

On the spectral gap of spherical spin glass dynamics

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Abstract. We consider the time to equilibrium for the Langevin dynamics of the spherical p -spin glass model of system size N . We show that the log-Sobolev constant and spectral gap are order 1 at sufficiently high temperatures whereas the spectral gap decays exponentially in N at sufficiently low temperatures. These verify the existence of a dynamical high temperature phase and a dynamical glass phase at the level of the spectral gap. Key to these results are the understanding of the extremal process and restricted free energy of Subag–Zeitouni and Subag.

Résumé. Nous considérons le temps d'atteinte de l'équilibre pour la dynamique de Langevin du modèle de verre de p -spin sphérique de taille N . Nous montrons que la constante de log-Sobolev et le trou spectral sont d'ordre 1 à température suffisamment grande, alors que le trou spectral décroît exponentiellement en N à température suffisamment basse. Ceci confirme l'existence d'une phase dynamique de haute température et d'une phase dynamique verre concernant le trou spectral. Les arguments clés de ces résultats sont la compréhension du processus extrémal et de l'énergie libre restreinte de Subag–Zeitouni et Subag.

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1. Introduction

In the study of glassy systems such as spin glasses and structural glasses [15,23,48] and constraint satisfaction problems [28,43,47,48], one of the fundamental objects of study is the time to relax to equilibrium. It is believed that natural dynamics for such systems undergo what is called a *glass transition* but the nature of such a transition is still unresolved in condensed matter physics [15,26]. At high temperature, one expects the system to reach equilibrium quickly as it is in a classical phase, e.g., paramagnetic. At low temperature, however, when the system is in a dynamical glassy phase, the equilibration time is expected to be far longer than observable timescales [15]. It is desirable to have a mathematically rigorous understanding of how these timescales to equilibrium change with temperature in well-studied models. In this paper, we rigorously study the timescales to equilibrium for an archetypal glassy model, namely the *spherical p -spin glass model*, defined as follows.

The state space for the spherical p -spin glass is the $(N - 1)$ -sphere in dimension N of radius \sqrt{N} ,

$$\mathcal{S}^N = S^{N-1}(\sqrt{N}) = \left\{ \sigma = (\sigma_1, \dots, \sigma_N) \in \mathbb{R}^N : \sum_{i=1}^N \sigma_i^2 = N \right\},$$

equipped with the induced metric g . For $p \geq 3$, define the p -spin Hamiltonian by,

$$H_{N,p}(\sigma) = \frac{1}{N^{(p-1)/2}} \sum_{i_1, \dots, i_p=1}^N J_{i_1 \dots i_p} \sigma_{i_1} \dots \sigma_{i_p}, \quad (1.1)$$

where J_{i_1, \dots, i_p} are i.i.d. standard Gaussian random variables. Throughout this paper we will drop the subscripts p and N when it is unambiguous. Corresponding to H , define the Gibbs measure, π_N , at inverse temperature $\beta > 0$ by

$$d\pi_N(\sigma) = \frac{e^{-\beta H}}{Z} dV(\sigma),$$

here dV is the normalized volume measure, and Z is chosen so that π_N is a probability measure. Define the Langevin dynamics as the heat flow

$$P_t = e^{t\mathcal{L}_N}$$

generated by the operator,

$$\mathcal{L}_N = \frac{1}{2}(\Delta - \beta g(\nabla H_N, \nabla \cdot)), \quad (1.2)$$

where ∇ is the covariant derivative, and Δ is the corresponding Laplacian. In more probabilistic terms, \mathcal{L}_N is the infinitesimal generator of a reversible Markov process whose invariant measure is π_N . (For a quick review of the properties of \mathcal{L} see Section 4.)

One of the defining features of spin glasses is the complexity of their energy landscape: they generally have exponentially many critical points that are separated by energy barriers of height diverging linearly in N . Although this complexity leads to rich phenomenological behavior, it is also at the heart of the difficulty of analyzing these systems. Indeed, even making this picture rigorous is a difficult problem. In our setting, it has been established rigorously in [4,5] for all $p \geq 3$.

Dynamically, the models are expected to have the following rich behavior that is a hallmark of dynamics for glassy systems. At small β , they are expected to be in the *high temperature phase* where P_t behaves similarly to the heat semigroup for the Laplacian on \mathcal{S}^N . For large β , however, this comparison breaks down and the system enters the *glass phase*. Here it is believed that P_t exhibits exponentially slow in N relaxation to equilibrium and *aging* (see the literature review below). A natural question, and the aim of this paper, is to make the relaxation picture rigorous.

A canonical way to analyze this from the point of view of Markov processes is through the analysis of the *spectral gap*, that is, the first nontrivial eigenvalue, called λ_1 , of $-\mathcal{L}$, which governs the time to equilibrium (see Section 1.1). Here the goal is to analyze the asymptotics of λ_1 in N as we vary β . From this framework the above expectation is natural as one expects *metastable* behavior leading to poor mixing due to the large energy barriers at low temperature (see e.g., Arrhenius's law). In the non-disordered setting, there is a vast and growing literature following this approach: central to this field is the differentiation of high and low temperature phases where the dynamics moves from an order 1 gap to an exponentially decaying gap. This phenomenon has been observed in lattice systems such as the 2D Ising model (see e.g., [2,29,37,44,49]), and in mean field models including the Curie–Weiss model [16,34,42].

The study of the spectral gap for natural spin glass dynamics has a much more limited history, though similar transitions are expected. For the “simplest” mean-field model of spin glasses, the random energy model (REM), it was found that there is only one dynamical phase in the natural local dynamics [31]. For models on the hypercube, there is an exponential lower bound on the spectral gap in terms of an intrinsic quantity [45]. In the short range setting, there are some results from e.g., [25,35]. However, for the classical mean-field models of the p -spin models on $\{\pm 1\}^N$ and \mathcal{S}^N , the study of the spectral gap of Glauber/Langevin dynamics has remained largely open.

In the mean-field spin glass dynamics literature, a different approach has been utilized to analyze off-equilibrium dynamics of the system. The aim here is to establish a set of equations for the evolution of certain observables in the large N limit – called the *Cugliandolo–Kurchan equations* – and observe a transition in the large t behavior as one varies β (see [24]). At low temperatures, this leads to the development of the theory of *aging*. The Cugliandolo–Kurchan equations were proven by Ben Arous, Dembo and Guionnet [10,11] for a “soft” relaxation of spherical p -spin

glass dynamics; furthermore, in the case $p = 2$ this led to a proof of aging [10]. At high temperature the same problem was studied as the relaxation goes to zero in [27], and similar analyses were undertaken in the study of related models in [12,13]. Such studies of off-equilibrium dynamics are restricted to time scales shorter than the relaxation time of the dynamics. Aging has also been extensively studied in related settings on the hypercube. In the REM, aging was established for the random hopping time dynamics, a randomly trapped random walk, in [8,9], in a local Glauber-type dynamics [46], and more recently Metropolis dynamics [20,32]. For the p -spin model on $\{\pm 1\}^N$, aging was studied, again for the random hopping time dynamics, in [7,14,17,18].

In this paper, we demonstrate, for the relaxation time, the existence of a dynamical high temperature and dynamical low temperature glass phase in the setting of Langevin dynamics for spherical p -spin glasses. In particular, we show that the spectral gap of $-\mathcal{L}_N$, has order 1 asymptotics in N for β small and exponential in N asymptotics for β large.

1.1. Statement of main results

The goal of this paper is to study the behavior of the spectral gap of the infinitesimal generator, \mathcal{L} defined in Eq. (1.2), of the Langevin dynamics for the $p \geq 3$ spherical spin glass model. Observe that $-\mathcal{L}$ is a non-negative essentially self-adjoint operator on $C^\infty(\mathcal{S}^N) \subset L^2(dV)$ and has pure point spectrum $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots$; we point the reader to Section 4 for a brief sketch of these facts.

The asymptotic rate of growth of λ_1 in N is of particular interest, as λ_1^{-1} , called the *relaxation time*, is a measure of the time to equilibrium in an L^2 sense. Our main result is to show that for all $p \geq 3$, the spectral gap of the pure spherical p -spin model dynamics is in a dynamical high temperature phase for small β and is in a dynamical glass phase for large β , suggesting the existence of a *dynamical glass transition* for the relaxation time:

Theorem 1. *For any $p \geq 3$, consider the Langevin dynamics of the pure spherical p -spin glass model at inverse temperature $\beta > 0$ with generator \mathcal{L} .*

(1) *There exists $0 < \beta_l(p) < \infty$ and constants $c_1(p, \beta), c_2(p, \beta) > 0$ such that for all $\beta > \beta_l$,*

$$\lim_{N \rightarrow \infty} \mathbb{P}\left(c_1 < -\frac{1}{N} \log \lambda_1 < c_2\right) = 1.$$

(2) *There exists a $\beta_h(p) > 0$ and a constant $c_3(p, \beta) > 0$ such that for all $\beta < \beta_h$,*

$$\lim_{N \rightarrow \infty} \mathbb{P}(\lambda_1 > c_3) = 1.$$

Remark 2. It is worth noting here that in the above, (1) holds for all β larger than the β_l necessary for the results of [51] to hold; in particular, that picture is expected to hold up to the static phase transition point β_s . Precise information about the relation between the constants c_1, c_2 in (1) and their dependence on β can be gleaned from the proofs, though the two do not match.

At the heart of the proof of item (1) are the recent results regarding the energy landscape, H , and the Gibbs measure, π , developed in a series of papers by Auffinger–Ben Arous–Cerny [5], Auffinger–Ben Arous [4], Subag–Zeitouni [52], and Subag [50,51]. In particular, the proof of part (1) of Theorem 1 relies on the restricted free estimates obtained by Subag [51] (see Proposition 12 below) in the recent study of the geometry of the Gibbs measure in spherical p -spin models.

The proof of item (2) follows from the following stronger result, namely that at high temperature, π admits a logarithmic Sobolev (log-Sobolev) inequality (see (4.1)).

Proposition 3. *There exists a $\beta_h(p) > 0$ and a constant $c_L(p, \beta) > 0$ such that for all $\beta < \beta_h$, π admits a log-Sobolev inequality with constant c_L with probability $1 - O(e^{-cN})$ for some $c > 0$.*

Remark 4. The proof of Proposition 3 and therefore item (2) of Theorem 1 also goes through for mixed p -spin glasses on \mathcal{S}^N .

This result does not follow by a tensorization argument as is common for short-range spin systems because H is non-local and \mathcal{S}^N is not a product space. Instead it follows by curvature dimension arguments after proving that the Hessian of the Hamiltonian is on the same order of magnitude as the Ricci tensor, uniformly over \mathcal{S}^N ; this follows by Gaussian comparison techniques.

Aside from its inherent interest, this also yields the following geometric analytic interpretation of Theorem 1. For β small, the curvature dimension of the system is positive and order 1, so that the effective geometry admits a comparison to Gaussian/spherical space. At low temperature, however, the energetic effects dominate and thus this comparison breaks down. One is then in a regime where the time to equilibrium is governed by passing between energy barriers.

Remark 5. The definition of $H_{N,p}$ extends naturally to $p = 2$, sometimes called the spherical Sherrington–Kirkpatrick model; we omit this case for the following reason. In contrast to all $p \geq 3$, the $p = 2$ Hamiltonian has exactly N critical points, yielding a very different structure to the energy landscape. The absence of exponentially many metastable states, a signature of the glassy phase, makes the $p = 2$ case less pertinent to the scope of this paper.

Phase boundaries in β

In light of the main theorem, it is natural to define the following two inverse temperatures. Let

$$\beta_{\text{para}} = \sup \left\{ \beta > 0 : \lim_{N \rightarrow \infty} \mathbb{P}(\lambda_1 \asymp 1) = 1 \right\},$$

$$\beta_{\text{dyn}} = \inf \left\{ \beta > 0 : \lim_{N \rightarrow \infty} \mathbb{P} \left(-\frac{1}{N} \log \lambda_1 \asymp 1 \right) = 1 \right\},$$

where $f(N) \asymp 1$ is to say there exist, $c, C > 0$ depending on p and β such that $c < f(N) < C$. These correspond to the thresholds for the dynamical high temperature and glassy phases, as discussed in the [Introduction](#). Evidently $\beta_h \leq \beta_{\text{para}}$ and $\beta_{\text{dyn}} \leq \beta_l$. We are led to the following question:

Question. Is $\beta_{\text{dyn}} = \beta_{\text{para}}$?

We expect that the equality is true, though we believe our method for part (1) of the theorem can only be extended to $\beta \geq \beta_s$ (where β_s is the static transition temperature obtained in [53]), because it relies heavily on information about the equilibrium measure in the static low temperature regime.

It is also natural to ask the question of whether the dynamical glass phase and the static low temperature (glass) phases are in fact distinct.

Question. Is $\beta_{\text{dyn}} < \beta_s$?

The answer to this question is expected to be yes [19,24].

Bearing in mind the results of [10,11] where they define a critical temperature for the aging phenomena, β_{aging} for a relaxation of the spherical p -spin model, it would also be interesting to prove the existence of aging for large but finite N in the spherical p -spin glass and determine the relation between β_{aging} , and the static and dynamical critical temperatures, β_s and β_{dyn} .

2. Preliminaries

In this section, we discuss basic properties of the energy landscape, H_N . We will prove an important regularity estimate regarding the operator norm of the Hessian of H_N to show that it is uniformly (over \mathcal{S}^N) order one. In particular, this regularity estimate (Lemma 8) will be the crux of the proof of item (2) of Theorem 1.

We will then proceed recall results and notation from [4,5,51,52] that will be important to the proofs of item (1) of Theorem 1.

Notation

In the following we drop the subscripts p, N whenever it is unambiguous, and we extend the definition of $H_{N,p}$ to $p = 1, 2$ in the natural way, when necessary. We say that $f(N) \lesssim_a g(N)$ if there is a constant $C(a)$ that depends only on a such that $f \leq Cg$ for all N . Whenever we use the notation $o(1)$, we mean by $f(N) = o(g(N))$ that $f(N)/g(N) \rightarrow 0$ as $N \rightarrow \infty$.

For a probability measure μ let L_μ^2 denote the space of functions that are square integrable with respect to μ . Let $C^\infty(M)$ be the space of smooth functions on a Riemannian manifold M . The notation ∇ will always refer to a covariant derivative and Δ the corresponding Laplacian.

Throughout the paper, let $R(\sigma, \sigma')$ be the normalized spin overlap: for $\sigma, \sigma' \in \mathcal{S}^N$,

$$R(\sigma, \sigma') = \frac{1}{N} \sum_i \sigma_i \sigma'_i.$$

Notice that $\mathbb{E}[H_{N,p}(\sigma)H_{N,p}(\sigma')] = NR(\sigma, \sigma')^p$.

2.1. Regularity of H

Before proving the uniform bound on the Hessian of H , we remind the reader that the maximum and minimum of the process H are order N .

Lemma 6. *For every $p \geq 1$, there exists $E(p) > 0$ and $c(p) > 0$, such that for every $\delta > 0$,*

$$\mathbb{P}\left(\max_{\sigma \in \mathcal{S}^N} H(\sigma) - NE \geq N\delta\right) \lesssim e^{-cN\delta^2}.$$

In particular, for every $p \geq 1$, we have,

$$\mathbb{E}\left[\max_{\sigma \in \mathcal{S}^N} |H(\sigma)|\right] \lesssim_p N.$$

The proof of the bound on $\mathbb{E}[\max_{\mathcal{S}^N} H]$ (and by symmetry also $\mathbb{E}[\max_{\mathcal{S}^N} |H|]$) in Lemma 6 is a classical application of Dudley's entropy integral; the tail estimate above then follows immediately from Borell's inequality [40].

Remark 7. The precise constant, call it $E_0(p)$, such that $\mathbb{E}[\min H] = -E_0N + o(N)$ was identified by Auffinger, Ben-Arous and Cerny [5] (see also [50]). Namely, in [5], Theorem 2.12, it is stated for p even as, at the time, the free energy had only been computed for those p 's rigorously. This has been done now by [22] for all $p \geq 3$ so the proof of [5], Theorem 2.12, holds for all $p \geq 3$. For $p = 2$, the estimate comes from the top eigenvalue of a GOE matrix [3].

We now turn to the estimate regarding the Hessian of H , central to the proof of item (2) of Theorem 1. In the following, for $f \in C^2$, we let $\text{Hess}(f(\sigma))$ denote the covariant Hessian of f with respect to \mathcal{S}^N at the point σ , and Hess_E denote the usual Euclidean Hessian on \mathbb{R}^N . Recall that the tangent space to \mathcal{S}^N at a point σ can then be thought of as the vector space $\{x \in \mathbb{R}^N : (x, \sigma)_E = 0\}$ where by $(\cdot, \cdot)_E$ we mean the usual Euclidean inner product. With this in mind, for $f \in C^2(\mathbb{R}^N)$ we have that at any point σ ,

$$\text{Hess}(f(\sigma)) = \text{Hess}_E(f(\sigma)) - \frac{1}{N}(\sigma, \nabla_E f(\sigma))_E \text{Id}, \quad (2.1)$$

where ∇_E is the Euclidean gradient, and $(\cdot, \cdot)_E$ is the usual Euclidean inner product in \mathbb{R}^N , and Id is the identity operator on $T_\sigma \mathcal{S}^N$. Define now the quantities

$$\bar{r}(H) = \sup_{\sigma \in \mathcal{S}^N} \sup_{\substack{v \in T_\sigma \mathcal{S}^N \\ g(v,v)=1}} \text{Hess}(H(\sigma))(v, v)$$

and

$$\underline{r}(H) = \inf_{\sigma \in \mathcal{S}^N} \inf_{\substack{v \in T_\sigma \mathcal{S}^N \\ g(v,v)=1}} \text{Hess}(H(\sigma))(v, v).$$

By separability of $T\mathcal{S}^N$ and the continuity of H , these random variables are measurable. Furthermore, by symmetry,

$$-\bar{r}(H) \stackrel{(d)}{=} \underline{r}(H).$$

Finally, define $r(H) = \bar{r} - \underline{r}$. Observe that $r(H)$ bounds the spectral radius of $\text{Hess}(H(\sigma))$ uniformly over $\sigma \in \mathcal{S}^N$.

Lemma 8. *For any $p \geq 3$, we have that*

$$\mathbb{E}[r(H)] \lesssim_p 1$$

and there exists a $c(p) > 0$ such that for all $\varepsilon > 0$,

$$\mathbb{P}(|r(H) - \mathbb{E}[r(H)]| > \varepsilon) \lesssim e^{-cN\varepsilon^2}.$$

Proof. By symmetry it suffices to prove the estimates for $\bar{r}(H)$. We begin by proving the first estimate. To this end, observe that H can be extended to all of \mathbb{R}^N by allowing σ to take values in \mathbb{R}^N and using the same definition of the Hamiltonian. Thus in the notation above,

$$(\sigma, \nabla_E H(\sigma))_E = p H(\sigma).$$

Combining this with Eq. (2.1) and the fact that H is smooth, we then see that for any $v \in S^{N-1}(1) \subset \mathbb{R}^N$, we have

$$\text{Hess}(H_N(\sigma))(v, v) = \frac{p(p-1)}{N^{\frac{p-1}{2}}} \sum_{l,m,i_1,\dots,i_{p-2}=1}^N J_{l,m,i_1,\dots,i_{p-2}} \sigma_{i_1} \cdots \sigma_{i_{p-2}} v_l v_m - \frac{p}{N} H_N(\sigma) \|v\|_2^2,$$

when viewed as an operator on $T_\sigma \mathcal{S}^N$.

Define the $\mathcal{S}^N \times S^{N-1}(1)$ -indexed Gaussian process, $\psi(\sigma, v)$, given by

$$\psi(\sigma, v) = \frac{p(p-1)}{N^{\frac{p-1}{2}}} \sum_{l,m,i_1,\dots,i_{p-2}=1}^N J_{l,m,i_1,\dots,i_{p-2}} \sigma_{i_1} \cdots \sigma_{i_{p-2}} v_l v_m - \frac{p}{N} H_N(\sigma).$$

As \mathcal{S}^N is given by induced metric, we have

$$\bar{r} = \sup_{\sigma \in \mathcal{S}^N} \sup_{v \in S^{N-1}(1) \cap T_\sigma \mathcal{S}^N} \psi \leq \sup_{\sigma \in \mathcal{S}^N} \sup_{v \in S^{N-1}(1)} \psi.$$

Define also the related process

$$\phi(\sigma, v) = \frac{p(p-1)}{N^{\frac{p-1}{2}}} \sum_{i_1,\dots,i_{p-2}=1}^N J'_{i_1,\dots,i_{p-2}} \sigma_{i_1} \cdots \sigma_{i_{p-2}} + \frac{p(p-1)}{\sqrt{N}} \sum_{l,m=1}^N J''_{lm} v_l v_m - \frac{p}{N} H_N(\sigma),$$

where $J'_{i_1 \dots i_{p-2}}$ and J''_{lm} are independent standard Gaussians. For any $\sigma, \sigma' \in \mathcal{S}^N$, $v, v' \in S^{N-1}(1)$, one sees that,

$$\begin{aligned} \mathbb{E}(\psi(\sigma, v) - \psi(\sigma', v'))^2 &\leq \frac{2p^2}{N^2} \mathbb{E}(H_N(\sigma) - H_N(\sigma'))^2 + \frac{2p^2(p-1)^2}{N^{p-1}} \sum (\sigma_{i_1} \cdots \sigma_{i_{p-2}})^2 (v_l v_m - v'_l v'_m)^2 \\ &\quad + \frac{2p^2(p-1)^2}{N^{p-1}} \sum ((\sigma_{i_1} \cdots \sigma_{i_{p-2}} - \sigma'_{i_1} \cdots \sigma'_{i_{p-2}}) v_l v_m)^2, \end{aligned}$$

where the above sums are over $l, m, i_1, \dots, i_{p-2} \in [N]$. The first term we leave as is and bound the sum of the latter two terms:

$$\frac{1}{N^{p-1}} \sum (\sigma_{i_1} \cdots \sigma_{i_{p-2}})^2 (v_l v_m - v'_l v'_m)^2 \lesssim_p \frac{1}{N} \sum_{l,m=1}^N (v_l v_m - v'_l v'_m)^2,$$

and similarly,

$$\frac{1}{N^{p-1}} \sum ((\sigma_{i_1} \cdots \sigma_{i_{p-2}} - \sigma'_{i_1} \cdots \sigma'_{i_{p-2}}) v_l v_m)^2 \lesssim_p \frac{1}{N^{p-1}} \sum_{i_1, \dots, i_{p-2}=1}^N (\sigma_{i_1} \cdots \sigma_{i_{p-2}} - \sigma'_{i_1} \cdots \sigma'_{i_{p-2}})^2.$$

Putting this together, we see that for any $\sigma, \sigma' \in \mathcal{S}^N$, $v, v' \in \mathcal{S}^{N-1}(1)$,

$$\mathbb{E}(\psi(\sigma, v) - \psi(\sigma', v'))^2 \lesssim_p \mathbb{E}(\phi(\sigma, v) - \phi(\sigma', v'))^2.$$

Thus by the Sudakov–Fernique inequality [41], we have that

$$\begin{aligned} \mathbb{E}[\bar{F}(H)] &= \mathbb{E} \left[\sup_{\sigma \in \mathcal{S}^N} \sup_{v \in \mathcal{S}^{N-1}(1)} \psi(\sigma, v) \right] \\ &\lesssim_p \mathbb{E} \left[\sup_{\sigma \in \mathcal{S}^N, v \in \mathcal{S}^{N-1}(1)} \phi(\sigma, v) \right] \\ &\lesssim_p \frac{1}{N} \mathbb{E} \left[\sup_{x \in \mathcal{S}^N} H_{N,p-2}(x) + \sup_{x \in \mathcal{S}^N} H_{N,2}(x) + \sup_{x \in \mathcal{S}^N} H_{N,p}(x) \right] \\ &\lesssim_p 1. \end{aligned}$$

The second to last inequality comes from scaling v , and the last inequality is a direct consequence of Lemma 6. Thus we have the first inequality in Lemma 8.

We now turn to proving the second inequality. To this end observe that for every $\sigma \in \mathcal{S}^N$, $v \in \mathcal{S}^{N-1}(1)$, we have that

$$\mathbb{E}[\psi(\sigma, v)^2] \leq \frac{2p^2(p-1)^2}{N^{p-1}} \sum_{l,m,i_1,\dots,i_{p-2}=1}^N (\sigma_{i_1} \cdots \sigma_{i_{p-2}} v_l v_m)^2 + \frac{2}{N^2} \mathbb{E}[H_{N,p}(\sigma)^2] \lesssim_p \frac{1}{N}.$$

The result then follows by Borell's inequality [40]. □

2.2. Previous results

We now remind the reader of several recent results that give a good understanding of the critical points of H with near-minimal energy. These will be important to the proof of item (1) of Theorem 1.

We begin by observing that the conditional law of H in a neighborhood of a critical point has a simple explicit form in terms of other p -spin models. This result follows by direct calculations as can be seen, for example in [51]. We state the result in the weakest form that we need. For each $x \in \mathcal{S}^N$ define the following conditional measure,

$$\mathbb{P}_u(\cdot) = \mathbb{P}(\cdot \mid H(x) = u, \nabla H \upharpoonright_x = 0),$$

with corresponding expectation \mathbb{E}_u , where the dependence on x is implicit. Dropping the dependence on x is justified as this law is invariant in x by isotropy. Evidently, this is the law of H conditioned on the event that x is a critical point of H with energy u .

Lemma 9. Let $u \in \mathbb{R}$ and $x \in \mathcal{S}^N$. Then, with respect to \mathbb{P}_u , $H_N(\sigma)$ satisfies

$$H_N(\sigma) \stackrel{(d)}{=} uR(\sigma, x)^p + Y_N(\sigma),$$

where $Y_N(\sigma)$ is a centered, smooth Gaussian process satisfying,

$$\text{Cov}(Y_N(\sigma), Y_N(\sigma')) = Nf(R(\sigma, \sigma')), \quad \text{and}$$

$$\mathbb{E} \left[\max_{\sigma \in \mathcal{S}^N} Y_N(\sigma) \right] < \infty,$$

where f is a polynomial of degree p whose coefficients depend only on p .

Proof. Recall that $(H_N(\sigma), \nabla H_N(\sigma))$ are jointly Gaussian. The distributional equality then follows by computing the conditional law of H given $\nabla H(x)$ and $H(x)$. See, for example, [51], Lemmas 14–15. Since Y is a.s. a continuous Gaussian process on a compact space, $\max_{\mathcal{S}^N} Y_N$ is a.s. finite. The last result follows from this, the covariance estimate and Borell’s inequality (see, e.g., [40]). \square

In the subsequent, it will be useful to understand basic properties of the local minima of the Hamiltonian. To this end, we introduce the following notation regarding the critical points of H . Observe that H is smooth, and almost surely Morse. (A function is Morse if its critical points are non-degenerate.) Furthermore, it has a global minimum that is a.s. unique for p odd and unique modulo the reflection symmetry $\sigma \mapsto -\sigma$ for p even, where we note that every smooth real-valued function on the sphere has finitely many critical points.

A natural question is to count the expected number of critical points of H . This was studied in [5]. Let

$$\Theta_p(E) = \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} [|\{x : \nabla H(x) = 0, H(x) \leq EN\}|].$$

In [5], it was shown that $\Theta_p(E)$ has the following explicit form.

$$\Theta_p(E) = \begin{cases} \frac{1}{2} + \frac{1}{2} \log(p-1) - \frac{E^2}{2} + \int_{-2}^2 \frac{1}{2\pi} \sqrt{4-x^2} \log|x-E| dx, & E < 0, \\ \frac{1}{2} \log(p-1), & E \geq 0. \end{cases}$$

(N.b. This result will not be used in our arguments in an essential way. We include it to clarify the exposition surrounding the following notions.)

With this in hand, we then observe the following important result of Subag–Zeitouni regarding the extremal process for H . For every fixed N , if p is even, order the locations of the local minima of H as $x_{\pm 1}, x_{\pm 2}, \dots \in \mathcal{S}^N$, where for x_i, x_j two local minima, $|i| < |j|$ if

$$H(x_i) \leq H(x_j),$$

and $x_i = -x_{-i}$; if p is odd, order them simply as $x_1, x_2, \dots \in \mathcal{S}^N$. Finally, let m_N be the quantity

$$m_N = -E_0 N + \frac{1}{2\Theta'_p(E_0)} \log N - K_0,$$

where K_0 is an explicitly defined constant (see [52], Eq. (2.6)), and E_0 is the unique zero of Θ_p . (We remark here that E_0 is the same constant mentioned in Remark 7.)

Proposition 10 ([52], Theorem 1). For any $p \geq 3$, we have that

$$\frac{2}{(3 + (-1)^p)} \sum_{\sigma: \nabla H|_{\sigma} = 0} \delta_{H(\sigma) - m_N} \xrightarrow[N \rightarrow \infty]{(d)} PPP(e^{\Theta'(-E_0)x} dx),$$

where $PPP(f(x) dx)$ denotes the Poisson point process of intensity $f(x)$, and the convergence is in distribution with respect to the vague topology.

In our paper, we do not need the full power of this deep result. Instead we only need the following simple corollary of Proposition 10.

Corollary 11. *For any $k \in \mathbb{N}$, if $x_1, \dots, x_k \in \mathcal{S}^N$ are the locations of the ground state to the k th smallest local minima, respectively, we have*

$$(H_N(x_l) - m_N)_{l \in [k]} \xrightarrow{(d)} Y,$$

where Y is a random variable supported on all of \mathbb{R}^k .

In order to obtain our low-temperature spectral estimates, we will need to control certain natural physical quantities, called free energies. Recall that the *free energy density* corresponding to the partition function $Z_N = Z_{N,\beta}$ defined in the Introduction, is given by

$$F_N = \frac{1}{N} \log Z_N = \frac{1}{N} \log \int_{\mathcal{S}^N} e^{-\beta H(\sigma)} dV(\sigma).$$

Then, for a Borel set $A \subset \mathcal{S}^N$, let

$$Z_N(A) = \int_A e^{-\beta H(\sigma)} dV(\sigma), \quad \text{and} \quad F_N(A) = \frac{1}{N} \log(Z_N(A)),$$

be the *restricted partition function* and *restricted free energy* of a set A , respectively, so that $F_N(\mathcal{S}^N) = F_N$. (This is called the *reduced free energy* in [51].)

The main estimate we use in the low temperature regime is the following result of Subag regarding the conditional law of the restricted free energy of bands around minima. More precisely, for any $x \in \mathcal{S}^N$, $q \in (0, 1)$, and any $\varepsilon > 0$, define the Borel sets

$$\begin{aligned} \text{Cap}(x, q) &= \{\sigma \in \mathcal{S}^N : R(x, \sigma) \geq q\}, \\ \text{Band}(x, q, \varepsilon) &= \{\sigma \in \mathcal{S}^N : R(x, \sigma) \in [q - \varepsilon, q + \varepsilon]\}, \end{aligned}$$

which are a cap and band respectively around a point x corresponding to an overlap q . These satisfy the following free energy estimates near critical points.

Proposition 12 ([51], Proposition 19, Lemma 20). *For every $p \geq 3$, there exists a $\beta_0(p)$ and a $0 < q_*(p, \beta) < 1$ such that for all $\beta \geq \beta_0$, the following holds:*

- (1) *Let $a_N = o(N)$ and $\varepsilon_N = o(1)$ be two sequences of positive numbers; then for $J_N = (m_N - a_N, m_N + a_N)$ we have for any $x \in \mathcal{S}^N$, $t > 0$,*

$$\lim_{N \rightarrow \infty} \sup_{u \in J_N} \left| \mathbb{P}_u \left(\frac{Z_N(\text{Band}(x, q_*, \varepsilon_N))}{\mathbb{E}_u[Z_N(\text{Band}(x, q_*, \varepsilon_N))]} \leq t \right) - \mathbb{P}(e^{Y_*} \leq t) \right| = 0,$$

for some Y_* , a normal random variable whose mean and variance are functions of p alone.

- (2) *Furthermore, there exists $0 < q_{**}(p, \beta) < q_*$ and $\Lambda(p, \beta) > 0$ such that for every $x \in \mathcal{S}^N$ and every $\eta > 0$,*

$$\limsup_{N \rightarrow \infty} \sup_{u \in J_N} \left| \frac{1}{N} \log(\mathbb{E}_u[Z_N(\text{Band}(x, q_*, \eta N^{-1/2})]) - \Lambda(p, \beta) \right| = 0, \quad (2.2)$$

and for any fixed $\varepsilon > 0$,

$$\limsup_{N \rightarrow \infty} \sup_{u \in J_N} \frac{1}{N} \log(\mathbb{E}_u[Z_N(\text{Cap}(x, q_{**}) \setminus \text{Band}(x, q_*, \varepsilon))]) < \Lambda(p, \beta). \quad (2.3)$$

Henceforth, $q_*(p, \beta)$, $q_{**}(p, \beta)$, and $\Lambda(p, \beta)$ will be those constants given by Proposition 12.

3. Free energy estimates

In this section, we prove the key equilibrium estimate for the proof of exponentially slow relaxation at low temperature. In particular, we compute ratios of Gibbs probabilities at the exponential level. We begin first with the a modification of a classical concentration estimate. We then turn to the main estimate in the following subsection. Finally we state as corollaries the precise applications of these results that we will use in the subsequent sections.

3.1. Concentration of restricted free energies

We begin by briefly recalling the fact that the restricted free energy of any Borel set concentrates under both \mathbb{P} and \mathbb{P}_u . This estimate is a modification of a classical concentration estimate for free energies. We include a proof for the reader's convenience.

Lemma 13. *For any Borel set $E \in \mathcal{B}(\mathcal{S}^N)$, the restricted free energy corresponding to H_N ,*

$$F_N(E) = \frac{1}{N} \log \int_E e^{-\beta H_N(\sigma)} dV(\sigma),$$

concentrates with respect to \mathbb{P} and, for any $x \in \mathcal{S}^N$, with respect to the conditional measure \mathbb{P}_u . That is, there is a constant $c > 0$ depending only on β and p such that for any N and any $E \in \mathcal{B}(\mathcal{S}^N)$,

$$\begin{aligned} \mathbb{P}(|F_N(E) - \mathbb{E}F_N(E)| > \varepsilon) &\lesssim e^{-cN\varepsilon^2}, \\ \mathbb{P}_u(|F_N(E) - \mathbb{E}_u F_N(E)| > \varepsilon) &\lesssim e^{-cN\varepsilon^2}. \end{aligned}$$

Proof. Without loss of generality, $V(E) > 0$, otherwise $F_N(E) = -\infty$ identically. Under \mathbb{P} and, by the equality in distribution in Lemma 9, under \mathbb{P}_u , the restricted free energy is equal in law to

$$\frac{1}{N} \log \int_E e^{-\beta X(\sigma)} dV(\sigma) \quad \text{where } X(\sigma) = \sum_{k=1}^p a_{p,k} H_{N,k}(\sigma) + g(\sigma)$$

for some deterministic, smooth, $g(\sigma)$ and coefficients $a_{p,k}$ suitably chosen depending on p . Consider this more general setup and denote this free energy by $F(J, E)$ to make the dependence on the coupling coefficients in $H_{N,k}$ explicit. Observe that

$$\frac{\partial}{\partial J_{i_1, \dots, i_k}} X(\sigma) = \frac{a_{p,k}}{N^{(k-1)/2}} \sigma_{i_1} \cdots \sigma_{i_k},$$

so that

$$\nabla_J F(J, E) = \frac{1}{N} \left(-\beta \frac{a_{p,k}}{N^{(k-1)/2}} \langle \sigma_{i_1} \cdots \sigma_{i_k} \rangle \right)_{i_1, \dots, i_k: k \leq p},$$

where $\langle \cdot \rangle$ denotes integration with respect to the Gibbs measure induced by X conditioned on the event E . (Since $V(E) > 0$ by assumption and H is continuous for each choice of J , $\pi(E) > 0$, so this is defined in the usual sense.) Thus F is c/\sqrt{N} -Lipschitz in J for some $c = c(\beta, a_{p,k}) > 0$. Since J is a collection of i.i.d. Gaussians, this implies the result by standard Gaussian concentration. \square

3.2. Refined free energy estimates

In this subsection, we prove the main estimate we need regarding π at low temperature. As is often the case, this result reduces to showing that certain free energy differences are negative. These results will come from combining the estimates from Section 2.2 with the concentration estimate from Section 3.1. The goal of this subsection is to prove the following proposition. Recall the notation $x_{\pm 1}, \dots \in \mathcal{S}^N$ regarding the lowest critical points from the end of Section 2.2.

Proposition 14. Fix any k and let $x_* = x_k$. Fix any $\eta > 0$ and let $A(x) = \text{Band}(x, q_*, \eta)$ and $B(x) = \text{Cap}(x, q_{**}) \setminus A(x)$, where the sets Band and Cap were defined in Section 2. Then there exists a $\beta_0(p)$ such that for every $\beta \geq \beta_0$, there exists $c(\beta) > 0$ such that,

$$\lim_{N \rightarrow \infty} \mathbb{P}(F_N(B(x_*)) - F_N(A(x_*)) < -c) = 1.$$

Before proving this proposition we will need estimates on $F_N(A(x))$ and $F_N(B(x))$ under \mathbb{P}_u . To this end, begin by observing that by Lemma 13, for any $x \in \mathcal{S}^N$, $F_N(B(x))$ and $F_N(A(x))$ concentrate around their respective means; in particular, there exists a constant $c(\beta, p) > 0$ such that for every $\delta > 0$,

$$\mathbb{P}_u(|F_N(B(x)) - \mathbb{E}_u F_N(B(x))| > \delta) \lesssim e^{-cN\delta^2}, \tag{3.1}$$

and similarly for $F_N(A)$.

We begin the proof with the following two lemmas. Recall the definitions of q_* , E_0 , and Λ from Section 2.2. The first lemma shows that the probability that $A(x)$ has free energy that is smaller than Λ is vanishing in the limit.

Lemma 15. Let a_N and J_N be as in Proposition 12 and η and $A(x)$ be as in Proposition 14. There exists $c(\beta, p) > 0$ such that for every $\delta > 0$,

$$\sup_{x \in \mathcal{S}^N} \sup_{u \in J_N} \mathbb{P}_u(F_N(A(x)) < \Lambda(p, \beta) - \delta) \lesssim e^{-cN\delta^2}. \tag{3.2}$$

Proof. Fix $x \in \mathcal{S}^N$ and define $\tilde{A}(x) = \text{Band}(x, q_*, \eta N^{-1/2})$. Observe that because $\tilde{A} \subset A$, we have $F(\tilde{A}) \leq F(A)$, from which it follows that

$$\limsup_{N \rightarrow \infty} F_N(A(x)) - \Lambda(p, \beta) \geq \limsup_{N \rightarrow \infty} F_N(\tilde{A}(x)) - \Lambda(p, \beta). \tag{3.3}$$

Define the set

$$V = \left[\frac{1}{N} \log \mathbb{E}_u [Z_N(\tilde{A}(x))] - \frac{K}{N}, \frac{1}{N} \log \mathbb{E}_u [Z_N(\tilde{A}(x))] + \frac{K}{N} \right].$$

By item (1) of Proposition 12, with the choice $\varepsilon_N = \eta N^{-1/2}$, and the Gaussian tails of Y_* (defined there), there is an absolute constant, $c > 0$ such that for any K sufficiently large,

$$\sup_{u \in J_N} \mathbb{P}_u(F_N(\tilde{A}(x)) \in V^c) \leq \exp(-cK^2) + o(1). \tag{3.4}$$

With these results in hand, observe that

$$\mathbb{E}_u [F_N(\tilde{A}(x))] = \mathbb{E}_u [F_N(\tilde{A}(x)) \mathbf{1}\{F_N(\tilde{A}(x)) \in V\}] + \mathbb{E}_u [F_N(\tilde{A}(x)) \mathbf{1}\{F_N(\tilde{A}(x)) \in V^c\}].$$

Combining this with Eq. (3.4) and the Cauchy–Schwarz inequality, we have that for each fixed K large enough, for every $u \in J_N$,

$$\left| \mathbb{E}_u [F_N(\tilde{A}(x))] - \frac{1}{N} \log \mathbb{E}_u [Z_{N,\beta}(\tilde{A}(x))] \right| \leq (\mathbb{E}_u [(F_N(\tilde{A}(x)))^2])^{\frac{1}{2}} (\exp(-cK^2) + o(1))^{\frac{1}{2}} + o(1).$$

We estimate the right hand side as follows. Splitting up the expectation, and using Eq. (3.1), we obtain for every $u \in J_N$,

$$\begin{aligned} \mathbb{E}_u [F_N(\tilde{A}(x))^2] &\leq \mathbb{E}_u [F_N(\tilde{A}(x))^2 (\mathbf{1}\{F_N(\tilde{A}(x)) \leq \mathbb{E}_u F_N(\tilde{A}(x))\} + \mathbf{1}\{F_N(\tilde{A}(x)) > \mathbb{E}_u F_N(\tilde{A}(x))\})] \\ &\leq (\mathbb{E}_u [F_N(\tilde{A}(x))])^2 + \sup_{u \in J_N} \int_0^\infty 2(\mathbb{E}_u F_N(\tilde{A}(x)) + t) \cdot e^{-cNt^2} dt. \end{aligned}$$

We now bound $\mathbb{E}_u[F_N(\tilde{A}(x))]$ uniformly in $u \in J_N$: letting Var_u denote the variance with respect to \mathbb{P}_u , we have that

$$\mathbb{E}_u[F_N(\tilde{A}(x))] \leq \frac{1}{N} \log \int_{S^N} \mathbb{E}_u[e^{-\beta H(\sigma)}] dV(\sigma) \leq \frac{\beta u}{N} + \frac{1}{N} \sup_{\sigma \in S^N} \frac{\beta^2}{2} \text{Var}_u(H(\sigma)),$$

where we use Jensen inequality for the first inequality, and the \mathbb{P}_u conditional distribution of $H(\sigma)$ given by Lemma 9 for the second.

Recall now, from the definition of J_N , that $\frac{u}{N}$ is bounded by some constant that depends only on p . Combining the above with the covariance bound obtained in Lemma 9 (independent of u) to bound $\text{Var}_u(H(\sigma)) \lesssim_p N$, we see that

$$\sup_{u \in J_N} (\mathbb{E}_u[(F_N(\tilde{A}(x)))^2])^{\frac{1}{2}} \lesssim_{p,\beta} 1 + o(1).$$

Altogether, we see that

$$\sup_{u \in J_N} \left| \mathbb{E}_u[F_N(\tilde{A}(x))] - \frac{1}{N} \log \mathbb{E}_u[Z_N(\tilde{A}(x))] \right| \lesssim_{p,\beta} e^{-cK^2} + o(1). \tag{3.5}$$

Although the $o(1)$ term is not uniform in K , for any $\delta > 0$, there exists a K such that for N sufficiently large the above difference is less than $\delta/6$. Moreover, by item (2) of Proposition 12,

$$\lim_{N \rightarrow \infty} \sup_{u \in J_N} \left| \frac{1}{N} \log \mathbb{E}_u[Z_N(\tilde{A}(x))] - \Lambda(p, \beta) \right| = 0, \tag{3.6}$$

so for any $\delta > 0$, for N sufficiently large the difference in Eq. (3.6) is less than $\delta/6$. By Eq. (3.5) and the finiteness of $\Lambda(p, \beta)$, for every $\delta > 0$, there exists K large enough that for all N sufficiently large,

$$\begin{aligned} \sup_{u \in J_N} \left| \mathbb{E}_u[F_N(\tilde{A}(x))] - \Lambda(p, \beta) \right| &\leq \sup_{u \in J_N} \left| \mathbb{E}_u[F_N(\tilde{A}(x))] - \frac{1}{N} \log \mathbb{E}_u[Z_N(\tilde{A}(x))] \right| \\ &\quad + \sup_{u \in J_N} \left| \frac{1}{N} \log \mathbb{E}_u[Z_{N,\beta}(\tilde{A}(x))] - \Lambda(p, \beta) \right| \\ &< \delta/3. \end{aligned}$$

Then by the triangle inequality and Lemma 13, for all such N ,

$$\begin{aligned} \sup_{u \in J_N} \mathbb{P}_u(|F_N(\tilde{A}(x)) - \Lambda(p, \beta)| > \delta) &\leq \sup_{u \in J_N} \mathbb{P}_u(|F_N(\tilde{A}(x)) - \mathbb{E}_u[F_N(\tilde{A}(x))]| > \delta/3) \\ &\lesssim e^{-cN\delta^2/9}. \end{aligned}$$

Combined with Eq. (3.3), and the observation that every estimate in this proof has been independent of $x \in S^N$, we obtain for every $\delta > 0$,

$$\sup_{x \in S^N} \sup_{u \in J_N} \mathbb{P}_u(F_N(A(x)) < \Lambda(p, \beta) - \delta) \lesssim e^{-cN\delta^2/9}. \quad \square$$

Now that we know that $F_N(A)$ is large with high probability, we want the corresponding estimate to show that the probability that $F_N(B)$ is larger than $\Lambda - \delta$ (for δ small enough) is small.

Lemma 16. *Let a_N and J_N be as in Proposition 12 and η and $B(x)$ be as in Proposition 14. There exists $c(\beta, p) > 0$ such that for every $\delta > 0$ sufficiently small,*

$$\sup_{x \in S^N} \sup_{u \in J_N} \mathbb{P}_u(F_N(B(x)) > \Lambda(p, \beta) - \delta) \lesssim e^{-cN\delta^2}.$$

Proof. For any $x \in \mathcal{S}^N$. By Jensen’s inequality, and item (2) of Proposition 12 (combined with the rotational invariance of H which implies that the estimate is uniform over \mathcal{S}^N), there exists a $\delta > 0$ such that,

$$\begin{aligned} \limsup_{N \rightarrow \infty} \sup_{x \in \mathcal{S}^N} \sup_{u \in J_N} \mathbb{E}_u[F_N(B(x))] &= \limsup_{N \rightarrow \infty} \sup_{u \in J_N} \mathbb{E}_u[F_N(B(x))] \\ &\leq \limsup_{N \rightarrow \infty} \sup_{u \in J_N} \frac{1}{N} \log \mathbb{E}_u[Z_{N,\beta}(B(x))] \\ &\leq \Lambda(p, \beta) - 3\delta. \end{aligned} \tag{3.7}$$

Thus for some sufficiently large N , the left hand side is less than $\Lambda(p, \beta) - 2\delta$. Combined with the concentration of the free energy under \mathbb{P}_u given by Eq. (3.1), we see that there exists a constant $c(\beta, p) > 0$ such that for sufficiently large N ,

$$\begin{aligned} \sup_{x \in \mathcal{S}^N} \sup_{u \in J_N} \mathbb{P}_u(F_N(B(x)) > \Lambda(p, \beta) - \delta) &\leq \sup_{x \in \mathcal{S}^N} \sup_{u \in J_N} \mathbb{P}_u(|F_N(B(x)) - \mathbb{E}_u F_N(B(x))| > \delta) \\ &\lesssim e^{-cN\delta^2}, \end{aligned} \tag{3.8}$$

as desired. □

In order to complete the proof of Proposition 14, it remains to move from free energy differences under \mathbb{P}_u for $u \in J_N$ to free energies under \mathbb{P} around the (random) point x_k .

For this, we will need the following result which is a standard application of the Kac–Rice formula combined with Proposition 10 (see [51], Lemma 38). Recall that in order to apply the Kac–Rice formula, one needs some basic smoothness criteria, called *tameness*. More precisely, a random field G is *tame* if it satisfies criteria, (a)–(g) in Theorem 12.1.1 of [1] and the random field $(H(x), G(x))_x$ is stationary random field. In [51], this was applied in the case where $G(x)$ is the restricted free energy of some set around x , using that such free energies are tame.

Lemma 17 (Lemma 38 of [51]). *Let G be tame, let $J_N = (m_N - a_N, m_N + a_N)$ for $a_N = o(N)$ and define $\mathcal{C}(J_N) = \{\sigma : \nabla H \upharpoonright_\sigma = 0, H(\sigma) \in J_N\}$. If D_N is an interval, there exist constants $C, c_p > 0$ given by [51], Eq. (2.8) such that for every $x \in \mathcal{S}^N$,*

$$\mathbb{E} \left[\sum_{\sigma \in \mathcal{C}(J_N)} \mathbf{1}\{G(\sigma) \in D_N\} \right] \leq C \int_{J_N} e^{c_p(u-m_N)} [\mathbb{P}_u(G(x))]^{1/2} du.$$

Proof of Proposition 14. For each $x \in \mathcal{S}^N$, $\delta > 0$, define the event

$$E(x, \delta) = \{F_N(B(x)) - F_N(A(x)) \leq -\delta\}.$$

We begin by finding a $\delta > 0$ for which

$$\sup_{x \in \mathcal{S}^N} \sup_{u \in J_N} \mathbb{P}_u(E^c(x, \delta)) \lesssim e^{-cN\delta^2}, \tag{3.9}$$

for some $c(\beta, p) > 0$. With this goal in mind, observe that for any $\delta > 0$, a union bound gives

$$\begin{aligned} \sup_{x \in \mathcal{S}^N} \sup_{u \in J_N} \mathbb{P}_u(F_N(A(x)) - F_N(B(x)) < \delta) &\leq \sup_{x \in \mathcal{S}^N} \sup_{u \in J_N} \mathbb{P}_u(F_N(A(x)) < \Lambda(p, \beta) - \delta) \\ &\quad + \sup_{x \in \mathcal{S}^N} \sup_{u \in J_N} \mathbb{P}_u(F_N(B(x)) > \Lambda(p, \beta) - 2\delta) \end{aligned}$$

whence using the sufficiently small $\delta > 0$ given by Lemma 16, combining Lemmas 15–16 with the definition of $E(x, \delta)$ yields the desired (3.9). In order to conclude the proof, we recall that $J_N = (m_N - a_N, m_N + a_N)$ for $a_N =$

$o(N)$ and $\mathcal{C}(J_N) = \{\sigma : \nabla H \upharpoonright_\sigma = 0, H(\sigma) \in J_N\}$. By Markov's inequality, we bound the quantity,

$$\mathbb{P}(\exists \sigma \in \mathcal{C}(J_N) : E^c(\sigma, \delta) \text{ holds}) \leq \mathbb{E} \left[\sum_{\sigma \in \mathcal{C}(J_N)} \mathbf{1}\{E^c(\sigma, \delta)\} \right].$$

Taking $G(x) = F_N(B(x)) - F_N(A(x))$ and $D_N = (-\delta, \infty)$ in Lemma 17, and noting that these restricted free energies are tame, so that G is tame, we obtain for $c(\beta, p) > 0$,

$$\begin{aligned} \mathbb{P}(\exists \sigma \in \mathcal{C}(J_N) : E^c(\sigma, \delta) \text{ holds}) &\leq 2C e^{c_p a_N} \sup_{u \in J_N} \sqrt{\mathbb{P}_u[E^c(x, \delta)]} \\ &\lesssim e^{c_p a_N - cN\delta^2/2}, \end{aligned}$$

which is exponentially small in N since $a_N = o(N)$. Specifically, for any fixed k , we have $\mathbb{P}(x_k \in \mathcal{C}(J_N), E^c(x_k, \delta)) = o(1)$ while by Corollary 11, for any fixed k ,

$$\lim_{N \rightarrow \infty} \mathbb{P}(x_k \in \mathcal{C}(J_N)) = \lim_{N \rightarrow \infty} \mathbb{P}(H(x_k) \in J_N) = 1,$$

so that by a union bound,

$$\mathbb{P}(E^c(x_*, \delta)) \leq \mathbb{P}(x_* \notin \mathcal{C}(J_N)) + \mathbb{P}(x_* \in \mathcal{C}(J_N), E^c(x_*, \delta)) = o(1). \quad \square$$

3.3. Ratios of Gibbs weights near local minima

Now that we have the free energy control from Proposition 14, we can control ratios of certain Gibbs probabilities. The corollaries capture the specific application of these estimates that we will need in the subsequent.

Recall the notation $x_{\pm 1}, \dots$, regarding the lowest critical points from Section 2.2. Following this convention, for any x_k , we define the following subsets of \mathcal{S}^N :

$$\begin{aligned} A_k &= \text{Cap}(x_k, q_{**} + \varepsilon N^{-1/2}) = \{\sigma \in \mathcal{S}^N : R(\sigma, x_k) > q_{**} + \varepsilon N^{-1/2}\}, \\ B_k &= \text{Band}\left(x_k, q_{**} + \frac{\varepsilon}{2} N^{-1/2}, \frac{\varepsilon}{2} N^{-1/2}\right) = \{\sigma \in \mathcal{S}^N : R(\sigma, x_k) \in [q_{**}, q_{**} + \varepsilon N^{-1/2}]\}, \\ B_k^* &= \text{Band}(x_k, q_*, \varepsilon) = \{\sigma \in \mathcal{S}^N : R(\sigma, x_k) \in [q_* - \varepsilon, q_* + \varepsilon]\} \end{aligned} \tag{3.10}$$

for some sufficiently small $\varepsilon = O(1)$ chosen such that $2\varepsilon < q_* - q_{**}$ (such a choice of $\varepsilon > 0$ exists since $q_* > q_{**}$). The first estimate shows that the ratio of Gibbs probabilities is exponential in N .

Corollary 18. *For every $p \geq 3$, there exists some $\beta_0(p)$ such that for all $\beta > \beta_0$, there exist $c_1(p, \beta), c_2(p, \beta) > 0$ such that for any fixed k , with \mathbb{P} -probability going to 1 as $N \rightarrow \infty$,*

$$\pi(B_k)\pi(A_k)^{-1} \leq c_1 \exp(-c_2 N), \tag{3.11}$$

and with \mathbb{P} -probability going to 1 as $N \rightarrow \infty$, $\pi(B_k) \leq c_1 \exp(-c_2 N)$.

Proof. The estimate is a direct consequence of Proposition 14 for the corresponding choice of k and the choice $\eta = \varepsilon$. To see this, first observe that H is smooth so that π is absolutely continuous with respect to dV and we do not need to worry about the mass of the boundaries of the sets A_k, B_k . Observe that the above ratio can be understood as a free energy difference:

$$\frac{1}{N} \log(\pi(B_k)\pi(A_k)^{-1}) = \frac{1}{N} \log\left(\frac{Z_N(B_k)}{Z_N(A_k)}\right) = F_N(B_k) - F_N(A_k).$$

Since $B_k^* \subset A_k$, we have that $\pi(A_k) \geq \pi(B_k^*)$; moreover, since $B_k \subset \text{Cap}(x_k, q_{**}) \setminus B_k^*$, we have that $\pi(B_k) \leq \pi(\text{Cap}(x_k, q_{**}) \setminus B_k^*)$. With these observations in hand, we see that Proposition 14 implies that for all $\beta \geq \beta_0$, where $\beta_0(p)$ is given by Proposition 14,

$$\begin{aligned} \pi(B_k)\pi(A_k)^{-1} &\leq \pi(\text{Cap}(x_k, q_{**}) \setminus B_k^*)\pi(B_k^*)^{-1} \\ &\leq \exp[N(F_N(\text{Cap}(x_k, q_{**}) \setminus B_k^*) - F_N(B_k^*))] \\ &\leq c_1 \exp(-c_2 N), \end{aligned}$$

for some $c_1(p, \beta), c_2(p, \beta) > 0$ with \mathbb{P} -probability going to 1 as $N \rightarrow \infty$. □

Corollary 19. For $k \in \{1, 2, 3\}$, the sets A_k, B_k defined in Eq. (3.10), satisfy

$$\lim_{N \rightarrow \infty} \mathbb{P}\left(\exists k \in \{1, 2, 3\} : \pi((A_k \cup B_k)^c) \geq \frac{1}{2}\right) = 1.$$

Proof. Each element of $\{x_k\}_{i \in \{1,2,3\}}$ has a corresponding $\text{Cap}(x_k, q_{**})$ and with probability going to 1 as $N \rightarrow \infty$, all three of $H(x_i) \in J_N$ so that the three caps are disjoint by the choice of q_{**}, ε and [51], Cor. 13. Then all three x_k 's having $\pi(\text{Cap}(x_k, q_{**})) \geq \frac{1}{2}$ would contradict $\pi(\mathcal{S}^N) = 1$. □

4. Spectral gap inequalities for Gibbs measures

Before turning to the proofs of the main results, we take a brief pause from the above probabilistic considerations and turn to the main analytical tools. Some of the results from this section are classical. We restate them for the completeness. We also prove an adaptation to our setting of a standard bound on the spectral gap.

The setting of this section is more general than that of other sections. Let M be a smooth compact boundaryless Riemannian manifold with metric g and normalized volume measure dV . Let $U \in C^\infty(M)$. As before, we define the Gibbs measure, π , by

$$d\pi(x) = \frac{e^{-\beta U(x)}}{Z} dV(x)$$

and the associated operator $\mathcal{L} = \frac{1}{2}\Delta - \frac{\beta}{2}g(\nabla U, \nabla)$ with domain $C^\infty(M) \subset L^2(M)$ where $\nabla = \nabla_g$ is again the co-variant derivative and Δ is the corresponding Laplacian. As $-\mathcal{L}$ is a uniformly elliptic operator with smooth and bounded coefficients, its eigenfunctions are C^∞ [30,33]. Thus by symmetry of $-\mathcal{L}$ on $C^\infty(M)$ with respect to π , it is essentially self-adjoint there [38]. Furthermore, it's domain, $H^1(\pi)$, is a compact subset of $L^2(\pi)$ so that it has pure point spectrum which we denote by $0 = \lambda_0 \leq \lambda_1 \leq \dots$. In particular, it has Markov semi-group $P_t = e^{t\mathcal{L}}$.

We say that a measure μ on M satisfies a *Poincaré inequality* with constant $C > 0$ if for every $f \in C^\infty(M)$,

$$\text{Var}_\mu(f) \leq C \int_M g(\nabla f, \nabla f) d\mu.$$

We say a measure μ satisfies a *log-Sobolev inequality* with constant $c > 0$ if for every $f \in C^\infty(M)$,

$$\int_M f^2 \log\left(\frac{f^2}{\int_M f^2 d\mu}\right) d\mu \leq 2c \int_M g(\nabla f, \nabla f) d\mu. \tag{4.1}$$

Corresponding to π_N , we define the *Dirichlet form* by

$$\mathcal{E}(f, h) = \int_{\mathcal{S}^N} g(\nabla f, \nabla h) d\pi_N. \tag{4.2}$$

By the Courant–Fischer min-max principle [38], the spectral gap λ_1 of the operator $-\mathcal{L}$ is given by the variational formula

$$\lambda_1 = \min_{f \in C^\infty, \|\nabla f\|_{L^2_\pi} \neq 0} \frac{\mathcal{E}(f, f)}{\text{Var}_\pi f}. \tag{4.3}$$

As a result, observe that if π satisfies a Poincaré inequality with constant $C > 0$ then the spectral gap of the corresponding operator, $-\mathcal{L}$, has $\lambda_1 \geq \frac{1}{C}$. We also remind the reader of the following classical fact.

Lemma 20. *If μ satisfies a log-Sobolev inequality with constant $c > 0$, then it also satisfies a Poincaré inequality with constant c .*

The proof of this result as well as the following two results is very classical and can be seen, for example, in [3,36]. There are many ways to verify that a Gibbs measure satisfies these inequalities. The two that we will be using are the following classical estimates. The first is a stability estimate for Poincaré inequalities.

Proposition 21 (Stability of Poincaré Inequalities). *Let M be a Riemannian manifold and suppose that $d\nu = \frac{e^{-U}}{Z} d\mu$ where $Z = \int_M e^{-U} d\mu$, μ, ν are two probability measures on M , and $U \in C_b(M)$. Then if μ satisfies the Poincaré inequality with constant $C > 0$, then ν satisfies the Poincaré inequality with constant $C e^{2\beta(\max U - \min U)}$.*

The next result is one of the foundational results regarding to Bakry and Emery’s curvature dimension.

Proposition 22 (Curvature-Energy Balance theorem, Bakry–Emery). *Let (M, π) be a Riemannian manifold with metric tensor g and Gibbs measure π corresponding to energy U . Let Ric denote the Ricci tensor on M and let Hess denote the covariant Hessian operator. If there exists a $c > 0$ such that at every point σ in M and every $v \in T_\sigma M$, the inequality*

$$\text{Ric}(v, v) + \text{Hess}(U)(v, v) \geq cg(v, v)$$

holds, then π admits a log-Sobolev inequality with constant c .

Before stating the final result of this section, we make the following definitions. For any Borel set A , define the ε -enlargement of A by

$$A_\varepsilon = \{x : d(x, A) \leq \varepsilon\},$$

where $d(x, A) = \inf_{y \in A} d(x, y)$, and for any $y \in M$, let $B_\varepsilon(y) = \{x : d(x, y) \leq \varepsilon\}$. We now turn to showing a conductance-type upper bound for the spectral gap, which is a standard adaptation of a canonical conductance bound for Markov processes to our setup.

Proposition 23 (Conductance bound). *Let $x \in M$ be a point with injectivity radius $R > 0$. Suppose that there is an $0 < r < R$ such that $\pi(B_r(x)) > 0$ and let $A = B_r(x)$. Then for all $\varepsilon > 0$ sufficiently small (i.e., $r + \varepsilon < R$), $\pi(A_\varepsilon^c) > 0$, and $\pi(A)\pi(A_\varepsilon^c) > 4\pi(A_\varepsilon \setminus A)$. Then,*

$$\lambda_1 \leq \frac{9\varepsilon^{-2}\pi(A_\varepsilon \setminus A)}{\pi(A)\pi(A_\varepsilon^c) - 4\pi(A_\varepsilon \setminus A)}.$$

Remark 24. Observe that this estimate cannot be sharp as its asymptotic order in ε is $O(\varepsilon^{-1})$ (under certain conditions on U and M). See for example [6,39].

Proof. Fix any $x \in M$ and an ε and r satisfying the above conditions and let $B = A_\varepsilon \setminus A \supset \partial A$. Consider the following test function:

$$f(\sigma) = \begin{cases} \pi(A) & \text{on } (A_\varepsilon)^c, \\ -\pi(A^c) & \text{on } A, \\ -\pi(A^c) + \eta(\varepsilon^{-1}d(\sigma, A)) & \text{else,} \end{cases}$$

where $\eta \in C^\infty([0, 1])$ and satisfies $\eta(0) = 0$, $\eta(1) = 1$ and $\sup_{[0,1]} |\frac{d\eta}{dx}| \leq 3$. For concreteness, we use the function

$$\eta(x) = \begin{cases} \exp(1 - \frac{1}{1-(x-1)^2}) & \text{for } x \in (0, 1], \\ 0 & \text{at } x = 0, \end{cases}$$

so that certainly, $\sup_{x \in [0,1]} |\frac{d\eta}{dx}(x)| \leq 3$.

First note that f is trivially smooth on A_ε^c because it is constant. Since $r + \varepsilon$ is less than the injectivity radius, it is canonical that $d(x, A)$ is smooth in $B_{r+\varepsilon}(x)$. By composition of η with d we see that $f \in C^\infty(M)$; moreover, it satisfies the gradient estimate $\sup_{\sigma \in \mathcal{S}^N} g(\nabla f, \nabla f) \leq 9 \cdot \varepsilon^{-2}$, and for $x \in B^c$ we have that $\nabla f \equiv 0$. By assumption, $\pi(B) > 0$, and on $B \setminus \partial B$, $g(\nabla f, \nabla f) > 0$ so that $\|\nabla f\|_{L_\pi^2} \neq 0$. Together, this implies that

$$\mathcal{E}(f, f) = \int_M g(\nabla f, \nabla f) d\pi = \int_B g(\nabla f, \nabla f) d\pi \leq 9 \cdot \varepsilon^{-2} \cdot \pi(B).$$

At the same time,

$$\left| \int_M f(\sigma) d\pi(\sigma) \right| \leq \left| \int_{M \setminus B} f d\pi \right| + \left| \int_M f d\pi \right| \leq 2\pi(B),$$

and moreover,

$$\int_M f(\sigma)^2 d\pi(\sigma) \geq \int_{M \setminus B} f(\sigma)^2 d\pi(\sigma) \geq \pi(A)\pi(A^c) - \pi(A)^2\pi(B).$$

Therefore,

$$\begin{aligned} \text{Var}_\pi f &= \int_M f(\sigma)^2 d\pi(\sigma) - \left(\int_M f(\sigma) d\pi(\sigma) \right)^2 \\ &\geq \pi(A)\pi(A^c) - \pi(A)^2\pi(B) - 4\pi(B)^2. \end{aligned}$$

Then, substituting $B = A_\varepsilon \setminus A$,

$$\pi(A)\pi(A^c) - \pi(A)^2\pi(B) - 4\pi(B)^2 \geq \pi(A)\pi(A_\varepsilon^c) - 4\pi(A_\varepsilon \setminus A).$$

Plugging in this choice of f as a test function in Eq. (4.3), and using the upper bound on the Dirichlet form and lower bound on the variance, we see the desired bound on λ_1 . \square

5. Proof of main theorem

In this section we prove the lower bound for the relaxation time (inverse of the spectral gap) of the Langevin dynamics of the of the spherical p -spin model at low temperatures, using the estimate on the free energy ratio obtained in Proposition 14 along with the conductance bound of the previous section. We also prove a matching (exponential in N) upper bound on the relaxation time which holds at all temperatures and prove that a much stronger $O(1)$ upper bound, along with a log-Sobolev inequality, holds at high temperatures as expected.

5.1. Low temperature

At sufficiently low temperatures we prove matching (up to constants) upper and lower bounds on λ_1 . We begin with the lower bound. Recall first the following classical fact which can be seen by an explicit calculation (see, e.g., [21]).

Fact 25. The spectral gap of $-\Delta$ on $\mathcal{S}^N = \mathcal{S}^{N-1}(\sqrt{N})$ is given by

$$\lambda_1 = 1 - \frac{1}{N},$$

and has eigenspace with multiplicity N . Furthermore, the Ricci tensor everywhere satisfies

$$\text{Ric} = \left(1 - \frac{1}{N}\right)g.$$

The above allow us to obtain the following lower bound on the gap of $-\mathcal{L}$ at all $\beta > 0$:

Lemma 26. For every $\beta > 0$, and all $p \geq 3$, there exists a $c(p) > 0$ such that the Langevin dynamics of the spherical p -spin model has,

$$\lim_{N \rightarrow \infty} \mathbb{P}(\lambda_1 \geq \exp(-c\beta N)) = 1.$$

Proof. Since the Laplacian on \mathcal{S}^N has spectral gap $1 - o(1)$ (see Fact 25), it follows from the variational form of the gap, Eq. (4.3), that $d\mu = dV$ on \mathcal{S}^N satisfies the Poincaré inequality with constant $1 - o(1)$. By Lemma 6, and the stability of the Poincaré Inequality under Gibbsian perturbations (taking $M = \mathcal{S}^N$ and $\nu = \pi, d\mu = dV$ in Proposition 21), there exists a $c > 0$ such that π satisfies the Poincaré inequality with constant

$$C_* = (1 - o(1)) \exp(4c\beta N),$$

with \mathbb{P} -probability tending to 1 as $N \rightarrow \infty$. We deduce that

$$\lim_{N \rightarrow \infty} \mathbb{P}\left(\lambda_1 \geq \frac{1}{2} \exp(-4c\beta N)\right) = 1. \quad \square$$

We now prove the following upper bound on the eigenvalue gap.

Lemma 27. For every $p \geq 3$, there exists a $\beta_0(p) > 0$ such that for all $\beta \geq \beta_0$, there exist $c_1(p, \beta), c_2(p, \beta) > 0$ such that the Langevin dynamics for the spherical p -spin model on \mathcal{S}^N satisfies,

$$\lim_{N \rightarrow \infty} \mathbb{P}(\lambda_1 \leq c_1 \exp(-c_2 N)) = 1.$$

Proof. For every N , every realization of the disorder $\{J_{i_1, \dots, i_p}\}_{\{i_1, \dots, i_p\} \subset [N]}$, choose the $k \in \{1, 2, 3\}$ given by Corollary 19 (on the complement of that event, choose $k = 1$) and define the sets $A = A_k, B = B_k$ for that choice of k , following Eq. (3.10). With \mathbb{P} -probability going to 1 as $N \rightarrow \infty$, Eq. (3.11) of Corollary 18 holds for such choice of k , independently of the realization of the disorder and N : observe that the constants in Corollary 18 are uniform over $k = 1, 2, 3$, because the estimate of Proposition 14 is uniform in all of the first $k = O(1)$ local minima.

We now use Proposition 23 to upper bound the spectral gap of $-\mathcal{L}$. To this end, let $r_* = \sqrt{N} \arccos(q_*)$ and $r_{**} = \sqrt{N} \arccos(q_{**})$. Observe that $\text{Cap}(x, q_{**} - \varepsilon N^{-1/2})$ is the ball of radius $r_{**} + \delta$ for a well chosen order one $\delta > 0$. Its easily seen that for all small $\delta, r_{**} + \delta$ is less than the injectivity radius of \mathcal{S}^N . Observe also that by Corollary 18 and Corollary 19, we have that $\pi(A_\delta^c) > 4 \frac{\pi(B)}{\pi(A)}$ so that for large enough N , the conditions of Proposition 23 are satisfied. Applying that proposition then yields

$$\lambda_1 \leq \frac{9\delta^{-2}\pi(A_\delta \setminus A)}{\pi(A)\pi(A_\delta^c) - 4\pi(A_\delta \setminus A)} = \frac{9\delta^{-2}\pi(B)}{\pi(A)\pi((A \cup B)^c) - 4\pi(B)}.$$

Then Corollary 18 and Corollary 19 together imply that with \mathbb{P} -probability going to 1 as $N \rightarrow \infty$,

$$\begin{aligned} \pi(A)\pi((A \cup B)^c) - 4\pi(B) &\geq \pi(A)\left(\frac{1}{2} - 4\frac{\pi(B)}{\pi(A)}\right) \\ &\geq \rho\pi(A), \end{aligned}$$

for some sufficiently small but fixed $\rho > 0$ (in particular, $\rho = \frac{1}{2} - \varepsilon$ certainly works for large enough N). Then, we see that with \mathbb{P} -probability approaching 1 as $N \rightarrow \infty$,

$$\lambda_1 \leq \frac{9\delta^{-2}\pi(B)}{\rho\pi(A)},$$

whence applying Corollary 18 again implies that there exists some $c_1(p, \beta), c_2(p, \beta) > 0$ such that

$$\lim_{N \rightarrow \infty} \mathbb{P}(\lambda_1 \leq c_1 \exp(-c_2 N)) = 1. \quad \square$$

With the above bounds in hand the proof of item (1) of Theorem 1 is immediate.

Proof of Theorem 1 part (1). The lower bound is obtained in Lemma 26 and the upper bound is obtained in Lemma 27. □

5.2. High temperature

It remains to prove the lower bound on the spectral gap of $-\mathcal{L}$ at high temperatures. This follows straightforwardly from Lemma 8.

Proof of Theorem 1 part (2) and Proposition 3. By Lemma 20, it suffices to prove Proposition 3. Recall that by the Curvature-Energy Balance theorem (Proposition 22), it suffices to show that there exists some $c > 0$ such that the inequality,

$$\text{Ric}_{\mathcal{S}^N}(v, v) + \beta \text{Hess}(H)(v, v) \geq cg(v, v)$$

holds uniformly over $\sigma \in \mathcal{S}^N$ and $v \in T_\sigma \mathcal{S}^N$ with probability tending to 1. By scaling, it suffices to check that this inequality holds for v such that $g(v, v) = 1$. Recall from Fact 25 that the Ricci tensor satisfies

$$\text{Ric}_{\mathcal{S}^N} = \left(1 - \frac{1}{N}\right)g.$$

Thus it suffices to show that there is a constant c such that with probability tending to 1, we have

$$1 - \frac{1}{N} + \beta \text{Hess}(H(\sigma))(v, v) \geq c.$$

To see this, observe that by Lemma 8, we have that on the complement of the event bounded there, with probability going to 1 as $N \rightarrow \infty$,

$$1 - \frac{1}{N} + \beta \text{Hess}(H(\sigma))(v, v) \geq 1 - \frac{1}{N} - \beta C_p,$$

holds for some constant $C_p > 0$. Choosing $\beta = \frac{\theta}{C_p}$ for any $\theta \in (0, 1)$, we have that the righthand side is bounded below by $1 - \theta - o(1)$, yielding the inequality for N sufficiently large. □

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