publication of Gordin's paper [4], martingale approximations have been an important tool in the research of limit theorems for stationary processes. Conditions for a martingale approximation can be found in many articles. Most of the results published before 1980

*Supported by Czech Science Foundation (project no. P201/11/P164).
†University of South Bohemia, Czech Republic. E-mail: janaklic@ef.jcu.cz
‡Université de Rouen, France. E-mail: dalibor.volny@univ-rouen.fr

can be found in [6], for more recent results see e.g. (the list being very incomplete) [14], [10], [1], [2], [3], [19], [11], [5], [17].

In 2004, Wu and Woodroofe ([18]) showed that for an L^2 process $(f \circ T^i)$ adapted to a filtration $(\mathcal{F}_i)$, there exists a triangular array of martingale differences $m_n \circ T^i$ ($P_0 m_n = m_n$) satisfying

$$\|S_n(f - m_n)\|_2 = o(\sigma_n)$$

if and only if

$$\|E(S_n(f)|\mathcal{F}_0)\|_2 = o(\sigma_n);$$

we denote $\sigma_n = \|S_n(f)\|_2 \to \infty$. The condition of Wu and Woodroofe, however, implies neither a martingale approximation, nor a CLT. In [8], Klicnarová and Volný found an L^2 process $(f \circ T^i)$ such that $\|S_n(f)\|_2/\sqrt{n} \to 1$, $\|E(S_n(f)|\mathcal{F}_0)\|_2 = O(\sqrt{n}/\log n)$, and the CLT does not take place, hence there is no martingale approximation.

It is thus a natural problem to find a supplementary condition which implies a martingale approximation. Such a condition has been found by Zhao and Woodroofe in [19]. They introduced the plus-norm

$$||f||_+^2 = \limsup_{n \to +\infty} \frac{1}{n} ||S_n(f)||_2^2$$

and a condition which we call Zhao-Woodroofe's condition (ZW):

Definition 1.2. *Let the process $(f \circ T^i)$ be adapted. We say that it (or, the function f) satisfies* ***Zhao-Woodroofe's condition (ZW)*** *if*

$$\lim_{m \to +\infty} \frac{1}{m} \sum_{k=1}^{m} ||E(f|\mathcal{F}_{-k})||_+^2 = 0.$$

Zhao and Woodroofe proved that a process $(f \circ T^i)$ satisfying the condition of Wu and Woodroofe has a martingale approximation if and only if it satisfies ZW. Later, Peligrad (see [11]) improved the result by showing that the Wu-Woodroofe's condition is redundant and found other equivalent conditions. We thus have:

Proposition 1.3. *An adapted process $(f \circ T^i)$ has a martingale approximation if and only if it satisfies ZW.*

Further results leading to invariance principles can be found in [5].

We will search a nonadapted version of ZW. Recall that a nonadapted version of the condition of Wu and Woodroofe has been found by Volný in [15]. For ZW, the same method as in [15] gives a condition presented below in Corollary 1 which is sufficient for a martingale approximation. As we will show in Proposition 2, this condition, however, is not necessary. We thus define the following condition.

Definition 1.4. *The function f satisfies* ***nonadapted Zhao-Woodroofe's condition (naZW)*** *if it satisfies:*

$$\lim_{m \to +\infty} \frac{1}{m} \sum_{k=1}^{m} ||E(f|\mathcal{F}_{-k}) + f - E(f|\mathcal{F}_{m-k})||_+^2 = 0. \tag{1.1}$$

Theorem 1.5. *A stationary process satisfies the nonadapted Zhao-Woodroofe's condition (1.1) if and only if it has a martingale approximation.*

Remark. As follows from [9], for f adapted, $||\mathrm{E}(f|\mathcal{F}_{-k})||_+ \to 0$ is not a necessary condition for a martingale approximation. Nevertheless, in Proposition 3 we shall give a necessary and sufficient condition for a martingale approximation using subsequences, both for the adapted and the nonadapted case.

Proof of the Theorem:

For $f \in L^2$ we have $f = f' + f^{(r)} + f''$ where $f' = E(f|\mathcal{F}_{-\infty})$ is the orthogonal projection of f onto $L^2(\mathcal{F}_{-\infty})$, $f'' = f - E(f|\mathcal{F}_\infty)$ is the orthogonal projection of f onto $L^2(\mathcal{A}) \ominus L^2(\mathcal{F}_\infty)$, and $f^{(r)} = E(f|\mathcal{F}_\infty) - E(f|\mathcal{F}_{-\infty})$ is the orthogonal projection of f onto $L^2(\mathcal{F}_\infty) \ominus L^2(\mathcal{F}_{-\infty})$ ($\mathcal{F}_\infty$ is the σ-algebra generated by the union of all $\mathcal{F}_k$ and $\mathcal{F}_{-\infty}$ is the intersection of all $\mathcal{F}_k$). The subspaces $L^2(F_{-\infty})$, $L^2(\mathcal{F}_\infty) \ominus L^2(\mathcal{F}_{-\infty})$, and $L^2(\mathcal{A}) \ominus L^2(\mathcal{F}_\infty)$ of $L^2(\mathcal{A})$ are mutually orthogonal and invariant with respect to the unitary operator U defined by $Uh = h \circ T$, $h \in L^2(\mathcal{A})$. For $m \in L^2$ with $P_0 m = m$ we thus have

$$||S_n(f-m)||_2^2 = ||S_n(f')||_2^2 + ||S_n(f^{(r)} - m)||_2^2 + ||S_n(f'')||_2^2$$

and for $1 \le k \le j$ we have

$$\mathrm{E}(f|\mathcal{F}_{-k}) + f - \mathrm{E}(f|\mathcal{F}_{j-k}) = f' + \mathrm{E}(f^{(r)}|\mathcal{F}_{-k}) + f^{(r)} - \mathrm{E}(f^{(r)}|\mathcal{F}_{j-k}) + f'',$$

$$\begin{aligned} ||S_n(\mathrm{E}(f|\mathcal{F}_{-k}) + f - \mathrm{E}(f|\mathcal{F}_{j-k}))||_2^2 &= ||S_n(f')||_2^2 + ||S_n(f'')||_2^2 \\ &\quad + ||S_n(\mathrm{E}(f^{(r)}|\mathcal{F}_{-k}) + f^{(r)} - \mathrm{E}(f^{(r)}|\mathcal{F}_{j-k}))||_2^2 \end{aligned}$$

hence $||S_n(f')||_2 = o(\sqrt{n})$ and $||S_n(f'')||_2 = o(\sqrt{n})$ are necessary conditions both for a martingale approximation with respect to the filtration $(\mathcal{F}_i)_{i=-\infty}^{+\infty}$ and for the nonadapted Zhao-Woodroofe's condition.

For proving Theorem 1 we can thus suppose that f is *regular*, i.e. $f = f^{(r)}$ (otherwise said, f is $\mathcal{F}_\infty$-measurable and $E(f|\mathcal{F}_{-\infty}) = 0$).

The naZW's condition implies MA. Let us suppose that the nonadapted ZW's condition is satisfied. Our aim is to prove that in such a case the process has a martingale approximation. By regularity, $f = \sum_{i=-\infty}^{\infty} P_i f$ hence the naZW's condition is equivalent to

$$\lim_{m \to +\infty} \frac{1}{m} \sum_{k=1}^{m} \left|\left| f - \sum_{i=-k+1}^{m-k} P_i f \right|\right|_+^2 = 0.$$

Such a condition implies (see Proposition 1.8) that there exist subsequences $n_k^{'}$ and $n_k^{''}$ (both tending to infinity) such that

$$\left|\left| f - \sum_{i=-n_k^{'}}^{n_k^{''}} P_i f \right|\right|_+^2 \to 0. \tag{1.2}$$

A regular L^2 function h has a martingale-coboundary decomposition $h = m + g - g \circ T$ with $m, g \in L^2$ and $P_0 m = m$ if and only if the series $\sum_{i=0}^{\infty} E(h \circ T^i|\mathcal{F}_{-1})$ and $\sum_{i=1}^{\infty} [h \circ T^{-i} - E(h \circ T^{-i}|\mathcal{F}_{-1})]$ converge in L^2; then, we can have $m = \sum_{i=-\infty}^{\infty} P_0(h \circ T^i)$, $g = \sum_{i=0}^{\infty} E(h \circ T^i|\mathcal{F}_{-1}) - \sum_{i=1}^{\infty}[h \circ T^{-i} - E(h \circ T^{-i}|\mathcal{F}_{-1})]$ (see [14], cf. [4]). From this we deduce that $\sum_{i=-n_k^{'}}^{n_k^{''}} P_i f - \sum_{i=-n_k^{''}}^{n_k^{'}} P_0(f \circ T^i)$ is a coboundary $g - g \circ T$ with an L^2-integrable transfer function g. More precisely, g is equal to

$$\sum_{i=1}^{n_k^{'}} \sum_{j=0}^{n_k^{'}-i} P_{-i}(f \circ T^j) - \sum_{i=0}^{n_k^{''}-1} \sum_{j=1}^{n_k^{''}-i} P_i(f \circ T^{-j})$$

and the martingale part m is zero.

Therefore, $\left|\left|\sum_{i=-n_k^{'}}^{n_k^{''}} P_i f - \sum_{i=-n_k^{''}}^{n_k^{'}} P_0(f\circ T^i)\right|\right|_+ = 0$, hence (notice that $\|.\|_+$ is a seminorm, cf. [19])

$$\left|\left|f - \sum_{i=-n_k^{''}}^{n_k^{'}} P_0(f\circ T^i)\right|\right|_+^2 \to 0$$

for some subsequences $n_k^{''}$ and $n_k^{'}$ both tending to infinity.

Denote $m_k = \sum_{i=-n_k^{''}}^{n_k^{'}} P_0(f\circ T^i)$; like in [14] we will show that there exists m such that $m_k \to m$ in L^2. Since $m_k = P_0(m_k)$ for every k and l we have $||m_k - m_l||_2 = ||m_k - m_l||_+$. Because $||m_k - m_l||_+ \le ||f - m_k||_+ + ||f - m_l||_+$, where both terms of the right-hand side tend to zero, we have $||m_k - m_l||_2 \to 0$. We deduce that there exists m such that $m = \lim_{k\to\infty} m_k$. $\|f-m\|_+ \le \|f-m_k\|_+ + \|m-m_k\|_+ = \|f-m_k\|_+ + \|m-m_k\|_2 \to 0$ (cf. [14]) hence $||f-m||_+ = 0$ and the function f has a martingale approximation.

The MA implies the naZW's condition. Let us suppose that the process $(f\circ T^i)$ has a martingale approximation $(m\circ T^i)$, i.e. $||S_n(f-m)||_2 = o(\sqrt{n})$; by [18] and [15],

$$||\mathrm{E}(S_n(f)|\mathcal{F}_0)||_2 = o(\sqrt{n}) \text{ and } ||S_n(f) - \mathrm{E}(S_n(f)|\mathcal{F}_n)||_2 = o(\sqrt{n}).$$

Therefore

$$||S_n(f-m) - \mathrm{E}(S_n(f)|\mathcal{F}_0) - (S_n(f) - \mathrm{E}(S_n(f)|\mathcal{F}_n))||_2 = o(\sqrt{n})$$

hence

$$||\mathrm{E}(S_n(f)|\mathcal{F}_n) - \mathrm{E}(S_n(f)|\mathcal{F}_0) - S_n(m)||_2 = o(\sqrt{n}).$$

We have

$$\mathrm{E}(S_n(f)|\mathcal{F}_n) - \mathrm{E}(S_n(f)|\mathcal{F}_0) = \sum_{k=1}^n P_k S_n(f).$$

Noticing that the mapping $Uh = h\circ T$ is a unitary operator in L^2 and that $UP_i = P_{i+1}U$ we get

$$\sum_{k=1}^n P_k S_n(f) = \sum_{k=1}^n U^k \sum_{j=1}^n P_0 U^{j-k} f.$$

We denote

$$\sum_{j=1}^n P_0 U^{j-k} f = \sum_{j=1}^{k-1} P_0 U^{-j} f + \sum_{j=0}^{n-k} P_0 U^j f = s_k^{''} + s_{n-k}^{'}.$$

Therefore,

$$\mathrm{E}(S_n(f)|\mathcal{F}_n) - \mathrm{E}(S_n(f)|\mathcal{F}_0) - S_n(m) = -\sum_{k=1}^n (m - s_{n-k}^{'} - s_k^{''})\circ T^k,$$

The functions $m - s_{n-k}^{'} - s_k^{''}$ are martingale differences, hence

$$||\mathrm{E}(S_n(f)|\mathcal{F}_n) - \mathrm{E}(S_n(f)|\mathcal{F}_0) - S_n(m)||_2^2 = \sum_{k=1}^n ||m - s_{n-k}^{'} - s_k^{''}||_2^2 = \sum_{k=1}^n ||m - s_{n-k}^{'} - s_k^{''}||_+^2 = o(n).$$

Let us note that the functions $\mathrm{E}(f|\mathcal{F}_0) - \mathrm{E}(f|\mathcal{F}_{k-n-1}) - s_{n-k}^{'}$ and $\mathrm{E}(f|\mathcal{F}_{k-1}) - \mathrm{E}(f|\mathcal{F}_0) - s_k^{''}$ are coboundaries. Indeed, for $h = \mathrm{E}(f|\mathcal{F}_0) - \mathrm{E}(f|\mathcal{F}_{k-n-1}) - s_{n-k}^{'}$ the condition for a martingale-coboundary decomposition is easily satisfied and the martingale part $m = \sum_{i=-\infty}^{\infty} P_0(h \circ T^i)$ is zero; the same reasoning applies to the second function. Therefore,

$$||\mathrm{E}(f|\mathcal{F}_0) - \mathrm{E}(f|\mathcal{F}_{k-n-1}) - s'_{n-k}||_+ = 0 \text{ and } ||\mathrm{E}(f|\mathcal{F}_{k-1}) - \mathrm{E}(f|\mathcal{F}_0) - s''_k||_+ = 0.$$

Using these equalities and $||f - m||_+ = 0$ we have

$$\begin{aligned} ||m - s'_{n-k} - s''_k||_+ &= ||f - s'_{n-k} - s''_k||_+ \\ &= ||f - (\mathrm{E}(f|\mathcal{F}_0) - \mathrm{E}(f|\mathcal{F}_{k-n-1})) - (\mathrm{E}(f|\mathcal{F}_{k-1}) - \mathrm{E}(f|\mathcal{F}_0))||_+ \\ &= ||f - \mathrm{E}(f|\mathcal{F}_{k-1}) + \mathrm{E}(f|\mathcal{F}_{n-k+1})||_+ \end{aligned}$$

hence

$$\frac{1}{n}\sum_{k=1}^{n} ||f - \mathrm{E}(f|\mathcal{F}_{k-1}) + \mathrm{E}(f|\mathcal{F}_{n-k+1})||_+^2 \to 0$$

and the naZW condition follows.

$\triangle$

From Theorem 1 we get the following corollary.

Corollary 1.6. *The conditions*

$$\lim_{m\to+\infty} \frac{1}{m}\sum_{k=1}^{m} ||E(f|\mathcal{F}_{-k})||_+^2 = 0$$

and

$$\lim_{m\to+\infty} \frac{1}{m}\sum_{k=1}^{m} ||f - E(f|\mathcal{F}_{k})||_+^2 = 0$$

imply MA.

The corollary resembles the nonadapted versions of the theorems of Wu and Woodroofe [18], Maxwell and Woodroofe [10] and Peligrad and Utev [13], cf. [16] and [7]. Here, however, the assumption of the corollary is not a necessary condition for a martingale approximation:

Proposition 1.7. *There exists a function f such that there is a martingale approximation but none of the conditions* $\lim_{m\to+\infty} \frac{1}{m}\sum_{k=1}^{m} ||E(f|\mathcal{F}_{-k})||_+^2 = 0$, $\lim_{m\to+\infty} \frac{1}{m}\sum_{k=1}^{m} ||f - E(f|\mathcal{F}_{k})||_+^2 = 0$ *is satisfied.*

Proof. Let (e_i) be a sequence of iid random variables, $E(e_i) = 0$, $\|e_i\|_2 = 1$, $e_i \circ T = e_{i+1}$. We define $N_0 = 0$ and $N_k = 2^k$, $k = 1, 2, \dots$. Let

$$f_k = \sum_{i=-N_k}^{N_k} a_{k,i} e_i, \quad k = 1, 2, \dots, \text{ where } a_{k,i} = \frac{\text{sign}(i)}{2kN_k};$$

sign(i) is defined as 1 if $i > 0$, -1 if $i < 0$, and 0 for $i = 0$. We define

$$\bar{f} = \sum_{k=1}^{\infty} f_k, \quad f = e_0 + \bar{f}.$$

Remark that $f = e_0 + \sum_{i\in\mathbb{Z}} a_i e_i$ where $a_i = \sum_{k:\ N_k \geq |i|} a_{k,i}$; because $N_k \geq k$, $\sum_{i\in\mathbb{Z}} a_i^2 < \infty$ hence $f \in L^2$.

By [18, proof of Theorem 1], if there is a martingale approximation $(m \circ T^i)$ for $E(f|\mathcal{F}_0)$ then m is the limit of $\frac{1}{n}\sum_{k=1}^{n}\sum_{i=0}^{k} P_0(f \circ T^i)$. Because $\sum_{i=-\infty}^{-1} a_i = -\infty$, the adapted part $\mathrm{E}(f|\mathcal{F}_0)$ of f does not have a martingale approximation. From $\sum_{i=1}^{\infty} a_i = \infty$ we get the same result for the nonadapted part $f - \mathrm{E}(f|\mathcal{F}_0)$. From Theorem 1 applied to $E(f|\mathcal{F}_0)$ and to $f - \mathrm{E}(f|\mathcal{F}_0)$ separately it follows that neither $\lim_{m\to+\infty} \frac{1}{m}\sum_{k=1}^{m} ||\mathrm{E}(f|\mathcal{F}_{-k})||_+^2 = 0$ nor $\lim_{m\to+\infty} \frac{1}{m}\sum_{k=1}^{m} ||f - \mathrm{E}(f|\mathcal{F}_k)||_+^2 = 0$ can be satisfied.

Now, let us show that for every $n < N_k$

$$\|S_n(\sum_{j=k}^{\infty} f_j)\|_2 \le 4\sqrt{\frac{n}{k}}$$

and for $n \ge N_k$,

$$\lim_{k\to\infty} \sup_{n\ge N_k} \frac{1}{\sqrt{n}} \|S_n(\sum_{j=1}^{k} f_j)\|_2 = 0.$$

This will imply that $\|S_n(f - e_0)\|_2 = o(\sqrt{n})$ hence $(e_0 \circ T^i)$ is a martingale approximation for f. For every k we have

$$\|S_n(\sum_{j=1}^{\infty} f_j)\|_2 \le \|S_n(\sum_{j=1}^{k-1} f_j)\|_2 + \|S_n(\sum_{j=k}^{\infty} f_j)\|_2$$

and that for each n there exists a k such that $N_{k-1} \le n < N_k$.

Let us prove the first estimation.

By orthogonality,

$$\|S_n(\sum_{j=k}^{\infty} f_j)\|_2^2 = \sum_{l=0}^{\infty} \sum_{N_l \le i \le N_{l+1}} \|P_i(S_n(\sum_{j=k}^{\infty} f_j)\|_2^2.$$

For $|i| \in \{N_{j-1}+1, \dots, N_j\}$ we have $|a_i| \le \frac{1}{jN_j}$. By direct computation we thus get

$$\sum_{l=0}^{\infty} \sum_{N_l \le i \le N_{l+1}} \|P_i(S_n(\sum_{j=k}^{\infty} f_j)\|_2^2 \le \sum_{|i|\le N_k} \|P_i(S_n(2f_k)\|_2^2 + \sum_{l=k}^{\infty} \sum_{N_l < |i| \le N_{l+1}} \|P_i(S_n(4f_l)\|_2^2 \le 8n^2 \sum_{j=k}^{\infty} \frac{1}{j^2 N_j}.$$

Since we suppose $n < N_k$,

$$\|S_n(\sum_{j=k}^{\infty} f_j)\|_2^2 \le 8n \sum_{j=k}^{\infty} \frac{1}{j^2} \le \frac{16n}{k}.$$

Now, we prove the second estimate.

For $j = 1, 2, \dots$ the functions f_j are coboundaries $f_j = g_j - g_j \circ T$ with a cobounding function

$$g_j = -\frac{1}{2jN_j} \sum_{i=0}^{N_j - 1} (N_j - i)e_i - \frac{1}{2jN_j} \sum_{i=1}^{N_j} (N_j - i + 1)e_{-i}.$$

$\sum_{j=1}^{k} f_j$ is thus a coboundary with a cobounding function $g = \sum_{i=-N_k}^{N_k - 1} b_i e_i$ where

$$0 \le b_i \le \frac{1}{2}\left(1 + \cdots + \frac{1}{k}\right) \quad \text{for} \quad 0 \le i \le N_1 - 1 = 1,$$
$$0 \le b_i \le \frac{1}{2}\left(\frac{1}{j} + \cdots + \frac{1}{k}\right) \quad \text{for} \quad N_{j-1} \le i \le N_j - 1,\ j = 2, \dots, k.$$

For $-N_k \le i \le -1$ we get a similar estimation for $|b_i|$.

For k sufficiently big and $\epsilon > 0$, $l = \epsilon k > 1$ we get, using $\frac{1}{2} + \cdots + \frac{1}{k} \le \log k$,

$$\begin{aligned}\|g\|_2^2 &\le 2\sum_{i=1}^{N_k} b_i^2 = 2\sum_{j=1}^{k} \sum_{i=N_{k-1}+1}^{N_k} b_i^2 \\ &\le N_{k-1}\left(\frac{1}{k}\right)^2 + \cdots + N_{k-l}\left(\frac{1}{k-l+1} + \cdots + \frac{1}{k}\right)^2 + 2N_{k-l}(1+\log k)^2 \\ &\le N_k\left(\frac{\epsilon}{1-\epsilon}\right)^2 + N_k 2^{-\epsilon k}(1+\log k)^2;\end{aligned}$$

choosing $\epsilon \to 0$ such that $\epsilon k/\log\log k \to \infty$ we see that for $k \to \infty$, $\|g\|_2^2/N_k \to 0$. Because $S_n(\sum_{j=1}^k f_j) = g \circ T - g \circ T^{n+1}$ we get the result.

$\triangle$

Proposition 1.8. *The nonadapted Zhao-Woodroofe's condition is equivalent to the existence of subsequences $(n_k^{'})$ and $(n_k^{''})$ such that both of them tend to infinity and*

$$||E(f|\mathcal{F}_{-n_k^{'}}) + f - E(f|\mathcal{F}_{n_k^{''}})||_+^2 \to 0.$$

In adapted case, the Zhao-Woodroofe's condition is equivalent to the existence of a subsequence (n_k) such that

$$||E(f|\mathcal{F}_{-n_k})||_+ \to 0.$$

Proof: Notice that $\lim_{N\to\infty}\inf_{k,l\ge N} ||\mathrm{E}(f|\mathcal{F}_{-n_k^{'}}) + f - \mathrm{E}(f|\mathcal{F}_{n_l^{''}})||_+^2 = 0$ if and only if there is a sequence of $(n_k^{'}, n_k^{''})$ such that $n_k^{'}$, $n_k^{''} \nearrow \infty$ and $||\mathrm{E}(f|\mathcal{F}_{-n_k^{'}}) + f - \mathrm{E}(f|\mathcal{F}_{n_k^{''}})||_+^2 \to 0$. If there are no subsequences $(n_k^{'})$ and $(n_k^{''})$ such that

$$||\mathrm{E}(f|\mathcal{F}_{-n_k^{'}}) + f - \mathrm{E}(f|\mathcal{F}_{n_k^{''}})||_+^2 \to 0$$

as $(n_k^{'})$ and $(n_k^{''})$ tend to infinity then there is a $\delta > 0$ such that for all N large enough and $k, m-k \ge N$: $||\mathrm{E}(f|\mathcal{F}_{-k}) + f - \mathrm{E}(f|\mathcal{F}_{m-k})||_+^2 > \delta$, hence the naZW condition cannot hold. This proves sufficiency. The same proof applies to the adapted case.

To prove necessity, we will treat the adapted case first. We will use the same idea as in the proof of Theorem 1. Let us suppose that there exists a subsequence (n_k) such that

$$||\mathrm{E}(f|\mathcal{F}_{-n_k})||_+ \to 0.$$

The function $(\sum_{i=-n_k+1}^{0} P_i f - \sum_{i=0}^{n_k-1} P_0(f\circ T^i))$ is a coboundary and for $m_k = \sum_{i=0}^{n_k-1} P_0(f\circ T^i)$, $(m_k \circ T^i)$ is a martingale difference sequence ($m_k = P_0 m_k$). Then $||f - m_k||_+ = \|\sum_{i=-n_k+1}^{0} P_i f - m_k + \mathrm{E}(f|\mathcal{F}_{-n_k})\|_+ \le \|\sum_{i=-n_k+1}^{0} P_i f - m_k\|_+ + \|\mathrm{E}(f|\mathcal{F}_{-n_k})\|_+ \to 0$ as $k \to \infty$. Using the same arguments as in the proof of Theorem 1 we get the existence of $m \in L^2$, $m = P_0 m$, such that $m_k \to m$ in L^2 and $||f - m||_+^2 = 0$. Therefore, the function f has a martingale approximation. From Theorem 1 it follows that the ZW condition is satisfied.

The proof of the nonadapted case is the same; in particular, $\sum_{i=-n_k^{'}}^{n_k^{''}} P_i f - \sum_{i=-n_k^{''}}^{n_k^{'}} P_0(f\circ T^i)$ is a coboundary, for $m_k = \sum_{i=-n_k^{''}}^{n_k^{'}} P_0(f\circ T^i)$ we have $P_0 m_k = m_k$ and m_k converge to a limit m in L^2.

$\triangle$

References

[1] Dedecker, J. and Rio, E. (2000): On the functional central limit theorem for stationary processes. *Ann. Inst. Henri Poincaré Prob. Statist.*, vol. 36, n. 1, p. 1-34. MR-1743095

[2] Dedecker and J., Merlevède, F. (2002): Necessary and sufficient conditions for the conditional central limit theorem. *Annals of Probability*, vol. 30, p. 1044-1081. MR-1920101

[3] Dedecker, J., Merlevède, F., and Volný, D. (2007): On the weak invariance principle for non adapted sequences under projective criteria. *Journal of Theoretical Probability*, vol. 20, p. 971-1004. MR-2359065

[4] Gordin, M.I. (1969): A central limit theorem for stationary processes. *Soviet Math. Dokl.*, vol. 17, p. 424-440. MR-0251785

[5] Gordin, M.I. and Peligrad, M. (2011): On the functional central limit theorem via martingale approximation. *Bernoulli*, vol. 10, p. 1174-1176. MR-2797997

[6] Hall, P. and Heyde, C.C. (1980): *Martingale limit theory and its application*. Academic Press, New York. MR-0624435

[7] Klicnarová, J. and Volný, D. (2007): An Invariance Principle for non Adapted Processes. *C.R. Acad. Sci. Paris*, vol. 345, n. 5, p. 283-287. MR-2353682

[8] Klicnarová, J. and Volný, D. (2009): On the exactness of the Wu-Woodroofe approximation. *Stochastic Processes and their Applications*, vol. 119, p. 2158-2165. MR-2531087

[9] Klicnarová, J. and Volný, D. (2010): Diagonal approximations by martingales. *Latin American Journal of Probability and Mathematical Statistics*, vol. 7, p. 257-276. MR-2732895

[10] Maxwell, M. and Woodroofe, M.(2000): Central limit theorems for additive functionals of Markov chains. *Ann. Probab.*, vol. 28, p. 713-724. MR-1782272

[11] Peligrad, M. (2010): Conditional central limit theorem via martingale approximation *in Dependence in Probability, Analysis and Number Theory, volume in memory of Walter Philipp, Kendrick Press* I. Berkes, R.C. Bradley, H. Dehling, M. Peligrad, R. Tichy Editors, p. 295-311. MR-2731064

[12] Peligrad, C. and Peligrad, M. (2011): On the invariance principle under martingale approximation. *Stochastics and Dynamics*, vol. 11, n. 1, p. 95-105. MR-2771344

[13] Peligrad, M. and Utev, S. (2005): A new maximal inequality and invariance principle for stationary sequences. *Ann. Probab.*, vol. 33, p. 798-815. MR-2123210

[14] Volný, D. (1993): Approximating martingales and the central limit theorem for strictly stationary processes. *Stochastic Proccesses and their Applications*, vol. 44, p. 41-74. MR-1198662

[15] Volný, D.(2006): Martingale approximation of non adapted stochastic processes with nonlinear growth of variance. *Dependence in Probability and Statistics Series: Lecture Notes in Statistics*, vol. 187, Bertail, Patrice; Doukhan, Paul; Soulier, Philippe (Eds.). MR-2283254

[16] Volný, D.(2007): A nonadapted version of the invariance principle of Peligrad and Utev. *C. R., Math., Acad. Sci. Paris*, vol. 345, n. 3, p. 167-169. MR-2344817

[17] Volný, D., Woodroofe, M. and Zhao, O. (2011): Central limit theorems for superlinear processes. *Stochastics and Dynamics*, vol. 11, n. 1, p. 71-80. MR-2771342

[18] Wu, W.B., Woodroofe, M.(2004): Martingale Approximations for Sums of Stationary Processes. *Ann. Prob.*, vol. 32, No. 2, p. 1674-1690. MR-2060314

[19] Zhao, O. and Woodroofe, M. (2008): On Martingale Approximations. *The Annals of Applied Probability*, vol. 18, n. 5, p. 1831-1847. MR-2462550

Acknowledgments. We thank to anonymous referee for carefully reading of our paper. His comments helped us to improve our paper.