

# EXPONENTIAL RANDOM GRAPHS BEHAVE LIKE MIXTURES OF STOCHASTIC BLOCK MODELS

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We study the behavior of exponential random graphs in both the sparse and the dense regime. We show that exponential random graphs are approximate mixtures of graphs with independent edges whose probability matrices are critical points of an associated functional, thereby satisfying a certain matrix equation. In the dense regime, every solution to this equation is close to a block matrix, concluding that the exponential random graph behaves roughly like a mixture of stochastic block models. We also show existence and uniqueness of solutions to this equation for several families of exponential random graphs, including the case where the subgraphs are counted with positive weights and the case where all weights are small in absolute value. In particular, this generalizes some of the results in a paper by Chatterjee and Diaconis from the dense regime to the sparse regime and strengthens their bounds from the cut-metric to the one-metric.

**1. Introduction.** With the emergent realization that large networks abound in science (e.g., metabolic networks), technology (e.g., the internet) and everyday life (e.g., social networks), there has been widespread interest in probabilistic models which capture the behavior of real life networks.

The simplest random graph is the Erdős–Rényi  $G(N, p)$  model of graphs with independent edges. While this model is well understood, real networks often exhibit dependencies between the edges: For example, in a social network, if two people have many mutual friends, it is more likely that they themselves are friends.

A natural and well-studied model which captures edge dependencies is the exponential random graph model, denoted here by  $G_N^f$ . In this model, the probability to obtain a graph  $G$  on  $N$  vertices

$$(1) \quad \Pr[G_N^f = G] = \exp(f(G))/Z,$$

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where  $f$  is a real functional on graphs called the “Hamiltonian” and  $Z$  is a normalizing constant. Typically,  $f$  is a “subgraph-counting function” of the form

$$f(G) = \sum_{i=1}^{\ell} \beta_i N(H_i, G),$$

where the function  $N(H_i, G)$  counts how many times the graph  $H_i$  appears as a subgraph of  $G$ . The parameters  $\beta_i$  are called “weights,” and may be either positive or negative. For a review of exponential graphs, see the papers in [8, 9].

Despite the simple definition of this distribution, many basic aspects about its behavior are far from being well understood. For example, there is at present no known explicit formula for the normalizing constant.

One of the first rigorous papers on the topic is due to Bhamidi, Bresler and Sly [1], which analyzes the mixing of the associated Glauber dynamics in the case that subgraphs are counted with positive weights, and gives a sufficient condition on those weights (referred to as the “high temperature regime”) under which any finite collection of edges are asymptotically independent.

Another significant advance toward understanding the dense case was done in a paper of Chatterjee and Diaconis [3], based on the technology developed in [4], which uses graph limit theory. They associate the normalizing constant with a variational problem, showing that every exponential graph distribution is close to the minimizing set of some functional on the space of graphons. Further, if the Hamiltonian of this distribution counts subgraphs only positively, then under the cut-metric the exponential random graph is close to a  $G(N, p)$  graph. In [11] and [10], the graphon framework also served the investigation of a similar problem, that of computing the asymptotic structure of graphs with constrained densities of subgraphs.

More recently, in [6], it was shown that an exponential graph is close in expectation to a mixture of independent graphs. Unfortunately, this result gives no information about the structure of those independent graphs.

*Our contributions.* In this work, we take one further step towards a better understanding of exponential random graphs. We strengthen the existing results in the following three ways:

1. We characterize the structure of the independent graphs of the mixture model in [6] by showing that the elements of the mixture approximately obey a certain fixed-point equation. In particular, we show that under certain conditions, exponential random graphs behave like mixtures of so-called stochastic block models.

2. We strengthen the results of both [3] and [1] by characterizing the graph structure in terms of the one-norm. This norm induces a stronger metric than the cut-metric on the space of graphons, and gives some information about the nature of dependence between the edges and other aspects which are not captured by the cut-metric.

3. Our characterization is meaningful not only in the dense regime, but also in a limited range of sparse graphs as well. In particular, several of our results hold for an edge density  $p$  which depends polynomially on  $N$ , for example,  $p \geq N^{-c}$  for some  $c > 0$ .

The following is an overview of our main theorems. An independent graph is a random graph whose edges are independent Bernoulli random variables. Denote by  $X$  the expected adjacency matrix of such a graph. In Theorem 10, we show that for every subgraph-counting function  $f$ , the corresponding exponential graph behaves like a mixture of independent graphs whose associated expectations satisfy

$$\|X - (\mathbf{1} + \tanh(\nabla f(X)))/2\|_1 = o(N^2),$$

where  $\mathbf{1}$  is the matrix with zero on the diagonal and whose off-diagonal entries are 1, the  $\tanh$  is applied entrywise, and  $\|X\|_1 = \sum_{i,j} |X_{ij}|$  is the one-norm. Using this result, we then characterize our mixtures in three different settings:

1. Theorem 14 shows that every subgraph-counting exponential random graph is  $o(N^2)$  close to a mixture of stochastic block models with a small number of blocks.

2. Theorem 18 roughly shows that if the subgraphs are counted only with positive weights, then there exists a constant matrix  $X_c$  so that for every mixture element  $X$ ,  $\|X - X_c\|_1 = o(N^2)$ . Thus, the graph behaves like  $G(N, p)$ .

3. Theorem 19 shows that if the absolute values of the weights  $\beta$  are small enough, then there exists a constant matrix  $X_c$  so that for every mixture element  $X$ ,  $\|X - X_c\|_1 = o(N^2)$ .

**2. Background and notation.** Throughout the entire paper,  $N > 0$  is an integer that represents the number of vertices and  $n = \binom{N}{2}$  represents the number of possible edges in an  $N$  vertex simple graph. For two vertices  $v$  and  $u$  in a graph,  $v \sim u$  denotes that  $v$  is adjacent to  $u$ . We denote the discrete hypercube by  $\mathcal{C}_n = \{0, 1\}^n$  and the continuous hypercube by  $\overline{\mathcal{C}}_n = [0, 1]^n$ .

For ease of notation, we identify the vectors  $\mathcal{C}_n$  with the family of symmetric matrices of size  $N \times N$  where the diagonal entries are 0 and the above diagonal entries are 0 or 1. Such matrices correspond to simple graphs: For  $X \in \mathcal{C}_n$ , the vertex  $i$  is connected to vertex  $j$  if and only if  $X_{ij} = 1$ . We therefore also identify the vector  $X$  with the graph it represents. For two graphs  $G, G'$  whose corresponding vectors are  $X, Y$ , we use the notation  $\|G - G'\|_1$  for  $\|X - Y\|_1$ .

This view extends also to vectors  $X \in \overline{\mathcal{C}}_n$ , by identifying with  $X$  the weighted graph whose edge weights are  $(X)_{ij}$ .

Thus, any function acting on a vector  $X \in \overline{\mathcal{C}}_n$  can also be seen as a function acting on a symmetric  $N \times N$  matrix with 0 diagonal or on a weighted graph on  $N$  vertices, and vice versa.

We denote by  $\mathbf{1}$  the matrix with zero on the diagonal and whose off-diagonal entries are 1.

2.1. *Subgraph counting functions.*

DEFINITION 1 (Injective homomorphism density). Let  $G$  be a simple graph on  $N$  vertices and let  $H$  be a simple graph on  $m$  vertices. Denote by  $\mathfrak{Inj}(H, G)$  the set of injective homomorphisms from  $H$  to  $G$ , that is, the set of functions  $\phi : V(H) \rightarrow V(G)$  such that if  $x, y \in H$  and  $x \sim y$ , then  $\phi(x) \sim \phi(y)$ , and if  $\phi(x) = \phi(y)$ , then  $x = y$ . Denote the number of such homomorphisms by  $\text{inj}(H, G) = |\mathfrak{Inj}(H, G)|$ . The “*injective homomorphism density*” of  $H$  is defined as

$$t(H, G) = \frac{\text{inj}(H, G)}{N(N - 1) \cdots (N - m + 1)}.$$

DEFINITION 2 (Subgraph-counting function). Let  $\ell, N > 0$  be integers. Let  $H_1, \dots, H_\ell$  be finite simple graphs and  $\beta_1, \dots, \beta_\ell$  be real numbers. The functional  $f$  on simple graphs with  $N$  vertices defined by

$$(2) \quad f(G) = N(N - 1) \sum_{i=1}^{\ell} \beta_i t(H_i, G)$$

is called a “*subgraph-counting function*.”

As we will see below (in Section 4), the normalization  $N(N - 1)$  is natural since under this normalization, the typical values of  $f$  are of the same order as the entropy of the graph.

REMARK 3. Subgraph counting functions are sometimes defined not by injective homomorphisms but by all *general* homomorphisms, denoted by  $\text{Hom}(H, G)$ . For our purposes, however, it is more convenient to use injective homomorphisms to count subgraphs. The difference between the injective homomorphism density and the general homomorphism density is asymptotically small, so this distinction will not matter in asymptotic calculations, and our results are equally valid for general homomorphism densities. See [12], Sections 5.2.1–5.2.3, for more details on such distinctions.

Depending on both the weights and the subgraphs that are counted, when using a subgraph-counting function as the Hamiltonian of an exponential random graph, the resulting graph can be either sparse or dense. For example, suppose that for a graph  $G = (V, E)$  we define

$$f(G) = |E| \log \frac{p}{1 - p}$$

for some  $p \in (0, 1)$ . Then

$$\exp(f(G)) = \exp\left(|E| \log \frac{p}{1 - p}\right) = p^{|E|} (1 - p)^{-|E|}.$$

The normalizing constant in this case is just  $Z = (1 - p)^{\binom{N}{2}}$ , so that

$$(3) \quad \Pr[G_N^f = G] = p^{|E|}(1 - p)^{\binom{N}{2} - |E|}.$$

This is exactly the  $G(N, p)$  distribution, and if  $p \rightarrow 0$  when  $N \rightarrow \infty$  we obtain a sparse graph.

DEFINITION 4 (Discrete gradient, Lipschitz constant). Let  $f : \mathcal{C}_n \rightarrow \mathbb{R}$  be a real function on the Boolean hypercube. The discrete derivative of  $f$  at coordinate  $i$  is defined as

$$\begin{aligned} \partial_i f(Y) = & \frac{1}{2}(f(Y_1, \dots, Y_{i-1}, 1, Y_{i+1}, \dots, Y_n) \\ & - f(Y_1, \dots, Y_{i-1}, 0, Y_{i+1}, \dots, Y_n)). \end{aligned}$$

With this we define both the the discrete gradient:

$$\nabla f(Y) = (\partial_1 f(Y), \dots, \partial_n f(Y)),$$

and the Lipschitz constant of  $f$ :

$$\text{Lip}(f) = \max_{i \in [n], Y \in \mathcal{C}_n} |\partial_i f(Y)|.$$

Note that subgraph-counting functions and their gradients were originally defined on simple graphs, or, alternatively, on vectors in  $\mathcal{C}_n$ . However, they can be naturally extended to weighted graphs, or, alternatively, to vectors in  $\overline{\mathcal{C}_n}$ , in the following way.

For a simple graph  $G$ , let  $X$  be its adjacency matrix. A subgraph-counting function  $f$  that counts only a single graph  $H = ([m], E)$  has the form (this is a slight variation from [6], Lemma 33):

$$(4) \quad f(G) = \frac{\beta}{(N - 2)(N - 3) \cdots (N - m + 1)} \sum_{\substack{q \in [N]^m \\ q \text{ has distinct elements}}} \prod_{(l, l') \in E} X_{q_l, q_{l'}}.$$

Further, for an edge  $e = \{i, j\}$ , the derivative satisfies

$$(5) \quad \begin{aligned} \partial f_{ij}(G) = & \frac{\beta}{(N - 2)(N - 3) \cdots (N - m + 1)} \\ & \cdot \sum_{(a, b) \in E} \sum_{\substack{q \in [N]^m \\ q \text{ has distinct elements} \\ q_a = i, q_b = j}} \prod_{\substack{(l, l') \in E \\ \{l, l'\} \neq \{a, b\}}} X_{q_l, q_{l'}}. \end{aligned}$$

As can be seen, both  $f(G)$  and each entry of  $\nabla f(G)$  are just polynomials in the entries of  $X$ . This notation allows us to extend  $f$ 's and  $\nabla f$ 's domain to  $[0, 1]^n$ , and thus to weighted matrices and graphs. Note that since we count injective homomorphisms and the entries of the vector  $q$  in the above calculation are distinct,

the degree of each variable is either 0 or 1. Further by equation (5), for every  $x \in [0, 1]$  we have that

$$\partial_{ij} f(x\mathbf{1}) = \beta |E| x^{|E|-1}.$$

2.2. *The variational approach.* To state the results of Chatterjee and Diaconis, we briefly present some definitions from graph limit theory; for a detailed exposition, see [12], part 3. Denote by  $\mathcal{W}$  the space of all measurable functions  $w : [0, 1]^2 \rightarrow [0, 1]$ , and by  $\tilde{\mathcal{W}}$  the space of equivalence classes of  $\mathcal{W}$  under the equivalence relation  $g \sim h \iff$  there exists a measure preserving bijection  $\sigma : [0, 1] \rightarrow [0, 1]$  such that  $g(x, y) = h(\sigma(x), \sigma(y)) = (\sigma h)(x, y)$ . The space  $\tilde{\mathcal{W}}$  is called the space of *graphons*.

For every graph  $G$  on  $N$  vertices, it is possible to assign a graphon  $\tilde{G}$  by

$$\tilde{G}(x, y) = \begin{cases} 1 & [xN] \sim [yN] \text{ in } G, \\ 0 & \text{o.w.} \end{cases}$$

With this correspondence, every distribution on graphs induces a distribution on graphons by the push-forward mapping.

For any continuous bounded function  $w : [0, 1]^2 \rightarrow \mathbb{R}$ , its cut-norm is defined as

$$\|w\|_{\square} = \sup_{S, T \subseteq [0, 1]} \left| \int_{S \times T} w(x, y) dx dy \right|.$$

This defines a metric on the space of graphons by  $d_{\square}(\tilde{g}, \tilde{h}) = \inf_{\sigma} \|\sigma g - h\|_{\square}$ , where the infimum is taken over all measure preserving bijections  $\sigma$  as above.

The results of Chatterjee and Diaconis can now be framed as follows.

**THEOREM 5** (Theorem 3.2 in [3]). *Let  $f : \tilde{\mathcal{W}} \rightarrow \mathbb{R}$  be a continuous bounded functional. Denote by  $G_N^f$  the exponential random graph whose Hamiltonian is  $f(\tilde{G})$ . Then there exists a bounded continuous functional  $\varphi_f : \tilde{\mathcal{W}} \rightarrow \mathbb{R}$  which depends on  $f$  with the following property. Denote by  $\tilde{F}^*$  the set of graphons maximizing  $\varphi_f$ . Then for any  $\eta > 0$  there exist  $C, \gamma > 0$  such that*

$$\Pr[d_{\square}(\tilde{G}_N^f, \tilde{F}^*) > \eta] \leq C e^{-N^2 \gamma}.$$

As a corollary, they show the following result for subgraph counting functions.

**THEOREM 6** (Theorem 4.2 in [3]). *Assume that  $H_1 = K_2$  is the complete graph on two vertices and that  $\beta_2, \dots, \beta_\ell$  are all nonnegative. Then the set of maximizers of  $\varphi_f$  consists of a finite set of constant graphons. Further,*

$$\min_{\tilde{u} \in \tilde{F}^*} d_{\square}(\tilde{G}_N^f, \tilde{u}) \rightarrow 0 \quad \text{in probability as } N \rightarrow \infty.$$

In other words, the exponential random graph  $G_N^f$  is close in the cut-distance to a distribution of Erdős–Rényi graphs  $G(N, p)$  where  $p$  is picked randomly from some probability distribution.

In a later paper, Chatterjee and Dembo [2] derived a variational framework which yields nontrivial estimates in the sparse regime. However, that framework does not seem to give strong enough bounds on the partition function in order to characterize the associated distribution.

2.3. *Mixture models.* In this paper, we are interested in approximating exponential random graphs by mixtures of independent graphs. The following definitions will be central to our results.

DEFINITION 7 ( $\rho$ -mixtures). For  $\vec{p} \in [0, 1]^{\binom{N}{2}}$ , denote by  $G(N, \vec{p})$  the random graph with independent edges such that the edge  $i \sim j$  appears with probability  $\vec{p}_{ij}$ . Let  $\rho$  be a measure on  $[0, 1]^{\binom{N}{2}}$ . We define the random vector  $G(N, \rho)$  by

$$\Pr[G(N, \rho) = G] = \int \Pr[G(N, \vec{p}) = G] d\rho(\vec{p}).$$

We say that  $G(N, \rho)$  is a  $\rho$ -mixture.

DEFINITION 8 (Approximate mixture decomposition). Let  $\delta > 0$  and let  $\rho$  be a measure on  $[0, 1]^{\binom{N}{2}}$ . A random graph  $G$  is called a  $(\rho, \delta)$ -mixture if there exists a coupling between  $G(N, \rho)$  and  $G$  such that

$$\mathbb{E}\|G(N, \rho) - G\|_1 \leq \delta n.$$

A complementary result, given in [6] roughly states that an exponential random graph  $G$  is close to a  $(\rho, o(1))$ -mixture in a way that most of the entropy comes from the individual  $G(N, \vec{p})$ 's rather than from the mixture.

For a random variable  $X$  with law  $\nu$ , we define the entropy of  $X$  as

$$\text{Ent}(X) = \int -\log(\nu(x)) d\nu.$$

THEOREM 9 (Theorem 9 in [6]). For any positive integers  $N, \ell$ , finite simple graphs  $H_1, \dots, H_\ell$ , real numbers  $\beta_1, \dots, \beta_\ell$  and  $\varepsilon \in (0, 1/2)$ , the exponential graph defined in 1, is a  $(\rho, \delta)$ -mixture, and such that

$$\delta \leq \frac{34n^{-1/12}}{\varepsilon^{1/3}} \left( \sum_{i=1}^{\ell} |\beta_i| |E(H_i)| \right)^{1/3}$$

with

$$\text{Ent}(G(N, \rho)) \leq \int \text{Ent}(G(N, \vec{p})) d\rho(\vec{p}) + \varepsilon \binom{N}{2}.$$

**3. Results.** The results of this paper are based on the following technical statement which is an application of the framework in [7]. This result gives a characterization of the measure  $\rho$  described above: With high probability with respect to  $\rho$ , the vector  $\vec{p}$  is nearly a critical point of a certain functional associated with  $f$ . In order to formulate this result, let us make some notation.

For every subgraph-counting function  $f$  of the form (2), define the constant

$$C_\beta = \max \left\{ 12 \sum_{i=1}^{\ell} |\beta_i| |E(H_i)|^2, 2 \right\}.$$

Remark that  $C_\beta$  depends only on the graph counting parameters, barring  $N$ . Denote by  $\mathcal{X}_f$  the set

$$(6) \quad \mathcal{X}_f = \left\{ X \in [0, 1]^n : \|X - (\mathbf{1} + \tanh(\nabla f(X)))/2\|_1 \leq 5000 C_\beta^2 n^{15/16} \right\},$$

with the tanh applied entrywise to the entries of  $\nabla f(X)$ .

**THEOREM 10** (Product decomposition of exponential random graphs). *Let  $f$  be a subgraph counting function. There exists a measure  $\rho$  on  $[0, 1]^n$  (which depends on  $n$  and on  $f$ ) such that  $G_n^f$  is a  $(\rho, 80 \frac{C_\beta}{n^{1/16}})$ -mixture with*

$$\rho(\mathcal{X}_f) \geq 1 - 80 \frac{C_\beta}{n^{1/16}}.$$

In other words, almost all the mass of the mixture resides on random graphs whose adjacency matrices  $X$  almost satisfy the fixed-point equation

$$(7) \quad X = \frac{\mathbf{1} + \tanh(\nabla f(X))}{2}.$$

**REMARK 11.** In fact, more is known about the structure of the measure  $\rho$ . Following the notation in [7], for a vector  $\theta \in \mathbb{R}^n$ , the tilt  $\tau_\theta \nu$  of a distribution  $\nu$  is defined by

$$\frac{d(\tau_\theta \nu)}{d\nu}(y) = \frac{e^{\langle \theta, y \rangle}}{\int_{\mathcal{C}_n} e^{\langle \theta, z \rangle} d\nu}.$$

As it turns out, the measure  $\rho$  in Theorem 10 is composed of small tilts, that is, there exists a measure  $m$  on  $\mathbb{R}^n$  supported on small vectors  $\theta$  such that  $\rho$  is the push-forward of  $m$  under the map  $\theta \mapsto \mathbb{E}_{X \sim \tau_\theta \nu} [X]$ . For more details, see [7].

**REMARK 12.** One can check that the solutions of the fixed-point equation are exactly the critical points of the functional  $f(X) + H(X)$  where  $H(X) = \sum_{i < j} X_{ij} \log X_{ij} + (1 - X_{ij}) \log(1 - X_{ij})$  is the entropy of  $X$ . This is a variant of the functional that arises in the variational problem in [3].

As described in [7], the solutions to the equation  $X = (\mathbf{1} + \tanh(\nabla f(X)))/2$  are critical points of a certain functional. Comparing our result to Theorem 3.2 in [3]: The latter shows that the exponential graphs are close to global *maxima* of the variational problem, while the former only shows that it is close to critical points; however, it gives a stronger, distributional description and works beyond the dense regime.

Our first main result shows that in the dense regime, the matrices obtained by Theorem 10 are close to matrices that can be decomposed into a small number of blocks, defined as follows.

**DEFINITION 13 (Stochastic block model).** Let  $N, k > 0$  be positive integers. A symmetric matrix  $X \in \mathbb{R}^{N \times N}$  is called a “*block matrix*” with  $k$  communities, if there exists a symmetric matrix  $P \in \mathbb{R}^{k \times k}$  and a partition of the indices  $1, \dots, N$  into  $k$  disjoint sets  $V_1, \dots, V_k$  such that for  $i \in V_{\ell_1}$  and  $j \in V_{\ell_2}$  with  $\ell_1, \ell_2 \in [k]$ ,

$$X_{ij} = P_{\ell_1, \ell_2}.$$

The sets  $V_1, \dots, V_k$  are called the “*communities*” of  $X$ . A random graph with independent edges whose expected adjacency matrix is a block matrix is called a “*stochastic block model*.”

**THEOREM 14 (Small number of communities for counting functions).** Let  $0 < \delta < 1$  and let  $f$  be a subgraph-counting function. Then there exists a constant  $C_\delta > 0$  (which depends on  $\delta$ , the subgraphs  $H_i$ , and their weights  $\beta_i$  but is otherwise independent of  $N$ ) such that for any  $X \in \mathcal{X}_f$ , there exists a block matrix  $X^*$  with no more than  $C_\delta$  communities such that

$$\|X - X^*\|_1 \leq \delta n + 5000C_\delta^2 n^{15/16}.$$

One can derive an explicit expression for the constant  $C_\delta$ , which is in general exponential in  $1/\delta^2$ . The explicit dependence in the case of triangle-counting functions is derived in the proof.

Theorems 10 and 14 combined give the following corollary.

**COROLLARY 15.** For any finite set of graphs  $H_1, \dots, H_\ell$ , constants  $\beta_1, \dots, \beta_\ell$  and any constant  $\delta > 0$  there exists a constant  $C_\delta$  such that the following holds. For every  $N$ , there exists a measure  $\rho$  supported on block matrices with at most  $C_\delta$  communities such that if  $G_N^f$  is the exponential random graph with the Hamiltonian  $f(g) = N(N - 1) \sum_{i=1}^\ell \beta_i t(H_i, g)$  then there is a coupling between  $G_N^f$  and  $G(N, \rho)$  which satisfies

$$\mathbb{E} \|G_N^f - G(n, \rho)\|_1 \leq \delta \binom{N}{2}.$$

We conjecture that Theorem 14 can be strengthened as follows.

CONJECTURE 16. *Let  $f$  be a subgraph-counting function. Then there is a constant  $c$  independent of  $N$  (but dependent on the weights  $\beta_i$ ) such that every  $X \in \mathcal{X}_f$  is  $o(n)$ -close to a block matrix with no more than  $c$  communities.*

Our second main result regarding the characterization of exponential graphs applies to subgraph-counting functions with positive weights. Its statement remains nontrivial for graphs with polynomially small density, for some range of exponents, as will be demonstrated in Example 23.

Following the notation of [1], we define  $\varphi_\beta : [0, 1] \rightarrow \mathbb{R}$  by

$$\varphi_\beta(x) = \frac{1 + \tanh(\sum_{i=1}^\ell \beta_i |E(H_i)|x^{|E(H_i)|-1})}{2}.$$

Note that  $\varphi_\beta(x)$  is equal to any off-diagonal entry of the constant matrix  $(\mathbf{1} + \tanh(\nabla f(x\mathbf{1}))) / 2$ . If the equation  $x = \varphi_\beta(x)$  has a unique fixed-point  $x_0$ , define the constant  $D_\beta = \sup_{\substack{x \in [0,1] \\ x \neq x_0}} \frac{|\varphi_\beta(x) - x_0|}{|x - x_0|}$ .

The following simple lemma gives a useful bound on  $D_\beta$ ; we present it without proof.

LEMMA 17. 1. *There exists an  $x_0 \in [0, 1]$  such that  $x_0 = \varphi_\beta(x_0)$ . Hence there always exists a constant solution  $X_c = x_0\mathbf{1}$  to the fixed-point equation (7).*

2. *Assume that  $\varphi_\beta(x)$  is increasing. If the solution  $x_0$  is unique and  $\varphi'_\beta(x_0) < 1$ , then  $D_\beta < 1$ .*

The condition in item (2) in the above lemma is referred to in [1] as the *high temperature regime*.

THEOREM 18 (Positive weights). *Let  $N > 3$  be an integer. Let  $H_1, \dots, H_\ell$  be graphs, let  $\alpha \in \mathbb{R}$  and  $\beta_1, \dots, \beta_\ell \in \mathbb{R}$  be real numbers and let  $f$  be a subgraph-counting function*

$$f(X) = \alpha \text{inj}(K_2, X) + N(N - 1) \sum_{i=1}^\ell \beta_i t(H_i, X),$$

where  $K_2$  is the complete graph on 2 vertices. Assume that  $\beta_i \geq 0$  are positive for all  $i$ , that the equation  $x = \varphi_\beta(x)$  has a unique solution  $x_0$  and that  $D_\beta < 1$ . Then for any  $X \in \mathcal{X}_f$  and any  $0 < \lambda < 1$ ,

$$(8) \quad \|X - x_0\mathbf{1}\|_1 \leq \lambda n + 10,000 C_\beta^3 \lambda^{\frac{\log C_\beta}{\log D_\beta}} n^{15/16}.$$

In particular, for any constants  $C_\beta$  and  $D_\beta$ , there exists constants  $0 < \gamma < 1/16$  and  $Q > 0$  such that

$$(9) \quad \|X - x_0\mathbf{1}\|_1 \leq Q \cdot n^{1-\gamma}.$$

Our third main result regarding the characterization of exponential graphs applies to subgraph-counting functions whose weights are small in absolute value: If all  $\beta$ 's are small enough, the only solution to equation (7) is the trivial one.

**THEOREM 19 (Small weights).** *Let  $N > 3$  be an integer. Let  $H_1, \dots, H_\ell$  be graphs, let  $\alpha \in \mathbb{R}$  and  $\beta_1, \dots, \beta_\ell \in \mathbb{R}$  be real numbers and let  $f$  be a subgraph-counting function*

$$f(X) = \alpha \operatorname{inj}(K_2, X) + N(N - 1) \sum_{i=1}^{\ell} \beta_i t(H_i, X),$$

where  $K_2$  is the complete graph on 2 vertices. Denote  $m_i = |E(H_i)|$  and define the sum

$$S_\beta = \sum_{i=1}^{\ell} |\beta_i| \binom{m_i}{2}.$$

If  $S_\beta < 1$ , then the constant solution  $X_c$  obtained from item (1) in Lemma 17 is the only solution to the fixed-point equation (7). Further, any  $X \in \mathcal{X}_f$  satisfies

$$\|X - X_c\|_1 \leq \frac{5000C_\beta^2}{1 - S_\beta} n^{15/16}.$$

**REMARK 20.** One should compare Theorem 18 and Theorem 19 to Theorems 4.2 and 6.2 in [3], respectively. There, similar conditions (positive  $\beta$ 's or  $S_\beta < 1$ ) imply that the exponential random graph is close in the cut metric to a finite set of constant graphons.

Finally, for the particular case of triangle-counts, it turns out that if  $\beta < 0$  is smaller than some universal constant, there exists at least one nontrivial solution in the form of two blocks.

**THEOREM 21 (Two block model).** *Let  $N > 3$  be an integer, let  $\beta \in \mathbb{R}$ , and let  $f(X) = \frac{\beta}{N-2} \operatorname{inj}(K_3, X)$ , where  $K_3$  is the triangle graph. There exists a  $\beta_0 < 0$  such that if  $\beta < \beta_0$ , there is a solution to equation (7) in the form of a block model with 2 communities. Specifically, the  $N$  vertices can be divided into two sets of equal size  $U$  and  $W$ , such that  $X_{ij} = c_1$  if  $(i, j) \in (U \times W) \cup (W \times U)$ , and  $X_{ij} = c_2$  if  $(i, j) \in (U \times U) \cup (W \times W)$  for  $i \neq j$ . Further, as  $\beta \rightarrow -\infty$ ,  $c_1 \rightarrow \frac{1}{2}$  and  $c_2 \rightarrow 0$ .*

**REMARK 22 (A remark on bounds and sparsity).** When considering subgraph counting functions, it is useful to think of the special case that the  $\beta_i$ 's are constants independent of  $N$ . In this case, the typical exponential graph will be dense, and inequalities involving the one-norm of matrices will yield meaningful information.

However, letting the  $\beta_i$ 's depend explicitly on  $N$  can lead to sparse graphs. The sparse case is typically harder analyze than the dense case, although there are some exact results in this regime (see, e.g., [15] where the partition function and two-edge correlations are derived for certain families of  $\beta$ 's).

Our theorems still hold true in the sparse regime, but for graphs which are too sparse they may only be trivially true. Consider Theorem 18 as an example. If the weights  $\beta_i$  are such that the expected number of edges in the exponential graph is smaller than the error term  $\inf_{\lambda \in (0,1)} \lambda n + 1000 C_\beta^3 \lambda^{\frac{\log C_\beta}{\log D_\beta}} n^{15/16}$ , then the weight matrix is trivially close to a constant matrix: namely, the zero matrix. In this case, the theorem tells us nothing new. The next example demonstrates that this is not always the case, and our results can give meaningful information in the sparse regime.

EXAMPLE 23. In this informal example, we give a sketch for the case of triangle counts. Let  $f$  be the function

$$f(X) = \alpha \operatorname{inj}(K_2, X) + \frac{\beta}{N-2} \operatorname{inj}(K_3, X),$$

where  $\alpha = \frac{1}{2} \log \frac{p}{1-p}$  and  $\frac{1}{200} |\alpha| \leq \beta \leq \frac{1}{100} |\alpha|$ . We will take  $p = p(N) = n^{-c}$  for some  $c > 0$ . This implies that  $\alpha \propto -\log N$  and  $\beta \propto \log N$ ; thus  $\alpha \rightarrow -\infty$  and  $\beta \rightarrow \infty$  as  $N \rightarrow \infty$ . We expect the typical number of edges in the resulting exponential graph to be  $\Omega(np)$ .

It can be verified that for large enough  $N$ , there is only a single solution to the equation  $x = \varphi_\beta(x)$ ; denote it by  $x_0$ . Our first task is to calculate  $D_\beta$ . By its definition, it is always smaller than the maximum of the derivative of  $\varphi_\beta(x) = \frac{1+\tanh(\alpha+3\beta x^2)}{2}$ . Thus, neglecting constants,

$$\begin{aligned} D_\beta &\leq \max_{x \in [0,1]} |\varphi'(x)| = \max_{x \in [0,1]} \frac{3\beta x^2}{\cosh^2(\alpha + 3\beta x^2)} \\ &\leq \frac{3\beta}{\cosh^2(\frac{1}{2}\alpha)} \\ &\lesssim |\alpha| e^\alpha. \end{aligned}$$

Hence for all  $N$  large enough, we have  $D_\beta < 1$ , and can apply Theorem 18: For any  $X \in \mathcal{X}_f$ , we have

$$\|X - x_0 \mathbf{1}\|_1 \leq \lambda n + 10,000 C_\beta^3 \lambda^{\frac{\log C_\beta}{\log D_\beta}} n^{15/16}.$$

Now, since  $D_\beta \leq |\alpha| e^\alpha < 1$ , we have that  $|\log D_\beta| \gtrsim |\alpha|$ , while  $C_\beta \approx |\alpha|$ , so  $\log C_\beta \approx \log |\alpha|$ ; this gives

$$\left| \frac{\log C_\beta}{\log D_\beta} \right| \lesssim \frac{\log |\log p|}{|\log p|} \approx \frac{\log \log n}{c \log n}.$$

Set  $\lambda = n^{-1} = e^{-\log n}$ . Then

$$\lambda \frac{\log C_\beta}{\log D} \approx e^{-\log n \frac{\log \log n}{-c \log n}} = e^{c' \log \log n} = (\log n)^{c'}$$

Since we want the error term to be smaller than the number of edges, then ignoring all logarithmic terms (i.e., those coming from  $\lambda$  and  $C_\beta$ ), we require the following inequality to hold:

$$np \gtrsim n^{15/16}$$

This indeed allows a polynomial dependence between  $p$  and  $n$ . For any  $p$  satisfying

$$p \gtrsim n^{-1/16},$$

we conclude that there exists a constant  $p'$  and a coupling between  $G(n, p')$  and  $G_N^f$  such that

$$\mathbb{E} \|G(n, p') - G_N^f\|_1 = o(np).$$

3.1. *Open questions and further directions.*

- Theorems 18 and 19 show that in some cases, the random graphs in the mixture are close to an actual fixed point of equation (7). It is natural to ask whether this is a general phenomenon. Let  $X \in \mathcal{X}_f$  and denote by  $S = \{Y : Y = (\mathbf{1} + \tanh(\nabla f(Y)))/2\}$  the set of solutions to the fixed-point equation (7). Is it true that

$$\inf_{Y \in S} \|X - Y\|_1 = o(n)?$$

In other words, is it true that approximately-fixed points are approximately fixed-points?

- How quickly can the the parameter  $\delta$  in Theorem 14 approach 0 while still keeping a meaningful bound? Can the theorem be improved to obtain a polynomial dependence on  $N$ ?
- Can Theorem 14 be formulated in a meaningful way for sparse exponential random graphs?
- Lubetzky and Zhao proposed in [13] a variant of subgraph-counting functions where the Hamiltonian is of the form

$$f(G) = N(N - 1) \left( \sum_{i=1}^{\ell} \beta_i t(H_i, G)^{\alpha_i} \right)$$

for some  $\alpha_1, \dots, \alpha_\ell > 0$ . Theorem 14 in [7] implies that this modified Hamiltonian also breaks up into a mixture of product measures. What are the components of this mixture? Is there a criterion on the exponents  $\alpha_i$  that enables/ensures symmetry-breaking?

- The fixed-point equation  $X = (\mathbf{1} + \tanh(\nabla f(X)))/2$  corresponds to the critical points of a variational problem. Classify these critical points; is it true that they are all maxima? If not, how does the mass of  $\rho$  distribute among the different types? In particular, is the mass always distributed on *global* maxima?
- Show that for the case of triangle counts, every solution to the exact fixed point equation  $X = (\mathbf{1} + \tanh(\frac{\beta}{N-2} X^2))/2$  is close to a stochastic block model with two communities. In other words, show an “only if” condition for Theorem 21.

*Organization.* The rest of this paper is organized as follows. The proof of Theorem 10 is given in Section 4. In Section 5, we prove the block model Theorem 14; we first show the proof for triangle-counting functions, and then generalize it to arbitrary counting functions. Finally, Section 6, Section 7 and Section 8 are devoted to proving the existence and uniqueness of solutions of the fixed-point equation in some special cases, as described in Theorems 18, 19 and 21.

**4. Proof of the mixture decomposition.** The proof of Theorem 10 will follow as a corollary from the main result of [7]. In order to formulate this result, we need the following definition.

DEFINITION 24 (Gaussian width, gradient complexity). The *Gaussian-width* of a set  $K \subseteq \mathbb{R}^n$  is defined as

$$\mathbf{GW}(K) = \mathbb{E} \left[ \sup_{X \in K} \langle X, \Gamma \rangle \right],$$

where  $\Gamma \sim N(0, \text{Id})$  is a standard Gaussian vector in  $\mathbb{R}^n$ . For a function  $f : \mathcal{C}_n \rightarrow \mathbb{R}$ , the *gradient complexity* of  $f$  is defined as

$$\mathcal{D}(f) = \mathbf{GW}(\{\nabla f(Y) : Y \in \mathcal{C}_n\} \cup \{0\}).$$

The main result of [7] reads the following.

THEOREM 25 (Theorem 9 in [7]). Let  $n > 0$ , let  $f : \mathcal{C}_n \rightarrow \mathbb{R}$ , and let  $X_n^f$  be a random vector given by the law

$$\Pr[X_n^f = X] = \exp(f(X))/Z,$$

where  $Z$  is a normalizing constant. Denote

$$D = \mathcal{D}(f),$$

$$L_1 = \max\{1, \text{Lip}(f)\},$$

$$L_2 = \max \left\{ 1, \max_{X \neq Y \in \mathcal{C}_n} \frac{\|\nabla f(X) - \nabla f(Y)\|_1}{\|X - Y\|_1} \right\}.$$

Denote by  $\mathcal{X}_f$  the set

$$\mathcal{X}_f = \left\{ X \in \overline{\mathcal{C}}_n : \left\| X - \frac{\mathbf{1} + \tanh(\nabla f(X))}{2} \right\|_1 \leq 5000L_1L_2^{3/4}D^{1/4}n^{3/4} \right\},$$

where  $\mathbf{1}$  is the  $N \times N$  matrix with zero on the diagonal and whose off-diagonal entries are 1,  $\nabla f(X)$  is extrapolated to  $\overline{\mathcal{C}}_n$  by equation (5) and with the tanh applied entrywise to the entries of  $\nabla f(X)$ . Then  $X_n^f$  is a  $(\rho, 80\frac{D^{1/4}}{n^{1/4}})$ -mixture such that

$$\rho(\mathcal{X}_f) \geq 1 - 80\frac{D^{1/4}}{n^{1/4}}.$$

We will prove Theorem 10 by applying the above theorem; this requires giving bounds on  $\mathcal{D}(f)$ ,  $\text{Lip}(f)$  and  $\max \frac{\|\nabla f(x) - \nabla f(y)\|_1}{\|x - y\|_1}$ . We bound the latter two quantities in the following three lemmas.

For a vector  $X \in \mathcal{C}_n$ , denote by  $X_j^+$  the vector  $X_j^+ = (X_1, X_2, \dots, X_{j-1}, 1, X_{j+1}, \dots, X_n)$ , and by  $X_j^-$  the vector  $X_j^- = (X_1, X_2, \dots, X_{j-1}, 0, X_{j+1}, \dots, X_n)$ . In terms of graphs,  $X_j^+$  is the graph  $X$  with the edge at index  $j$  added (if it is not already there), while  $X_j^-$  is the graph  $X$  with the edge at index  $j$  removed.

The first lemma states that such subgraph-counting functions have bounded Lipschitz constants.

LEMMA 26. *Let  $f$  be a subgraph-counting function of the form (2). Then for every  $X \in \mathcal{C}_n$  and for every index  $j$ ,  $|\partial_j f(X)| \leq \sum_{i=1}^\ell |\beta_i| |E(H_i)|$ . In other words,  $f$  is  $\sum_{i=1}^\ell |\beta_i| |E(H_i)|$ -Lipschitz.*

PROOF. By definition, for any graph  $H$ ,

$$\partial_j \text{inj}(H, X) = \frac{\text{inj}(H, X_j^+) - \text{inj}(H, X_j^-)}{2}.$$

The graphs  $X_j^+$  and  $X_j^-$  differ by only one edge, which we call  $e$ . Now look at  $\text{inj}(H, X_j^+) - \text{inj}(H, X_j^-)$ . All homomorphisms which do not send at least one edge of  $H$  into the edge  $e$  cancel out in this sum. Hence it is equal to

$$\#\{\phi \in \mathfrak{I}\text{nj}(H, X_j^+) : e \in E(\phi(H))\}.$$

To bound the number of such homomorphisms, we construct them as follows: first map one of the edges of  $H$  to the edge  $e$ , and then injectively map the remaining vertices of  $H$  to vertices of  $G$ . There are  $2|E(H)|$  ways to do the former and  $(N - 2)(N - 3) \cdots (N - m + 1)$  ways to do the latter, so overall:

$$\begin{aligned} (10) \quad \partial_j \text{inj}(H, X) &= \frac{\text{inj}(H, X_j^+) - \text{inj}(H, X_j^-)}{2} \\ &\leq |E(H)|(N - 2)(N - 3) \cdots (N - m + 1). \end{aligned}$$

This means that

$$\begin{aligned}
 |\partial_j f(X)| &= \left| \partial_i N(N-1) \sum_{i=1}^{\ell} \frac{\beta_i \operatorname{inj}(H_i, X)}{N(N-1) \cdots (N-m+1)} \right|, \\
 (\text{triangle ineq.}) &\leq \sum_{i=1}^{\ell} |\beta_i| \left| \partial_i \frac{\operatorname{inj}(H_i, X)}{(N-2) \cdots (N-m+1)} \right|, \\
 [\text{by (10)}] &\leq \sum_{i=1}^{\ell} |\beta_i| |E(H_i)|
 \end{aligned}$$

as needed.  $\square$

The second lemma tells us that that if  $X$  and  $Y$  differ by only one index, then  $\nabla f(X)$  and  $\nabla f(Y)$  are close to each other.

LEMMA 27. *Let  $f$  be a subgraph-counting function. Let  $X, Y \in \mathcal{C}_n$  be two vectors that differ only in a single index  $k$ . Let  $j$  be an index,  $e_j$  be the edge that corresponds to index  $j$ , and  $e_k$  be the edge that corresponds to index  $k$ . If  $e_j$  and  $e_k$  share a common vertex, then*

$$|\partial_j f(X) - \partial_j f(Y)| \leq \sum_{i=1}^{\ell} \frac{2|\beta_i| |E(H_i)|^2}{\sqrt{n}}.$$

If  $e_j$  and  $e_k$  do not share a common vertex, then

$$|\partial_j f(X) - \partial_j f(Y)| \leq \sum_{i=1}^{\ell} \frac{6|\beta_i| |E(H_i)|^2}{n}.$$

PROOF. Assume without loss of generality that  $X_k = 1$  while  $Y_k = 0$ . This means that  $X$  contains the edge  $e_k$  while  $Y$  does not. Then for every graph  $H$ ,

$$\begin{aligned}
 &\partial_j \operatorname{inj}(H, X) - \partial_j \operatorname{inj}(H, Y) \\
 &= \frac{\operatorname{inj}(H, X_j^+) - \operatorname{inj}(H, X_j^-)}{2} - \frac{\operatorname{inj}(H, Y_j^+) - \operatorname{inj}(H, Y_j^-)}{2}.
 \end{aligned}$$

We can assume that  $j \neq k$ : If they were equal, then  $X_j^+$  and  $X_j^-$  would be equal to  $Y_j^+$  and  $Y_j^-$ , respectively, and the difference  $\partial_j \operatorname{inj}(H, X) - \partial_j \operatorname{inj}(H, Y)$  would just be 0.

Similar to the proof of Lemma 26, the first term  $\operatorname{inj}(H, X_j^+) - \operatorname{inj}(H, X_j^-)$  counts the number of homomorphisms from  $H$  to  $X$  that map an edge of  $H$  into the edge  $e_j$ , while the second term  $\operatorname{inj}(H, Y_j^+) - \operatorname{inj}(H, Y_j^-)$  counts the number of homomorphisms from  $H$  to  $Y$  that map an edge of  $H$  into the edge  $e_j$ . However,

the homomorphisms in the first term may map edges from  $H$  into the edge  $e_k$ , while those of the second term may not, since  $e_k$  does not exist in  $Y$ . Thus, their difference is equal to

$$\begin{aligned} & \partial_j \operatorname{inj}(H, X) - \partial_j \operatorname{inj}(H, Y) \\ &= \frac{\#\{\phi \in \mathfrak{I}nj(H, X_j^+) : \{e_j, e_k\} \subseteq E(\phi(H))\}}{2}. \end{aligned}$$

To bound the number of such homomorphisms, we construct them as follows: first map two of the edges of  $H$  to the edges  $e_j$  and  $e_k$ , and then injectively map the remaining vertices of  $H$  to vertices of  $G$ . There are less than  $(2|E(H)|)^2$  ways to do the former. For the latter, it depends on whether  $e_j$  and  $e_k$  have a vertex in common. If they do not, then the edges in  $H$  mapping to  $e_j$  and  $e_k$  must also be disjoint, and mapping them involves choosing 4 vertices to map to the vertices of  $e_j$  and  $e_k$ . This gives  $(N - 4) \cdots (N - m + 1)$  ways to map the remaining vertices of  $H$ . If  $e_j$  and  $e_k$  do have a vertex in common, then it is possible to map the corresponding edges of  $H$  by mapping only 3 vertices to the vertices of  $e_j$  and  $e_k$ . This gives  $(N - 3) \cdots (N - m + 1)$  ways to map the remaining vertices of  $H$ .

So overall, we get that

$$\begin{aligned} e_j \cap e_k = \emptyset & \\ \implies \partial_j \operatorname{inj}(H, X) - \partial_j \operatorname{inj}(H, Y) &\leq 2|E(H)|^2(N - 4) \cdots (N - m + 1), \\ e_j \cap e_k \neq \emptyset & \\ \implies \partial_j \operatorname{inj}(H, X) - \partial_j \operatorname{inj}(H, Y) &\leq 2|E(H)|^2(N - 3) \cdots (N - m + 1). \end{aligned}$$

This means that for  $e_j \cap e_k = \emptyset$ , we get

$$\begin{aligned} |\partial_j f(X) - \partial_j f(Y)| &= \left| \partial_j N(N - 1) \sum_{i=1}^{\ell} \beta_i \frac{\operatorname{inj}(H_i, X)}{N \cdots (N - m + 1)} \right. \\ &\quad \left. - \partial_j N(N - 1) \sum_{i=1}^{\ell} \beta_i \frac{\operatorname{inj}(H_i, Y)}{N \cdots (N - m + 1)} \right| \\ \text{(triangle ineq.)} &\leq \sum_{i=1}^{\ell} \frac{|\beta_i|}{(N - 2) \cdots (N - m + 1)} |\partial_j \operatorname{inj}(H_i, X) - \partial_j \operatorname{inj}(H_i, Y)| \\ &\leq \sum_{i=1}^{\ell} \frac{|\beta_i|}{(N - 2) \cdots (N - m + 1)} \\ &\quad \cdot (2|E(H_i)|^2(N - 4) \cdots (N - m + 1)) \\ &= \sum_{i=1}^{\ell} \frac{2|\beta_i||E(H_i)|^2}{(N - 2)(N - 3)} \leq \sum_{i=1}^{\ell} \frac{6|\beta_i||E(H_i)|^2}{n}, \end{aligned}$$

while for  $e_j \cap e_k \neq \emptyset$ , we get

$$\begin{aligned} & |\partial_j f(X) - \partial_j f(Y)| \\ & \leq \sum_{i=1}^{\ell} \frac{|\beta_i|}{(N-2) \cdots (N-m+1)} (2|E(H_i)|^2(N-3) \cdots (N-m+1)) \\ & = \sum_{i=1}^{\ell} \frac{2|\beta_i||E(H_i)|^2}{N-2} \leq \sum_{i=1}^{\ell} \frac{2|\beta_i||E(H_i)|^2}{\sqrt{n}} \end{aligned}$$

as needed.  $\square$

This result can be generalized to arbitrary  $X, Y$ , giving us a bound for the one-norm  $\|\nabla f(X) - \nabla f(Y)\|_1$ .

LEMMA 28. *Let  $f$  be a subgraph-counting function. Let  $X, Y \in \overline{C}_n$  be two vectors. Then*

$$\|\nabla f(X) - \nabla f(Y)\|_1 \leq C \|X - Y\|_1,$$

where  $C = 12 \sum_{i=1}^{\ell} |\beta_i||E(H_i)|^2$ .

PROOF. First, assume that  $X$  and  $Y$  differ only in single coordinate  $k$ . Then for each coordinate  $j$ , either the edge  $e_j$  intersects with  $e_k$  or not. Holding all other coordinates fixed,  $\nabla f$  is linear as a function of the  $k$ th coordinate. Then using Lemma 27, we can write

$$\begin{aligned} & \|\nabla f(X) - \nabla f(Y)\|_1 \\ & = \sum_{j=1}^n |\partial_j f(X) - \partial_j f(Y)| |X_k - Y_k| \\ & \leq \sum_{i=1}^{\ell} \sum_{j=1}^n \left( \mathbf{1}_{e_j \cap e_k = \emptyset} \frac{6}{n} + \mathbf{1}_{e_j \cap e_k \neq \emptyset} \frac{2}{\sqrt{n}} \right) |\beta_i||E(H_i)|^2 |X_k - Y_k|. \end{aligned}$$

The edge  $e_k$  can intersect at most  $N$  different edges at each of its endpoints, so the number of indices  $j$  for which  $e_j \cap e_k \neq \emptyset$  is bounded by  $2N \leq 2\sqrt{2n}$ . The number of indices  $j$  for which  $e_j \cap e_k = \emptyset$  is trivially bounded by  $n$ , giving

$$\begin{aligned} & \|\nabla f(X) - \nabla f(Y)\|_1 \\ & \leq \sum_{i=1}^{\ell} \sum_{j=1}^n \left( \mathbf{1}_{e_j \cap e_k = \emptyset} \frac{6}{n} + \mathbf{1}_{e_j \cap e_k \neq \emptyset} \frac{2}{\sqrt{n}} \right) |\beta_i||E(H_i)|^2 |X_k - Y_k| \end{aligned}$$

$$\begin{aligned} &\leq \sum_{i=1}^{\ell} \left( 6 \frac{n}{n} + \frac{2 \cdot 2\sqrt{2}\sqrt{n}}{\sqrt{n}} \right) |\beta_i| |E(H_i)|^2 |X_k - Y_k| \\ &\leq 12 \sum_{i=1}^{\ell} |\beta_i| |E(H_i)|^2 |X_k - Y_k|. \end{aligned}$$

The above reasoning is valid for  $X$  and  $Y$  which differ by one coordinate; by the triangle inequality we achieve the desired result for arbitrary  $X, Y \in \overline{C}_n$ . □

**PROOF OF THEOREM 10.** By [6], Section 5, the Gaussian-width of the image of  $\nabla f$  is bounded by

$$\mathcal{D}(f) \leq \sum_i |\beta_i| |E(H_i)| N^{3/2} \leq C_{\beta} n^{3/4}.$$

By Lemma 26,  $\text{Lip}(f) \leq C_{\beta}$ , and by Lemma 28,

$$\max_{X, Y \in C_n} \frac{\|\nabla f(X) - \nabla f(Y)\|_1}{\|X - Y\|_1} \leq C_{\beta}$$

as well. Plugging these bounds into Theorem 25, we obtain the desired results. □

**5. Approximate block model for the dense regime.** In this section, we prove Theorem 14. It will be instructive to first prove the theorem for triangle-counting functions, as this case is simple and gives easy-to-calculate bounds. The same techniques will then be used to give a sketch of the proof for general subgraph-counting functions.

The proof technique uses random orthogonal projections in order to perform some of the calculations in a low-dimensional space. For this, we will need the following results concerning concentration of measure of orthogonal random projections.

**LEMMA 29** (Orthogonal projections preserve distance. Due to [5], page 62). *Let  $0 < \delta < 1$ , let  $d, k > 0$  be positive integers, let  $\pi : \mathbb{R}^d \rightarrow \mathbb{R}^k$  be an orthogonal projection into a uniformly random  $k$  dimensional subspace, and let  $g : \mathbb{R}^d \rightarrow \mathbb{R}^k$  be defined as  $g(v) = \sqrt{\frac{d}{k}} \pi(v)$ . Then for any vector  $v \in \mathbb{R}^d$ ,*

$$\Pr[(1 - \delta)\|v\|^2 \leq \|g(v)\|^2 \leq (1 + \delta)\|v\|^2] \leq 2e^{-k(\delta^2/2 - \delta^3/3)/2}.$$

From this lemma about the magnitude of vectors, it is possible to obtain similar bounds on the scalar product between two vectors.

**LEMMA 30** (Preserving scalar products). *Let  $0 < \delta < 1$ , let  $d, k > 0$  be positive integers, and let  $g : \mathbb{R}^d \rightarrow \mathbb{R}^k$  be a linear transformation. Let  $u, v \in \mathbb{R}^d$  be*

two vectors of norm smaller than 1 such that  $(1 - \delta)\|u \pm v\|^2 \leq \|g(u \pm v)\|^2 \leq (1 + \delta)\|u \pm v\|^2$ . Then

$$|\langle g(v_1), g(v_2) \rangle - \langle v_1, v_2 \rangle| \leq 2\delta.$$

The proof is postponed to the [Appendix](#).

5.1. *Counting triangles.*

PROOF OF THEOREM 14 (FOR THE CASE OF TRIANGLE-COUNTING FUNCTIONS). Let  $N$  be a positive integer, let  $\alpha, \beta \in \mathbb{R}$  be real numbers, and let  $f$  be of the form

$$f(X) = \alpha \operatorname{inj}(K_2, X) + \frac{\beta}{N - 2} \operatorname{inj}(K_3, X),$$

where  $K_2$  is the complete graph on two vertices and  $K_3$  is the triangle graph. Let  $X \in \mathcal{X}_f$ . It can be verified by direct calculation that

$$f(X) = \alpha \operatorname{Tr}(X^2) + \frac{\beta}{N - 2} \operatorname{Tr}(X^3)$$

and

$$(11) \quad \nabla f(X) = \alpha \mathbf{1} + \frac{3\beta}{N - 2} \overline{X^2},$$

where  $\overline{X^2}$  is the matrix with zero on the diagonal and whose off-diagonal entries are those of  $X^2$ . We then have by Theorem 10 that

$$(12) \quad \left\| X - \frac{\mathbf{1} + \tanh(\alpha \mathbf{1} + \frac{3\beta}{N-2} \overline{X^2})}{2} \right\|_1 \leq 5000 C_\beta^2 n^{15/16}.$$

We proceed to show that the term  $\frac{3\beta}{N-2} \overline{X^2}$  is close to a block matrix with a small number of communities. This is done roughly as follows: Each entry in the matrix  $\frac{3\beta}{N-2} \overline{X^2}$  can be written as the scalar product of two vectors in  $\mathbb{R}^N$ ; namely, the column vectors of  $\sqrt{\frac{3\beta}{N-2}} X$ . It is possible to project these vectors into a low-dimensional space, so that their scalar products are almost preserved. This low-dimensional projection can then be rounded to a  $\delta$ -net, whose size depends only on  $\delta$  and on the dimension. Thus if the dimension is small, then the  $\delta$ -net is small. The matrix  $\frac{3\beta}{N-2} \overline{X^2}$  can then be approximated by scalar products of elements from the  $\delta$ -net, and each element in the net defines a community. Applying  $\tanh$  entry-wise, adding the constant  $\mathbf{1}$  and dividing by 2 does not change the block model parameters, implying that  $X$  itself is close to a block matrix.

Denote by  $v_i$  the  $i$ th column of  $X$  multiplied by  $1/\sqrt{N}$ , so that

$$(v_i)_j = \frac{1}{\sqrt{N}} X_{ij}.$$

Since all the entries of  $X$  are in  $[0, 1]$ , each  $v_i$  lies within the unit ball:

$$(13) \quad \|v_i\|^2 = \frac{1}{N} \sum_{j=1}^N X_{ij}^2 \leq 1.$$

Let  $f$  be a triangle-counting function, and assume that  $\beta = 1$ . Then for two distinct vertices  $i$  and  $j$ , the derivative  $\partial_{ij} f$  is equal to

$$\partial_{ij} f(X) = \frac{N}{N-2} \langle v_i, v_j \rangle = \frac{N}{N-2} \frac{1}{N} \sum_k X_{ik} X_{kj} = \left( \frac{1}{N-2} \overline{X^2} \right)_{ij}.$$

This is because the difference between  $\text{inj}(K_3, G)$  with  $G$  containing the edge  $ij$  and  $\text{inj}(K_3, G)$  where  $G$  does not contain the edge  $ij$  is exactly the sum of weights of all the triangles of the form  $ijk$  for  $k = 1, \dots, n$ .

Let  $k > 0$  be a positive integer to be chosen later, let  $U \subseteq \mathbb{R}^N$  be a uniformly random subspace of dimension  $k$ , and denote by  $\pi : \mathbb{R}^N \rightarrow U$  an orthogonal projection from  $\mathbb{R}^N$  into  $U$ . Let  $g : \mathbb{R}^N \rightarrow U$  be defined as  $g(v) = \sqrt{\frac{N}{k}} \pi(v)$ . For every two indices  $i \neq j$ , denote

$$\mathcal{B}_{ij} = \{(1 - \delta)\|x\|^2 \leq \|g(x)\|^2 \leq (1 + \delta)\|x\|^2 \text{ for } x \in \{v_i, v_j, v_i + v_j, v_i - v_j\}\},$$

the event that  $g$  almost preserves the squared norm of both of the original vectors  $v_i$  and  $v_j$  and of their sum and difference  $v_i + v_j$  and  $v_i - v_j$ . By Lemma 29, the probability for  $\mathcal{B}_{ij}$  to occur is at least

$$(14) \quad \Pr[\mathcal{B}_{ij}] \geq 1 - 8e^{-k(\delta^2/2 - \delta^3/3)/2}.$$

Under this event, since  $\delta < 1$ , both  $g(v_i)$  and  $g(v_j)$  are contained inside a ball of radius 2 around the origin. Further, by Lemma 30, the scalar product between  $v_i$  and  $v_j$  is also almost preserved:

$$(15) \quad |\langle g(v_i), g(v_j) \rangle - \langle v_i, v_j \rangle| \leq 2\delta.$$

Let  $T$  be a  $\delta$ -net of the ball of radius 2 around the origin in  $k$  dimensions. By [14], Lemma 2.6, there exists such a net of size smaller than  $(1 + 4/\delta)^{k+1}$ . For every vertex  $i$ , denote by  $w_i = \text{argmin}_{w \in T} \|g(v_i) - w\|$  the vector in  $T$  that is closest to  $g(v_i)$ , and denote by  $\Delta w_i = w_i - g(v_i)$  the difference between the two. Then under  $\mathcal{B}_{ij}$ , since  $g(v_i)$  is in the ball of radius 2, the magnitude of the difference  $\|\Delta w_i\|$  is smaller than  $\delta$ . In this case,

$$\begin{aligned} |\langle w_i, w_j \rangle - \langle g(v_i), g(v_j) \rangle| &= |\langle g(v_i) + \Delta w_i, g(v_j) + \Delta w_j \rangle - \langle g(v_i), g(v_j) \rangle| \\ &= |\langle g(v_i), \Delta w_j \rangle + \langle \Delta w_i, g(v_j) \rangle + \langle \Delta w_i, \Delta w_j \rangle| \\ &\quad (\text{since } \|\Delta w\| \leq \delta) \leq 6\delta. \end{aligned}$$

Thus, under  $\mathcal{B}_{ij}$  and together with equation (15), we almost surely have that

$$|\langle w_i, w_j \rangle - \langle v_i, v_j \rangle| \leq 8\delta.$$

Denote by  $\tilde{X}$  the matrix defined by  $(\tilde{X})_{ij} = \langle w_i, w_j \rangle$  for  $i \neq j$  and with 0 on the diagonal. It is clear that the matrix  $\tilde{X}$  is a block matrix, with the communities in correspondence with the elements of the  $\delta$ -net  $T$ ; hence there are no more than  $(1 + 4/\delta)^{k+1}$  communities in  $\tilde{X}$ .

The expected value of the one-norm between  $\frac{1}{N}\overline{X^2}$  and  $\tilde{X}$  is

$$\begin{aligned} \mathbb{E} \left\| \frac{1}{N}\overline{X^2} - \tilde{X} \right\|_1 &= \mathbb{E} \sum_{i,j} \left| \frac{1}{N}(\overline{X^2})_{ij} - (\tilde{X})_{ij} \right| \\ (16) \qquad \qquad \qquad &= \sum_{i \neq j} \mathbb{E} |\langle v_i, v_j \rangle - \langle w_i, w_j \rangle|. \end{aligned}$$

Each expectation term of the form  $\mathbb{E}|\langle v_i, v_j \rangle - \langle w_i, w_j \rangle|$  can be controlled by conditioning on the event  $\mathcal{B}_{ij}$ . Keeping in mind that in the general case  $|\langle v_i, v_j \rangle - \langle w_i, w_j \rangle| \leq 5$  since the norm of  $v_i$  and  $v_j$  is bounded by 1 and the norm of  $w_i$  and  $w_j$  is bounded by 2, we can bound the expectation by

$$\begin{aligned} \mathbb{E}|\langle v_i, v_j \rangle - \langle w_i, w_j \rangle| &= \mathbb{E}[|\langle v_i, v_j \rangle - \langle w_i, w_j \rangle| \mid \mathcal{B}_{ij}] \Pr[\mathcal{B}_{ij}] \\ &\quad + \mathbb{E}[|\langle v_i, v_j \rangle - \langle w_i, w_j \rangle| \mid \neg\mathcal{B}_{ij}] \Pr[\neg\mathcal{B}_{ij}] \\ &\leq 8\delta \cdot 1 + 5 \cdot 8e^{-k(\delta^2/2 - \delta^3/3)/2}. \end{aligned}$$

Choosing  $k = \lceil 2 \log(1/\delta)(\delta^2/2 - \delta^3/3)^{-1} \rceil$ , we have

$$\mathbb{E}|\langle v_i, v_j \rangle - \langle w_i, w_j \rangle| \leq 48\delta.$$

Plugging this into equation (16), we obtain the bound

$$\mathbb{E} \left\| \frac{1}{N}\overline{X^2} - \tilde{X} \right\|_1 \leq 48\delta n.$$

Hence, there exists a block matrix  $\hat{X}$  with no more than  $(1 + 4/\delta)^{k+1}$  communities such that

$$\left\| \frac{1}{N}\overline{X^2} - \hat{X} \right\|_1 \leq 48\delta n.$$

Multiplying both sides by  $3\beta N/(N - 2)$ , we have that

$$\left\| \frac{3\beta}{N - 2}\overline{X^2} - \frac{3\beta N}{N - 2}\hat{X} \right\|_1 \leq 144 \frac{N}{N - 2} \beta \delta n \leq 450\beta \delta n.$$

Note that the function  $\tanh$  is contracting; that is,

$$(17) \qquad \qquad \qquad |\tanh(x) - \tanh(y)| \leq |x - y|.$$

This implies that

$$\begin{aligned} &\left\| \frac{\mathbf{1} + \tanh(\alpha \mathbf{1} + \frac{3\beta}{N-2}\overline{X^2})}{2} - \frac{\mathbf{1} + \tanh(\alpha \mathbf{1} + \frac{3\beta N}{N-2}\hat{X})}{2} \right\|_1 \\ &= \frac{1}{2} \left\| \frac{3\beta}{N - 2}\overline{X^2} - \frac{3\beta N}{N - 2}\hat{X} \right\|_1 \leq 225\beta \delta n. \end{aligned}$$

Finally, by equation (12),

$$\left\| X - \frac{\mathbf{1} + \tanh(\alpha\mathbf{1} + \frac{3\beta}{N}\overline{X^2})}{2} \right\|_1 \leq 5000C_\beta^2 n^{15/16},$$

and so by the triangle inequality, denoting  $X^* = \frac{\mathbf{1} + \tanh(\alpha\mathbf{1} + 3\beta\hat{X})}{2}$ ,

$$\|X - X^*\|_1 \leq 225\beta\delta n + 5000C_\beta^2 n^{15/16}. \quad \square$$

*5.2. Counting general subgraphs.* In this section, we give a proof sketch of general form of Theorem 14. The proof relies on the same techniques as those in the previous subsection, which gave block matrix bounds for the specific case of triangles.

Let  $X \in \mathcal{X}_f$ . The main argument in the previous proof was as follows: For triangles, each entry in the gradient  $\nabla f(X) = \frac{3\beta}{N-2}\overline{X^2}$  was written as a scalar product between two vectors. These vectors were then projected to a low dimensional space, yielding a block matrix form.

We will generalize the above procedure, and show that the gradient  $\nabla f$  of any subgraph-counting function  $f$  can be written as a sum of scalar products of vectors: There exist an integer  $S > 0$ , a family of constants  $c_r, r = 1, \dots, S$ , and two families of vectors  $v_i^r$  and  $u_j^r$  of norm smaller than 1, such that

$$(18) \quad \partial_{ij} f = \sum_{r=1}^S c_r \langle v_i^r, u_j^r \rangle.$$

The number of scalar products  $S$  and the constants  $c_r$  depend on the subgraphs  $H_k$  that  $f$  counts and their weights  $\beta_k$ , but do not grow explicitly with  $N$ . Repeating the reasoning in the previous proof, these vectors can all be simultaneously projected by an orthogonal projection  $g$  to a low dimensional space, so that

$$\partial_{ij} f \approx \sum_{r=1}^S c_r \langle g(v_i^r), g(u_j^r) \rangle.$$

Taking a  $\delta$ -net of the sphere in the new space will give us an approximation of these sums: For every  $r$  we will obtain a block matrix  $W^r$  whose  $ij$ th entry approximates the scalar product  $\langle g(v_i^r), g(u_j^r) \rangle$ . As before, the number of communities of  $W^r$  will depend only on  $\delta$ . Finally, since the sum of  $S$  block matrices is also a block matrix (albeit with a number of communities exponential in  $S$ ), the sum  $\sum_{r=1}^S c_r \langle g(v_i^r), g(u_j^r) \rangle$  is itself a block matrix, with a number of communities that depends only on the subgraphs  $H_k$ , their weights  $\beta_k$ , and on  $\delta$ .

Let us now fill in some of the details for this proof sketch. Let  $H = ([m], E(H))$  be a finite simple graph on  $m$  vertices with edge set  $E(H)$ . This simple edge set can also be viewed as a directed edge set, with two directed edges replacing every original simple edge:  $D(H) = \bigcup_{\{x,y\} \in E(H)} \{(x, y), (y, x)\}$ . The essential part

of the proof is showing that  $\partial_{i,j} \text{inj}(H, G)$  can be obtained by scalar products as above; the rest will follow from linearity.

Let  $i$  be a vertex of  $G$  and let  $e = (x, y) \in D(H)$  be an oriented edge of  $H$ . Denote by  $\Phi_e$  the set of all injective maps from  $H \setminus \{x, y\}$  to  $G$ . The vectors  $v_i^e$  and  $u_i^e$  will have one entry for every function  $\phi \in \Phi_e$ . For  $v_i^e$ , the entry  $v_i^e(\phi)$  contains the weight of edges from  $i$  to the image  $\phi(H \setminus \{y\})$ , times the square root of the weight of the image  $\phi(H \setminus \{x, y\})$ . For  $u_i^e$ , the entry  $u_i^e(\phi)$  contains the weight of edges from  $i$  to the image  $\phi(H \setminus \{x\})$ , times the square root of the weight of the image  $\phi(H \setminus \{x, y\})$ . More formally, for every  $\phi \in \Phi_e$ ,

$$v_i^e(\phi) = \prod_{a \text{ s.t. } \{x,a\} \in E(H \setminus \{y\})} X_{i,\phi(a)} \prod_{\{a,b\} \in E(H \setminus \{x,y\})} \sqrt{X_{\phi(a),\phi(b)}},$$

$$u_i^e(\phi) = \prod_{a \text{ s.t. } \{y,a\} \in E(H \setminus \{x\})} X_{i,\phi(a)} \prod_{\{a,b\} \in E(H \setminus \{x,y\})} \sqrt{X_{\phi(a),\phi(b)}}.$$

For two different vertices  $i \neq j$ , the scalar product between two vectors becomes

$$\langle v_i^e, u_j^e \rangle = \sum_{\phi \in \Phi_e} \left( \prod_{\{x,a\} \in E(H \setminus \{y\})} X_{i,\phi(a)} \prod_{\{y,a\} \in E(H \setminus \{x\})} X_{j,\phi(a)} \cdot \prod_{\{a,b\} \in E(H \setminus \{x,y\})} X_{\phi(a),\phi(b)} \right).$$

Let us inspect this scalar product. For each fixed  $\phi$ , the summand is the edge weight of the image of the homomorphism  $\psi : H \rightarrow G$ , where

$$\psi(z) = \begin{cases} i & z = x, \\ j & z = y, \\ \phi(z) & \text{o.w.} \end{cases}$$

The mapping  $\psi$  is in general not an injection: Although  $\phi$  itself was chosen to be an injection, the function  $\psi$  is not one-to-one when  $\phi(a) = i$  or  $\phi(a) = j$  for some  $a \in H$ . But in this case, either  $X_{i,\phi(a)}$  or  $X_{j,\phi(a)}$  are 0, since the diagonal entries of  $X$  are 0. Thus, summing over all  $\phi$  effectively means summing over all injective mappings that send the particular (directed) edge  $(x, y)$  in  $H$  to  $(i, j)$  in  $G$ . By the discussion in the proof of Lemma 26, summing over all possible edges  $e$  that can map to  $(i, j)$  exactly gives the definition of the discrete derivative:

$$\partial_{ij} \text{inj}(H, G) = \frac{1}{2} \sum_{e \in D(H)} \langle v_i^e, u_j^e \rangle.$$

The gradient of a subgraph-counting function that counts a single subgraph  $H$  with weight  $\beta$  can then be written as

$$\partial_{ij} f = \frac{\beta}{2(N-2) \cdots (N-m+1)} \sum_{e \in D(H)} \langle v_i^e, u_j^e \rangle.$$

When we proved the theorem for the case of triangles, it was important that the vectors were of unit length—this meant that the projection was contained in a ball of radius 2, and this is what allowed us to take a  $\delta$ -net that did not depend on  $N$ . This is the case here as well: Each entry of  $v_i^e$  and  $u_i^e$  is bounded by 1. Their norm is therefore bounded by the square root of the number of entries, which is the number of injective mappings from  $H \setminus \{x, y\}$  to  $G$ . Thus,

$$\|v_i^e\|^2 \leq |\Phi_e| < N(N - 1) \cdots (N - m + 3).$$

This means that  $v_i^e/\sqrt{N \cdots (N - m + 3)}$  and  $u_i^e/\sqrt{N \cdots (N - m + 3)}$  have their norm bounded by 1.

Finally, for the case of general subgraph-counting functions that count the subgraphs  $H_1, \dots, H_\ell$  with weights  $\beta_1, \dots, \beta_\ell$ , we have that

$$\begin{aligned} \partial_{ij} f &= \sum_{k=1}^{\ell} \frac{N(N - 1)}{(N - m_k + 2)(N - m_k + 1)} \\ &\cdot \left( \frac{\beta_k}{2} \sum_{r=1}^{2|E(H_k)|} \left\langle \frac{v_i^{k,r}}{\sqrt{N \cdots (N - m_k + 3)}}, \frac{u_j^{k,r}}{\sqrt{N \cdots (N - m_k + 3)}} \right\rangle \right). \end{aligned}$$

This shows that  $\partial_{ij} f$  can indeed be written in the form of equation (18).

**6. Positive weights.**

6.1. *The exact case.* We would like to first give some intuition regarding the proof of Theorem 18: We will show that if all the weights  $\beta_i$  are positive and if  $x = \varphi(x)$  has a unique solution, then the fixed-point equation  $X = (\mathbf{1} + \tanh(\nabla f(X)))/2$  has a single solution  $x_0 \mathbf{1}$ . The proof that any  $X \in \mathcal{X}_f$  is close to  $x_0 \mathbf{1}$  will be more involved but analogous.

For clarity, we will assume that  $f$  counts edges and triangles. Let  $\alpha, \beta \in \mathbb{R}$  with  $\beta > 0$ , and let  $f$  be of the form

$$f(X) = \alpha \operatorname{inj}(K_2, X) + \frac{\beta}{N - 2} \operatorname{inj}(K_3, X),$$

where  $K_2$  is an edge and  $K_3$  is the triangle graph. Direct calculation shows that  $\nabla f(X) = \alpha \mathbf{1} + \frac{3\beta}{N-2} \overline{X^2}$ . In terms of the adjacency matrix, the fixed-point equation is then

$$(19) \quad X = \frac{\mathbf{1} + \tanh(\alpha \mathbf{1} + \frac{3\beta}{N-2} \overline{X^2})}{2}.$$

Let  $X$  be a solution to equation (19). Denote by  $a$  the minimum off-diagonal entry of  $X$  and by  $b$  the maximum off-diagonal entry of  $X$ . For every index  $i$  and  $j$  with  $i \neq j$ , we have

$$\frac{3\beta}{N - 2} (\overline{X^2})_{ij} = \frac{3\beta}{N - 2} \sum_{k=1}^N X_{ik} X_{kj}.$$

For  $k = i$  and  $k = j$ , we have  $X_{ii} = X_{jj} = 0$ . For all other indices  $k$ ,  $X_{ik} \leq b$  by definition, so

$$(20) \quad \frac{3\beta}{N-2}(\overline{X^2})_{ij} \leq 3\beta b^2.$$

This is where the condition  $\beta > 0$  comes into play: The inequality would have been reversed had  $\beta$  been negative. The maximum element of the right-hand side of equation (19) is

$$\max \frac{1 + \tanh(\alpha \mathbf{1} + \frac{3\beta}{N-2} \overline{X^2})}{2} \leq \frac{1 + \tanh(\alpha \mathbf{1} + 3\beta b^2)}{2}.$$

Taking the maximum of both sides of equation (19), we get

$$b \leq \frac{1 + \tanh(\alpha \mathbf{1} + 3\beta b^2)}{2}.$$

By similar argument, we get that

$$\frac{3\beta}{N-2}(\overline{X^2})_{ij} \geq 3\beta a^2,$$

and hence

$$a \geq \frac{1 + \tanh(\alpha \mathbf{1} + 3\beta a^2)}{2}.$$

Putting both of these together, we must solve the two inequalities

$$(21) \quad \begin{aligned} 2a - 1 &\geq \tanh(\alpha \mathbf{1} + 3\beta a^2), \\ 2b - 1 &\leq \tanh(\alpha \mathbf{1} + 3\beta b^2). \end{aligned}$$

By assumption, there is exactly one solution  $x_0$  to the equation  $2x - 1 = \tanh(\alpha \mathbf{1} + 3\beta x^2)$ . By equation (21), we would then need that  $a \geq x_0$  and  $b \leq x_0$ . But  $a$  is the minimum off-diagonal entry of  $X$  and  $b$  is the maximum off-diagonal entry of  $X$ , so they must be equal. Hence the constant solution  $x_0 \mathbf{1}$  of Lemma 17 is the only solution; see Figure 1 for an illustration.

In order to generalize this argument to any subgraph-counting function, recall that every entry of  $\nabla f(x \mathbf{1})$  is just some polynomial  $p(x)$ . If all the weights are  $\beta_i$  are positive then the preceding argument can be repeated for  $p(x)$  with the inequalities all intact.

### 6.2. Closeness.

**PROOF OF THEOREM 18.** Let  $X \in \mathcal{X}_f$ . We would have liked to use an argument in the same vein as that of Section 6.1 and claim that the solution  $X$  is close to a constant solution because its minimum and maximum entries are close to each other. However, this is not in general true: A matrix  $X$  can easily have  $\min X = 0$

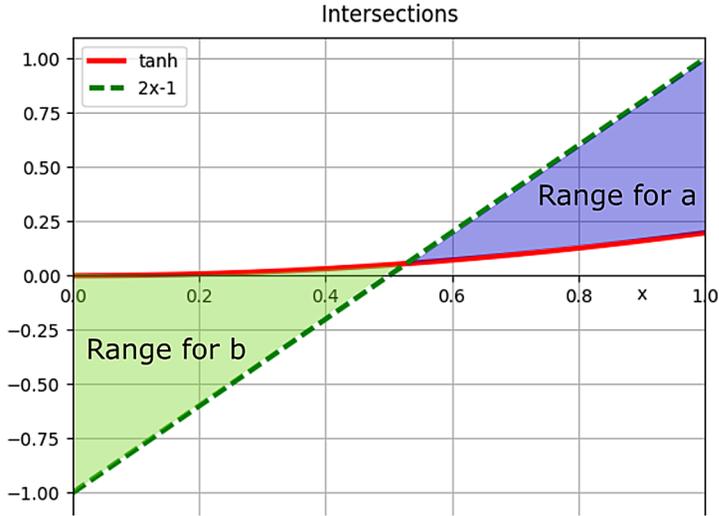


FIG. 1. An illustration of the permissible range for  $a$  and  $b$ .

and  $\max X = 1$  while still satisfying the equation  $\|X - (\mathbf{1} + \tanh(\nabla f(X)))/2\|_1 = o(n)$ , since the equation is not sensitive to changes in a small number of entries.

To overcome this, we will iterate the function  $\frac{1 + \tanh(\nabla f(X))}{2}$ , showing that each time we do so, the minimum and maximum values tend closer to a constant.

Define the sequence of functions  $\{\varphi_i\}_{i=1}^\infty$  by  $\varphi_1(x) = \varphi_\beta(x)$  and  $\varphi_{i+1}(x) = \varphi(\varphi_i(x))$  for  $i \geq 1$ . Denote  $k = \lceil \frac{\log \lambda}{\log D_\beta} \rceil = \lceil \log_{D_\beta}(\lambda) \rceil$ . By assumption, for all  $x_0 \in [0, 1]$  we have

$$|\varphi_\beta(x) - x_0| \leq D_\beta |x - x_0|.$$

This implies that

$$(22) \quad |\varphi_k(x) - x_0| \leq D_\beta^k |x - x_0| \leq \lambda.$$

Denote by  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  the function  $\Phi(X) = \frac{1 + \tanh(\nabla f(X))}{2}$ , let  $Y_0 = X$  and recursively define  $Y_{i+1} = \Phi(Y_i)$ . Then  $\|Y_k - x_0 \mathbf{1}\|_\infty \leq \lambda$ . To see this, observe that since all  $\beta$ 's are positive,

$$\begin{aligned} \min Y_1 &= \min \Phi(X) = \min \frac{1 + \tanh(\nabla f(X))}{2} \\ &\geq \min \frac{1 + \tanh(\nabla f((\min X) \mathbf{1}))}{2} \\ &= \varphi(\min X). \end{aligned}$$

Iterating, we have that

$$\min Y_k \geq \varphi_k(\min X).$$

But by equation (22),  $|\varphi_k(x) - x_0| < \lambda$  for every  $x \in [0, 1]$ , and in particular for  $\min X$ . Hence

$$\min Y_k \in [x_0 - \lambda, x_0 + \lambda].$$

The same argument can be applied to  $\max Y_k$ , showing that all of  $Y_k$ 's entries are in  $[x_0 - \lambda, x_0 + \lambda]$ . Consequently,

$$(23) \quad \|Y_k - x_0 \mathbf{1}\|_1 \leq \lambda n.$$

The distance between  $X$  and  $Y_k$  can be bounded as follows. By Lemma 28, we have that for any two matrices  $A$  and  $B$ ,

$$\|\Phi(A) - \Phi(B)\|_1 \leq C_\beta \|A - B\|_1.$$

This gives a bound on consecutive iterations:

$$\begin{aligned} \|Y_i - Y_{i-1}\|_1 &= \|\Phi(Y_{i-1}) - \Phi(Y_{i-2})\|_1 \\ &\leq C_\beta \|Y_{i-1} - Y_{i-2}\|_1, \end{aligned}$$

and so by induction,

$$\|Y_i - Y_{i-1}\|_1 \leq C_\beta^i \|X - Y_1\|_1 = C_\beta^i \|X - \Phi(X)\|_1.$$

Using this bound, we have

$$\begin{aligned} (24) \quad \|X - Y_k\|_1 &= \left\| \sum_{i=1}^k Y_i - Y_{i-1} \right\|_1 \\ &\leq \sum_{i=1}^k \|Y_i - Y_{i-1}\|_1 \\ &\leq \sum_{i=1}^k C_\beta^i \|X - \Phi(X)\|_1 \\ &\leq 2C_\beta^k \|X - \Phi(X)\|_1. \end{aligned}$$

Combining equations (23), (24) and Theorem 10, we have

$$\begin{aligned} \|X - x_0 \mathbf{1}\|_1 &\leq \lambda n + C_\beta^{\frac{\log \lambda}{\log D_\beta} + 1} 10,000 C_\beta^2 n^{15/16} \\ &= \lambda n + 10,000 C_\beta^3 \lambda^{\frac{\log C_\beta}{\log D_\beta}} n^{15/16}. \end{aligned}$$

Optimizing over  $\lambda$  gives the dependence described in equation (9).  $\square$

**7. Small weights.** In this section, we prove Theorem 19.

PROOF. We will show that the function

$$\Phi_f(X) = \frac{1 + \tanh(\nabla f(X))}{2}$$

is contracting if  $S_\beta < 1$ . For that, we will need the following lemma, whose proof is postponed to the Appendix.

LEMMA 31. *Let  $f$  be a subgraph counting function. Then for any two matrices  $X, Y \in \overline{C}_n$ ,*

$$\|\nabla f(X) - \nabla f(Y)\|_1 \leq \sum_{i=1}^{\ell} |\beta_i| m_i (m_i - 1) \|X - Y\|_1.$$

Using this lemma, we have that

$$\begin{aligned} \|\Phi_f(X) - \Phi_f(Y)\|_1 &= \left\| \frac{\mathbf{1} + \tanh(\nabla f(X))}{2} - \frac{\mathbf{1} + \tanh(\nabla f(Y))}{2} \right\|_1 \\ &\text{[by equation (17)]} \leq \frac{1}{2} \|\nabla f(X) - \nabla f(Y)\|_1 \\ (25) \quad &\text{(by Lemma 31)} \leq \frac{1}{2} \sum_{i=1}^{\ell} |\beta_i| m_i (m_i - 1) \|X - Y\|_1 \\ &= \sum_{i=1}^{\ell} |\beta_i| \binom{m_i}{2} \|X - Y\|_1 \\ &= S_\beta \|X - Y\|_1. \end{aligned}$$

If  $S_\beta < 1$ , then  $\Phi_f(X)$  is contracting, and by Banach’s fixed-point theorem it has a unique fixed point in the compact space of all matrices with entries in  $[0, 1]$ ; we already know by Lemma 17 that it is a constant solution  $X_c = c \cdot \mathbf{1}$ . This shows the first part of Theorem 19. For the second part, let  $X \in \mathcal{X}_f$ . Then by a simple calculation,

$$\begin{aligned} \|X - X_c\|_1 &= \|X - \Phi_f(X) + \Phi_f(X) - X_c + \Phi_f(X_c) - \Phi_f(X_c)\|_1 \\ &\leq \|X - \Phi_f(X)\|_1 + \|\Phi_f(X) - \Phi_f(X_c)\|_1 + \|X_c - \Phi_f(X_c)\|_1 \\ &= \|X - \Phi_f(X)\|_1 + \|\Phi_f(X) - \Phi_f(X_c)\|_1 \end{aligned}$$

[by equation (25)]  $\leq \|X - \Phi_f(X)\|_1 + S_\beta \|X - X_c\|_1$ .

Rearranging, we get the desired result:

$$\|X - X_c\|_1 \leq \frac{\|X - \Phi_f(X)\|_1}{1 - S_\beta} \leq \frac{5000C_\beta^2}{1 - S_\beta} n^{15/16}.$$

□

**8. Two block model.** The proof of Theorem 21 is rather technical. It goes roughly as follows: We assume that there exists a fixed point of the form

$$X = \alpha_1 v_1 v_1^T + \alpha_2 v_2 v_2^T - I(\alpha_1 + \alpha_2),$$

where  $v_1$  is the vector  $(1, 1, \dots, 1)$  whose entries are all 1, and  $v_2$  is the vector  $(-1, \dots, -1, 1, \dots, 1)$  whose first  $N/2$  entries are  $-1$  and whose second  $N/2$  entries are 1. From this assumption, we arrive at pair of nonlinear scalar equations for  $\alpha_1$  and  $\alpha_2$ ; nontrivial solutions of these equations guarantee a nontrivial block model for  $X$ . We then show by direct calculation that for large enough  $|\beta|$ , such a solution does indeed exist.

We postpone the proof to the [Appendix](#).

### APPENDIX

**PROOF OF LEMMA 30.** We will show the proof only for the inequality  $\langle g(v_1), g(v_2) \rangle - \langle v_1, v_2 \rangle \leq 2\delta$ ; the inequality  $\langle v_1, v_2 \rangle - \langle g(v_1), g(v_2) \rangle \leq 2\delta$  follows a similar calculation.

The scalar product between any two vectors  $x$  and  $y$  can be written as a function of  $x + y$  and  $x - y$ :

$$\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2).$$

We can now calculate:

$$\begin{aligned} \langle g(v_1), g(v_2) \rangle &= \frac{1}{4}(\|g(v_1) + g(v_2)\|^2 - \|g(v_1) - g(v_2)\|^2) \\ &= \frac{1}{4}(\|g(v_1 + v_2)\|^2 - \|g(v_1 - v_2)\|^2) \\ &\leq \frac{1}{4}((1 + \delta)\|v_1 + v_2\|^2 - (1 - \delta)\|v_1 - v_2\|^2) \\ &= \frac{1}{4}(4\langle v_1, v_2 \rangle + \delta\|v_1 + v_2\|^2 + \delta\|v_1 - v_2\|^2) \\ (\text{because } \|v_1 \pm v_2\|^2 &\leq 4) \leq \frac{1}{4}(4\langle v_1, v_2 \rangle + 8\delta) \\ &= \langle v_1, v_2 \rangle + 2\delta. \end{aligned}$$

This implies that  $\langle g(v_1), g(v_2) \rangle - \langle v_1, v_2 \rangle \leq 2\delta$ .  $\square$

**LEMMA 32.** *Let  $I \subseteq [n]$  be a set of indices. Then for any  $X, Y \in \overline{C}_n$ ,*

$$\left| \prod_{\alpha \in I} X_\alpha - \prod_{\alpha \in I} Y_\alpha \right| \leq \sum_{\alpha \in I} |X_\alpha - Y_\alpha|.$$

PROOF. By induction on  $|I|$ . Let  $\beta \in I$ . Then

$$\begin{aligned} \left| \prod_{\alpha \in I} X_\alpha - \prod_{\alpha \in I} Y_\alpha \right| &= \left| \prod_{\alpha \in I} X_\alpha - \prod_{\alpha \in I} Y_\alpha + Y_\beta \prod_{\substack{\alpha \in I \\ \alpha \neq \beta}} X_\alpha - Y_\beta \prod_{\substack{\alpha \in I \\ \alpha \neq \beta}} Y_\alpha \right| \\ &= \left| \left( \prod_{\substack{\alpha \in I \\ \alpha \neq \beta}} X_\alpha \right) (X_\beta - Y_\beta) + Y_\beta \left( \prod_{\substack{\alpha \in I \\ \alpha \neq \beta}} X_\alpha - \prod_{\substack{\alpha \in I \\ \alpha \neq \beta}} Y_\alpha \right) \right| \\ &\leq |X_\beta - Y_\beta| + \left| \prod_{\substack{\alpha \in I \\ \alpha \neq \beta}} X_\alpha - \prod_{\substack{\alpha \in I \\ \alpha \neq \beta}} Y_\alpha \right|, \end{aligned}$$

where the last inequality is because  $|X_\alpha| \leq 1$  for all  $\alpha$ .  $\square$

PROOF OF LEMMA 31. It is enough to show the result for a function  $f$  that counts just a single subgraph  $H = (V, E)$  with  $m := |E|$ ; the general result follows by linearity of the derivative and the triangle inequality. By equation (5),

$$\begin{aligned} \partial f_{ij}(X) &= \frac{\beta}{(N-2)(N-3)\cdots(N-m+1)} \\ &\cdot \sum_{(a,b) \in E} \sum_{\substack{q \in [N]^m \\ q \text{ has distinct elements} \\ q_a=i, q_b=j}} \prod_{\substack{(l,l') \in E \\ \{l,l'\} \neq \{a,b\}}} X_{q_l, q_{l'}}. \end{aligned}$$

The difference between the gradients is then

$$\begin{aligned} &\|\nabla f(X) - \nabla f(Y)\|_1 \\ &= \sum_{ij} \left| \frac{|\beta|}{(N-2)(N-3)\cdots(N-m+1)} \right. \\ &\quad \cdot \left. \sum_{(a,b) \in E} \sum_{\substack{q \in [N]^m \\ q \text{ has distinct elements} \\ q_a=i, q_b=j}} \left( \prod_{\substack{(l,l') \in E \\ \{l,l'\} \neq \{a,b\}}} X_{q_l, q_{l'}} - \prod_{\substack{(l,l') \in E \\ \{l,l'\} \neq \{a,b\}}} Y_{q_l, q_{l'}} \right) \right|. \end{aligned}$$

By Lemma 32, this can be bounded by

$$\begin{aligned} \|\nabla f(X) - \nabla f(Y)\|_1 &\leq \sum_{ij} \frac{|\beta|}{(N-2)(N-3)\cdots(N-m+1)} \\ &\quad \cdot \sum_{(a,b) \in E} \sum_{\substack{q \in [N]^m \\ q \text{ has distinct elements} \\ q_a=i, q_b=j}} \sum_{(l,l') \in E} |X_{q_l, q_{l'}} - Y_{q_l, q_{l'}}|. \end{aligned}$$

Fix a pair of vertices  $\alpha, \beta$ . By symmetry, as  $i$  and  $j$  span over all possible pairs of vertices, the term  $|X_{\alpha,\beta} - Y_{\alpha,\beta}|$  appears  $m(m - 1)(N - 2)(N - 3) \cdots (N - m + 1)$  times. Thus

$$\|\nabla f(X) - \nabla f(Y)\|_1 \leq |\beta|m(m - 1)\|X - Y\|_1. \quad \square$$

LEMMA 33. For every  $\alpha \in \mathbb{R}$ , the equation

$$2x - 1 = \tanh(\alpha x^2)$$

has a unique solution with  $x \in (0, 1)$ .

PROOF. Denote  $g(x) = 2x - 1$  and  $h(x) = \tanh(\alpha x^2)$ ; we must then show that there is a unique point  $x \in (0, 1)$  such that  $g(x) = h(x)$ :

- The case  $\alpha = 0$  is solved by  $x = \frac{1}{2}$ .
- The case  $\alpha < 0$ : The function  $g(x)$  is strictly increasing with  $g(0) = -1$  and  $g(1) = 1$ , while  $h(0) = 0$  and is strictly decreasing. A solution exists as both functions are continuous.
- The case  $\alpha > 0$ : The function  $g(x)$  is increasing with  $g(0) = -1$  and  $g(1) = 1$ , while  $h(0) = 0$  and  $h$  is strictly bounded by 1; hence by continuity a solution exists. For uniqueness of this solution, denote the smallest point of intersection of  $g$  and  $h$  by  $x_1$ . Note that  $x_1 > \frac{1}{2}$ , since  $g(\frac{1}{2}) = 0$  and  $h(\frac{1}{2}) > 0$ . Since  $g(x) < h(x)$  in the interval  $[0, x_1)$ , the derivative  $h'$  must be no greater than  $g' = 2$  at  $x_1$ . But in order for there to be another point of intersection, the derivative must be larger than 2 at some point in the interval  $[x_1, 1]$ . Differentiating, we have

$$(26) \quad h'(x) = \frac{2\alpha x}{\cosh^2(\alpha x^2)}.$$

Differentiating again, we have

$$h''(x) = \frac{2\alpha}{\cosh^2(\alpha x^2)}(1 - 4\alpha x^2 \tanh(\alpha x^2)).$$

The maximum of the derivative is attained when the second derivative is 0, that is,  $1 - 4\alpha x^2 \tanh(\alpha x^2) = 0$ . This implies that  $2\alpha x = \frac{1}{2 \tanh(\alpha x^2)}$ . Substituting this into equation (26), we get that for all  $\frac{1}{2} < x < 1$ ,

$$\begin{aligned} h'(x) &= \frac{2\alpha x}{\cosh^2(\alpha x^2)} \\ &\leq \frac{1}{x \cdot 2 \tanh(\alpha x^2) \cosh^2(\alpha x^2)} \\ &= \frac{1}{x \cdot 2 \sinh(\alpha x^2) \cosh(\alpha x^2)} \\ &= \frac{1}{x \cdot \sinh(2\alpha x^2)} < 2 \end{aligned}$$

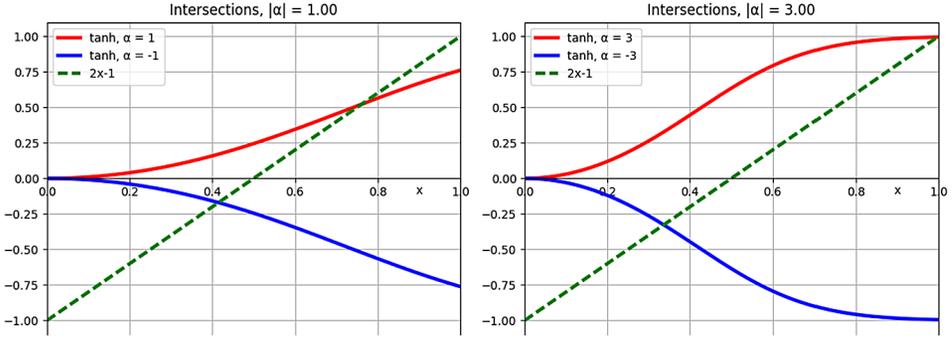


FIG. 2. Two examples showing that there is only one intersection between  $2x - 1$  and  $\tanh(\alpha x^2)$ .

since  $x > \frac{1}{2}$  and  $\sinh(2\alpha x^2) > 1$ . Hence no other intersection point exists.

See Figure 2 for a visual illustration of  $g$  and  $h$ .  $\square$

**PROOF OF THEOREM 21.** For simplicity, instead of solving the equation  $X = \frac{1 + \tanh(\frac{3\beta}{N-2}\overline{X^2})}{2}$  for negative  $\beta$ , we will solve the equation  $X = \frac{1 - \tanh(\frac{\beta}{N-2}\overline{X^2})}{2}$  for positive  $\beta$  (where we assimilated the factor of 3 inside  $\beta$ ).

Denote by  $v_1$  the vector  $(1, 1, \dots, 1)$  whose entries are all 1, and by  $v_2$  the vector  $(-1, \dots, -1, 1, \dots, 1)$  whose first  $N/2$  entries are  $-1$  and whose second  $N/2$  entries are 1. Let

$$X = \alpha_1 v_1 v_1^T + \alpha_2 v_2 v_2^T - I(\alpha_1 + \alpha_2).$$

Then  $X$  is a symmetric matrix with 0 on the diagonal,  $\alpha_1 + \alpha_2$  in the top left and bottom right quarters, and  $\alpha_1 - \alpha_2$  in the top right and bottom left quarters. Squaring  $X$ , we get

$$\begin{aligned} X^2 &= (\alpha_1 v_1 v_1^T + \alpha_2 v_2 v_2^T - I(\alpha_1 + \alpha_2))^2 \\ &= \alpha_1^2 (v_1 v_1^T)^2 + \alpha_2 (v_2 v_2^T)^2 + I(\alpha_1 + \alpha_2)^2 \\ &\quad - 2\alpha_1(\alpha_1 + \alpha_2)v_1 v_1^T - 2\alpha_2(\alpha_1 + \alpha_2)v_2 v_2^T \\ &= (\alpha_1^2(N - 2) - 2\alpha_1\alpha_2)v_1 v_1^T \\ &\quad + (\alpha_2^2(N - 2) - 2\alpha_1\alpha_2)v_2 v_2^T + I(\alpha_1 + \alpha_2)^2. \end{aligned}$$

Setting the diagonal to zero, we have

$$\begin{aligned} \overline{X^2} &= (\alpha_1^2(N - 2) - 2\alpha_1\alpha_2)v_1 v_1^T + (\alpha_2^2(N - 2) - 2\alpha_1\alpha_2)v_2 v_2^T \\ &\quad - I(\alpha_1^2(N - 2) + \alpha_2^2(N - 2) - 4\alpha_1\alpha_2). \end{aligned}$$

So  $\frac{\beta}{N-2}\overline{X^2}$  is a symmetric matrix with 0 on the diagonal,  $\frac{\beta}{N-2}((\alpha_1^2 + \alpha_2^2)(N - 2) - 4\alpha_1\alpha_2)$  in the top left and bottom right quarters, and  $\beta(\alpha_1^2 - \alpha_2^2)$  in the top right and bottom left quarters. The matrix  $\tanh(\frac{\beta}{N-2}\overline{X^2})$  can then also be written as a sum of the form  $av_1v_1^T + bv_2v_2^T - I(a + b)$ , where

$$(27) \quad \begin{aligned} \tanh\left(\beta\left((\alpha_1^2 + \alpha_2^2)(N - 2) - \frac{4}{N - 2}\alpha_1\alpha_2\right)\right) &= a + b, \\ \tanh(\beta(\alpha_1^2 - \alpha_2^2)) &= a - b. \end{aligned}$$

The expression  $\frac{1 - \tanh(\frac{\beta}{N-2}\overline{X^2})}{2}$  can then be written as

$$\frac{1 - \tanh(\frac{\beta}{N-2}\overline{X^2})}{2} = \frac{1 - a}{2}v_1v_1^T - \frac{b}{2}v_2v_2^T + I\left(\frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}\right).$$

Equating this with  $X$ , we get

$$\begin{aligned} \alpha_1 &= \frac{1 - a}{2}, \\ \alpha_2 &= -\frac{b}{2}. \end{aligned}$$

Rearranging and plugging into equation (27), we obtain the following two equations in two variables:

$$(28) \quad \begin{aligned} \tanh\left(\beta\left((\alpha_1^2 + \alpha_2^2) - \frac{4}{N - 2}\alpha_1\alpha_2\right)\right) &= 1 - 2\alpha_1 - 2\alpha_2, \\ \tanh(\beta(\alpha_1^2 - \alpha_2^2)) &= 1 - 2\alpha_1 + 2\alpha_2. \end{aligned}$$

We will now show that for large enough  $\beta$ , these equations have at least two solutions. As shown in Lemma 17, there is always a constant  $X = c \cdot \mathbf{1}$  is solution to the fixed-point equation (7). It corresponds to the case  $\alpha_2 = 0$ ; in this case the two equations both identify to  $\tanh(\beta\alpha_1^2) = 1 - 2\alpha_1$ . We must therefore show that that for large enough  $\beta$ , there is a solution with  $\alpha_2 \neq 0$ .

Let us change variables in order to bring the equations to a more friendly form. Denote  $x = \alpha_1 + \alpha_2$  and  $y = \alpha_1 - \alpha_2$ . Then  $\alpha_1^2 - \alpha_2^2 = xy$ ,  $\alpha_1^2 + \alpha_2^2 = \frac{1}{2}(x^2 + y^2)$  and  $\alpha_1\alpha_2 = \frac{1}{4}(x^2 - y^2)$ , and (28) can be rewritten as

$$(29) \quad \begin{aligned} \tanh\left(\frac{\beta}{N - 2}\left(\frac{N - 4}{2}x^2 + \frac{N}{2}y^2\right)\right) &= 1 - 2x, \\ \tanh(\beta xy) &= 1 - 2y. \end{aligned}$$

We now need to show that there exists a solution with  $x \neq y$ .

The matrix  $X$  has entries in  $[0, 1]$ , so we know that

$$\begin{aligned} 0 &\leq \alpha_1 - \alpha_2 \leq 1, \\ 0 &\leq \alpha_1 + \alpha_2 \leq 1. \end{aligned}$$

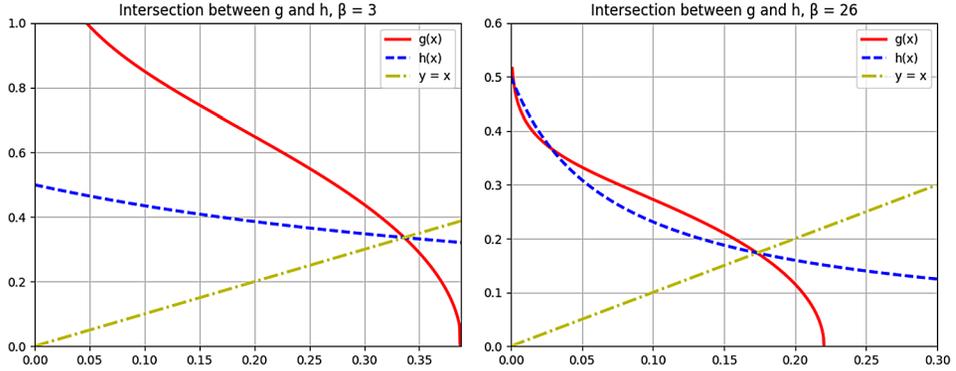


FIG. 3. Two examples illustrating the behavior of  $g$  and  $h$ . For small  $\beta$ , there is only one intersection between them, while for large  $\beta$  there are 3. For making these images,  $N$  was assumed large enough so that  $N - 4 \approx N - 2 \approx N$ .

Hence  $x$  and  $y$  are also in  $[0, 1]$ . For the first equation in (29), if  $x$  is small enough, then there is a unique  $y \in \mathbb{R}$  the satisfies it. Denote this  $y$  by  $g(x)$ ; its range and domain will be calculated later. For the second equation, a unique  $y \in (0, 1)$  exists for all  $x \in [0, 1]$  since  $\tanh(\beta xy)$  is an increasing function of  $y$  while  $1 - 2y$  is a decreasing function  $y$ . Denote this  $y$  by  $h(x) : [0, 1] \rightarrow (0, 1)$ .

Showing that a nonconstant solution exists therefore requires showing that  $g$  and  $h$  intersect at a point for which  $x \neq y$ . Figure 3 shows that this is indeed the case for large enough  $\beta$  (by numerical calculations, the solution first appears at around  $\beta \approx 22$ , if we approximate  $N - 4 \approx N - 2 \approx N$ ).

Let us now grit our teeth and show this result analytically. First, consider  $h$ . It satisfies the functional equation

$$\tanh(\beta x h(x)) - 1 + 2h(x) = 0.$$

At  $x = 0$ , we must have  $h(0) = \frac{1}{2}$ . Differentiating, we get

$$\frac{\beta(h(x) + xh'(x))}{\cosh^2(\beta x h(x))} + 2h'(x) = 0.$$

Isolating  $h'$ , we obtain

$$h'(x) = -\frac{\beta h(x)}{\beta x + 2 \cosh^2(\beta x h(x))}.$$

Thus  $h$  is decreasing. Forgoing calculations, differentiating again shows that  $h''$  is positive. Hence  $h'$  is increasing, so we can bound  $h'$  by

$$\begin{aligned} h'(x) &\geq h'(0) \\ (30) \quad &= -\frac{\beta h(0)}{\beta \cdot 0 + 2 \cosh^2(\beta \cdot 0 \cdot h(0))} \\ &= -\beta/2. \end{aligned}$$

Now consider  $g$ . It satisfies the functional equation

$$(31) \quad \tanh\left(\frac{\beta}{N-2}\left(\frac{N-4}{2}x^2 + \frac{N}{2}g^2(x)\right)\right) - 1 + 2x = 0.$$

First, let us calculate its domain.

There exists an  $x_1 > 0$  such that  $g(x_1) = 1$ . Indeed, setting  $g(x) = 1$ , we have

$$\tanh\left(\frac{\beta}{2}\left(\frac{N-4}{N-2}x^2 + \frac{N}{N-2}\right)\right) = 1 - 2x.$$

At  $x = 0$ , the left-hand side is equal to  $\tanh(\frac{\beta}{2}\frac{N}{N-2})$ , which is smaller than 1. The left-hand side is increasing as a function of  $x$ , while the right-hand side is decreasing as a function of  $x$ , with derivative  $-2$ . Hence a solution  $x_1$  exists, with

$$x_1 \leq \frac{1 - \tanh(\frac{\beta}{2})}{2}.$$

Using  $\tanh(z) = \frac{1-e^{-2z}}{1+e^{2z}}$ , this can also be written as

$$x_1 \leq \frac{1 - \frac{1-e^{-\beta}}{1+e^{-\beta}}}{2} = \frac{\frac{2e^{-\beta}}{1+e^{-\beta}}}{2} = \frac{2e^{-\beta}}{1+e^{-\beta}} \leq 2e^{-\beta}.$$

There exists an  $x_2$  such that  $g(x_2) = 0$ . Indeed, setting  $g(x) = 0$ , we get

$$\tanh\left(\beta\frac{N-4}{2(N-2)}x^2\right) = 1 - 2x,$$

and a unique solution exists by Lemma 33. It is clear that for all  $x_1 < x < x_2$ , a unique solution exists for  $g(x)$ . Differentiating equation (31), we get

$$\frac{\frac{\beta}{N-2}((N-4)x + Ng(x)g'(x))}{\cosh^2(\frac{\beta}{N-2}(\frac{N-4}{2}x^2 + \frac{N}{2}g^2(x)))} + 2 = 0,$$

and isolating  $g'$ , we obtain

$$g'(x) = \frac{-2 \cosh^2(\frac{\beta}{N-2}(\frac{N-4}{2}x^2 + \frac{N}{2}g^2(x))) - \frac{\beta}{N-2}(N-4)x}{\beta g(x)}.$$

This is negative, and so  $g$  is decreasing. The domain of  $g$  is therefore  $[x_1, x_2]$ , and its range is  $[0, 1]$ .

We may now finally inspect the intersection of  $g$  and  $h$ . Let  $\varepsilon = \frac{2}{\beta^2}$ , and let  $\beta$  be large enough so that  $\frac{1}{2}\varepsilon = \frac{1}{\beta^2} > 2e^{-\beta} > x_1$ ; this implies that  $\varepsilon - x_1 \geq \frac{1}{\beta^2}$ . By (30) and the fact that  $h(0) = \frac{1}{2}$ , we have that

$$\begin{aligned} h(\varepsilon) &\geq \frac{1}{2} - \varepsilon\beta/2 \\ &= \frac{1}{2} - \frac{1}{\beta}. \end{aligned}$$

Assume by contradiction that in the interval  $[x_1, \varepsilon]$ , there is no intersection between  $g$  and  $h$ . Since  $g(x_1) = 1 > h(x_1)$ , this means that in  $g(x) > h(x)$  for the entire interval  $[x_1, \varepsilon]$ . In particular, we have  $g(x) > \frac{1}{2} - \frac{1}{\beta}$ . We can then give a bound on the derivative  $g'$ :

$$\begin{aligned} g'(x) &= \frac{-2 \cosh^2(\frac{\beta}{N-2}(\frac{N-4}{2}x^2 + \frac{N}{2}g^2(x))) - \frac{\beta}{N-2}(N-4)x}{\beta g(x)} \\ &\leq \frac{-2 \cosh^2(\beta g^2(x))}{\beta} \\ &\leq \frac{-2 \cosh^2(\beta(\frac{1}{2} - \frac{1}{\beta})^2)}{\beta} \\ (\text{for } \beta > 4) &\leq \frac{-2 \cosh^2(\beta(\frac{1}{4})^2)}{\beta} \\ &= \frac{-2 \cosh^2(\frac{\beta}{16})}{\beta}. \end{aligned}$$

We then have

$$\begin{aligned} g(\varepsilon) &\leq g(x_1) + (\varepsilon - x_1) \frac{-2 \cosh^2(\frac{\beta}{16})}{\beta} \\ &\leq 1 + \frac{1}{\beta^2} \frac{-2 \cosh^2(\frac{\beta}{16})}{\beta} \\ &= 1 - \frac{2 \cosh^2(\frac{\beta}{16})}{\beta^3}. \end{aligned}$$

This quantity goes to  $-\infty$  as  $\beta \rightarrow \infty$ . This is a contradiction, as we assumed  $g(x) \geq \frac{1}{2} - \frac{1}{\beta}$  in the interval  $[x_1, \varepsilon]$ . Thus for  $\beta$  large enough, the curves  $g$  and  $h$  intersect at a point  $x^* \in [x_1, \frac{2}{\beta^2}]$ . This intersection point satisfies  $g(x^*) \geq \frac{1}{2} - \frac{1}{\beta}$ ; for  $\beta > 4$ , we have  $y = g(x^*) > \frac{1}{4}$ . However,  $x^* \leq \frac{2}{\beta^2} < \frac{1}{4}$ . This intersection point does not satisfy  $x = y$  and, therefore, does not correspond to the constant solution.

Finally, as  $\beta \rightarrow \infty$ , it is clear that  $x^* \rightarrow 0$  and  $y^* = g(x^*) \rightarrow \frac{1}{2}$ , implying that  $\alpha_1 \rightarrow \frac{1}{4}$  and  $\alpha_2 \rightarrow -\frac{1}{4}$ , meaning that  $X$  tends to the adjacency matrix of a bipartite graph.  $\square$

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