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# TRACY–WIDOM FLUCTUATIONS FOR PERTURBATIONS OF THE LOG-GAMMA POLYMER IN INTERMEDIATE DISORDER

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The free-energy fluctuations of the discrete directed polymer in $1+1$ dimensions is conjecturally in the Tracy–Widom universality class at all finite temperatures and in the intermediate disorder regime. Seppäläinen's log-gamma polymer was proven to have GUE Tracy–Widom fluctuations in a restricted temperature range by Borodin, Corwin and Remenik [*Comm. Math. Phys.* **324** (2013) 215–232]. We remove this restriction, and extend this result into the intermediate disorder regime. This result also identifies the scale of fluctuations of the log-gamma polymer in the intermediate disorder regime, and thus verifies a conjecture of Alberts, Khanin and Quastel [*Ann. Probab.* **42** (2014) 1212–1256]. Using a perturbation argument, we show that any polymer that matches a certain number of moments with the log-gamma polymer also has Tracy–Widom fluctuations in intermediate disorder.

**1. Introduction.** In 1999–2000, Baik, Deift and Johansson [8] and Johansson [21] proved that the asymptotic fluctuations of the maximal energy (passage-time) in certain point-to-point last-passage problems were governed by the same Tracy–Widom law which arises in the large $N$ limit of the top eigenvalue of an $N \times N$ matrix from the Gaussian Unitary Ensemble (GUE). It was then conjectured that this holds for very general distributions, and furthermore that it extends to the asymptotic free energy fluctuations of directed polymers in $1+1$ dimensions; that is, the *positive temperature* case. Here, the free energy takes the form

$$F(\beta, N) = \log \sum_{\mathbf{x}} \exp\left(\beta \sum_{i=1}^{N} \xi_{\mathbf{x}(i)}\right), \tag{1}$$

where the up-right lattice paths $\mathbf{x}$ go from $(1,1)$ to $(N,N) \in \mathbb{Z}^2$, $\beta > 0$ is the inverse temperature, and the $\{\xi_{i,j}\}_{(i,j)\in\mathbb{Z}^2}$ are independent identically distributed (i.i.d.) random variables, collectively referred to as the disorder.

To date, the only progress that has been made on the positive temperature conjecture is: (1) It has been verified for the special, exactly-solvable log-gamma case [25] in a certain low temperature range ($\beta > \beta^*$) [13], and (2) It has been shown to hold under certain double scaling regimes for long thin rectangles [5], and in a special case of the intermediate disorder limit [3].

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In $1+1$ dimensions, the directed random polymers are in the strong disorder regime for all values of inverse temperature $\beta > 0$ [18], and thus the free energy is conjectured to have Tracy-Widom GUE fluctuations. When $\beta = 0$, the polymer is in the weak disorder regime and behaves like a simple random walk. The intermediate disorder regime takes $\beta \to 0$ as $N \to \infty$ to probe the transition from strong to weak disorder: The more slowly $\beta$ is taken to 0, the closer one is to the Tracy–Widom asymptotics at fixed $\beta > 0$. The special case $\beta_N = \hat{\beta} N^{-1/4}$ was studied in detail in [3]. This is a double scaling regime involving the Kardar–Parisi–Zhang (KPZ) equation, where the fluctuations crossover from the Gaussian (Edwards–Wilkinson) regime to the Tracy–Widom regime as $\hat{\beta} \to \infty$.

In this article, we use a standard perturbation argument (Theorem 2.4) which shows the universality of the Tracy–Widom GUE distribution when $1 \gg \beta_N \gg \mathcal{O}(N^{-1/4})$. If we fix some sequence $\beta_N$ in the last regime, the perturbation theorem says that if two disorder distributions have moments that are sufficiently close up to a certain explicitly identified order, then they have the same asymptotic free energy fluctuations. In principle, one would like to use this to prove the universality of the Tracy–Widom law in intermediate disorder for free energies of the form (1). However, the only case in which the Tracy–Widom law is known, the log-gamma polymer, is not really of the form (1).

A log-gamma random variable is the log of a gamma distributed random variable; that is, it has the exp-gamma distribution. The exp-gamma distributions form a two parameter family corresponding to the scale and shape parameters of the gamma distribution. The scale is a trivial parameter in the directed polymer since it corresponds to centering the weights. The shape parameter ($\theta$) affects the properties of the exp-gamma distribution nonlinearly. However, at least at high temperature ($\theta \to \infty$), the shape parameter approximately controls the centered moments just like the inverse temperature $\beta$ does in the standard polymer [see (6)]. Since the shape/temperature parameter of the log-gamma random variable does not appear multiplicatively, the statement (Corollary 2.5) is not as simple as it would be if there were a solvable model of the form (1). Nevertheless, the result shows that the log-gamma polymer can be significantly perturbed in the intermediate disorder regime without changing the Tracy–Widom fluctuations (see Remark 1).

Finally, it turns out that the free energy fluctuations of the log-gamma polymer in intermediate disorder is outside of the range of the best available result [13], which requires $\beta \geq \beta^* > 0$. Most of the present article is devoted to removing this restriction, caused by the form of the contours employed in the exact formula for the log-gamma polymer in [13]. We start with a different exact formula from [12] that has more convenient contours, and we thank I. Corwin for pointing us toward this paper. We also thank an anonymous reviewer and I. Corwin for comments about a small error in the Theorem from [12]. Since we rely on this theorem, we sketch a way to fix their oversight in Remark 6.

In this way, we obtain the Tracy–Widom GUE law for the point-to-point log-gamma polymer for all temperature parameter values, and appropriate "nearby" distributions in intermediate disorder.

**2. Main results.** We now describe precisely the discrete random polymer model. The disorder is a random field given by variables $\xi_{i,j}(\beta)$, $i, j \in \{1, 2, \dots\}$ which are independent for each $\beta > 0$. The polymer is represented as an up-right directed lattice path $\mathbf{x}$ from $(1, 1)$ to $(N, N)$. The energy of such a path is given by

$$H_\beta(\mathbf{x}) = -\sum_{(i,j)\in\mathbf{x}} \xi_{i,j}(\beta).$$

The partition function is given by

$$Z_\beta^N = \sum_{\mathbf{x}} e^{-H_\beta(\mathbf{x})}. \tag{2}$$

Typically, one would have $\xi_{i,j}(\beta) = \beta\xi_{i,j}$, but since we want to also consider the log-gamma polymer, we allow for a more complicated dependence on $\beta$. The limiting free energy is given by

$$F(\beta) = \lim_{N\to\infty} \frac{1}{N} \log Z_\beta^N.$$

Carmona and Hu [14], Proposition 1.4, proved that the limit exists when the $\xi_{i,j}$ are i.i.d. Gaussians. Comets, Shiga and Yoshida [17], Proposition 2.5, extended their result to general i.i.d. weights with exponential moments.

The scaled and centered free energy fluctuations are given by

$$h_N(\beta) := \frac{\log Z_\beta^N - NF(\beta)}{\sigma(\beta)N^{1/3}}, \tag{3}$$

where in general, one expects the right scaling for $\sigma(\beta)$ to be

$$\sigma(\beta) \approx C\beta^{4/3} \qquad \text{as } \beta \searrow 0, \tag{4}$$

with a constant $C$ depending only on the distribution of the weights $\xi$. This scaling was conjectured in [2], and we prove it in this paper for the log-gamma and nearby polymers.

We will primarily be interested in the intermediate disorder regime, in which $\beta$ goes to zero as $N \to \infty$, but $\lim_{N\to\infty} \sigma(\beta)N^{1/3} > 0$. In particular, if

$$\lim_{N\to\infty} \sigma(\beta)N^{1/3} = \infty, \tag{5}$$

we expect the fluctuations to be Tracy–Widom GUE. If $\lim_{N\to\infty} \sigma(\beta)N^{1/3} = 0$ the fluctuations are Gaussian, as can be seen by doing a chaos expansion in $\beta$ and checking that only the leading term, linear in the noise, survives. If $\lim_{N\to\infty} \sigma(\beta)N^{1/3} = \hat{\beta} \in (0, \infty)$, the partition function converges to the solution of the stochastic heat equation.

In case (5), the limiting fluctuations are supposed to have the Tracy–Widom GUE law in wide generality, but the only case where there are any results is the special log-gamma polymer. Here, $e^{-\xi(\beta)}$ have the gamma distribution [or $-\xi(\beta)$ have the exp-gamma distribution], which is supported on $x > 0$ with density

$$P\big(e^{-\xi(\beta)} \in dx\big) = \frac{1}{\Gamma(\theta)} x^{\theta-1} e^{-x}\, dx,$$

where

$$\theta = \beta^{-2}. \tag{6}$$

We show in Section 3 that the $k$th centered moment of the exp-gamma distribution satisfies $\mathbb{E}[(\xi(\beta) - \mathbb{E}[\xi(\beta)])^k] = \Theta(\beta^k)$ for all $k \geq 1$ as $\beta \to 0$. Here, $\Theta(\beta^k)$ means that there exist constants $c_1, c_2$ such that the quantity in question is bounded above and below by $c_1\beta^k$ and $c_2\beta^k$, respectively, for all small enough $\beta$. This mimics the way in which the inverse temperature $\beta$ would enter the standard polymer $\xi(\beta) = \beta\xi$. In other words, choosing $\beta = \theta^{-1/2}$ *in* (6) *ensures that at high-temperature, $\beta$ plays the role of inverse temperature in the log-gamma model.*

For the log-gamma polymer,

$$F(\beta) = -2\Psi(\theta/2), \qquad \sigma(\beta) = \big(-\Psi''(\theta/2)\big)^{1/3}, \tag{7}$$

where

$$\Psi(\theta) = \frac{\Gamma'(\theta)}{\Gamma(\theta)} \tag{8}$$

is the digamma function. The limiting free-energy was identified in [25] and the variance in [13].

Our first theorem concerns the fluctuations of the log-gamma model.

THEOREM 2.1. *Let $-\xi_{i,j}(\beta)$ have the exp-gamma distribution and $\beta_N \to \beta \in [0, \infty)$ such that $\sigma(\beta_N)N^{1/3} \to \infty$. Then*

$$\lim_{N \nearrow \infty} \mathbb{P}\big(h_N(\beta_N) < r\big) = F_{\mathrm{GUE}}(r),$$

*where $h_N$ is the scaled-centered log partition function in* (3), $\mathbb{P}$ *is the probability of the disorder and $F_{\mathrm{GUE}}$ is the GUE Tracy–Widom distribution.*

Borodin, Corwin and Remenik [13] proved Theorem 2.1 for $\beta \geq \beta^* > 0$, where $\beta^*$ is some unidentified but finite number. Our result removes this restriction, and further extends it into the intermediate disorder regime.

Our next result extends the $\beta_N \searrow 0$ part of this result to "nearby" distributions.

DEFINITION 2.2 (Moment matching condition). Two parametrized families of weights $\xi = \xi(\beta)$ and $\tilde{\xi} = \tilde{\xi}(\beta)$ are said to *match moments up to order $k$* if for some $C < \infty$ and for all sufficiently small $\beta$,

$$\big|\mathbb{E}[\xi^n] - \mathbb{E}[\tilde{\xi}^n]\big| \leq C\beta^k, \qquad n = 1, \dots, k-1,$$

and

$$|\mathbb{E}[\xi^k]|, |\mathbb{E}[\tilde{\xi}^k]| \le C\beta^k. \tag{9}$$

Let $C^k(\mathbb{R})$ be the space of functions on $\mathbb{R}$ whose derivatives up to order $k$ are uniformly bounded on all of $\mathbb{R}$.

LEMMA 2.3. *Suppose two families of weights $\xi(\beta)$ and $\tilde{\xi}(\beta)$ match moments up to order $k$ (as in Definition 2.2) and let $\varphi \in C^k(\mathbb{R})$. Let $h_N(\beta)$ and $\tilde{h}_N(\beta)$ be the scaled-centered partition functions corresponding to $\xi$ and $\tilde{\xi}$. Then there is a $C < \infty$ depending only on $\varphi$ and $k$ such that*

$$|\mathbb{E}[\varphi(h_N)] - \mathbb{E}[\varphi(\tilde{h}_N)]| \le C\frac{N^2\beta^k}{\sigma(\beta)N^{1/3}}. \tag{10}$$

Lemma 2.3 is proved in Section 3, and the following theorem is a consequence of Lemma 2.3 and the fact that weak convergence is equivalent to convergence of expectations of all $C^k(\mathbb{R})$ functions [9], Theorem 2.1.

THEOREM 2.4 (Perturbation theorem). *Suppose $\xi(\beta)$ and $\tilde{\xi}(\beta)$ match moments up to order $k$ (as in Definition 2.2) and $\beta_N \searrow 0$ with*

$$\lim_{N\nearrow\infty} \frac{N^2\beta_N^k}{\sigma(\beta_N)N^{1/3}} = 0. \tag{11}$$

*Then*

$$\lim_{N\to\infty} \mathbb{P}(h_N(\beta_N) \le r) = \lim_{N\to\infty} \mathbb{P}(\tilde{h}_N(\beta_N) \le r).$$

We do not expect this perturbation technique to extend to positive temperature. The reason it works here is because the $k$th term of the Taylor expansion of the log-partition function is of order $\beta_N^k$ [see (9) and (16)], and $\beta_N \to 0$ in intermediate disorder.

COROLLARY 2.5. *Suppose $\xi(\beta)$ and $\tilde{\xi}(\beta)$ match moments up to order $k$ with $-\tilde{\xi}(\beta)$ having centered exp-gamma distribution as above and (11) holds. Then*

$$\lim_{N\to\infty} \mathbb{P}(h_N(\beta_N) \le r) = F_{\mathrm{GUE}}(r).$$

For example, if $\beta_N = N^{-\alpha}$, $\alpha \in (0, 1/4]$, it is sufficient if $\xi(\beta)$ matches moments up to order

$$k > \frac{5}{3\alpha} + \frac{4}{3}$$

with $\tilde{\xi}$.

REMARK 1. It is not obvious that the exp-gamma distribution satisfies the moment bound in (9). This is addressed in the proof of Corollary 2.5. Given this fact, let $\{X_{i,j}\}$ be a family of independent random variables with $k$ bounded moments. Let $\{\tilde{\xi}_{i,j}\}$ be an i.i.d. family of centered exp-gamma random variables with parameter $\theta = \beta^{-2}$ that are independent of the $\{X_{ij}\}$. Then the polymer with weights

$$\xi_{i,j} = \tilde{\xi}_{i,j}(1 + X_{i,j}\beta^k)$$

satisfies the moment matching condition, and hence its log-partition function has asymptotic GUE Tracy–Widom fluctuations when (10) holds.

REMARK 2. In the standard polymer, it is known when $\alpha = 1/4$ that 6 moments suffice to get the crossover law governed by the KPZ equation, so the result is slightly suboptimal. Note that our result requires an increasing number of moments to match as $\alpha \searrow 0$, whereas 5 moments are thought to suffice when $\alpha = 0$ [10].

REMARK 3. The perturbation theorem can clearly be stated in higher dimensions; we only need to modify Lemma 2.3. However, the critical temperature is positive in dimensions higher than 2, and our perturbation argument would have to be modified. We do not pursue these issues here.

Theorem 2.1 is proved in Section 4. Lemma 2.3 and Corollary 2.5 are both proved in Section 3. The Appendix contains the proof of the Fredholm determinant formula we use in Section 4, and fixes a small error in Borodin et al. [12], Theorem 2.1.

**3. Proof of the perturbation theorem.** The Lindeberg proof of the central limit theorem is now a standard argument for proving universality [22, 23]. Let $f$ be a function on $\mathbb{R}^n$ and consider two sets of i.i.d. random variables $\xi = (\xi_1, \dots, \xi_n)$ and $\tilde{\xi} = (\tilde{\xi}_1, \dots, \tilde{\xi}_n)$ that match some number of moments. For any bounded smooth function $\varphi$, the Lindeberg strategy allows one to show $|\mathbb{E}[\varphi \circ f(\xi)] - \mathbb{E}[\varphi \circ f(\tilde{\xi})]| = o(n)$ for all smooth and bounded functions $\varphi$. This is shown by replacing the $\xi$ variables one by one with $\tilde{\xi}$ variables, and using Taylor expansion to control the error. This estimate controls the weak-$*$ distance between $f(\tilde{\xi})$ and $f(\xi)$, and thus shows that they converge to the same distributional limit if it exists for either one of them. The technique has been applied to show, for example, that the limiting free energy of the Sherrington–Kirkpatrick (SK) spin glass, and the semicircle distribution in Wigner random matrices are *universal*; that is, they are independent of the distributions of the variables involved [15].

There is another related technique in spin glass theory called Guerra's interpolation method that also relies on Taylor expansion. It uses the Ornstein–Uhlenbeck process to interpolate between a vector of i.i.d. random variables $\tilde{\xi}$ and an independent i.i.d. *Gaussian* vector $\xi$. In the SK model, the partition function is of the form

$Z = \sum_{\sigma \in \{-1,1\}^N} \exp(-\beta_N H(\sigma))$ where $H(\sigma) = -\sum_{ij} \xi_{ij}\sigma_i\sigma_j$ and $\beta_N = N^{-1/2}$. Again, $N^{-1}\log Z$ has a deterministic limit called the free-energy as $N \to \infty$. For i.i.d. Gaussian weights, the limit was shown to be given by the celebrated Parisi formula by Talagrand [27]. The limit was shown to be the same for all i.i.d. families of symmetric random variables with four moments by Guerra and Toninelli [20]. They used the aforementioned interpolation technique and the so-called approximate integration by parts for weights that match the moments of a Gaussian up to some order. Their ideas were extended by Carmona and Hu [14] to include distributions that match two moments with the Gaussian and have finite third moment. Chatterjee used truncation and the Lindeberg technique to remove the finite third moment requirement [15]. Other results like [6] and [19] extend the interpolation technique using higher order Taylor expansions.

In particular, Auffinger and Chen [6] showed that the *Gibbs' measure*—the random measure on configurations given by $P_\xi(\sigma) = Z^{-1}\exp(\beta_N H(\sigma))$—also converged to a universal limit as long as the weights matched a certain number of moments with the Gaussian. Since their results are for general spin-systems and not just for the SK model, they also apply to polymer models in intermediate disorder. In a personal communication, [4] applied the theorems in [6] to show that the limiting Gibbs' measure associated with the polymer path is universal: Let $\beta_N = \beta N^{-\alpha}$, and let $\gamma = (\gamma_i)_{i=1}^{Nd}$ be a directed path from the origin to $N(1,\dots,1) \in \mathbb{Z}^d$, where $\gamma_i$ are the vertices along path. Suppose the weights $\tilde{\xi}$ in the polymer match the first $k$ moments of the standard Gaussian such that

$$\mathbb{E}[\tilde{\xi}^{k+1}] < \infty, \qquad k > \frac{d+1}{\alpha}. \tag{12}$$

For $n \in \mathbb{N}$, let $\gamma^a$, $a = 1,\dots,n$ be any $n$ directed paths from the origin to $N(1,\dots,1)$.

THEOREM 3.1 (Auffinger [4]). *Let $L$ be a function depending on $n$ paths $(\gamma^a)_{1\le a\le n}$ where $n$ is fixed, and suppose $\|L\|_\infty \le 1$. Then*

$$\left|\mathbb{E}_{\tilde{\xi}}\big[\langle L(\gamma)\rangle\big] - \mathbb{E}_{\xi}\big[\langle L(\gamma)\rangle\big]\right| \to 0 \qquad \text{as } N \to \infty.$$

*Here, $\langle\cdot\rangle$ represents the average over the Gibbs' measure on paths, and $\mathbb{E}.$ represents expectation over the corresponding set of weights.*

This allowed him to show that various quantities of interest were universal, including the transversal fluctuation exponent of the path measure that is defined as follows: Let $\gamma_{N/2}$ be the midpoint of a path $\gamma$ sampled from the Gibbs' measure. The polymer has transversal fluctuation exponent $\alpha$ if for any $\alpha' < \alpha < \alpha''$ and $C > 0$,

$$\begin{aligned} &\mathbb{P}_{\tilde{\xi}}\big(|\gamma_{N/2} - N/2(1,\dots,1)| \le CN^{\alpha'}\big) \to 0, \\ \text{and} \quad &\mathbb{P}_{\tilde{\xi}}\big(|\gamma_{N/2} - N/2(1,\dots,1)| \le CN^{\alpha''}\big) \to 1 \qquad \text{as } N \to \infty. \end{aligned} \tag{13}$$

The transversal exponent has a relationship with $\chi$, the fluctuation exponent defined by the asymptotic behavior of the variance: $\mathrm{Var}(\log Z^N(\beta)) \sim N^{2\chi}$. Under certain strong hypotheses on the *existence* of these exponents (they are not known to exist for any standard polymer), it is known that $\chi = 2\alpha - 1$ [7, 16]. Therefore, [4] also indicates that polymers ought to have the same fluctuation exponent as the Gaussian polymer under (12). Without assuming $\chi = 2\alpha - 1$, we prove that $\chi$ is universal for the log-gamma and nearby polymers in intermediate disorder, thereby verifying a conjecture of [2] in this special case.

In our setting, there are a few problems with Guerra's interpolation technique. *In its current form, it only allows one to match moments with the Gaussian*. Hence one can only look at polymers where the weights are of the form $\xi_{i,j}(\beta) = \beta\xi_{i,j}$ and only match moments with the Gaussian distribution. This by itself is not a very serious shortcoming and may be overcome with some work.

But for the Gaussian polymer, it is not known whether the fluctuation exponents discussed above exist, and whether the fluctuations of $\log Z^N(\beta)$ are in the GUE Tracy–Widom universality class. In fact, very little is known about the Gaussian polymer other than the fact that the limiting free energy exists.

On the other hand, a lot more is known about the log-gamma polymer. The scale of the variance [25], and the limiting fluctuations are known in a large parameter range [13]. Moreover, the Lindeberg replacement technique is well suited to comparing other polymers with the log-gamma polymer. Since we need more terms in the Taylor expansion (than [15], e.g.), and since Guerra's Gaussian integration by parts technique does not apply directly to prove Lemma 2.3, we reproduce the fairly standard Lindeberg argument for completeness.

PROOF OF LEMMA 2.3. For a fixed vertex $x = (i, j)$, we write the paritition function in (2) as

$$Z(y) = Z_{x^c} + Z_x e^y, \tag{14}$$

where $Z_{x^c}$ represents the sum in (2) over paths that do not pass through $x$, and $Z_x$ is the sum over paths that do pass through $x$, but whose terms do not include weight at $x$. Let $h(y) = N^{-1/3}\sigma^{-1}(\log Z(y) - NF)$. $Z(y)$ and $h(y)$ do indeed depend on the other weights $\xi_z$ for $z \neq x$, but the dependence is suppressed in the notation because we want to isolate the effect of replacing $\xi_x$ by $\tilde{\xi}_x$. For any function $\varphi \in C^k$, we will show that

$$\bigl|\mathbb{E}[\varphi(h(\xi_x))] - \mathbb{E}[\varphi(h(\tilde{\xi}_x))]\bigr| \le C(\sigma N^{1/3})^{-1}\beta^k, \tag{15}$$

where the expectation is over the disorder. We obtain (10) by replacing $\xi_x$ by $\tilde{\xi}_x$, $N^2$ times for each $x \in \{1, \dots, N\}^2$.

Fix all the other weights in the disorder, and write Taylor's theorem for $\varphi(h(\xi_x))$:

$$\varphi(h(\xi_x)) = \sum_{j=0}^{k-1} \frac{\partial_y^j \varphi(h(0))}{j!}\xi_x^j + \frac{\partial_y^k \varphi(h(\zeta))}{k!}\xi_x^k,$$

with $\zeta$ between 0 and $\xi_x$. Taking expectation and using the independence of $\{\xi_z\}_{z\in\mathbb{R}^2}$, we get

$$\mathbb{E}[\varphi(h(\xi_x))] = \sum_{j=0}^{k-1} \frac{a_j}{j!}\mathbb{E}[\xi_x^j] + \frac{a_k}{k!}\mathbb{E}[\xi_x^k], \tag{16}$$

where $a_j = \mathbb{E}[\partial_y^j \varphi(h(0))]$, $j = 1, \dots, k-1$ and $a_k = \mathbb{E}[\partial_y^k \varphi(h(\zeta))]$. One has an analogous expression for $\mathbb{E}[\varphi(h(\tilde{\xi}_x))]$, but note that in fact $a_j = \tilde{a}_j$ for $j = 1, \dots, k-1$ since they both do not depend on $\xi_x$ and $\tilde{\xi}_x$, and all the other weights they depend on are the same. Hence, from the moment matching condition (9),

$$\bigl|\mathbb{E}[\varphi(h(\xi_x))] - \mathbb{E}[\varphi(h(\tilde{\xi}_x))]\bigr| \le \Bigl(\sum_{j<k} |a_j| + (|a_k| + |\tilde{a}_k|)\Bigr) C\beta^k.$$

To control the error term, we will show that for any $k \ge 1$ and all $y \in \mathbb{R}$,

$$\bigl|\partial_y^k \varphi(h(y))\bigr| \le C_{k,\varphi}(\sigma N^{1/3})^{-1}, \tag{17}$$

where $C_{k,\varphi}$ is a constant dependent only on $\varphi$, $k$ and the constant from the moment matching condition. The estimates (16), (17) and the moment matching condition in Definition 2.2 together imply (15).

To prove (17), we expand the derivative of a composition (à la Faa di Bruno)

$$\partial^k \varphi(h) = \sum_{\sum s m_s = k} C_{m_1\cdots m_k} \partial^{\sum m_s}\varphi \prod_{r=1}^{k} (\partial^r h)^{m_r},$$

where the $C_{m_1\cdots m_k}$ are multinomial coefficients, and $m_s \ge 0$ for $s = 1, \dots, k$. Since $\varphi$ is smooth with bounded derivatives up to order $k$, we only need to control $\partial^r h(0)$ for $r \ge 1$. Computing derivatives in (14),

$$\partial_y \log Z(y) = \frac{Z_x e^y}{Z_{x^c} + Z_x e^y} =: p(y),$$

$$\partial_y^i \log Z(y) = \mathcal{P}_i(p(y)), \qquad i > 1,$$

where $\mathcal{P}_i$ is the polynomial given by the recurrence

$$\mathcal{P}_{i+1}(p) = \mathcal{P}_i'(p)p(1-p), \qquad \mathcal{P}_1(p) = p, \qquad i \ge 1.$$

The recurrence follows from the chain rule and $p'(y) = p(y)(1 - p(y))$. Since $0 \le p(y) \le 1$ for all $y \in \mathbb{R}$, we can bound each of the polynomials $\mathcal{P}_i$ by constants for $i = 1, \dots, k$. Putting the last few observations together, we get (17) for $k \ge 1$. $\square$

PROOF OF COROLLARY 2.5. Theorem 2.1 shows that if the $-\tilde{\xi}(\beta)$ are exp-gamma random variables, and $\beta_N \to \infty$ such that (11) holds, then

$$\lim_{N\to\infty} \mathbb{P}(h_N \le r) = F_{\mathrm{GUE}}(r).$$

We only need to show that $-\tilde{\xi}(\beta)$ satisfies the moment bound in (9) (see Remark 1), which says that for $\theta = \beta^{-2}$

$$\left|\mathbb{E}\left[\left(\tilde{\xi}(\beta) - \mathbb{E}\tilde{\xi}(\beta)\right)^k\right]\right| \le C_k \frac{1}{\theta^{\lceil k/2 \rceil}}, \tag{18}$$

for some constant $C_k$. We will in fact show that for all $k > 1$,

$$\mathbb{E}\left[\left(\tilde{\xi}(\beta) - \mathbb{E}\tilde{\xi}(\beta)\right)^k\right] = \Theta\left(\frac{1}{\lceil \theta^{k/2} \rceil}\right), \qquad \theta \to \infty. \tag{19}$$

The cumulant generating function of the exp-gamma distribution is given by

$$\log \mathbb{E}\left[\exp(tX)\right] = \log\left(\frac{\Gamma(t+\theta)}{\Gamma(\theta)}\right),$$

where $X$ is a gamma distributed random variable. Differentiating this $k$ times with respect to $t$, we see that the $k$th cumulant of the exp-gamma distribution is given by $\Psi^{(k-1)}(\theta)$, the $(k-1)$th derivative of the digamma function (8). The digamma function can be written as [1], 6.3.16,

$$\Psi(z) = -\gamma_{EM} + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+z}\right), \tag{20}$$

where $\gamma_{EM}$ is the Euler–Mascheroni constant. Note that the series is absolutely convergent when $z$ is bounded away from the nonpositive integers. It follows that

$$\kappa_k := \Psi^{(k-1)}(\theta) = \Theta\left(\frac{1}{\theta^{k-1}}\right), \qquad k > 1, \theta \to \infty.$$

For any random variable, the moments $\mu_n$ are related to the cumulants $\kappa_n$ via the following combinatorial expansion (see [26] for this formula and its famous Möbius inversion). If $\pi$ is a (set) partition of $\{1, \dots, k\}$, then we represent $\pi$ as the union of disjoint sets $\pi = \{B_i\}_{i=1}^{n(\pi)}$ where $B_i \subset \{1, \dots, k\}$, and $\bigcup B_i = \{1, \dots, k\}$. Then

$$\mu_k = \sum_{\pi \in \mathcal{L}} \prod_{B \in \pi} \kappa_{|B|},$$

where $|B|$ represents the cardinality of the set $B$, and $\mathcal{L}$ is the set of all partitions of $\{1, \dots, k\}$. Since we are considering centered random variables, we simply set the first cumulant to zero ($\kappa_1 = 0$) and therefore, get

$$\begin{aligned} |\mu_k| &= \left|\sum_{\pi \in \mathcal{L}} \prod_{i=1}^{n(\pi)} \kappa_{|B_i|} 1_{|B_i| \ne 1}\right| \\ &\le \sum_{\pi \in \mathcal{L}} \prod_{i=1}^{n(\pi)} \frac{C_{|B_i|}}{\theta^{|B_i|-1}} 1_{|B_i| \ne 1}, \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\pi \in \mathcal{L}} \frac{C'_\pi}{\theta^{\sum_{i=1}^{n(\pi)} |B_i| - n(\pi)}} 1_{|B_i| \neq 1}, \\
&\leq \frac{C''_k}{\theta^{k - \max_{\pi, |B_i| \neq 1} n(\pi)}} \\
&= \frac{C''_k}{\theta^{\lceil k/2 \rceil}},
\end{aligned} \tag{21}
$$

where $C'_\pi = \prod_{i=1}^{n(\pi)} C_{|B_i|}$ and $C''_k = 2^k \max_\pi C'_\pi$. In (21), we used the following observation: when $\pi$ contains no sets with only one element, it follows that $n(\pi) \leq \lfloor k/2 \rfloor$. This proves (18).

When $k$ is even, the leading order term for $\mu_k$ comes from partitions whose $B_i$ have exactly 2 elements for all $i$. If there is even one $B_i$ with more than 2 elements, then $n(\pi) < k/2$. Thus, all the other partitions result in terms that have strictly smaller order as $\theta \to \infty$ and get $|\mu_k| \geq c_k \theta^{-k/2}$. A similar argument for $k$ odd applies; here, partitions that have one $|B_i| = 3$ and all the rest having two elements provide the leading order term. This proves (19). □

## 4. Tracy–Widom fluctuations for the log-gamma polymer.

In this section, we prove Theorem 2.1. We begin with the Fredholm determinant formula for the Laplace transform of the partition function.

### 4.1. *Fredholm determinant formula.*

THEOREM 4.1. *For $N \geq 9$, let $Z_\beta^N$ be the partition function of the log-gamma polymer with $\theta = \beta^{-2}$. Then for* $\mathrm{Re}(u) > 0$,

$$
\mathbb{E}\big[e^{-u Z_\beta^N}\big] = \det\big(I + K_u^N\big)_{L^2(\mathcal{C}_\varphi)}, \tag{22}
$$

*where*

$$
\begin{aligned}
K_u^N(v, v') = {} & \frac{1}{2\pi \mathrm{i}} \int_{\ell_{z_{\mathrm{crit}}+\delta}} -\frac{\pi}{\sin(\pi(w - v))} \left( \frac{\Gamma(v)}{\Gamma(w)} \frac{\Gamma(\theta - w)}{\Gamma(\theta - v)} \right)^N \frac{u^{w-v}}{w - v'} \, dw \\
& + \sum_{j=1}^{q(v)} \mathrm{Res}_{u,j}(v, v'),
\end{aligned} \tag{23}
$$

*where for $1 \leq j \leq q(v)$, the residues are*

$$
\mathrm{Res}_{u,j}(v, v') = (-1)^j \left( \frac{\Gamma(v)}{\Gamma(v + j)} \frac{\Gamma(\theta - v - j)}{\Gamma(\theta - v)} \right)^N \frac{u^j}{v + j - v'}, \tag{24}
$$

*and*

$$
q(v) = \lfloor z_{\mathrm{crit}} + \delta - \mathrm{Re}(v) \rfloor, \qquad z_{\mathrm{crit}} = \theta/2, \qquad 0 < \delta \leq \frac{z_{\mathrm{crit}}}{2}. \tag{25}
$$

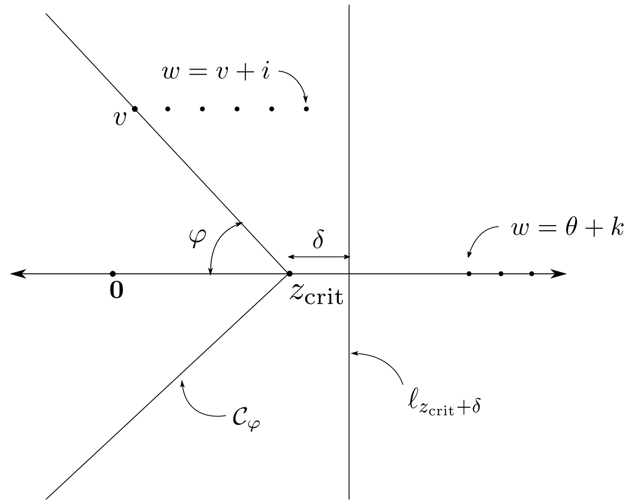


FIG. 1. *Contours in Theorem 4.1. The critical point is at $z_{\mathrm{crit}} = \frac{\theta}{2}$. The triangular contour is $\mathcal{C}_\varphi$, and the vertical contour $\ell_{z_{\mathrm{crit}}+\delta}$ has real part $z_{\mathrm{crit}} + \delta$. There are two sets of poles that one needs to watch out for. The poles of the sine are shown as dots to the right of $v$ and are of the form $w = v + i$. The poles of $\Gamma(\theta - w)$ are of the form $w = \theta + k$, $k = 0, 1, \ldots$.*

*The contours are defined as follows: For any $\varphi \in (0, \pi/4]$, the $\mathcal{C}_\varphi$ contour is given by $\{z_{\mathrm{crit}} + e^{\mathrm{i}(\pi+\varphi)} y\}_{y \in \mathbb{R}^+} \cup \{z_{\mathrm{crit}} + e^{\mathrm{i}(\pi-\varphi)} y\}_{y \in \mathbb{R}^+}$, where $\mathbb{R}^+$ is the set of nonnegative reals. The $\ell_x$ contour is a vertical straight line with real part $x$ (see Figure 1). Both $\ell_x$ and $\mathcal{C}_\varphi$ are oriented to have increasing imaginary parts.*

REMARK 4. Theorem 4.1 is proved by setting $\tau = 0$ in [12], Theorem 2.1, as suggested in Remark 2.9 of the same paper. This requires a new estimate, and this is done in the Appendix.

We will see in the next section that the critical point of the integrand of $K_u^N$ in (23) is at $z_{\mathrm{crit}}$. The contours $\ell_{z_{\mathrm{crit}}+\delta}$ (as $\delta \to 0$) and $\mathcal{C}_\varphi$ are located at the critical point.

4.2. *Estimates along the contours.* We are interested in the asymptotic probability distribution of (3). The trick is to rewrite the left-hand side of (22) as

$$\mathbb{E}\big[\exp\big(-e^{\sigma N^{1/3}(h_N - t)}\big)\big] \tag{26}$$

by taking

$$u = e^{-NF - t\sigma N^{1/3}}. \tag{27}$$

As $N \nearrow \infty$, by (5) $\sigma N^{1/3} \nearrow \infty$, and (26) becomes $\lim_{N \nearrow \infty} \mathbb{P}(h_N \le t)$ (see [13], Proof of Theorem 1, and [11], Lemma 4.1.38). Now we consider the same limit of the right-hand side of (22). We start with a formal critical point analysis of the integral in (23), which can be rewritten as

$$-\frac{1}{2\pi\mathrm{i}}\int_{\ell_{z_{\mathrm{crit}}+\delta}}\frac{\pi}{\sin(\pi(w-v))}e^{N[G(v)-G(w)]+t\sigma N^{1/3}(v-w)}\frac{dw}{w-v'},\tag{28}$$

where we have ignored the residues, and let

$$G(z)=\log\Gamma(z)-\log\Gamma(\theta-z)+F(\beta)z.\tag{29}$$

We have $G'(z)=\Psi(z)+\Psi(\theta-z)+F(\beta)$. From (7), it follows that the critical point, $G'(z_{\mathrm{crit}})=0$, is at $z_{\mathrm{crit}}=\theta/2$, and $G''$ vanishes there as well. Therefore, the exponent is cubic near the critical point and it is natural to define

$$\tilde{v}=\sigma N^{1/3}(v-z_{\mathrm{crit}}),\qquad \tilde{w}=\sigma N^{1/3}(w-z_{\mathrm{crit}}),\tag{30}$$

and let $\tilde{K}^N(\tilde{v},\tilde{v}')=K^N(v,v')$ in (23), where we have dropped the subscript $u$ in the kernel to indicate that we have set $u$ as in (27). The change of variable introduces a Jacobian factor of $(\sigma N^{1/3})^{-1}$ into the Fredholm expansion (36). Then it is easy to prove the following lemma.

LEMMA 4.2.

$$\lim_{N\to\infty}\frac{1}{(\sigma N^{1/3})}\tilde{K}^N(\tilde{v},\tilde{v}')=K_{\mathrm{Ai}}(\tilde{v},\tilde{v}'),\tag{31}$$

*where the Airy kernel is defined as*

$$K_{\mathrm{Ai}}(\tilde{v},\tilde{v}'):=\frac{1}{2\pi\mathrm{i}}\int\frac{\exp\{-\frac{1}{3}\tilde{v}^3+t\tilde{v}\}}{\exp\{\frac{1}{3}\tilde{w}^3+t\tilde{w}\}}\frac{d\tilde{w}}{(\tilde{v}-\tilde{w})(\tilde{w}-\tilde{v}')}.\tag{32}$$

*The Airy kernel acts on the contour* $\{e^{-2\pi\mathrm{i}/3}\mathbb{R}^+\cup e^{2\pi\mathrm{i}/3}\mathbb{R}^+\}$ *and the integral in* $\tilde{w}$ *is on the contour* $\{e^{-\pi\mathrm{i}/3}\mathbb{R}^++\delta\}\cup\{e^{\pi\mathrm{i}/3}\mathbb{R}^++\delta\}$ *for any horizontal shift* $\delta>0$. *Both are oriented to have increasing imaginary part.*

The Fredholm determinant of the right-hand side of (32) is $F_{\mathrm{GUE}}(t)$ [13]. We prove Lemma 4.2 and flesh out the details of this sketch in the rest of this section.

Next, we upgrade the pointwise convergence in Lemma 4.2 to prove that $\det(\mathbf{1}+K^N)\to\det(\mathbf{1}+K_{\mathrm{Ai}})$. Recall the kernel (23)

$$K^N(v,v')=\frac{1}{2\pi\mathrm{i}}\int_{\ell_{z_{\mathrm{crit}}+\delta(\sigma N)^{-1/3}}}I(v,v',w-v)\,dw+\sum_{i=1}^{q(v)}\mathrm{Res}_i(v,v'),\tag{33}$$

where $I(v,v',w-v)$ is the integrand in (28). Again, we have dropped the $u$ subscript on $K^N$, $I$ and $\mathrm{Res}_i$. The kernel acts on the $\mathcal{C}_\varphi$ contour as before, and we set $\varphi=\pi/4$. The little extra displacement of the $\ell_{z_{\mathrm{crit}}+\delta(\sigma N)^{-1/3}}$ is a necessary technicality that we will address in due course. As a shorthand, we will drop the extra $(\sigma N)^{-1/3}$ from the $\ell_{z_{\mathrm{crit}+\delta}}$.

For $(v, v')$ on the $\mathcal{C}_{\pi/4}$ contour, we show that

$$|K^N(v, v')| \le f(v, N), \tag{34}$$

where $f(v, N)$ is defined in Lemma 4.8. Then, from the Hadamard inequality for determinants, we get for $m > 1$,

$$|\det(K^N(v_i, v_j))_{1\le i,j\le m}| \le \prod_{i=1}^{m} f(v_i, N) m^{m/2}. \tag{35}$$

$f(v, N)$ depends on $v$ and $N$ in such a way that it integrates over $\mathcal{C}_{\pi/4}$ to a quantity that is bounded above by a constant independent of $N$. It follows that the Fredholm expansion of the determinant

$$\begin{aligned} &\det(I + K^N)_{L^2(\mathcal{C}_{\pi/4})} \\ &\quad = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\mathcal{C}_{\pi/4}} dv_1 \cdots \int_{\mathcal{C}_{\pi/4}} dv_m \det(K^N(v_i, v_j))_{1\le i,j\le m}, \end{aligned} \tag{36}$$

is absolutely summable uniformly in $N$ [see (50)]. Thus, we can take the $N \to \infty$ limit inside the series and integrals and replace $K^N$ by its pointwise limit. This is similar to what was done in Borodin, Corwin and Remenik [13], but now the quantity in (35) must be bounded uniformly in $\theta$ as well as $N$, since $\theta$ can go to infinity with $N$ (see Theorem 2.1). The rigorous estimates are shown in Lemma 4.8. Lemma 4.2 and (34) require several estimates on contours that appear in Lemmas 4.5, 4.6 and 4.7. These estimates are summarized in the following 4 steps below.

Recall the function $G(z)$ defined in (29). Using (7), we write it as

$$G(z) = \log\Gamma(z) - \log\Gamma(\theta - z) - 2\Psi(z_{\text{crit}})z. \tag{37}$$

The bound in (34) will follow from an analysis of this function along the contours $\mathcal{C}_{\pi/4}$ and $\ell_{z_{\text{crit}}}$, and an estimate on the residues $\text{Res}_i(v, v')$. The constants in the estimates are independent of $N$, but they do depend on $\theta$. However, as long as $\theta \ge \theta_0 > 0$, these constants are well behaved; this is guaranteed by $\sigma(\beta)N^{1/3} \to \infty$ in (11).

In the following, we set $\tilde{\sigma} = \sigma(\beta)N^{1/3}$, and note that

$$\tilde{\sigma} = \Theta((N z_{\text{crit}}^{-2})^{1/3}). \tag{38}$$

1. In Proposition 4.3, we first show that the Taylor approximation is effective in the region $|z - z_{\text{crit}}| \le c\tilde{\sigma}^{-1}$. Although $G$ is analytic, one must be careful because the derivatives of $G$ are a function of $\theta$, which is allowed to go to infinity with $N$.

Using the Taylor expansion, we may also arrange for an estimate of the form (recall $z_{\text{crit}} = \theta/2$)

$$\operatorname{Re}\big(G(z) - G(z_{\text{crit}})\big) \le -\frac{C}{z_{\text{crit}}^2}|z - z_{\text{crit}}|^3, \qquad |z - z_{\text{crit}}| \le \frac{z_{\text{crit}}}{2},$$

where $C > 0$ can be explicitly chosen. This is needed to show that the pointwise limit of $K^N$ is $K_{\text{Ai}}$.

2. In Lemma 4.5, we show that the real part of $G$ decreases sufficiently rapidly away from the critical point on the $\mathcal{C}_{\pi/4}$ contour. The upper and lower halves of the $\mathcal{C}_{\pi/4}$ contour are parametrized as

$$z(r) = z_{\text{crit}} + r\hat{e}_{\pm}, \qquad r \ge 0, \tag{39}$$

where $\hat{e}_{\pm} = -1 \pm \mathrm{i}$. On both halves of the contour, the derivative of $G$ satisfies

$$\frac{d}{dr}\operatorname{Re}\big(G\big(z(r)\big) - G(z_{\text{crit}})\big) \le -\frac{2r^2}{(1 + z_{\text{crit}} + 2r)^2}. \tag{40}$$

This captures the cubic behavior of $G$ near the critical point, and the linear decay for large $r$: for some constants $C$ and $r_0$ independent of $z_{\text{crit}}$ and $N$,

$$\begin{aligned}\operatorname{Re}\big(G\big(z(r)\big) - G(z_{\text{crit}})\big) &\le -H(r)\\ &:= -\begin{cases} C z_{\text{crit}}^{-2} r^3, & r \le r_0,\\ C\big(z_{\text{crit}}^{-2} r_0^3 + z_{\text{crit}}^{-2}(r - r_0)\big), & r > r_0.\end{cases}\end{aligned} \tag{41}$$

3. In Lemma 4.6, we estimate $G$ on the $\ell_{z_{\text{crit}}+\delta}$ contour. We use the parametrization

$$w(r) = z_{\text{crit}} + r\hat{e}_{\pm} + \delta\tilde{\sigma}^{-1}, \qquad r \ge 0, \tag{42}$$

where $\hat{e}_{\pm} = \pm\mathrm{i}$. Since $\ell_{z_{\text{crit}}+\delta}$ is not a steep-descent contour, we cannot show that the derivative of $\operatorname{Re}(G)$ is strictly positive [cf. (40)]. So we first modify the $\ell_{z_{\text{crit}}+\delta}$ contour as follows. It begins at $z_{\text{crit}} + \delta'\tilde{\sigma}^{-1}$ for some $\delta' < \delta$, and then locally follows the Airy contours $\{e^{-\pi\mathrm{i}/3}\mathbb{R}^+ + z_{\text{crit}} + \delta'\tilde{\sigma}^{-1}\} \cup \{e^{\pi\mathrm{i}/3}\mathbb{R}^+ + z_{\text{crit}} + \delta'\tilde{\sigma}^{-1}\}$ until its real part becomes $z_{\text{crit}} + \delta\tilde{\sigma}^{-1}$. Suppose this happens at $r'$ in the parametrization in (42). After that, it follows the $\ell_{z_{\text{crit}}+\delta}$ contour once again. Then we use the Taylor expansion and obtain an estimate analogous to the first equation in (41) for $|w(r) - z_{\text{crit}}| \le r'$. For the rest of the contour, we show in Lemma 4.6 that

$$\operatorname{Re}\big(G\big(w(r)\big) - G\big(w(r')\big)\big) \ge 0, \qquad r \ge r'.$$

4. Finally, in Lemma 4.7, we estimate the contribution of the residues to the bound in (34). The bound also shows that the residues vanish in the limit, and helps prove Lemma 4.2.

PROOF OF LEMMA 4.2. Using the estimates in steps 1–4, we first show that the pointwise limit of the integral in (33) is the Airy kernel. We will use both the rescaled variables $\tilde{v}, \tilde{v}'$ from (30) and $v, v'$ in the following.

First, consider the integral term in (31), and split the integral over the top half of the contour into three parts,

$$\tilde{\sigma}^{-1}\tilde{K}^N(\tilde{v},\tilde{v}') = \int_0^{M\tilde{\sigma}^{-1}} + \int_{M\tilde{\sigma}^{-1}}^{r_0} + \int_{r_0}^{\infty} \tilde{\sigma}^{-1} I\big(v, v', w(r) - v\big)\, dw(r) \tag{43}$$
$$:= I_1 + I_2 + I_3,$$

where $w(r)$ is the parametrization in (42) of the $\ell_{z_{\text{crit}}}$ contour, and $M > 0$ is a parameter that will eventually go to infinity. Since the integrand is analytic in a tiny ball of size $M\tilde{\sigma}^{-1}$, we modify the contours so that they are locally aligned with the Airy contours when $r \le M\tilde{\sigma}^{-1}$.

We first estimate the absolute value of $I_3$. Note that there is a $C'$ such that $C'r \le Cr^3 - r$ for $r \ge r_0$. We will use this estimate in both the $v$ and $w$ variables:

$$\begin{aligned} |I_3| &= \left| -\frac{1}{2\pi \mathrm{i}} \int_{r_0}^{\infty} \frac{\pi\tilde{\sigma}^{-1}}{\sin(\pi(w - v))} e^{N(G(v)-G(w)) + t\tilde{\sigma}(v-w)} \frac{dw}{w - v'} \right| \\ &\le C\delta^{-1} r_0^{-1} e^{-NH(|v - z_{\text{crit}}|) + \tilde{\sigma}|v - z_{\text{crit}}|} e^{-C'M^3} \int_{r_0}^{\infty} e^{-\pi r}\, dr, \\ &\le C\delta^{-1} r_0^{-1} e^{-C'|\tilde{v}|} e^{-C'M^3}, \end{aligned}$$

where $H(r)$ is defined in (41), and we have used (38), $|\sin(\pi(w(r) - v))|^{-1} \le c\delta^{-1}\tilde{\sigma} e^{-\pi r}$, and $|w(r) - v'|^{-1} \le c r_0^{-1}$ for some constant $c$.

To estimate $I_2$, we use $e^{-\pi r} \le 1$, make the change of variable $\tilde{r} = \tilde{\sigma} r$, and use the bound $|w(r) - v'|^{-1} \le c\delta^{-1}\tilde{\sigma}$; all the other estimates are the same as the ones used for $I_3$. Thus, we obtain

$$|I_2| \le C\delta^{-2} e^{-C|\tilde{v}|} \int_M^{r_0\tilde{\sigma}} e^{-C'\tilde{r}}\, d\tilde{r} \le C\delta^{-2} e^{-C'|\tilde{v}|} e^{-M}.$$

In $I_1$, we make the change of variable in (30), and take $N \to \infty$. By dominated convergence, the limit can be taken inside the integral, and by the argument in the first part of Section 4.2, the first integral goes to the integrand of $K_{\text{Ai}}$ in the rescaled variables $(\tilde{v}, \tilde{v}')$. Letting $M \to \infty$ shows that the integral term in (33) goes to $K_{\text{Ai}}$. From Lemma 4.7, it follows that residues converge pointwise to 0. □

Next, we flesh out the details of the estimates in steps 1 through 4.

*Step 1. Taylor expansion near the critical point.* We have already seen that the first two derivatives of $G$ [defined in (29)] vanish at the critical point. If the third and fourth derivatives of $G$ were well behaved, the Taylor expansion for $G(z)$ is an effective approximation when $z$ is close to the critical point:

$$G(z) - G(z_{\text{crit}}) = \frac{G^{(3)}}{3!}(z_{\text{crit}})(z - z_{\text{crit}})^3 + \frac{G^{(4)}}{4!}(\xi)(z - z_{\text{crit}})^4,$$

for $\xi \in \{y: |y - z_{\mathrm{crit}}| < |z - z_{\mathrm{crit}}|\}$. Since $G$ is an analytic function, it is clear that $G^{(3)}$ and $G^{(4)}$ are well behaved for fixed $z_{\mathrm{crit}}$. However, we allow $z_{\mathrm{crit}} \to \infty$; Proposition 4.3 shows roughly that $\frac{G^{(4)}(z)}{G^{(3)}(z_{\mathrm{crit}})} \approx \frac{C}{z_{\mathrm{crit}}}$ for a constant $C > 0$, when $|z(r) - z_{\mathrm{crit}}| \le z_{\mathrm{crit}}/2$ and $z_{\mathrm{crit}} \to \infty$.

PROPOSITION 4.3. *When* $|z - z_{\mathrm{crit}}| \le z_{\mathrm{crit}}/2$,

$$\frac{2}{(2 + z_{\mathrm{crit}})^2} \le -G^{(3)}(z_{\mathrm{crit}}) - \frac{4}{z_{\mathrm{crit}}^3} \le \frac{2}{z_{\mathrm{crit}}^2}, \tag{44}$$

$$\left|G^{(4)}(z)\right| \le \frac{96}{z_{\mathrm{crit}}^4} + \frac{32}{z_{\mathrm{crit}}^3}. \tag{45}$$

PROOF OF PROPOSITION 4.3. Recall the series expansion for the digamma function in (20). Differentiating (37) thrice, we obtain

$$-G^{(3)}(z_{\mathrm{crit}}) = -2\Psi^{(2)}(z_{\mathrm{crit}}) = \frac{4}{z_{\mathrm{crit}}^3} + 4\sum_{n=1}^{\infty} \frac{1}{(n + z_{\mathrm{crit}})^3}.$$

Estimating the sum with an integral, we get

$$4\int_1^{\infty} \frac{1}{(x + z_{\mathrm{crit}})^3}\,dx \le -G^{(3)}(z_{\mathrm{crit}}) - \frac{4}{z_{\mathrm{crit}}^3} \le 4\int_0^{\infty} \frac{1}{(x + z_{\mathrm{crit}})^3}\,dx$$

which proves (44). We estimate $G^{(4)}(z)$ similarly: To apply the integral test as before, we first show that $|x + z|$ is increasing with $x \in \mathbb{R}^+$. It is clear that if $|z - z_{\mathrm{crit}}| \le z_{\mathrm{crit}}/2$, then $z$ has positive real part and consequently, so does $x + z$ for all $x > 0$. It follows that $|x+z|$ increases with $x$. Then, using (20) and $|x+z| \ge 2^{-1}(x + z_{\mathrm{crit}})$,

$$\left|\Psi^{(3)}(z)\right| \le \sum_{n=0}^{\infty} \frac{6}{|n + z|^4} \le \frac{96}{|z_{\mathrm{crit}}|^4} + \int_0^{\infty} \frac{96}{|x + z_{\mathrm{crit}}|^4}\,dx,$$

which proves (45). □

*Step 2. Decay of $G$ along the $\mathcal{C}_{\pi/4}$ contour.* In the following lemma, we first compute the derivative of $G$ along a general contour. We will use this computation repeatedly to estimate $G$ along the $\mathcal{C}_{\pi/4}$ and $\ell_{z_{\mathrm{crit}}}$ contours (Lemma 4.6) and to estimate the residues in Lemma 4.7.

LEMMA 4.4. *Let $z(r) = z_{\mathrm{crit}} + v(r)$ be a contour. Then the derivative of* $\mathrm{Re}(G)$ *is*

$$\frac{d}{dr}\mathrm{Re}\big(G\big(z(r)\big)\big) = 2\sum_{n=0}^{\infty} \frac{-\mathrm{Re}(v'(r)v(r)^2)(n + z_{\mathrm{crit}})^2 + \mathrm{Re}(v'(r))|v(r)|^4}{(n + z_{\mathrm{crit}})|(n + z_{\mathrm{crit}})^2 - v(r)^2|^2}$$

PROOF. From (37),

$$\frac{d}{dr}G\big(z(r)\big)=z'(r)\big(\Psi\big(z_{\mathrm{crit}}+v(r)\big)-\Psi(z_{\mathrm{crit}})\big)+z'(r)\big(\Psi\big(z_{\mathrm{crit}}-v(r)\big)-\Psi(z_{\mathrm{crit}})\big).$$

Using (20),

$$\begin{aligned}\frac{d}{dr}G\big(z(r)\big)&=z'(r)\sum_{n=0}^{\infty}\frac{2}{n+z_{\mathrm{crit}}}-\frac{1}{n+z(r)}-\frac{1}{n+\theta-z(r)}\\&=v'(r)\sum_{n=0}^{\infty}\frac{2}{n+z_{\mathrm{crit}}}-\frac{2(n+z_{\mathrm{crit}})}{(n+z_{\mathrm{crit}}+v(r))(n+z_{\mathrm{crit}}-v(r))}\\&=2\sum_{n=0}^{\infty}\frac{-v'(r)v^2(n+z_{\mathrm{crit}})^2+v'(r)|v(r)|^4}{(n+z_{\mathrm{crit}})|(n+z_{\mathrm{crit}})^2-v^2|^2}.\end{aligned}$$

□

LEMMA 4.5. $\mathrm{Re}(G)$ *in* (37) *satisfies the following derivative bound*:

$$\frac{d}{dr}\mathrm{Re}\big(G\big(z(r)\big)\big)\le-\frac{2r^2}{(1+z_{\mathrm{crit}}+2r)^2},\qquad r>0,$$

*where $z(r)$ is the parametrization of $\mathcal{C}_{\pi/4}$ given in* (39).

PROOF. We parametrize the upper-half of the $\mathcal{C}_{\pi/4}$ contour as in Lemma 4.4 with $v(r)=r\hat{e}$ where $\hat{e}=-1+\mathrm{i}$. Then

$$\begin{aligned}\frac{d}{dr}\mathrm{Re}\big(G\big(z(r)\big)\big)&=2\sum_{n=0}^{\infty}\frac{-2r^2(n+z_{\mathrm{crit}})^2-4r^4}{(n+z_{\mathrm{crit}})|(n+z_{\mathrm{crit}})^2-2r^2\mathrm{i}|^2}\\&\le-4\sum_{n=0}^{\infty}\frac{r^2}{(n+z_{\mathrm{crit}}+2r)^3}\le-4\int_1^{\infty}\frac{r^2}{(x+z_{\mathrm{crit}}+2r)^3}\,dx\\&=-2\frac{r^2}{(1+z_{\mathrm{crit}}+2r)^2}.\end{aligned}$$

This captures the behavior of $G$ along the steep-descent contours rather well: cubic near the critical point, and then linear decay. $G$ behaves symmetrically in the lower half plane, and hence satisfies a similar estimate on the lower half of the $\mathcal{C}_{\pi/4}$ contour. □

*Step 3. Decay along the $\ell_{z_{\mathrm{crit}}}$ contour.*

LEMMA 4.6. $\mathrm{Re}(G)$ *in* (37) *increases away from the critical point along the $\ell_{z_{\mathrm{crit}}}$ contour.*

PROOF. Recall that the $\ell_{z_{\mathrm{crit}}}$ contour starts off at a distance $\hat{\delta}_N = \delta\tilde{\sigma}^{-1}$ away from $z_{\mathrm{crit}}$. Let $w(r)$ be the parametrization of $\ell_{z_{\mathrm{crit}}}$ in (42); as before we focus on the upper half of the contour. Using Lemma 4.4 and $v(r) = r\mathrm{i} + \hat{\delta}_N$, we get

$$\begin{aligned}\frac{d}{dr}\operatorname{Re}\big(G(w(r))\big) &= 2\sum_{n=0}^{\infty}\frac{-\operatorname{Re}(\mathrm{i}(\hat{\delta}_N - r^2 + 2\hat{\delta}_N r\mathrm{i}))(n+z_{\mathrm{crit}})^2 + \operatorname{Re}(\mathrm{i})|v|^4}{(n+z_{\mathrm{crit}})|(n+z_{\mathrm{crit}})^2 - v^2|^2} \\ &\geq 0.\end{aligned}$$ □

*Step 4. Triviality of the residues*

LEMMA 4.7. *There exist constants $c_1, C > 0$ independent of $N$ and $z_{\mathrm{crit}}$ such that for $j = 1, \dots, \lfloor|\operatorname{Im}(v)|\rfloor$, the residues in (33) satisfy*

$$\log\big|\operatorname{Res}_j(v, v')\big| \leq \begin{cases} -c_1 N j^2 \dfrac{|\operatorname{Im}(v)|}{z_{\mathrm{crit}}^2}, & 1 \leq |\operatorname{Im}(v)| \leq C z_{\mathrm{crit}}, \\ -c_1 N j \log\Big(1 + \dfrac{|\operatorname{Im}(v)|}{z_{\mathrm{crit}}}\Big), & |\operatorname{Im}(v)| > C z_{\mathrm{crit}}, \end{cases}$$

*when $v, v' \in \mathcal{C}_{\pi/4}$. If $Cz_{\mathrm{crit}} < 1$, the first bound holds vacuously.*

Lemma 4.7 helps show the estimate on the kernel in (34) that is used in the Hadamard bound in Lemma 4.8. It also shows that the residues go to 0 as $\tilde{\sigma} \to \infty$.

PROOF OF LEMMA 4.7. From (24) and (27), we can write the residues in the form

$$\begin{aligned}\big|\operatorname{Res}_j(v, v')\big| &= \left|\left(\frac{\Gamma(v)}{\Gamma(v+j)}\frac{\Gamma(\theta - v - j)}{\Gamma(\theta - v)}\right)^N \frac{e^{2\Psi(z_{\mathrm{crit}})Nj - tj\tilde{\sigma}}}{v + j - v'}\right| \\ &\leq e^{N(G(v) - G(v+j)) + |t|j\tilde{\sigma}}\end{aligned} \tag{46}$$

since $j \geq 1$. We estimate $G(v) - G(v+j)$ using Lemma 4.4 again. Let $v(r) = k\hat{e}_+ + r$ in the parametrization of the contour in Lemma 4.4, where $\hat{e}_+ = -1 + \mathrm{i}$. Since $k = |\operatorname{Im}(v)|$, (25) implies that we only need to consider $r$ in the range $0 \leq r \leq k$ (assuming $\delta$ is small enough). We interpolate between $G(v)$ and $G(v+l)$ by computing the following derivative:

$$\begin{aligned}\frac{d}{dr}\operatorname{Re}\big(G(z(r))\big) &= 2\sum_{n=0}^{\infty}\frac{r(2k-r)(n+z_{\mathrm{crit}})^2 + ((r-k)^2+k^2)^2}{(n+z_{\mathrm{crit}})|(n+z_{\mathrm{crit}})+v|^2|(n+z_{\mathrm{crit}})-v|^2} \\ &\geq 2\sum_{n=0}^{\infty}\frac{r(2k-r)(n+z_{\mathrm{crit}})^2 + ((r-k)^2+k^2)^2}{(n+z_{\mathrm{crit}})(n+z_{\mathrm{crit}}+2k-r)^4} \\ &\geq 2\int_{1+\max(k,z_{\mathrm{crit}})}^{\infty}\frac{r(2k-r)x}{(x+2k-r)^4}\,dx \\ &\quad + 2\int_{1+z_{\mathrm{crit}}}^{\infty}\frac{((r-k)^2+k^2)^2}{x(x+2k-r)^4}\,dx := I_1 + I_2.\end{aligned}$$

The limits of integration in $I_1$ have been chosen as above because $f(x) = x/(x + 2k - r)^4$ is decreasing for $x \geq \max(k, z_{\mathrm{crit}})$. This lets us use the integral test to estimate the sum.

Our bounds on the integrals $I_j$, $j = 1, 2$ will ensure:

$$G(v+j) - G(v) \geq \begin{cases} c_1 j^2 \dfrac{|\operatorname{Im}(v)|}{z_{\mathrm{crit}}^2}, & |\operatorname{Im}(v)| \leq C z_{\mathrm{crit}}, \quad (47) \\[2ex] c_1 j \log\left(1 + \dfrac{|\operatorname{Im}(v)|}{z_{\mathrm{crit}}}\right), & |\operatorname{Im}(v)| > C z_{\mathrm{crit}}. \quad (48) \end{cases}$$

1. The first bound (47) ensures that the exponent in (46) contains a negative term of order at least $N z_{\mathrm{crit}}^{-2}$, and thus overwhelms $\tilde{\sigma} = \Theta((N z_{\mathrm{crit}}^{-2})^{1/3})$ as they both go to infinity [by (5)].

2. The second bound (48) ensures that the residue is integrable in $v$ on the contour $\mathcal{C}_{\pi/4}$ over the range $|\operatorname{Im}(v)| \in [C z_{\mathrm{crit}}, \infty)$.

Explicitly computing $I_1$, we get

$$I_1 = \frac{r(2k-r)(3(1+\max(k, z_{\mathrm{crit}})) + 2k - r)}{3(1+\max(k, z_{\mathrm{crit}}) + 2k - r)^3} \geq \frac{rk}{3(1+\max(k, z_{\mathrm{crit}}) + 2k)^2},$$

for $r \leq k$. For the second integral,

$$I_2 \geq k^4 \left( \frac{1}{(2k-r)^4} \log\left(1 + \frac{2k-r}{1+z_{\mathrm{crit}}}\right) - \frac{2}{(2k-r)^3(1+z_{\mathrm{crit}} + (2k-r))} \right),$$

where we have used the elementary integral

$$\int_a^\infty \frac{dx}{x(x+c)^4} = -\frac{6a^2 + 15ac + 11c^2}{6c^3(a+c)^3} + \frac{1}{c^4} \log\left(1 + \frac{c}{a}\right).$$

Here, for $k \geq C z_{\mathrm{crit}}$ where $C$ is some constant, the first term in $I_2$ dominates the second for all $r \leq k$. Therefore, integrating over $r$, we get (47) and (48) for some constants $c_1$ and $C$. $\square$

Finally, we prove the inequality used in the Hadamard bound in (34).

LEMMA 4.8. *There exist constants $c_1, c_2, C > 0$ that are independent of $N$ and $z_{\mathrm{crit}}$ such that for all $N$ large enough,*

$$(49) \quad |K^N(v, v')| \leq \begin{cases} c_1 \tilde{\sigma} \exp(-c_2 N z_{\mathrm{crit}}^{-2} |\operatorname{Im}(v)|^3), & |\operatorname{Im}(v)| < 1, \\ c_1 \tilde{\sigma} \exp(-c_2 N z_{\mathrm{crit}}^{-2} |\operatorname{Im}(v)|), & 1 \leq |\operatorname{Im}(v)| \leq C z_{\mathrm{crit}}, \\ c_1 \left(1 + \dfrac{|\operatorname{Im}(v)|}{z_{\mathrm{crit}}}\right)^{-c_2 N}, & |\operatorname{Im}(v)| > C z_{\mathrm{crit}} \end{cases}$$

$$=: f(v, N)$$

*on the contour $\mathcal{C}_{\pi/4}$. Consequently, the $m$th term of the Fredholm series for $K^N$ in* (36) *satisfies*

$$\frac{1}{m!}\int_{\mathcal{C}_{\pi/4}} dv_1 \cdots \int_{\mathcal{C}_{\pi/4}} dv_m \det\bigl(K^N(v_i, v_j)\bigr)_{1\le i,j\le m} \le \frac{C}{m^{(m-1)/2}}. \tag{50}$$

PROOF. There are many undetermined constants in this proof, and we allow them to change from line to line. Using the technique of splitting up the integral term as in (43), and from the estimates in Lemma 4.5 and Lemma 4.6, it follows that the integral in (33) has two regimes of behavior: for constants $c_1, c_2, C > 0$

$$\int_{\ell_{z_{\mathrm{crit}}}} I(v, v', \omega)\, dw \le \begin{cases} c_1\tilde{\sigma}\exp\bigl(-c_2 N z_{\mathrm{crit}}^{-2}|\mathrm{Im}(v)|^3\bigr), & |\mathrm{Im}(v)| \le C z_{\mathrm{crit}}, \\ c_1\tilde{\sigma}\exp\bigl(-c_2 N|\mathrm{Im}(v)|\bigr), & |\mathrm{Im}(v)| > C z_{\mathrm{crit}}. \end{cases}$$

The $\ell_{z_{\mathrm{crit}}}$ contour is a small distance $\tilde{\delta}_N = \delta(\sigma N^{1/3})^{-1}$ away from $z_{\mathrm{crit}}$. From (25), it follows there are about $|\mathrm{Im}(v)| + \tilde{\delta}_N$ residues. Hence, when $|\mathrm{Im}(v)| < 1$ and when $N$ large enough, there are no residues. We estimate the contribution of the residues when $|\mathrm{Im}(v)| > 1$. For $N$ large enough, and $1 \le |\mathrm{Im}(v)| \le C z_{\mathrm{crit}}$, Lemma 4.7 implies

$$\begin{aligned}\sum_{j=1}^{\lfloor|\mathrm{Im}(v)|\rfloor} \mathrm{Res}_j(v, v') &\le \sum_{j=1}^{\lfloor|\mathrm{Im}(v)|\rfloor} c_1 \exp\Bigl(-c_1 N j \frac{|\mathrm{Im}(v)|}{z_{\mathrm{crit}}^2}\Bigr) \\ &\le c_1' \exp\bigl(-c_2 N z_{\mathrm{crit}}^{-2}|\mathrm{Im}(v)|\bigr).\end{aligned}$$

When $|\mathrm{Im}(v)| > C z_{\mathrm{crit}}$,

$$\sum_{j=1}^{\lfloor|\mathrm{Im}(v)|\rfloor} \mathrm{Res}_j(v, v') \le \sum_{j=1}^{\lfloor|\mathrm{Im}(v)|\rfloor} c_1\Bigl(1 + \frac{|\mathrm{Im}(v)|}{z_{\mathrm{crit}}}\Bigr)^{-c_2 N l} \le c_1' \frac{1}{(1 + |\mathrm{Im}(v)|/z_{\mathrm{crit}})^{c_2 N}}.$$

Thus, (49) follows, and we can integrate the bound over the $\mathcal{C}_{\pi/4}$ contour to obtain

$$\begin{aligned}\int_{\mathcal{C}_{\pi/4}} f(v, N)\, dv &\le C\Bigl(\tilde{\sigma}\int_0^1 e^{-c_2 N z_{\mathrm{crit}}^{-2} x^3}\, dx + \tilde{\sigma}\int_1^{C z_{\mathrm{crit}}} e^{-c_2 N z_{\mathrm{crit}}^{-2} x}\, dx \\ &\qquad + \int_{C z_{\mathrm{crit}}}^{\infty}\Bigl(1 + \frac{x}{z_{\mathrm{crit}}}\Bigr)^{-c_2 N}\Bigr) \\ &\le C\Bigl(\frac{\tilde{\sigma}}{(N z_{\mathrm{crit}}^{-2})^{1/3}} + \tilde{\sigma} e^{-c_2 N z_{\mathrm{crit}}^{-2}} + \frac{(1+C)^{-c_2 N}}{N z_{\mathrm{crit}}^{-1}}\Bigr).\end{aligned}$$

Since $\tilde{\sigma} = \Theta((N z_{\mathrm{crit}}^{-2})^{1/3})$, it is clear that the integral is bounded above by a constant independent of $N$ and $z_{\mathrm{crit}}$. The Hadamard inequality now implies (50). $\square$

# APPENDIX: FREDHOLM DETERMINANT AS A LIMIT OF THE FORMULA OF BORODIN–CORWIN–FERRARI–VETO

**A.1. The BCFV theorem.** Borodin et al. [12] consider a mixed polymer that consists of Seppäläinen's log-gamma polymer [25] and the O'Connell–Yor semi-discrete polymer [24]. For $N \geq 1$, the paths $\mathbf{x}$ consist of a discrete portion $\mathbf{x}^d$ adjoined to a semidiscrete portion $\mathbf{x}^{sd}$, both of which only go up and right. The discrete portion is a nearest-neighbor up–right path on $\mathbb{Z}^2$ from $(-N, 1)$ to $(-1, n)$ for some $1 \leq n \leq N$. The semidiscrete path goes from $(0, n)$ to $(\tau, N)$: For $0 \leq s_n < \cdots < s_{N-1} \leq \tau$, it consists of horizontal segments on $(s_i, s_{i+1})$ for $i = n, \ldots, N-2$ and a final interval $(s_{N-1}, \tau)$ connected by vertical jumps of size 1 at each $s_i$. For $1 \leq m, n \leq N$ let $\xi_{-m,n}$ be independent exp-gamma random variables with parameter $\theta$, and for all $1 \leq n \leq N$ let $B_n$ be independent Brownian motions. The paths have energy

$$H_\beta(\mathbf{x}) = -\sum_{(i,j)\in\mathbf{x}^d} \xi_{i,j} + B_n(s_n) + \big(B_{n+1}(s_{n+1}) - B_{n+1}(s_n)\big) + \cdots + \big(B_N(\tau) - B_N(s_{N-1})\big).$$

The partition function is given by

$$\mathbf{Z}^N(\tau) = \sum_{i=1}^{N} \sum_{\mathbf{x}^d:(-N,1)\nearrow(-1,i)} \int_{\mathbf{x}^{sd}:(0,i)\nearrow(\tau,N)} e^{-H_\beta(\mathbf{x})}\, d\mathbf{x}^{sd},$$

where $d\mathbf{x}^{sd}$ represents the Lebesgue measure on the simplex $0 \leq s_n < s_{n+1} < \cdots < s_{N-1} \leq \tau$.

REMARK 5. When $\tau = 0$, the polymer is simply the standard point-to-point log-gamma polymer. There is no semidiscrete part, and the discrete path is forced to end at $(-1, N)$.

THEOREM A.1 (Borodin, Corwin and Remenik [13], Theorem 2.1). *Fix $N \geq 9$, $\tau > 0$ and $\theta > 0$. For all $u \in \mathbb{C}$ with positive real part,*

$$\mathbb{E}\big[e^{-u\mathbf{Z}^N(\tau)}\big] = \det\big(\mathbf{1} + K^N_{u,\tau}\big)_{L^2(\mathcal{C}_\varphi)}, \tag{51}$$

*where*

$$\begin{aligned} K^N_{u,\tau}(v, v') &= \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{D}_v} \frac{1}{\sin(\pi s)} \left(\frac{\Gamma(v)}{\Gamma(s+v)} \frac{\Gamma(\theta - v - s)}{\Gamma(\theta - v)}\right)^N \frac{e^{\tau(sv + s^2/2)}}{v + s - v'} u^s\, ds \\ &=: \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{D}_v} I_{u,\tau}(v, v', s)\, ds. \end{aligned} \tag{52}$$

The $\mathcal{C}_\varphi$ contour is the wedge-shaped contour defined in Theorem 4.1, and depends on the angle $\varphi$ and the parameter $\theta$. The $\mathcal{D}_v$ contour depends on $v$ and parameters $R$ and $d$. For every $v \in \mathcal{C}_\varphi$, we choose $R = -\operatorname{Re}(v) + 3\theta/4$, $d > 0$ and let $\mathcal{D}_v$ consist of straight lines from $R - \mathrm{i}\infty$ to $R - \mathrm{i}d$ to $\theta/8 - \mathrm{i}d$ to $\theta/8 + \mathrm{i}d$ to $R + \mathrm{i}d$ to $R + \mathrm{i}\infty$. The parameter $d$ must be taken small enough so that $v + \mathcal{D}_v$ does not intersect $\mathcal{C}_\varphi$. Both contours are oriented to have increasing imaginary part.

REMARK 6. In Borodin et al. [12], the $\mathcal{D}_v$ contour consisting of straight lines from $R - \mathrm{i}\infty$ to $R - \mathrm{i}d$ to $1/2 - \mathrm{i}d$ to $1/2 + \mathrm{i}d$ to $R + \mathrm{i}d$. The formula only holds if the poles in $s$ of $\Gamma(\theta - v - s)$ lie strictly to the right of the contour $\mathcal{D}_v$, and this imposes a lower bound $\theta > 1$ that was not noticed by them. We remove the restriction $\theta > 1$ as follows.

Note first of all that both sides of (51) are analytic functions of $\theta$ in some region of $\mathbb{C}$ containing the ray $(1, \infty)$, on which they coincide, by Borodin et al. [12]: The left-hand side is actually analytic in a region containing the ray $\theta \in (0, \infty)$ because the expectation is an $N^2$-fold integral of a function $e^{-u\mathbf{Z}^N(\tau)}$ of the variables $\xi_{i,j}$, $i, j = 1, \dots, N$ with

$$\mathbb{E}[e^{-u\mathbf{Z}^N(\tau)}] = \int F(\xi_{ij}) \prod_{i,j} \frac{e^{-\xi_{i,j}} \xi_{i,j}^{\theta-1}}{\Gamma(\theta)} \, d\xi_{i,j}, \qquad F(\xi_{ij}) = \mathbb{E}[e^{-u\mathbf{Z}^N(\tau)} | \xi_{i,j}].$$

Fix $1/2 > \theta^* > 0$. The right-hand side is analytic for $\theta > \theta^*$ because the Fredholm determinant of a kernel analytic in $\theta$ is analytic in $\theta$, as long as one has a uniform bound for the kernel for $\theta > \theta^*$.

Call $\mathcal{D}_{v,\eta}$ the contour consisting of straight lines from $R - \mathrm{i}\infty$ to $R - \mathrm{i}d$ to $\eta - \mathrm{i}d$ to $\eta + \mathrm{i}d$ to $R + \mathrm{i}d$. We can use Cauchy's theorem to deform the contour in (52) to $\mathcal{D}_{v,\theta^*/4}$ without changing the kernel, since the only obstacle is the zero of the sine in the denominator, which is at the origin. Proposition A.2 gives us a uniform bound on the kernel in the region, say $[\theta^*/16, 2]$, that can be used together with Hadamard's bound to control the Fredholm series. Using this new representation of the kernel, the determinant in (51) is an analytic function of $\theta$, now in a region containing $[\theta^* - \gamma, 1 + \gamma]$ for some $\gamma > 0$. Because the kernel is unchanged, this coincides with the old determinant for $\theta \in (1, 1 + \gamma)$. By the formula from Borodin et al. [12], the determinant coincides with the left-hand side of (51) for $\theta \in (1, 1 + \gamma)$. But the left-hand side is analytic in a region containing the ray $(0, \infty)$, hence the determinant with the new, deformed kernel is equal to the left-hand side on a region containing the interval $[\theta^* - \gamma, 1 + \gamma]$, including at the value $\theta^*$.

REMARK 7. We write $\mathcal{D}_v$ contour as a union of the vertical contour $\ell_{-Re(v)+R}$ defined in Theorem 4.1 and the "sausage" $\mathcal{D}_{v,\square}$ which consists of $(\mathcal{D}_v \setminus \ell_{-Re(v)+R}) \cup [R + \mathrm{i}d, R - \mathrm{i}d]$ and forms a clockwise loop. The integral of the kernel $K_{u,\tau}^N$ over $\mathcal{D}_{v,\square}$ consists of residues due to the sine that we will estimate separately. For each $v$, there is some wiggle room in the $R$ parameter that allows the vertical contour $\ell_{-\operatorname{Re}(v)+R}$ to avoid the singularity of the sine function in $I_{u,\tau}(v, v', s)$.

**A.2. Proof of Theorem 4.1.** In this section, we obtain Theorem 4.1 by letting $\tau \searrow 0$ in Theorem A.1. We may do so if we can truncate the series that defines the Fredholm determinant of $\mathbf{Z}^N(\tau)$ uniformly in $\tau$. This is done by proving an estimate on $K^N_{u,\tau}(v, v')$ that depends favorably with $\tau$, and then using Hadamard's inequality in the usual way [see (50)]. The constants in the following propositions may depend on the angle of the contour $\varphi \in (0, \pi/4]$.

PROPOSITION A.2 (BCFV kernel estimate). *When* $|\operatorname{Im}(v)| > \max((2e^{\tau\theta/2} \times |u|)^{1/2N}, c_3\theta))$, *for some constants* $c_1, c_2, c_3 > 0$, *the kernel in* (52) *satisfies the bound*

$$\left|K^N_{u,\tau}(v, v')\right| \le 2\frac{e^{\tau\theta/2}|u|}{|\operatorname{Im}(v)|^{2N}} e^{-c_1\tau|\operatorname{Im}(v)|} + e^{C(\theta,N,\tau,\varphi)} e^{-c_2 N|\operatorname{Im}(v)|(\log|v|-c_3\theta)},$$

*where* $|C(\theta, N, \tau, \varphi)| \le C'(\theta^*, \varphi)(N(\theta + |\log\theta| + \theta^{-1} + 1) + \tau\theta)$ *for* $\theta \ge \theta^*$. *Here*, $\theta^* > 0$ *is any fixed number.*

Proposition A.2 is proved by estimating $I_u$ on the closed rectangular contour $\mathcal{D}_{v,\square}$, and the vertical line $\ell_R$. These two estimates are done separately in Proposition A.3 and Proposition A.4.

PROPOSITION A.3 (Integral over the $\mathcal{D}_{v,\square}$ contour). *When* $|\operatorname{Im}(v)| > \max((2e^{\tau\theta/2}|u|)^{\frac{1}{2N}}, c_3\theta)$,

$$\left|\int_{\mathcal{D}_{v,\square}} I_{u,\tau}(v, v', s)\, ds\right| \le \frac{e^{\tau\theta/2}|u|}{|\operatorname{Im}(v)|^{2N}} e^{-c\tau|\operatorname{Im}(v)|},$$

*where* $c > 0$ *is some constant*, *and* $c_3$ *is the same constant that appears in Proposition* A.2.

PROPOSITION A.4 (Integral over the $\ell_R$ contour). *For* $|\operatorname{Im}(v)| \ge c_3\theta$,

$$\int_{\ell_R} I_{u,\tau}(v, v', s)\, ds \le e^{C(\theta,N,\tau,\varphi)} e^{-c_1 N|\operatorname{Im}(v)|(\log|v|-c_2\theta)},$$

*where* $c_1, c_2, c_3$ *and* $C(\theta, N, \tau, \varphi)$ *are the same constants that appear in Proposition* A.2.

Before proving the Propositions, we complete the proof of Theorem 4.1.

PROOF OF THEOREM 4.1. In Theorem A.1, the vertical contour $\ell_R$ is at $R = -\operatorname{Re}(v) + 3\theta/4$. This must be moved to the critical point so that $R = -\operatorname{Re}(v) +$

$\theta/2+\delta$ for small $\delta$. Using the bound on $I_{u,\tau}(v,v',s)$ in (65), we can truncate the vertical contour at large $|\operatorname{Im}(s)|$ and use Cauchy's theorem to move over the vertical contour.

Next, we have to take a limit $\tau \searrow 0$ in (51). Since $u$ has positive real part, $e^{-uZ_\beta^N}$ is absolutely bounded, and we can take the limit inside the integral by bounded convergence. For the right-hand side, we can use the Hadamard inequality argument in Section 4.2 and the bounds in Proposition A.3 and Proposition A.4 to show that it converges to the Fredholm determinant of the pointwise limit of the kernel $K_{u,\tau}^N$ as $\tau \searrow 0$. $\square$

PROOF OF PROPOSITION A.3. The integral over $\mathcal{D}_{v,\square}$ simply collects residues from the poles of $\sin^{-1}(\pi s)$. Then

$$\begin{aligned}\int_{\mathcal{D}_{v,\square}} I_{u,\tau}(v,v',s)\,ds &= \sum_{i=1}^{q(v)}\left(\frac{\Gamma(v)\Gamma(\theta-v-i)}{\Gamma(v+i)\Gamma(\theta-v)}\right)^N u^i \frac{e^{\tau(\operatorname{Re}(v)i+i^2/2)}}{|v+i-v'|}(-1)^i\\ &= \sum_{i=1}^{q(v)} \operatorname{Res}_{u,\tau,i}(v,v'),\end{aligned}$$

where $q(v)\le R$ is the number of zeros of the sine caught inside the sausage (25). Since $v,v'\in\mathcal{C}_\varphi$, we have

$$\operatorname{Re}(v)=\frac{\theta}{2}-\cot(\varphi)\big|\operatorname{Im}(v)\big|. \tag{53}$$

Then, we may estimate $q(v)$ as follows:

$$q(v)\le R=-\operatorname{Re}(v)+\frac{\theta}{2}+\delta=\cot(\varphi)\big|\operatorname{Im}(v)\big|+\delta,$$

where $0<\delta\le\frac{\theta}{4}$. For our bound, the number of residues does not matter, and the contribution of the first residue dominates. The ratio of gamma functions in the residues become $\Gamma(v)/\Gamma(v+i)=\prod_{j=0}^{i-1}(v+j)^{-1}$ and $\Gamma(\theta-v-i)/\Gamma(\theta-v)=\prod_{j=1}^{i}(\theta-v-j)^{-1}$. It is clear that $|v+i|\ge|\operatorname{Im}(v)|$ and $|\theta-v-i|\ge|\operatorname{Im}(v)|$. The $|v-v'+i|^{-1}$ term can be bounded above by a constant since $v+i$ lies to the right of the $\mathcal{D}_v$ contour. Therefore, for $|\operatorname{Im}(v)|>(e^{\tau\theta/2}|u|/2)^{\frac{1}{2N}}$,

$$\begin{aligned}\sum_{i=1}^{q(v)}\big|\operatorname{Res}_{u,\tau,i}(v,v')\big| &\le \sum_{i=1}^{q(v)}\frac{1}{|\operatorname{Im}(v)|^{2Ni}}|u|^i e^{i\tau\theta/2}e^{-\tau i(\cot(\varphi)|\operatorname{Im}(v)|-i/2)}\\ &\le \sum_{i=1}^{q(v)}\left(\frac{e^{\tau\theta/2}|u|}{|\operatorname{Im}(v)|^{2N}}\right)^i e^{-\tau c|\operatorname{Im}(v)|},\\ &\le 2\frac{e^{\tau\theta/2}|u|}{|\operatorname{Im}(v)|^{2N}}e^{-\tau c|\operatorname{Im}(v)|},\end{aligned}$$

where $c$ is a $\varphi$-dependent constant that comes from bounding the $i(\cot(\varphi)|\operatorname{Im}(v)|-i/2)$ term on the interval $1 \le i \le \cot(\varphi)|\operatorname{Im}(v)| + \delta$. We choose the constant $c_3$ to ensure that $\cot(\varphi)|\operatorname{Im}(v)| \ge \cot(\varphi)c_3\theta > \theta/4 \ge \delta$. The same constant $c_3$ appears in Proposition A.4. $\square$

PROOF OF PROPOSITION A.4. We will focus first on estimating the product of gamma functions in $I_{u,\tau}(v, v', s)$. For $s \in \ell_{-\operatorname{Re}(v)+R}$,

$$\operatorname{Re}(s) = \delta + \cot(\varphi)\big|\operatorname{Im}(v)\big|. \tag{54}$$

Stirling's formula holds whenever $\arg(z)$ remains bounded away from $\pm\pi$ [1], 6.1.41,

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi + \mathcal{O}\left(\frac{1}{|z|}\right),$$

and

$$\operatorname{Re}\big(\log\Gamma(z)\big) = -\operatorname{Im}(z)\arg(z) + \operatorname{Re}(z)\big(\log|z| - 1\big) - \frac{1}{2}\log|z| + \mathcal{O}\left(\frac{1}{|z|}\right).$$

This gives

$$\begin{aligned}
&\log\left(\frac{\Gamma(v)}{\Gamma(v+s)}\right) + \log\left(\frac{\Gamma(\theta - v - s)}{\Gamma(\theta - v)}\right)\\
&\quad = -\operatorname{Im}(v)\arg(v) + \operatorname{Im}(\theta - v)\arg(\theta - v) \qquad (55)\\
&\qquad + \operatorname{Re}(v)\big(\log|v| - 1\big) - \operatorname{Re}(\theta - v)\big(\log|\theta - v| - 1\big) \qquad (56)\\
&\qquad + \operatorname{Im}(v+s)\arg(v+s) - \operatorname{Im}(\theta - v - s)\arg(\theta - v - s) \qquad (57)\\
&\qquad - \operatorname{Re}(v+s)\big(\log|v+s| - 1\big) + \operatorname{Re}(\theta - v - s)\big(\log|\theta - v - s| - 1\big) \qquad (58)\\
&\qquad - \frac{1}{2}\log\left|\frac{v(\theta - v - s)}{(v+s)(\theta - v)}\right| + \mathcal{O}\big(|\theta|^{-1}\big) \qquad (59)
\end{aligned}$$

since $|v|, |\theta - v|, |\theta - v - s|, |v + s| \ge c\theta$ for some constant $c > 0$. Here and in the following, $a = \mathcal{O}(b)$ means that there is a constant $C$ dependent only on $\theta^*$ and $\varphi$ such that $|a| \le Cb$ for all $\theta > \theta^*$. We will estimate the numbered terms in the above display one-by-one.

We first estimate (56):

$$\begin{aligned}
&\operatorname{Re}(v)\big(\log|v| - 1\big) - \operatorname{Re}(\theta - v)\big(\log|\theta - v| - 1\big)\\
&\qquad = \frac{\theta}{2}\log\frac{|v|}{|\theta - v|} - \cot(\varphi)\big|\operatorname{Im}(v)\big|\big(\log\big(|v||\theta - v|\big) - 2\big).
\end{aligned}$$

Since the ratio inside the logarithm is $\mathcal{O}(1)$ for all $v \in \mathcal{C}_\varphi$ we have for some $\varphi$-dependent constant $c$,

$$(56) \le \mathcal{O}(\theta) - c\big|\operatorname{Im}(v)\big|\log\big(|v||\theta - v|\big). \tag{60}$$

Thus, the terms in (56) dominate the terms in (55) and this gives us the exponential decay in $v$ that we need.

Since $\operatorname{Im}(\theta - v - s) = -\operatorname{Im}(v + s)$, (57) becomes $\operatorname{Im}(v + s)(\arg(v + s) + \arg(\theta - v - s))$. From (53) and (54), we get $\theta/2 + \delta = \operatorname{Re}(v + s) \geq \operatorname{Re}(\theta - (v + s)) = \theta/2 - \delta$. It follows that $\operatorname{Im}(v + s)$ and $\arg(v + s) + \arg(\theta - v - s)$ have opposite signs, and hence

$$(57) \leq \operatorname{Im}(v+s)\bigl(\arg(v+s) + \arg(\theta - v - s)\bigr) \leq 0. \tag{61}$$

For some constant $c > 0$,

$$\begin{aligned}(58) &= -\frac{\theta}{2}\log\frac{|v+s|}{|\theta - v - s|} - \delta\log|v+s||\theta - v - s| \\ &\leq -\frac{\theta}{2}c - \delta\log\Bigl(\frac{\theta}{2} - \delta\Bigr)\Bigl(\frac{\theta}{2} + \delta\Bigr) \\ &= \mathcal{O}(\theta) + \mathcal{O}\bigl(|\log(\theta)|\bigr).\end{aligned} \tag{62}$$

The term (59) is split into two terms: the first term $-\log|v/(\theta - v)|$ is $\mathcal{O}(|\log\theta|)$ for small $v$, and $\mathcal{O}(1)$ for $|v| \geq c\theta$. The second term $-\log|(\theta - v - s)/(v + s)|$ behaves similarly, and we get

$$(59) = \mathcal{O}(1) + \mathcal{O}\bigl(|\log\theta|\bigr). \tag{63}$$

From (53) and (54), we get $|v + s - v'|^{-1} \leq \frac{2}{\theta}$. Finally, to analyze $\exp(\tau(sv + s^2/2))$, we look at the real part of $sv + s^2/2$:

$$\begin{aligned}&\operatorname{Re}\bigl(sv + s^2/2\bigr) \\ &\quad = \operatorname{Re}(s)\operatorname{Re}(v) - \operatorname{Im}(s)\operatorname{Im}(v) + \frac{\operatorname{Re}(s)^2 - \operatorname{Im}(s)^2}{2} \\ &\quad = \operatorname{Re}(s)\operatorname{Re}(v) + \frac{\operatorname{Re}(s)^2}{2} + \frac{\operatorname{Im}(v)^2}{2} - \frac{(\operatorname{Im}(s) + \operatorname{Im}(v))^2}{2} \\ &\quad = \bigl(\delta + \cot(\varphi)|\operatorname{Im}(v)|\bigr)\Bigl(\frac{\theta}{2} - \cot(\varphi)|\operatorname{Im}(v)|\Bigr) + \frac{(\delta + \cot(\varphi)|\operatorname{Im}(v)|)^2}{2} \\ &\qquad + \frac{\operatorname{Im}(v)^2}{2} - \frac{(\operatorname{Im}(s) + \operatorname{Im}(v))^2}{2} \\ &\quad = \frac{\theta\delta + \delta^2}{2} + \cot(\varphi)\frac{\theta}{2}|\operatorname{Im}(v)| + \bigl(1 - \cot(\varphi)^2\bigr)\frac{\operatorname{Im}(v)^2}{2} - \frac{(\operatorname{Im}(s) + \operatorname{Im}(v))^2}{2} \\ &\quad \leq C\theta + \cot(\varphi)\frac{\theta}{2}|\operatorname{Im}(v)|,\end{aligned} \tag{64}$$

using $\cot(\varphi) \geq 1$.

Putting (60), (61), (62), (63) and (64) together with $|\sin(\pi s)|^{-1} \le Ce^{-\pi|\operatorname{Im}(s)|}$, we get

$$I_{u,\tau}(v, v', s) \le e^{C(\theta,N,\tau,\varphi)} e^{-N|\operatorname{Im}(v)|(\log|v| - c\tau\cot(\varphi)\theta)} e^{-\pi|\operatorname{Im}(s)|}, \tag{65}$$

where

$$C(\theta, N, \tau, \varphi) = \mathcal{O}(N(\theta + |\log\theta| + \theta^{-1} + 1) + \tau\theta).$$

Integrating this over $s$ completes the proof. □

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