

OPTIMAL DIVIDEND AND INVESTMENT PROBLEMS UNDER SPARRE ANDERSEN MODEL

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In this paper, we study a class of optimal dividend and investment problems assuming that the underlying reserve process follows the Sparre Andersen model, that is, the claim frequency is a “renewal” process, rather than a standard compound Poisson process. The main feature of such problems is that the underlying reserve dynamics, even in its simplest form, is no longer Markovian. By using the *backward Markovization* technique, we recast the problem in a Markovian framework with expanded dimension representing the time elapsed after the last claim, with which we investigate the regularity of the value function, and validate the dynamic programming principle. Furthermore, we show that the value function is the unique *constrained viscosity solution* to the associated HJB equation on a cylindrical domain on which the problem is well defined.

1. Introduction. The problem of maximizing the cumulative discounted dividend payout can be traced back to the seminal work of de Finetti [17] in 1957, when he proposed to measure the performance of an insurance portfolio by looking at the maximum possible dividend paid during its lifetime, instead of focussing only on the safety aspect measured by its ruin probability. Although other criteria such as the so-called Gordon model [21] as well as the simpler model by Miller–Modigliani [34] have been proposed over the years, to date the cumulative discounted dividend is still widely accepted as an important and useful performance index, and various approaches have been employed to find the optimal strategy that maximizes such index. The solution of the optimal dividend problem under the classical Cramér–Lundberg model has been obtained in various forms. Gerber [19] first showed that an optimal dividend strategy has a “band” structure. Since then the optimal dividend policies, especially the barrier strategies, have been investigated in various settings, sometimes under more general reserve models (see, e.g., [2, 3, 6, 20, 25, 27, 31, 35, 39], to mention a few). We refer the interested reader to the excellent 2009 survey by Albrecher–Thonhauser [4] and the exhaustive references cited therein for the past developments on this issue.

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The more general optimization problems for insurance models involving the possibility of investment and/or reinsurance have also been studied quite extensively in the past two decades. In 1995, Browne [15] first considered the problem of minimizing the probability of ruin under a diffusion approximated Cramér–Lundberg model, where the insurer is allowed to invest some fraction of the reserve dynamically into a Black–Scholes market. Hipp–Plum [22] later considered the same problem with a compound Poisson claim process. The problems involving either proportional or excess-of-loss reinsurance strategies have also been studied under the Cramér–Lundberg model or its diffusion approximations (see, e.g., [23–25, 38]). The optimal dividend and reinsurance problem with transaction cost and taxes was studied by the first author of this paper with various co-authorships [10–12]; whereas the ruin problems, reinsurance problems and universal variable insurance problems involving investment in the more general jump diffusion framework have been investigated by the second author [30, 32, 33], from the stochastic control perspective. We should remark that the two references that are closest to the present paper are Azcue–Muler [8, 9], obtained in 2005 and 2010, respectively. The former concerns the optimal dividend-reinsurance, and the latter concerns the optimal dividend-investment. Both papers followed the dynamic programming approach, and the analytic properties of the value function, including its being the viscosity solution to the associated Hamilton–Jacobi–Bellman (HJB) equation became the main purpose.

It is worth noting, however, that all aforementioned results are based on the Cramér–Lundberg type of surplus dynamics or its variations within the Markovian paradigm, whose analytical structure plays a fundamental role. A well-recognized generalization of such model is one in which the Poisson claim number process is replaced by a *renewal* process, known as the Sparre Andersen risk model [41]. The dividend problem under such a model is much subtler due to its *non-Markovian* nature in general, and the literature is much more limited. In this context, Li–Garrido [29] first studied the properties of the renewal risk reserve process with a barrier strategy. Later, after calculating the moments of the expected discounted dividend payments under a barrier strategy in [1], Albrecher–Hartinger [2] showed that, unlike the classical Cramér–Lundberg model, even in the case of Erlang (2) distributed interclaim times and exponentially distributed claim amounts, the horizontal barrier strategy is no longer optimal. Consequently, the optimal dividend problem under the Sparre Andersen models has since been listed as an open problem that requires attention (see [4]), and to the best of our knowledge, it remains unsolved to this day.

The main technical difficulties, from the stochastic control perspective, for a general optimal dividend problem under the Sparre Andersen model can be roughly summarized into two major points: the non-Markovian nature of the model, and the random duration of the insurance portfolio. We note that although the former would seemingly invalidate the dynamical programming approach, a Markovization is possible, by extending the dimension of the state space of the

risk process, taking into account the time elapsed since the last claim (see [4]). It turns out that such an extra variable would cause some subtle technical difficulties in analyzing the regularity of the value function. For example, as we shall see later, unlike the compound Poisson cases studied in [8, 9], even the continuity of the value function requires some heavy arguments, much less the Lipschitz properties, which play a fundamental role in a standard argument. For the latter issue, since we are focusing on the life of the portfolio until ruin, the optimization problem naturally has a random terminal time. While it is known in theory that such a problem can often be converted to one with a fixed (deterministic) terminal time (see, e.g., [14]) once the distribution of the random terminal is known, finding the distribution for the ruin time under Sparre Andersen model is itself a challenging problem, even under very explicit strategies (see, e.g., [1, 20, 29]), which makes the optimization problem technically prohibitive along this line.

This paper is our first attempt to attack this open problem. We will start with a rather simplified renewal reserve model but allowing both investment and dividend payments. As was suggested in [4], our plan is to first “Markovize” the model and then study the optimal dividend problem via the dynamic programming approach. Specifically, we shall first investigate the property of the value function and then validate the *dynamic programming principle* (DPP), from which we can formally derive the associated HJB equation to which the value function is a solution in some sense. An important observation, however, is that the value function could very well be discontinuous at the boundary of a region on which it is well defined, and no explicit boundary condition can be established directly from the information of the problem. Among other things, the lack of boundary information of the HJB equation will make the comparison principle, whence uniqueness, particularly subtle, if not impossible. To overcome this difficulty, we shall invoke the notion of *constrained viscosity solution* for the exit problems (see, e.g., Soner [40]), and as it turns out we can prove that the value function is indeed a constrained viscosity solution to the associated HJB equation on an appropriately defined domain, completing the dynamic programming approach on this problem. To the best of our knowledge, these results are novel.

The rest of the paper is organized as follows. In Section 2, we establish the basic setting, formulate the problem and introduce the backward Markovization technique. In Section 3, we study the properties of the value function and prove the continuity of the value function in the temporal variable. In Sections 4 and 5, we prove the continuity of the value function in variables x and w , respectively. In Section 6, we validate the Dynamic Programming Principle (DPP), and in Section 7 we show that the value function is a constrained viscosity solution to the HJB equation. Finally, in Section 8 we prove the comparison principle, hence prove that the value function is the unique constrained viscosity solution among a fairly general class of functions.

2. Preliminaries and problem formulation. Throughout this paper, we assume that all uncertainties come from a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is defined d -dimensional Brownian motion $B = \{B_t : t \geq 0\}$, and a *renewal* counting process $N = \{N_t\}_{t \geq 0}$, independent of B . More precisely, we denote $\{\sigma_n\}_{n=1}^\infty$ to be the jump times ($\sigma_0 := 0$) of the counting process N , and $T_i = \sigma_i - \sigma_{i-1}$, $i = 1, 2, \dots$ to be its waiting times (the time elapses between successive jumps). We assume that T_i 's are independent and identically distributed, with a common distribution $F : \mathbb{R}_+ \mapsto \mathbb{R}_+$; and that there exists an *intensity function* $\lambda : [0, \infty) \mapsto [0, \infty)$ such that $\bar{F}(t) := \mathbb{P}\{T_1 > t\} = \exp\{-\int_0^t \lambda(u) du\}$. In other words, $\lambda(t) = f(t)/\bar{F}(t)$, $t \geq 0$, where f is the common density function of T_i 's.

Further, throughout the paper we will denote, for a generic Euclidean space \mathbb{X} , regardless of its dimension, (\cdot, \cdot) and $|\cdot|$ to be its inner product and norm, respectively. Let $T > 0$ be a given time horizon, we denote the space of continuous functions taking values in \mathbb{X} with the usual sup-norm by $C([0, T]; \mathbb{X})$, and we shall make use of the following notation:

- For any sub- σ -field $\mathcal{G} \subseteq \mathcal{F}$ and $1 \leq p < \infty$, $L^p(\mathcal{G}; \mathbb{X})$ denotes the space of all \mathbb{X} -valued, \mathcal{G} -measurable random variables ξ such that $\mathbb{E}|\xi|^p < \infty$. As usual, $\xi \in L^\infty(\mathcal{G}; \mathbb{X})$ means that it is a bounded, \mathcal{G} -measurable random variable.
- For a given filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ in \mathcal{F} , and $1 \leq p < \infty$, $L^p_{\mathbb{F}}([0, T]; \mathbb{X})$ denotes the space of all \mathbb{X} -valued, \mathbb{F} -progressively measurable processes ξ satisfying $\mathbb{E} \int_0^T |\xi_t|^p dt < \infty$. The meaning of $L^\infty_{\mathbb{F}}([0, T]; \mathbb{X})$ is defined similarly.

2.1. *Backward Markovization and delayed renewal process.* An important ingredient of the Sparre Andersen model is the following “compound renewal process” that will be used to represent the *claim process* in our reserve mode: $Q_t = \sum_{i=1}^{N_t} U_i$, $t \geq 0$, where N is the renewal process representing the frequency of the incoming claims, whereas $\{U_i\}_{i=1}^\infty$ is a sequence of random variables representing the “size” of the incoming claims. We assume that $\{U_i\}$ are i.i.d. with a common distribution $G : \mathbb{R}_+ \mapsto \mathbb{R}_+$, independent of (N, B) .

The main feature of the Sparre Andersen model, which fundamentally differentiates this paper with all existing works is that the process Q is non-Markovian in general (unless the counting process N is a Poisson process), consequently we cannot directly apply the dynamic programming approach. We shall therefore first apply the so-called *Backward Markovization* technique (cf., e.g., [37]) to overcome this obstacle. More precisely, we define a new process $W_t = t - \sigma_{N_t}$, $t \geq 0$, be the time elapsed since the last jump. Then clearly, $0 \leq W_t \leq t \leq T$, for $t \in [0, T]$, and it is known (see, e.g., [37]) that the process (t, Q_t, W_t) , $t \geq 0$, is a piecewise deterministic Markov process (PDMP). We note that at each jump time σ_i , the jump size $|\Delta W_{\sigma_i}| = \sigma_i - \sigma_{i-1} = T_i$.

Throughout this paper, we consider the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, in which $\mathcal{F}_t := \mathcal{F}_t^B \vee \mathcal{F}_t^Q \vee \mathcal{F}_t^W$, $t \geq 0$. Here, $\{\mathcal{F}_t^\xi : t \geq 0\}$ denotes the natural filtration gener-

ated by process $\xi = B, Q, W$, respectively, with the usual \mathbb{P} -augmentation such that it satisfies the *usual hypotheses* (cf., e.g., [36]).

A very important element in the study of the dynamic optimal control problem with final horizon is to allow the starting point to be any time $s \in [0, T]$. In fact, this is one of the main subtleties in the Sparre Andersen model, which we now describe. Suppose that, instead of starting the clock at $t = 0$, we start from $s \in [0, T]$, such that $W_s = w$, \mathbb{P} -a.s. Let us consider the *regular conditional probability distribution* (RCPD) $\mathbb{P}_{sw}(\cdot) := \mathbb{P}[\cdot | W_s = w]$ on (Ω, \mathcal{F}) , and consider the “shifted” version of processes (B, Q, W) on the space $(\Omega, \mathcal{F}, \mathbb{P}_{sw}; \mathbb{F}^s)$, where $\mathbb{F}^s = \{\mathcal{F}_t\}_{t \geq s}$. We first define $B_t^s := B_t - B_s, t \geq s$. Clearly, since B is independent of (Q, W) , B^s is an \mathbb{F}^s -Brownian motion under \mathbb{P}_{sw} , defined on $[s, T]$, with $B_s^s = 0$. Next, we restart the clock at time $s \in [0, T]$ by defining the new counting process $N_t^s := N_t - N_s, t \in [s, T]$. Then, under \mathbb{P}_{sw} , N^s is a “delayed” renewal process, in the sense that while its waiting times $T_i^s, i \geq 2$, remain independent, identically distributed as the original T_i 's, its “time-to-first jump,” denoted by $T_1^{s,w} := T_{N_s+1} - w = \sigma_{N_s+1} - s$, should have the survival probability

$$(2.1) \quad \mathbb{P}_{sw}\{T_1^{s,w} > t\} = \mathbb{P}\{T_1 > t + w | T_1 > w\} = e^{\int_w^{w+t} \lambda(u) du}.$$

In what follows, we shall denote $N_t^s |_{W_s=w} := N_t^{s,w}, t \geq s$, to emphasize the dependence on w as well. Correspondingly, we shall denote $Q_t^{s,w} = \sum_{i=1}^{N_t^{s,w}} U_i$ and $W_t^{s,w} := w + W_t - W_s = w + [(t - s) - (\sigma_{N_t} - \sigma_{N_s})], t \geq s$. It is readily seen that $(B_t^s, Q_t^{s,w}, W_t^{s,w}), t \geq s$, is an \mathbb{F}^s -adapted process defined on $(\Omega, \mathcal{F}, \mathbb{P}_{sw})$, and it is Markovian.

2.2. *Optimal dividend-investment problem with the Sparre Andersen model.*

In this paper, we assume that the dynamics of surplus of an insurance company, denoted by $X = \{X_t\}_{t \geq 0}$, in the absence of dividend payments and investment, is described by the following *Sparre Andersen model* on the given probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$:

$$(2.2) \quad X_t = x + pt - Q_t := x + pt - \sum_{i=1}^{N_t} U_i, \quad t \in [0, T],$$

where $x = X_0 \geq 0, p > 0$ is the constant premium rate, and $Q_t = \sum_{i=1}^{N_t} U_i$ is the (renewal) claim process. We shall assume that the insurer is allowed to both invest its surplus in a financial market and will also pay dividends, and will try to maximize the dividend received before the ruin time of the insurance company. To be more precise, we shall assume that the financial market is described by the standard Black–Scholes model. That is, the prices of the risk-free and risky assets satisfy the following SDEs:

$$(2.3) \quad dS_t^0 = rS_t^0 dt, \quad dS_t = \mu S_t dt + \sigma S_t dB_t, \quad t \in [0, T],$$

where $B = \{B_t\}_{t \geq 0}$ is the given Brownian motion, r is the interest rate and $\mu > r$ is the appreciation rate of the stock.

With the same spirit, in this paper we shall consider a portfolio with only one risky asset and one bank account and define the control process by $\pi = (\gamma_t, L_t)$, $t \geq 0$, where $\gamma \in L^2_{\mathbb{F}}([0, T])$ is a self-financing strategy, representing the proportion of the surplus invested in the stock at time t (hence $\gamma_t \in [0, 1]$, for all $t \in [0, T]$), and $L \in L^2_{\mathbb{F}}([0, T])$ is the cumulative dividends the company has paid out up to time t (hence L is increasing). Throughout this paper, we will consider the the filtration $\mathbb{F} = \mathbb{F}^{(B, Q, W)}$, and we say that a control strategy $\pi = (\gamma_t, L_t)$ is *admissible* if it is \mathbb{F} -predictable with càdlàg paths, and square-integrable (i.e., $\mathbb{E}[\int_0^T |\gamma_t|^2 dt + |L_T|^2] < \infty$) and we denote the set of all admissible strategies restricted to $[s, t] \subseteq [0, T]$ by $\mathcal{U}_{ad}[s, t]$. Furthermore, we shall often use the notation $\mathcal{U}_{ad}^{s,w}[s, T]$ to specify the probability space $(\Omega, \mathcal{F}, \mathbb{P}_{sw})$, and denote $\mathcal{U}_{ad}^{0,0}[0, T]$ by $\mathcal{U}_{ad}[0, T] = \mathcal{U}_{ad}$ for simplicity.

By a standard argument using the self-financing property, one shows that, for $\pi \in \mathcal{U}_{ad}$ and initial surplus x , the dynamics of the controlled risk process X satisfies the following SDE:

$$(2.4) \quad \begin{aligned} dX_t^\pi &= p dt + r X_t^\pi dt + (\mu - r)\gamma_t X_t^\pi dt + \sigma \gamma_t X_t^\pi dB_t - dQ_t - dL_t, \\ X_0^\pi &= x, t \in [0, T]. \end{aligned}$$

We shall denote the solution to (2.4) by $X_t = X_t^\pi = X_t^{\pi,x}$, whenever the specification of (π, x) is necessary. Moreover, for any $\pi \in \mathcal{U}_{ad}$, we denote $\tau^\pi = \tau^{\pi,x} := \inf\{t \geq 0; X_t^{\pi,x} < 0\}$ to be the ruin time of the insurance company. We shall make use of the following *standing assumptions*.

ASSUMPTION 2.1. (a) The interest rate r , the volatility σ and the insurance premium p are all positive constants.

(b) The distribution functions F (of T_i 's) and G (of U_i 's) are continuous on $[0, \infty)$. Furthermore, F is absolutely continuous, with density function f and intensity function $\lambda(t) := f(t)/\bar{F}(t) > 0$, $t \in [0, T]$.

(c) The cumulative dividend process L is absolutely continuous with respect to the Lebesgue measure, that is, there exists $a \in L^2_{\mathbb{F}}([0, T]; \mathbb{R}_+)$, such that $L_t = \int_0^t a_s ds$, $t \geq 0$. We assume further that for some constant $M \geq p > 0$, it holds that $0 \leq a_t \leq M$, $dt \times d\mathbb{P}$ -a.e.

REMARK 2.2. Assumption 2.1(c) is merely technical, and it is not unusual; see, for example, [4, 20, 27]. But this assumption will certainly exclude the possibility of having “singular” type of strategies, which could very well be the form of an optimal strategy in this kind of problem. However, since in this paper our main focus is to deal with the difficulty caused by the renewal feature of the model, we are content with such an assumption.

We should note that the surplus dynamics (2.4) with Assumption 2.1(a) is in the simplest form. More general dynamics with carefully posed assumptions is clearly possible, but not essential for the main results of this paper. In fact, as we can see later, even in this simple form the technical difficulties are already significant. We therefore prefer not to pursue the generality of the surplus dynamics in the current paper so as not to disturb the already lengthy presentation. In the rest of the paper, we shall consider, for given $s \in [0, T]$, the following SDE [recall (2.4) and Remark 2.2 on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}_{sw}; \mathbb{F}^s)$]: for $(\gamma, a) \in \mathcal{U}_{ad}^{s,w}[s, T]$, and $t \in [s, T]$,

$$(2.5) \quad \begin{cases} X_t^\pi = x + p(t-s) + \int_s^t [r + (\mu - r)\gamma_u] X_u^\pi du \\ \quad + \sigma \int_s^t \gamma_u X_u^\pi dB_u - Q_t^{s,w} - \int_s^t a_u du, \\ W_t := w + (t-s) - (\sigma_{N_t} - \sigma_{N_s}). \end{cases}$$

We denote $(X^\pi, W) = (X^{\pi,s,x,w}, W^{s,w})$ when the dependence on (s, x, w) needs to be emphasized.

We now describe our optimization problem. Given an admissible strategy $\pi \in \mathcal{U}_{ad}^{s,w}[s, T]$, we define the *cost functional*, for the given initial data (s, x, w) and the state dynamics (2.5), as

$$(2.6) \quad \begin{aligned} J(s, x, w; \pi) &= \mathbb{E}_{sw} \left\{ \int_s^{\tau_s^\pi \wedge T} e^{-c(t-s)} dL_t \mid X_s^\pi = x \right\} \\ &:= \mathbb{E}_{s,x,w} \left\{ \int_s^{\tau_s^\pi \wedge T} e^{-c(t-s)} dL_t \right\}. \end{aligned}$$

Here, $c > 0$ is the *discounting factor* and $\tau_s^\pi = \tau_s^{\pi,s,x,w} := \inf\{t > s : X_t^{\pi,s,x,w} < 0\}$ is the ruin time of the insurance company. Namely, $J(s, x, w; \pi)$ is the expected total discounted dividend received until ruin. Our objective is to find the optimal strategy $\pi^* \in \mathcal{U}_{ad}[s, T]$ such that

$$(2.7) \quad V(s, x, w) := \sup_{\pi \in \mathcal{U}_{ad}[s, T]} J(s, x, w; \pi).$$

We note that the value function should be defined for $(s, x, w) \in D$ where $D = \{(s, x, w) : 0 \leq s \leq T, x \geq 0, 0 \leq w \leq s\}$. Here, $0 \leq w \leq s$ is due to the fact that we are considering the *ordinary* renewal case so that the clock process W satisfies $W_t \leq t$ for $t \in [0, T]$ ($W_t = t$ only if there is no claims in $[0, t]$). We make the convention that $V(s, x, w) = 0$, for $(s, x, w) \notin D$. We shall frequently carry out our discussion on the following two sets:

$$(2.8) \quad \begin{aligned} \mathcal{D} &:= \text{int}D = \{(s, x, w) \in D : 0 < s < T, x > 0, 0 < w < s\}; \\ \mathcal{D}^* &:= \{(s, x, w) \in D : 0 \leq s < T, x \geq 0, 0 \leq w \leq s\}. \end{aligned}$$

We note that $\mathcal{D} \subset \mathcal{D}^* \subseteq \bar{\mathcal{D}} = D$, the closure of \mathcal{D} , and \mathcal{D}^* does not include boundary $s = T$.

To end this section, we list two technical lemmas that will be useful in our discussion. The proofs of these lemmas are very similar to the Brownian motion case (cf., e.g., [42], Chapter 3), along the lines of *Monotone Class Theorem* and *Regular Conditional Probability Distribution* (RCPD), we therefore omit them. Let us denote $D_T^m := D([0, T]; \mathbb{R}^m)$, the space of all \mathbb{R}^m -valued càdlàg functions on $[0, T]$, endowed with the sup-norm, and $\mathcal{B}_T^m := \mathcal{B}(D_T^m)$, the topological Borel field on D_T^m . Let $D_t^m := \{\zeta_{\cdot \wedge t} | \zeta \in D_T^m\}$, $\mathcal{B}_t^m := \mathcal{B}(D_t^m)$, $t \in [0, T]$, and $\mathcal{B}_{t+}^m := \bigcap_{s>t} \mathcal{B}_s^m$, $t \in [0, T]$. For a generic Euclidean space \mathbb{X} , we denote $\mathcal{A}_T^m(\mathbb{X})$ to be the set of all $\{\mathcal{B}_{t+}^m\}_{t \geq 0}$ -progressively measurable process $\eta : [0, T] \times D_T^m \rightarrow \mathbb{X}$, that is, for any $\phi \in \mathcal{A}_T^m(\mathbb{X})$, it holds that $\phi(t, \eta) = \phi(t, \eta_{\cdot \wedge t})$, for $t \in [0, T]$ and $\eta \in D_T^m$. As usual, we denote $\mathcal{A}_T^m = \mathcal{A}_T^m(\mathbb{R})$ for simplicity.

LEMMA 2.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and $\zeta : \Omega \rightarrow D_T^m$ be a D^m -valued process. Let $\mathcal{F}_t^\zeta = \sigma\{\zeta(s) : 0 \leq s \leq t\}$. Then $\phi : [0, T] \times \Omega \mapsto \mathbb{X}$ is $\{\mathcal{F}_t^\zeta\}_{t \geq 0}$ -adapted if and only if there exists an $\eta \in \mathcal{A}_T^m(\mathbb{X})$ such that $\phi(t, \omega) = \eta(t, \zeta_{\cdot \wedge t}(\omega))$, \mathbb{P} -a.s.- $\omega \in \Omega$, $\forall t \in [0, T]$.*

LEMMA 2.4. *Let $(s, x, w) \in D$ and $\pi = (\gamma, a) \in \mathcal{U}_{ad}[s, T]$. Then for any stopping time $\tau \in [s, \tau^\pi]$, \mathbb{P} -a.s., and any \mathcal{F}_τ -measurable random variable (ξ, η) taking values in $[0, \infty) \times [0, T]$, it holds that*

$$(2.9) \quad J(\tau, \xi(\omega), \eta(\omega); \pi) = \mathbb{E} \left\{ \int_\tau^{\tau^\pi} e^{-c(t-\tau)} a_t dt \middle| \mathcal{F}_\tau \right\}(\omega)$$

for \mathbb{P} -a.s. $\omega \in \Omega$.

3. Basic properties of the value function. In this section, we present some results that characterize the regularity of the value function V . We note that the presence of the renewal process, and consequently the clock process W , changes the nature of the dynamics significantly. In fact, even in this simple setting, many well-known properties of the value function becomes either invalid, or much less obvious.

We begin by making some simple but important observations, which will be used throughout the paper. First, we note that in the absence of claims (or in between the jumps of N), for a given $\pi = (\gamma, a) \in \mathcal{U}_{ad}[s, T]$, the dynamics of the surplus follows a nonhomogeneous linear SDE (2.5) with $Q^{s,w} \equiv 0$, and has the explicit form (cf. [28], page 361)

$$(3.1) \quad X_t^\pi = Z_t^s \left[X_s^\pi + \int_s^t [Z_u^s]^{-1} (p - a_u) du \right], \quad t \in [s, T],$$

where $Z_t^s := \exp\{\int_s^t [r + (\mu - r)\gamma_u] du + \sigma \int_s^t \gamma_u dB_u - \frac{\sigma^2}{2} \int_s^t |\gamma_u|^2 du\}$. From (2.2) and (3.1), it is clear that in the absence of claims, the surplus $X_t < 0$ would never

happen if one does not over pay the dividend whenever $X_t = 0$. For example, if we consider only those $\pi \in \mathcal{U}_{ad}$ such that $(p - a_t)\mathbf{1}_{\{X_t^\pi=0\}} \geq 0$, \mathbb{P} -a.s., then we have $dX_t^\pi \geq 0$, whenever $X_t^\pi = 0$, which implies that $X_t^\pi \geq 0$ holds for all $t \geq 0$. Such an assumption, however, would cause some unnecessary complications on the well-posedness of the SDE (2.2). We shall argue slightly differently.

Since it is intuitively clear that the dividends should only be paid when reserve is positive, we suspect that any $\pi \in \mathcal{U}_{ad}$ such that τ^π occurs in between claim times (i.e., caused by overpaying dividends) can never be optimal. The following result justifies this point.

LEMMA 3.1. *Suppose that $\pi \in \mathcal{U}_{ad}$ is such that $\mathbb{P}\{\sigma_i \wedge T < \tau^\pi < \sigma_{i+1} \wedge T\} > 0$, for some $i \in \mathbb{N}$, where σ_i 's are the jump times of N , then there exists $\tilde{\pi} \in \mathcal{U}_{ad}$ such that $\mathbb{P}\{\tau^{\tilde{\pi}} \in \bigcup_{i=1}^\infty \sigma_i\} = 1$, and $J(s, x, w; \tilde{\pi}) > J(s, x, w; \pi)$.*

PROOF. Without loss of generality, we assume $s = w = 0$. We first note from (3.1) that on the set $\{\sigma_i \wedge T < \tau^\pi < \sigma_{i+1} \wedge T\}$, one must have $X_{\tau^\pi-}^\pi = X_{\tau^\pi}^\pi = 0$, and for some $\delta > 0$, $a_t > p$ for $t \in [\tau^\pi, \tau^\pi + \delta]$. Now define $\tilde{\pi}_t := \pi_t \mathbf{1}_{\{t < \tau^\pi\}} + (0, p)\mathbf{1}_{\{t \geq \tau^\pi\}}$, and denote $\tilde{X} := X^{\tilde{\pi}}$. Then clearly, $\tilde{X}_t = X_t^\pi$ for all $t \in [0, \tau^\pi]$, \mathbb{P} -a.s., and $d\tilde{X}_{\tau^\pi} = (p - \tilde{a}_{\tau^\pi}) dt = 0$. Consequently, $\tilde{X}_t \equiv 0$ for $t \in [\tau^\pi, \sigma_{i+1} \wedge T]$ and $\tilde{X}_{\sigma_{i+1}} < 0$ on $\{\sigma_{i+1} < T\}$. In other words, $\tau^{\tilde{\pi}} = \sigma_{i+1}$, and thus,

$$\begin{aligned} J(0, x, 0; \tilde{\pi}) &= \mathbb{E}\left[\int_0^{\tau^{\tilde{\pi}} \wedge T} e^{-ct} a_t dt\right] \\ &\geq J(0, x, 0; \pi) + \mathbb{E}\left[\int_{\tau^\pi}^{\sigma_{i+1} \wedge T} p e^{-ct} dt : \sigma_i \wedge T < \tau^\pi < \sigma_{i+1} \wedge T\right] \\ &> J(0, x, 0; \pi), \end{aligned}$$

since $\mathbb{P}\{\sigma_i \wedge T < \tau^\pi < \sigma_{i+1} \wedge T\} > 0$, proving the lemma. \square

We remark that Lemma 3.1 amounts to saying that for an optimal policy it is necessary that ruin only occurs at the arrival of a claim. Thus, in the sequel we shall consider a slightly fine-tuned set of *admissible strategies*:

$$(3.2) \quad \tilde{\mathcal{U}}_{ad} := \{\pi = (\gamma, a) \in \mathcal{U}_{ad} : \Delta X_{\tau^\pi}^\pi \mathbf{1}_{\{\tau^\pi < T\}} < 0, \mathbb{P}\text{-a.s.}\}.$$

The set $\tilde{\mathcal{U}}_{ad}[s, T]$ is defined similarly for $s \in [0, T]$, and we shall often drop the “ $\tilde{}$ ” for simplicity.

We now list some generic properties of the value function.

PROPOSITION 3.2. *Assume that Assumption 2.1 is in force. Then the value function V enjoys the following properties:*

- (i) $V(s, x, w)$ is increasing with respect to x ;

- (ii) $V(s, x, w) \leq \frac{M}{c}(1 - e^{-c(T-s)})$ for any $(s, x, w) \in D$, where $M > 0$ is the constant given in Assumption 2.1; and
- (iii) $\lim_{x \rightarrow \infty} V(s, x, w) = \frac{M}{c}[1 - e^{-c(T-s)}]$, for $0 \leq s \leq T, 0 \leq w \leq s$.

PROOF. (i), (ii) follow from (3.1) and the estimate $V(s, x, w) \leq \int_s^T e^{-c(t-s)} M dt = \frac{c}{M}[1 - e^{-c(T-s)}]$.

To see (iii), we consider a simple strategy: $\pi^0 := (\gamma, a) \equiv (0, M)$. Then we can write

$$(3.3) \quad \begin{aligned} X_t^{\pi^0, x, w} &= e^{r(t-s)}x + \frac{p - M}{r}(1 - e^{-r(t-s)}) - \int_s^t e^{r(t-u)} dQ_u^{s, w}, \\ t &\in [s, T], \end{aligned}$$

and it is obvious that $\lim_{x \rightarrow \infty} \tau_s^{\pi^0, x, w} \wedge T = T, \mathbb{P}$ -a.s. Thus we have

$$\begin{aligned} V(s, x, w) &\geq J(s, x, w; \pi^0) = \mathbb{E} \left[\int_s^{\tau_s^{\pi^0, x, w} \wedge T} e^{-c(t-s)} M dt \right] \\ &= \frac{M}{c} \mathbb{E} [1 - e^{-c(\tau_s^{\pi^0, x, w} \wedge T - s)}]. \end{aligned}$$

By the bounded convergence theorem, we have $\lim_{x \rightarrow \infty} V(s, x, w) \geq \frac{M}{c}(1 - e^{-c(T-s)})$. This, combined with (ii), leads to (iii). \square

In the rest of this subsection, we study the continuity of the value function $V(s, x, w)$ on the temporal variable s , for fixed initial state (x, w) . We have the following result.

PROPOSITION 3.3. Assume Assumption 2.1. Then $\forall (s, x, w), (s + h, x, w) \in D, h > 0$, it holds:

- (a) $V(s + h, x, w) - V(s, x, w) \leq 0$;
- (b) $V(s, x, w) - V(s + h, x, w) \leq Mh$, where $M > 0$ is the constant in Assumption 2.1.

PROOF. We note that the main difficulty here is that, given (s, x, w) , the process $Q_t^{s, w} = \sum_{i=1}^{N_t^{s, w}} U_i$ and the ‘‘clock’’ process $W_t^{s, w}$ cannot be controlled, thus it is not possible to keep the process W ‘‘frozen’’ at the initial state w during the time interval $[s, s + h]$ by any control strategy. We shall try to get around this by adopting the idea of ‘‘time shift’’ so as to freeze the w -clock.

To be more precise, let us assume that $s = 0$ and $w = 0$, other cases can be argued similarly. For $h \in (0, T)$, let $\pi \in \mathcal{U}_{ad}^{h, 0}[h, T]$. We define $\bar{\pi}_t^h = (\bar{\gamma}_t, \bar{a}_t) := (\gamma_{h+t}, a_{h+t}), t \in [0, T - h]$. Then $\bar{\pi}^h$ is adapted to the filtration $\bar{\mathbb{F}}^h := \{\mathcal{F}_{h+t}\}_{t \geq 0}$. Consider the optimization problem on the new probability set-up

$(\Omega, \mathcal{F}, \mathbb{P}_{h0}, \bar{\mathbb{F}}^h; \bar{B}^h, \bar{Q}^h, \bar{W}^h)$, where $(\bar{B}_t^h, \bar{Q}_t^h, \bar{W}_t^h) = (B_{h+t}^h, Q_{h+t}^h, W_{h+t}^h), t \geq 0$. Let us denote the corresponding admissible control set by $\mathcal{U}_{ad}^{\bar{\pi}^h}[0, T - h]$. Then $\bar{\pi}^h \in \mathcal{U}_{ad}^{\bar{\pi}^h}[0, T - h]$, and the corresponding surplus process, denoted by $\bar{X}^{\bar{\pi}^h}$, should satisfy the SDE

$$(3.4) \quad \begin{aligned} \bar{X}_t^{\bar{\pi}^h} = & x + pt + \int_0^t [r + (\mu - r)\bar{\gamma}_u] \bar{X}_u^{\bar{\pi}^h} du \\ & + \sigma \int_0^t \bar{\gamma}_u \bar{X}_u^{\bar{\pi}^h} d\bar{B}_u^h - \bar{Q}_t^h - \int_0^t \bar{a}_u du, \quad t \geq 0. \end{aligned}$$

Since the SDE is obviously pathwisely unique, whence unique in law, we see that the laws of $\{\bar{X}_t^{\bar{\pi}^h}\}_{t \geq 0}$ and that of $\{X_{h+t}^\pi\}_{t \geq 0}$ [which satisfies (2.5), with $s = h$], under \mathbb{P}_{h0} , are identical. In other words, if we specify the time duration in the cost functional, then we should have

$$(3.5) \quad \left\{ \begin{aligned} J_{h,T}(h, x, 0; \pi) &:= \mathbb{E}_{h0} \left[\int_h^{\tau^\pi \wedge T} e^{-c(t-h)} a_t dt \mid X_h^\pi = x \right] \\ &= \mathbb{E}_{h0} \left[\int_0^{\tau^{\bar{\pi}^h} \wedge (T-h)} e^{-ct} \bar{a}_t dt \mid \bar{X}_0^{\bar{\pi}^h} = x \right] \\ &=: \bar{J}_{0,T-h}(0, x, 0; \bar{\pi}^h), \\ V(h, x, 0) &= \sup_{\bar{\pi} \in \mathcal{U}_{ad}^{\bar{\pi}^h}[0, T-h]} \bar{J}_{0,T-h}(0, x, 0; \bar{\pi}). \end{aligned} \right.$$

Now, for the given $\bar{\pi}^h \in \mathcal{U}_{ad}^{\bar{\pi}^h}[0, T - h]$ we apply Lemma 2.3 to find $\eta \in \mathcal{A}_{T-h}^3(\mathbb{R}^2)$, such that $\bar{\pi}_t^h = \eta(t, \bar{B}_{\cdot \wedge t}^h, \bar{Q}_{\cdot \wedge t}^h, \bar{W}_{\cdot \wedge t}^h), t \in [0, T]$. We now define

$$\tilde{\pi}_t^h := \eta(t, B_{\cdot \wedge t \wedge (T-h)}, Q_{\cdot \wedge t \wedge (T-h)}, W_{\cdot \wedge t \wedge (T-h)}), \quad t \in [0, T - h].$$

Then $\tilde{\pi}^h \in \mathcal{U}_{ad}[0, T]$. Furthermore, since the law of $(\bar{B}_t^h, \bar{Q}_t^h, \bar{W}_t^h), t \in [0, T - h]$, under \mathbb{P}_{h0} , and that of $(B_t, Q_t, W_t), t \in [0, T - h]$, under \mathbb{P} , are identical, by the pathwise uniqueness (whence uniqueness in law) of the solutions to SDE (2.5), the processes $\{(X_t^{\tilde{\pi}^h}, W_t, \tilde{\pi}_t^h)\}_{t \in [0, T-h]}$ and $\{(X_t^{\bar{\pi}^h}, \bar{W}_t^h, \bar{\pi}_t^h)\}_{t \in [0, T-h]}$ are identical in law. Thus, by (3.5),

$$J_{h,T}(h, x, 0, \pi) = \bar{J}_{0,T-h}(0, x, 0; \bar{\pi}^h) = \mathbb{E}_{0x0} \left[\int_0^{\tau^{\tilde{\pi}^h} \wedge (T-h)} e^{-ct} \tilde{a}_t dt \right] \leq V(0, x, 0).$$

Since $\pi \in \mathcal{U}_{ad}^{h,0}[h, T]$ is arbitrary, we obtain $V(h, x, 0) \leq V(0, x, 0)$, proving (a).

To prove (b), let $\pi \in \mathcal{U}_{ad}[0, T]$. For any $h \in (0, T)$, we define $\pi_t^h := \pi_{t-h}$ for $t \in [h, T]$. Then clearly, $\pi^h \in \mathcal{U}_{ad}^{h,0}[h, T]$. Furthermore, we have

$$\begin{aligned} J(0, x, 0; \pi) - J(h, x, 0; \pi^h) &= \mathbb{E}_{0x0} \left[\int_0^{\tau^\pi} e^{-ct} a_t dt : \tau^\pi \leq T - h \right] \end{aligned}$$

$$\begin{aligned}
 (3.6) \quad & + \mathbb{E}_{0;x0} \left[\int_0^{\tau^\pi \wedge T} e^{-ct} a_t dt : \tau^\pi > T - h \right] \\
 & - \mathbb{E}_{hx0} \left[\int_h^{\tau^\pi} e^{-c(t-h)} a_{t-h} dt : \tau^{\pi^h} \leq T \right] \\
 & - \mathbb{E}_{hx0} \left[\int_h^T e^{-c(t-h)} a_{t-h} dt : \tau^{\pi^h} > T \right].
 \end{aligned}$$

By definition of the strategy π^h , it is easy to check that

$$\begin{aligned}
 \mathbb{E}_{0;x0} \left[\int_0^{\tau^\pi} e^{-ct} a_t dt : \tau^\pi \leq T - h \right] &= \mathbb{E}_{hx0} \left[\int_h^{\tau^\pi} e^{-c(t-h)} a_{t-h} dt : \tau^{\pi^h} \leq T \right], \\
 \mathbb{E}_{0;x0} \left[\int_0^T e^{-ct} a_t dt : \tau^\pi > T - h \right] &= \mathbb{E}_{hx0} \left[\int_h^T e^{-c(t-h)} a_{t-h} dt : \tau^{\pi^h} > T \right],
 \end{aligned}$$

we deduce from (3.6) that

$$\begin{aligned}
 (3.7) \quad J(0, x, 0; \pi) &\leq J(h, x, 0; \pi^h) + \mathbb{E}_{0;x0} \left[\int_{T-h}^T e^{-ct} a_t dt \right] \\
 &\leq V(h, x, 0) + Mh.
 \end{aligned}$$

Since $\pi \in \mathcal{U}_{ad}[0, T]$ is arbitrary, we obtain (b), proving the proposition. \square

We complete this section with an estimate that is quite useful in our discussion. First, note that (3.1) implies that in the absence of claims, the surplus without investment and dividend [i.e., $\pi \equiv (0, 0)$] is $X_t^{0,s,x} = e^{r(t-s)} [x + \frac{p}{r} (1 - e^{-r(t-s)})]$.

PROPOSITION 3.4. *Let $(s, x, w) \in D$. Then, for any $(s + h, X_{s+h}^{0,s,x}, w + h) \in D, h > 0$, it holds that*

$$(3.8) \quad V(s + h, X_{s+h}^{0,s,x}, w + h) \leq e^{ch + \int_w^{w+h} \frac{f(u)}{F(u)} du} V(s, x, w).$$

PROOF. For any $\varepsilon > 0$, we choose $\pi^{h,\varepsilon} \in \mathcal{U}_{ad}^{s+h,w+h}[s + h, T]$ such that

$$J(s + h, X_{s+h}^{0,s,x}, w + h; \pi^{h,\varepsilon}) \geq V(s + h, X_{s+h}^{0,s,x}, w + h) - \varepsilon.$$

Now define a new strategy: $\bar{\pi}_t^{h,\varepsilon} = \pi_t^{h,\varepsilon} \mathbf{1}_{\{T_1^{s,w} > h\}} \mathbf{1}_{[s+h, T]}(t), t \in [s, T]$, where $T_1^{s,w}$ is the first jump time of the delayed renewal process $N^{s,w}$. Then, clearly, $\bar{\pi}^{h,\varepsilon} \in \mathcal{U}_{ad}^{s,w}[s, T]$, and $X_{s+h}^{\bar{\pi}^{h,\varepsilon}} = X_{s+h}^{0,x}$ on the set $\{T_1^{s,w} > h\} \in \mathcal{F}_{s+h}$. Thus, using (2.1) we have

$$\begin{aligned}
 V(s, x, w) &\geq J(s, x, w; \bar{\pi}^{h,\varepsilon}) \\
 &= \mathbb{E}_{s;xw} \left[\int_{s+h}^{\bar{\tau}^{h,\varepsilon} \wedge T} e^{-c(t-s)} a_t^{h,\varepsilon} dt \cdot \mathbf{1}_{\{T_1^{s,w} > h\}} \right]
 \end{aligned}$$

$$\begin{aligned}
 &= e^{-ch} J(s+h, X_{s+h}^{0,s,x}, w+h; \pi^{h,\varepsilon}) \mathbb{P}_{s,x,w} \{T_1^{s,w} > h\} \\
 &\geq [V(s+h, X_{s+h}^{0,s,x}, w+h) - \varepsilon] e^{-ch - \int_w^{w+h} \frac{f(u)}{F(u)} du}.
 \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ we obtain the result. \square

We note that a direct consequence of (3.8) is the following inequality:

$$\begin{aligned}
 (3.9) \quad &V(s+h, X_{s+h}^{0,s,x}, w+h) - V(s, x, w) \\
 &\leq [e^{ch + \int_w^{w+h} \frac{f(u)}{F(u)} du} - 1] V(s, x, w).
 \end{aligned}$$

This gives a kind of one-sided continuity of the value function, although it is a far cry from a true joint continuity which we will study in the next sections.

4. Continuity of the value function on x . In this section, we investigate the continuity of value function on initial surplus x . As in all “exit-type” problems, the main subtle point here is that the ruin time τ^π is generally not continuous in the initial state x . We borrow the idea of *penalty method* (see, e.g., [18]), which we now describe.

To begin with, we recall the domain $D = \{(s, x, w) : 0 \leq s \leq T, x \geq 0, 0 \leq w \leq s\}$. Let $d(x, w) := (-x) \vee 0$ for $(x, w) \in \mathbb{R} \times [0, T]$, and for $\pi \in \mathcal{U}_{ad}^{s,w}[s, T]$ we define a *penalty function* by

$$\begin{aligned}
 (4.1) \quad \beta(t, \varepsilon) &= \beta^{\pi, s, x, w}(t, \varepsilon) \\
 &= \exp\left\{-\frac{1}{\varepsilon} \int_s^t d(X_r^{\pi, s, x, w}, W_r^{s, w}) dr\right\}, \quad t \geq 0.
 \end{aligned}$$

Then clearly $\beta(t, \varepsilon) = 1$ for $t \leq \tau_s^\pi$. Thus we have

$$\begin{aligned}
 (4.2) \quad V^\varepsilon(s, x, w) &= \sup_{\pi \in \mathcal{U}_{ad}[s, T]} J^\varepsilon(s, x, w; \pi) \\
 &:= \sup_{\pi \in \mathcal{U}_{ad}[s, T]} \mathbb{E} \left[\int_s^T \beta^{\pi, s, x, w}(t, \varepsilon) e^{-c(t-s)} a_t dt \right] \\
 &= \sup_{\pi \in \mathcal{U}_{ad}[s, T]} \mathbb{E} \left[\int_s^{\tau_s^\pi} e^{-c(t-s)} a_t dt + \int_{\tau_s^\pi}^T \beta^{\pi, s, x, w}(t, \varepsilon) e^{-c(t-s)} a_t dt \right] \\
 &\geq V(s, x, w).
 \end{aligned}$$

We have the following lemma.

LEMMA 4.1. $V^\varepsilon(s, x, w)$ is continuous in x , uniformly for (s, x, w) in any compact set $K \subset D$.

PROOF. For $\pi \in \mathcal{U}_{ad}^{s,w}[s, T]$, and $x_1, x_2 \in [0, \infty)$ we have

$$\begin{aligned}
 & \mathbb{E}|\beta^{\pi,x_1}(t, \varepsilon) - \beta^{\pi,x_2}(t, \varepsilon)| \\
 &= \mathbb{E}\left|e^{-\frac{1}{\varepsilon} \int_s^t d(X_r^{\pi,x_1}, W_r) dr} - e^{-\frac{1}{\varepsilon} \int_s^t d(X_r^{\pi,x_2}, W_r) dr}\right| \\
 &\leq \frac{1}{\varepsilon} \mathbb{E}\left|\int_s^t d(X_r^{\pi,x_1}, W_r) - d(X_r^{\pi,x_2}, W_r) dr\right| \\
 (4.3) \quad &\leq \frac{1}{\varepsilon} \int_s^t \mathbb{E}|X_r^{\pi,x_1} - X_r^{\pi,x_2}| dr \\
 &\leq \sqrt{T} \frac{1}{\varepsilon} \left(\int_s^t \mathbb{E}|X_r^{\pi,x_1} - X_r^{\pi,x_2}|^2 dr\right)^{\frac{1}{2}} \\
 &\leq \frac{T}{\varepsilon} |x_1 - x_2|.
 \end{aligned}$$

In the above, the last inequality is due to a standard estimate of the SDE (2.2). It then follows that V^ε is continuous in x . Since K is compact, the continuity is uniform for $(s, x, w) \in K$. \square

We should note that the estimate (4.3) indicates that the continuity of V^ε (in x), while uniformly on compacta, is not uniform in ε (!). Therefore, we are to argue that, as $\varepsilon \rightarrow 0$, $V^\varepsilon \rightarrow V$ on any compact set $K \subset D$, uniformly in all $(s, x, w) \in K$, which would in particular imply that V is continuous on D . In other words, we are aiming at the following main result of this section.

THEOREM 4.2. *For any compact set $K \subset D$, the mapping $x \mapsto V(s, x, w)$ is continuous, uniformly for $(s, x, w) \in K$. In particular, the value function V is continuous in x , for $x \in [0, \infty)$.*

To prove Theorem 4.2, we shall introduce an intermediate problem. For each $\theta > 0$, we denote $D_\theta := \{(s, x, w) : s \in [0, T], x \in (-\theta, \infty), w \in [0, s]\}$. Clearly, $D_\theta \subset D_{\theta'}$ for $\theta < \theta'$, and $\bigcap_{\theta>0} D_\theta = D$. For $(s, x, w) \in K$ and $\pi \in \mathcal{U}_{ad}[s, T]$, we denote $\tau_s^{\pi,\theta} = \tau_{s,x,w}^{\pi,\theta}$ (resp. $\tau_s^{\pi,0}$) to be the exit time of the process $(t, X_t^{\pi,s,x,w}, W_t^{s,w})$ from D_θ (resp., D) before T . For notational simplicity, we shall write $(X^\pi, W) := (X^{\pi,s,x,w}, W^{s,w})$, $\tau := \tau_s^{\pi,0}$, and $\tau^\theta := \tau_s^{\pi,\theta}$, when the context is clear. It is worth noting that the function $\beta(t, \varepsilon)$ satisfies a SDE:

$$\beta(t, \varepsilon) = 1 - \frac{1}{\varepsilon} \int_s^t d(X_r^\pi, W_r) \beta(r, \varepsilon) dr, \quad t \in [s, T].$$

Thus, together with the underlying process (X^π, W) , we see that the optimization problem in (4.2) is a standard stochastic control problem with jumps and fixed terminal time T , therefore, the standard Dynamic Programming Principle (DPP)

holds for V^ε . To be more precise, for any stopping time $\hat{\tau} \in [s, T]$, it holds that

$$\begin{aligned}
 (4.4) \quad V^\varepsilon(s, x, w) &= \sup_{\pi \in \mathcal{U}_{ad}[s, T]} \mathbb{E}_{s,x,w} \left\{ \int_s^{\hat{\tau}} \beta(t, \varepsilon) e^{-c(t-s)} a_t dt \right. \\
 &\quad \left. + e^{-(\hat{\tau}-s)} \beta(\hat{\tau}, \varepsilon) V^\varepsilon(\hat{\tau}, X_{\hat{\tau}}^\pi, W_{\hat{\tau}}) \right\}.
 \end{aligned}$$

We are now ready to prove Theorem 4.2.

PROOF OF THEOREM 4.2. We first note that, for any $(s, x, w) \in K$ and $\pi \in \mathcal{U}_{ad}[s, T]$, by DPP (4.4) and the fact (4.2) we have

$$\begin{aligned}
 (4.5) \quad V(s, x, w) &\leq V^\varepsilon(s, x, w) \\
 &= \sup_{\pi \in \mathcal{U}_{ad}[s, T]} \mathbb{E}_{s,x,w} \left\{ \int_s^{\tau^\theta} \beta(t, \varepsilon) e^{-c(t-s)} a_t dt \right. \\
 &\quad \left. + e^{-(\tau^\theta-s)} \beta(\tau^\theta, \varepsilon) V^\varepsilon(\tau^\theta, X_{\tau^\theta}, W_{\tau^\theta}) \right\} \\
 &= \sup_{\pi \in \mathcal{U}_{ad}[s, T]} \mathbb{E}_{s,x,w} \left\{ \int_s^\tau e^{-c(t-s)} a_t dt + \int_\tau^{\tau^\theta} \beta(t, \varepsilon) e^{-c(t-s)} a_t dt \right. \\
 &\quad \left. + e^{-(\tau^\theta-s)} \beta(\tau^\theta, \varepsilon) V^\varepsilon(\tau^\theta, X_{\tau^\theta}^\pi, W_{\tau^\theta}) \right\} \\
 &\leq V(s, x, w) + M \sup_{\pi \in \mathcal{U}_{ad}[s, T]} \mathbb{E}_{s,x,w} (\tau^\theta - \tau) + h^\theta(\varepsilon),
 \end{aligned}$$

where $h^\theta(\varepsilon) := \mathbb{E}_{s,x,w}[V^\varepsilon(\tau^\theta, X_{\tau^\theta}^\pi, W_{\tau^\theta})]$, and $M > 0$ is the constants in Assumption 2.1. We first argue that $\sup_{\pi \in \mathcal{U}_{ad}[s, T]} \mathbb{E}_{s,x,w} |\tau - \tau^\theta| \rightarrow 0$, as $\theta \rightarrow 0$, uniformly in $(s, x, w) \in K$.

To see this, first note that $\sup_{\pi \in \mathcal{U}_{ad}^{s,w}[s, T]} \mathbb{E}_{s,x,w} |\tau - \tau^\theta| \leq \sup_{\pi \in \mathcal{U}_{ad}^{s,w}[s, T]} T \mathbb{P}\{\tau \neq \tau^\theta\}$, here and in what follows $\mathbb{P} := \mathbb{P}_{s,x,w}$, if there is no danger of confusion. On the other hand, recall that τ must happen at a claim arrival time on $\{\tau \neq \tau^\theta\}$, and $\Delta X_t^\pi = \Delta Q_t^{s,w}$, it is easy to check that

$$\begin{aligned}
 \mathbb{P}\{\tau \neq \tau^\theta\} &= \mathbb{P}\{\Delta X_\tau^\pi \in (X_{\tau^-}^\pi, X_{\tau^-}^\pi + \theta)\} \\
 &= \int_0^\infty \mathbb{P}\{\Delta Q_\tau^{s,w} \in (y, y + \theta) | X_{\tau^-}^\pi = y\} F_{X_{\tau^-}^\pi}(dy) \\
 &= \int_0^\infty [G(y + \theta) - G(y)] F_{X_{\tau^-}^\pi}(dy),
 \end{aligned}$$

where G is the common distribution function of the claim sizes U^i 's. Since G is uniformly continuous on $[0, \infty)$, thanks to Assumption 2.1(b), for any $\eta > 0$ we

can find $\theta_0 > 0$, depending only on η , such that $|G(y + \theta_0) - G(y)| < \frac{\eta}{2T}$, for all $y \in [0, \infty)$,

$$(4.6) \quad \begin{aligned} & \sup_{\pi \in \mathcal{U}_{ad}[s, T]} \mathbb{E}_{sxx} \{ |\tau^\theta - \tau| \} \\ & \leq \sup_{\pi \in \mathcal{U}_{ad}[s, T]} T \int_0^\infty |G(y + \theta_0) - G(y)| F_{X_{\tau^-}^\pi}(dy) < \frac{\eta}{2}. \end{aligned}$$

Plugging (4.6) into (4.5), we obtain that

$$(4.7) \quad V(s, x, w) \leq V^\varepsilon(s, x, w) \leq V(s, x, w) + \frac{\eta}{2} + h^{\theta_0}(\varepsilon).$$

We claim that $\lim_{\varepsilon \rightarrow 0} h^{\theta_0}(\varepsilon) = 0$, and that the limit is uniform in $(s, x, w) \in K$. To this end, we define, for the given $\pi \in \mathcal{U}_{ad}[s, T]$, and $\theta = \theta_0$,

$$(4.8) \quad \begin{aligned} \bar{\tau}_\theta & := \inf\{t > \tau^\theta, d(X_t^\pi, W_t) < \theta/2\} \wedge T; \\ \bar{\tau}_\theta^c & := \inf\{t > \tau^\theta, d(X_t^{\pi, \theta, c}, W_t) < \theta/2\} \wedge T, \end{aligned}$$

where $X^{\pi, \theta, c}$ is the continuous part of X^π , for $t \geq \tau^\theta$, given $X_{\tau^\theta}^{\pi, \theta, c} = X_{\tau^\theta}^\pi$. Since X^π only has negative jumps, we have $\Delta X_t^\pi \leq 0, \forall t \in [0, T]$. Thus $\bar{\tau}_\theta^c \leq \bar{\tau}_\theta$ and $d(X_t^{\pi, \theta, c}, W_t) \leq d(X_t^\pi, W_t)$, for all $t \in [s, T]$, \mathbb{P} -a.s. Furthermore, we note that $d(X_t^{\pi, \theta, c}, W_t) \geq \frac{\theta}{2}$ for $t \in [\tau^\theta, \bar{\tau}_\theta^c]$, \mathbb{P} -a.s.

Now, denoting $\mathbb{E}_{\tau^\theta}[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_{\tau^\theta}]$ and $X^c = X^{\pi, \theta, c}$ we have, \mathbb{P} -almost surely,

$$(4.9) \quad \begin{aligned} & J^\varepsilon(\tau^\theta, X_{\tau^\theta}^\pi, W_{\tau^\theta}; \pi) \\ & = \mathbb{E}_{\tau^\theta} \left[\int_{\tau^\theta}^T e^{-\frac{1}{\varepsilon} \int_{\tau^\theta}^t d(X_r^\pi, W_r)} dr e^{-c(t-\tau^\theta)} a_t dt \right] \\ & \leq \mathbb{E}_{\tau^\theta} \left[\int_{\tau^\theta}^T e^{-\frac{1}{\varepsilon} \int_{\tau^\theta}^t d(X_r^c, W_r)} dr e^{-c(t-\tau^\theta)} a_t dt \right] \\ & \leq \mathbb{E}_{\tau^\theta} \left[\int_{\tau^\theta}^T e^{-\frac{1}{\varepsilon} \int_{\tau^\theta}^{t \wedge \bar{\tau}_\theta^c} d(X_r^c, W_r)} dr e^{-c(t-\tau^\theta)} a_t dt \right] \\ & \leq \mathbb{E}_{\tau^\theta} \left[\int_{\tau^\theta}^T e^{-\frac{1}{\varepsilon} \frac{\theta}{2} [(t \wedge \bar{\tau}_\theta^c) - \tau^\theta]} e^{-c(t-\tau^\theta)} a_t dt \right] \\ & \leq M \mathbb{E}_{\tau^\theta} \left[\int_{\tau^\theta}^{\bar{\tau}_\theta^c} e^{-\frac{\theta}{2\varepsilon}(t-\tau^\theta)} dt + \int_{\bar{\tau}_\theta^c}^T e^{-\frac{\theta}{2\varepsilon}(\bar{\tau}_\theta^c - \tau^\theta)} dt \right] \\ & \leq M \mathbb{E}_{\tau^\theta} \left[\int_{\tau^\theta}^T e^{-\frac{\theta}{2\varepsilon}(t-\tau^\theta)} dt \right] + M \mathbb{E}_{\tau^\theta} [e^{-\frac{\theta}{2\varepsilon}(\bar{\tau}_\theta^c - \tau^\theta)}] \\ & \triangleq A_\theta(\varepsilon) + B_\theta(\varepsilon), \end{aligned}$$

where $A_\theta(\cdot)$ and $B_\theta(\cdot)$ are defined in an obvious way. Clearly, for fixed $\theta = \theta_0$,

$$(4.10) \quad 0 \leq A_\theta(\varepsilon) \leq \frac{2\varepsilon M}{\theta} [1 - e^{-\frac{\theta}{2\varepsilon} T}] \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \mathbb{P}\text{-a.s.}$$

and the limit is uniform in (s, x, w) and $\pi \in \mathcal{U}_{ad}^{s,w}[s, T]$. We shall argue that $B_\theta(\varepsilon) \rightarrow 0$, as $\varepsilon \rightarrow 0$, in the same manner. Indeed, note that $X_{\tau^\theta} \leq -\theta$, for $\delta > 0$ we have

$$(4.11) \quad \begin{aligned} \mathbb{P}_{\tau^\theta}(|\bar{\tau}_\theta^c - \tau^\theta| < \delta) &\leq \mathbb{P}_{\tau^\theta} \left\{ \sup_{\tau^\theta \leq t \leq \tau^\theta + \delta} X_t^{\pi, \theta, c} > -\frac{\theta}{2} \right\} \\ &\leq \mathbb{P}_{\tau^\theta} \left\{ \sup_{\tau^\theta \leq t \leq \tau^\theta + \delta} [X_t^{\pi, \theta, c} - X_{\tau^\theta}^\pi] > \frac{\theta}{2} \right\} \\ &\leq \frac{4}{\theta^2} \mathbb{E}_{\tau^\theta} \left\{ \sup_{\tau^\theta \leq t \leq \tau^\theta + \delta} |X_t^{\pi, \theta, c} - X_{\tau^\theta}^\pi|^2 \right\} \leq C_\theta \delta, \end{aligned}$$

for some generic constant $C_\theta > 0$ depending only on p, r, σ, T, M , and θ . Here, we have applied the Chebyshev inequality, as well as some standard SDE estimates. Consequently, we derive from (4.11) that $\sup_\pi \mathbb{P}_{\tau^\theta}(|\bar{\tau}_\theta^c - \tau^\theta| < \delta) \leq C_\theta \delta$, \mathbb{P} -a.s., and thus for fixed θ , and any $\eta > 0$, we can find $\delta_0(\eta, \theta) > 0$, such that $\mathbb{P}_{\tau^\theta}(|\bar{\tau}_\theta^c - \tau^\theta| < \delta_0) < \frac{\eta}{2T}$. Then

$$(4.12) \quad \begin{aligned} B_\theta(\varepsilon) &= M \{ \mathbb{E}_{\tau^\theta} [e^{-\frac{\theta}{2\varepsilon}(\bar{\tau}_\theta^c - \tau^\theta)} : \bar{\tau}_\theta^c - \tau^\theta \geq \delta_0] \\ &\quad + \mathbb{E}_{\tau^\theta} [e^{-\frac{\theta}{2\varepsilon}(\bar{\tau}_\theta^c - \tau^\theta)} : \bar{\tau}_\theta^c - \tau^\theta < \delta_0] \} \\ &\leq M \{ e^{-\frac{\theta}{2\varepsilon}\delta_0} + \mathbb{P}_{\tau^\theta}(\bar{\tau}_\theta^c - \tau^\theta < \delta_0) \} < M e^{-\frac{\theta}{2\varepsilon}\delta_0} + \frac{\eta}{2}. \end{aligned}$$

Therefore, for fixed $\theta = \theta_0$, one has $\overline{\lim}_{\varepsilon \rightarrow 0} B_\theta(\varepsilon) \leq \frac{\eta}{2}$, \mathbb{P} -a.s. This, together with (4.9) and (4.10), then implies that $\overline{\lim}_{\varepsilon \rightarrow 0} J^\varepsilon(\tau^\theta, X_{\tau^\theta}^\pi, W_{\tau^\theta}; \pi) \leq \frac{\eta}{2}$, uniformly in $(s, x, w) \in K$ and $\pi \in \mathcal{U}_{ad}[s, T]$, which in turn implies that, for $\theta = \theta_0$, $\overline{\lim}_{\varepsilon \rightarrow 0} h^\theta(\varepsilon) = \overline{\lim}_{\varepsilon \rightarrow 0} \mathbb{E}_{s,x,w} [V^\varepsilon(\tau^\theta, X_{\tau^\theta}^\pi, W_{\tau^\theta})] \leq \frac{\eta}{2}$, and the limit is uniformly in $(s, x, w) \in K$. Combining this with (4.6), we derive from (4.7) that

$$V(s, x, w) \leq \liminf_{\varepsilon \rightarrow 0} V^\varepsilon(s, x, w) \leq \limsup_{\varepsilon \rightarrow 0} V^\varepsilon(s, x, w) \leq V(s, x, w) + \eta.$$

Since η is arbitrary, we have $\lim_{\varepsilon \rightarrow 0} V^\varepsilon(s, x, w) = V(s, x, w)$, uniformly in $(s, x, w) \in K$. Finally, note that V^ε is continuous in x , uniformly in $(s, x, w) \in K$, thanks to Lemma 4.1, thus so is V . In particular, V is continuous in x for $x \in [0, k]$, for all $k > 0$, proving the theorem. \square

5. Continuity of the value function on w . We now turn our attention to the continuity of value function V in the variable w . We should note that this is the most technical part of the paper, as it involves the study of the delayed renewal

process that has not been fully explored in the literature. We begin by a proposition that extends Proposition 3.3. Recall the intensity of the interclaim times T_i 's: $\lambda(t) = \frac{f(t)}{F(t)}, t \geq 0$.

PROPOSITION 5.1. *Assume that Assumption 2.1 is in force. Then, for $0 \leq s < s + h < T$, one has:*

- (i) $V(s + h, x, w + h) - V(s, x, w) \leq [1 - e^{-(ch + \int_w^{w+h} \lambda(u) du)]V(s + h, x, w + h)$;
- (ii) $V(s, x, w + h) - V(s, x, w) \leq Mh + [1 - e^{-(ch + \int_w^{w+h} \lambda(u) du)]V(s + h, x, w + h)$.

PROOF. (i) For any $\pi = (\gamma, a) \in \mathcal{U}_{ad}^{s+h, w+h}[s + h, T]$, we define, for $t \in [s, T]$, $\tilde{\pi}_t^h = (\tilde{\gamma}_t, \tilde{a}_t)$ by

$$(5.1) \quad \begin{aligned} (\tilde{\gamma}_t, \tilde{a}_t) &= (0, (p + rX_t^h) \wedge M) \\ &+ [(\gamma_t, a_t) - (0, (p + rX_t^h) \wedge M)] \mathbf{1}_{\{T_1^{s,w} > h\}} \mathbf{1}_{[s+h, T]}(t), \end{aligned}$$

where $T_1^{s,w}$ is the first jump time of $N^{s,w}$, and $X^h := X^{\tilde{\pi}^h, s, x, w}$. Since $T_1^{s,w}$ is a $\{\mathcal{F}_t^s\}_{t \geq 0} = \{\mathcal{F}_{s+t}\}_{t \geq 0}$ -stopping time, it is clear that $\tilde{\pi}^h \in \mathcal{U}_{ad}^{s,w}[s, T]$. Let us denote $\tau^h := \tau_{s,x,w}^{\tilde{\pi}^h}$ and consider the following two cases:

Case 1. $x \leq \frac{M-p}{r}$. In this case, for $s \leq t < s + T_1^{s,w}$, we have $X_t^h \equiv x$ and $\tilde{a}_t \equiv p + rx \leq M$. In particular, we note that by definition of $\tilde{\pi}^h$, given $T_1^{s,w} > h$ it must hold that $X_{s+h}^h = x$, $W_{s+h}^{s,w} = w + h$, and $T_1^{s+h, w+h} = T_1^{s,w}$, $\mathbb{P}_{s,x,w}$ -a.s. Thus it is not hard to check that

$$(5.2) \quad \begin{aligned} V(s, x, w) &\geq J(s, x, w; \tilde{\pi}^h) \\ &\geq \mathbb{E}_{s,x,w} \left[\int_s^{\tau^{\tilde{\pi}^h} \wedge T} e^{-c(t-s)} \tilde{a}_t dt \mid T_1^{s,w} > h \right] \mathbb{P}_{s,x,w} \{ T_1^{s,w} > h \} \\ &\geq e^{-\int_w^{w+h} \lambda(u) du} \mathbb{E}_{s,x,w} \left[\int_{s+h}^{\tau^{\tilde{\pi}^h} \wedge T} e^{-c(t-s)} \tilde{a}_t dt \mid T_1^{s,w} > h \right] \\ &= e^{-(ch + \int_w^{w+h} \lambda(u) du)} J(s + h, x, w + h; \pi). \end{aligned}$$

Since $\pi \in \mathcal{U}_{ad}[s + h, T]$ is arbitrary, we obtain that $V(s, x, w) \geq e^{-(ch + \int_w^{w+h} \lambda(u) du)} V(s + h, x, w + h)$ which, with an argument similar to the one led to (3.9), implies (a).

Case 2. $x > \frac{M-p}{r}$. In this case, we have $\tilde{a}_s = M < p + rx = p + rX_s^h$, thus, by (3.1) $dX_s^h > 0$. Namely, on the set $\{T_1^{s,w} > h\}$, X^h will be continuous and increasing, so that $X_{s+h}^h = e^{rh}x + \frac{p-M}{r}(1 - e^{-rh}) =: x(h)$ [see (3.3)]. Thus, noting

that $W_{s+h}^{s,w} = w + h$ and $T_1^{s+h,w+h} = T_1^{s,w}$ on $\{T_1^{s,w} > h\}$, a similar argument as (5.2) would lead to that

$$\begin{aligned} V(s, x, w) &\geq J(s, x, w; \tilde{\pi}^h) \\ &\geq e^{-(ch + \int_w^{w+h} \lambda(u) du)} V(s + h, x(h), w + h). \end{aligned}$$

Now note that $x(h) > x$, it follows from Proposition 3.2(a) that $V(s + h, x(h), w + h) \geq V(s + h, x, w + h)$, proving (a) again.

Finally, (ii) follows from (i) and Proposition 3.3(b). This completes the proof. □

The next result concerns the uniform continuity of V on the variables (s, w) . We have the following result.

PROPOSITION 5.2. *Assume that Assumption 2.1 is in force. Then it holds that*

$$\lim_{h \downarrow 0} [V(s + h, x, w + h) - V(s, x, w)] = 0 \quad \text{uniformly in } (s, x, w) \in D.$$

PROOF. From Proposition 5.1(i) and the boundedness of V , we see that

$$\begin{aligned} (5.3) \quad \limsup_{h \downarrow 0} [V(s + h, x, w + h) - V(s, x, w)] &\leq 0, \\ &\text{uniformly in } (s, x, w) \in D. \end{aligned}$$

We need only prove the opposite inequality. We shall keep all the notation as in the previous proposition. For any $h \in (0, T - s)$, and $\pi = (\gamma_t, a_t) \in \mathcal{U}_{ad}[s, T]$, we still consider the strategy $\tilde{\pi}^h \in \mathcal{U}_{ad}^{s,w}[s, T]$ defined by (5.1). (Note that $\tilde{\pi}^h$ depends on π only for $t \in [s + h, T]$.) We again consider two cases, and denote $\tau_1 := T_1^{s,w}$ for simplicity.

Case 1. $x \leq \frac{M-p}{r}$. In this case, we first write

$$\begin{aligned} (5.4) \quad J(s, x, w; \tilde{\pi}^h) &= \mathbb{E}_{s,x,w} \left[\int_s^{s+h} e^{-c(t-s)} \tilde{a}_t dt \mid \tau_1 > h \right] \mathbb{P}(\tau_1 > h) \\ &\quad + \mathbb{E}_{s,x,w} \left[\int_{s+h}^{\tau^h \wedge T} e^{-c(t-s)} \tilde{a}_t dt \mid \tau_1 > h \right] \mathbb{P}(\tau_1 > h) \\ &\quad + \mathbb{E}_{s,x,w} \left[\int_s^{\tau^h \wedge T} e^{-c(t-s)} \tilde{a}_t dt \mid \tau_1 \leq h \right] \mathbb{P}(\tau_1 \leq h) \\ &:= I_1 + I_2 + I_3, \end{aligned}$$

where I_i 's are defined as the three terms on the right-hand side above, respectively. Clearly, by (5.1), on the set $\{\tau_1 > h\}$, $\tilde{\gamma} \equiv 0$, $X_t^h = x$, and $\tilde{a}_t = p + rx \leq M$ for

$t \in [s, s + h]$, thus

$$\begin{aligned}
 I_1 &= e^{-\int_w^{w+h} \lambda(u) du} \mathbb{E}_{s,x,w} \left[\int_s^{s+h} e^{-c(t-s)} (p + rX_t^h) dt \mid \tau_1 > h \right] \\
 (5.5) \quad &\leq (p + rx)h;
 \end{aligned}$$

$$I_2 \leq e^{-ch - \int_w^{w+h} \lambda(u) du} V(s + h, x, w + h) \leq V(s + h, x, w + h).$$

To estimate I_3 , we first note that on the set $\{\tau_1 \leq h\}$, by (5.1), $\tilde{\gamma}_t \equiv 0$, for all $t \in [s, T]$. Thus $X_t^h = x$ and $\tilde{a}_t = p + rx$ for $t \in [s, s + \tau_1]$. We also note that $\tau^h \geq s + \tau_1$ and $\{\tau^h > s + \tau_1\} = \{U_1 \leq x\}$. Bearing these in mind, we now write

$$(5.6) \quad I_3 = \mathbb{E}_{s,x,w} \left[\left(\int_s^{s+\tau_1} + \int_{s+\tau_1}^{\tau^h \wedge T} \right) e^{-c(t-s)} \tilde{a}_t dt : \tau_1 \leq h \right] := I_3^1 + I_3^2,$$

where I_3^1 and I_3^2 are defined in an obvious way. For simplicity, let us denote the density function of $T_1^{s,w}$ by $p_{\tau_1}(z) = \lambda(w + z)e^{-\int_w^{w+z} \lambda(v) dv}$, $z \geq 0$. Clearly, given $\tau_1 \leq h$ we have

$$\begin{aligned}
 I_3^1 &= \int_0^h \mathbb{E}_{s,x,w} \left[\int_s^{s+\tau_1} e^{-c(t-s)} (p + rX_t^h) dt \mid \tau_1 = z \right] p_{\tau_1}(z) dz \\
 (5.7) \quad &= \int_0^h \left[\int_s^{s+z} e^{-c(t-s)} (p + rx) dt \right] p_{\tau_1}(z) dz \\
 &\leq \int_s^{s+h} e^{-c(t-s)} (p + rx) dt (1 - e^{-\int_w^{w+h} \lambda(v) dv}) \\
 &\leq (1 - e^{-\int_w^{w+h} \lambda(v) dv}) (p + rx)h.
 \end{aligned}$$

Further, we note that $(X_{s+\tau_1}^h, W_{s+\tau_1}^{s,w}) = (x - U_1, 0)$, \mathbb{P} -a.s., thus

$$\begin{aligned}
 I_3^2 &= \int_0^h \mathbb{E}_{s,x,w} \left[\int_{s+z}^{\tau^h \wedge T} e^{-c(t-s)} (p + rX_t^h) dt \mathbf{1}_{\{\tau^h > s+z\}} \mid \tau_1 = z \right] p_{\tau_1}(z) dz \\
 &= \int_0^h \int_0^x \mathbb{E}_{s,x,w} \left[\int_{s+z}^{\tau^h \wedge T} e^{-c(t-s)} (p + rX_t^h) dt \mid \tau_1 = z, U_1 = u \right] \\
 (5.8) \quad &\times p_{\tau_1}(z) dG(u) dz \\
 &\leq \int_0^h \int_0^x e^{-cz} V(s + z, x - u, 0) p_{\tau_1}(z) dG(u) dz \\
 &\leq \frac{M}{c} (1 - e^{-\int_w^{w+h} \lambda(v) dv}).
 \end{aligned}$$

Here, the last inequality is due to Proposition 3.2(ii). Now, combining (5.7) and (5.8) we have

$$(5.9) \quad I_3 \leq (1 - e^{-\int_w^{w+h} \lambda(v) dv}) \left((p + rx)h + \frac{M}{c} \right),$$

and consequently we obtain from (5.4)–(5.9) that, for $x < \frac{M-p}{r}$,

$$(5.10) \quad \begin{aligned} J(s, x, w; \tilde{\pi}^h) &\leq (p + rx)h + V(s + h, x, w + h) \\ &\quad + (1 - e^{-\int_w^{w+h} \lambda(v) dv})((p + rx)h + M/c). \end{aligned}$$

Case 2. $x \geq \frac{M-p}{r}$. In this case, using the strategy $\tilde{\pi}^h$ as in (5.1) with a similar argument as in Case 1 we can derive that

$$(5.11) \quad \begin{aligned} J(s, x, w; \tilde{\pi}^h) &\leq Mh + V\left(s + h, e^{rh}\left(x + \frac{p - M}{r}(1 - e^{-rh})\right), w + h\right) \\ &\quad + (1 - e^{-\int_w^{w+h} \lambda(v) dv})\left(M\left(h + \frac{1}{c}\right)\right). \end{aligned}$$

To complete the proof, we are to replace the left-hand side of (5.10) and (5.11) by $J(s, x, w, \pi)$, which would lead to the desired inequality, as $\pi \in \mathcal{U}_{ad}[s, T]$ is arbitrary. To this end, we shall argue along a similar line as those in the previous section.

Recall the penalty function $\beta^{\pi, s}(t, \varepsilon) := \beta^{\pi, s, x, w}(t, \varepsilon)$ defined by (4.1), and define

$$J^\varepsilon(s, w, x; \pi) = \mathbb{E}_{s,wx} \left[\int_s^T \beta^{\pi, s}(t, \varepsilon) e^{-c(t-s)} a_t dt \right].$$

We first write

$$(5.12) \quad \begin{aligned} &|J^\varepsilon(s, x, w; \pi) - J^\varepsilon(s, x, w; \tilde{\pi}^h)| \\ &\leq \mathbb{E}_{s,xw} \left| \int_s^{s+h} e^{-c(t-s)} [\beta^{\pi, s}(t, \varepsilon) a_t - \beta^{\tilde{\pi}^h, s}(t, \varepsilon) \tilde{a}_t] dt \right| \\ &\quad + \mathbb{E}_{s,xw} \left| \int_{s+h}^T e^{-c(t-s)} [\beta^{\pi, s}(t, \varepsilon) a_t - \beta^{\tilde{\pi}^h, s}(t, \varepsilon) \tilde{a}_t] dt \right| \\ &:= I_1 + I_2 \end{aligned}$$

It is easy to see that $I_1 < 2Mh$, thanks to Assumption 2.1. We shall estimate I_2 . Note that

$$(5.13) \quad \begin{aligned} I_2 &= \mathbb{E}_{s,xw} \left\{ \left| \int_{s+h}^T e^{-c(t-s)} (\beta^{\pi, s}(t, \varepsilon) - \beta^{\tilde{\pi}^h, s}(t, \varepsilon)) a_t dt \right| \middle| \tau_1 > h \right\} \mathbb{P}(\tau_1 > h) \\ &\quad + \mathbb{E}_{s,xw} \left\{ \left| \int_{s+h}^T e^{-c(t-s)} [\beta^{\pi, s}(t, \varepsilon) a_t - \beta^{\tilde{\pi}^h, s}(t, \varepsilon) \tilde{a}_t] dt \right| \middle| \tau_1 \leq h \right\} \\ &\quad \times \mathbb{P}(\tau_1 \leq h) \\ &:= I_2^1 + I_2^2. \end{aligned}$$

Since $X_t^\pi, X_t^h \geq 0$ for $t \leq s + h$ on the set $\tau_1 > h$ (i.e., ruin occurs only at arrival of a claim), we have $d(X_t^\pi, W_t) = d(X_t^h, W_t) = 0$ for $t \in [s, s + h]$, that is,

$\beta^{\pi,s}(t, \varepsilon) = \beta^{\pi,s+h}(t, \varepsilon)$, $\beta^{\tilde{\pi}^h,s}(t, \varepsilon) = \beta^{\tilde{\pi}^h,s+h}(t, \varepsilon)$, for $t \in [s + h, T]$. Thus, by the similar arguments as in Lemma 4.1 one shows that

$$\begin{aligned}
 I_2^1 &= \mathbb{E}_{s,x,w} \left\{ \left| \int_{s+h}^T (\beta^{\pi,s+h}(\varepsilon, t) - \beta^{\tilde{\pi}^h,s+h}(\varepsilon, t)) e^{-c(t-s)} a_t dt \right| \middle| \tau_1 > h \right\} \\
 (5.14) \quad &\times \mathbb{P}(\tau_1 > h) \\
 &\leq C \mathbb{E}_{s,x,w} |X_{s+h}^\pi - X_{s+h}^h|,
 \end{aligned}$$

where $C > 0$ is a generic constant depending only on ε and T . Furthermore, since $\mathbb{P}(\tau_1 \leq h) = (1 - e^{-\int_w^{w+h} \lambda(v) dv}) = O(h)$, we have $I_2^2 = O(h)$. It then follows from (5.13) and (5.14) that $I_2 \leq C \mathbb{E}_{s,x,w} |X_{s+h}^\pi - X_{s+h}^h| + O(h)$. The standard result of SDE then leads to $\lim_{h \rightarrow 0} I_2 = 0$, whence $\lim_{h \rightarrow 0} |J^\varepsilon(s, x, w; \pi) - J^\varepsilon(s, x, w; \tilde{\pi}^h)| = 0$, and the convergence is obviously uniform for $(s, x, w) \in D$ and $\pi \in \mathcal{U}_{ad}^{s,w}[s, T]$.

To complete the proof we note that, with exactly the same argument as that in Theorem 4.2 one shows that, for any $\eta > 0$, there exists $\varepsilon_0 > 0$, such that

$$\begin{aligned}
 &|J^{\varepsilon_0}(s, x, w; \pi) - J(s, x, w; \pi)| + |J^{\varepsilon_0}(s, x, w; \tilde{\pi}^h) - J(s, x, w; \tilde{\pi}^h)| \\
 &< \eta \quad \forall (s, x, w) \in D.
 \end{aligned}$$

Then, for the fixed ε_0 , we choose $h_0 > 0$, independent of $\pi \in \mathcal{U}_{ad}^{s,w}[s, T]$, such that

$$|J^{\varepsilon_0}(s, x, w; \pi) - J^{\varepsilon_0}(s, x, w; \tilde{\pi}^h)| < \eta, \quad \forall (s, x, w) \in D, \forall 0 < h < h_0.$$

Thus, if $x < \frac{M-p}{r}$, for all $0 < h < h_0$, we derive from (5.10) that

$$\begin{aligned}
 &J(s, x, w; \pi) - V(s + h, x, w + h) \\
 &\leq |J(s, x, w; \pi) - J^{\varepsilon_0}(s, x, w; \pi)| + |J^{\varepsilon_0}(s, x, w; \pi) - J^{\varepsilon_0}(s, x, w; \tilde{\pi}^h)| \\
 &\quad + |J^{\varepsilon_0}(s, x, w; \tilde{\pi}^h) - J(s, x, w; \tilde{\pi}^h)| \\
 &\quad + J(s, x, w; \tilde{\pi}^h) - V(s + h, x, w + h) \\
 &\leq 2\eta + (p + rx)h + (1 - e^{-\int_w^{w+h} \lambda(v) dv})((p + rx)h + M/c) \leq 2\eta + g_1(h),
 \end{aligned}$$

where $g_1(h) := Mh + (1 - e^{-\int_w^{w+h} \lambda(v) dv})(Mh + M/c)$. Since $\pi \in \mathcal{U}_{ad}[s, T]$ is arbitrary, we have

$$(5.15) \quad V(s, x, w) - V(s + h, x, w + h) \leq 2\eta + g_1(h).$$

First, sending $h \rightarrow 0$ and then $\eta \rightarrow 0$ we obtain the desired opposite inequality of (5.3).

The case for $x \geq \frac{M-p}{r}$ can be argued similarly. We apply (5.11) to get the analogue of (5.15):

$$\begin{aligned}
 & V(s, x, w) - V(s + h, x, w + h) \\
 (5.16) \quad & \leq 2\eta + g_1(h) + V\left(s + h, e^{rh}\left(x + \frac{p-M}{r}(1 - e^{-rh})\right), w + h\right) \\
 & \quad - V(s + h, x, w + h).
 \end{aligned}$$

Fixing $x \geq \frac{M-p}{r}$, and sending $h \rightarrow 0$, then $\eta \rightarrow 0$, we get $\liminf_{h \downarrow 0} [V(s + h, x, w + h) - V(s, x, w)] \geq 0$, thanks to the uniform continuity $V(s, x, w)$ in x [uniformly in (s, w)]. This, together with (5.3), yields that, for given $x \geq 0$,

$$(5.17) \quad \lim_{h \downarrow 0} [V(s + h, x, w + h) - V(s, x, w)] = 0 \quad \text{uniformly in } (s, w).$$

Then, combining (5.17) and Proposition 5.1, one shows that $V(s, x, w)$ is continuous in (s, w) for fixed x . It remains to argue that (5.17) holds uniformly in $(s, x, w) \in D$.

To this end, we note that, by Proposition 3.2 and Theorem 4.2, $V(s, x, w)$ is increasing in x , continuous in (s, w) , and with a continuous limit function $\frac{M}{c}(1 - e^{-(T-s)})$ [in (s, w)]. Thus $V(s, x, w)$ converges uniformly to $\frac{M}{c}(1 - e^{-(T-s)})$ as $x \rightarrow \infty$, uniformly in (s, w) , thanks to Dini's theorem. That is, for $\eta > 0$, there exists $N = N(\eta) > \frac{M-p}{r}$, such that

$$\begin{aligned}
 & V\left(s + h, e^{rh}\left(x + \frac{p-M}{r}(1 - e^{-rh})\right), w + h\right) - V(s + h, x, w + h) \\
 & < \eta, \quad x > N.
 \end{aligned}$$

On the other hand, for $\frac{M-p}{r} \leq x \leq N$, by Theorem 4.2, there exists $\delta(\eta) = \delta(N(\eta)) > 0$, such that for $h < \delta(N)$, it holds that

$$V\left(s + h, e^{rh}\left(x + \frac{p-M}{r}(1 - e^{-rh})\right), w + h\right) - V(s + h, x, w + h) < \eta.$$

Thus, we see from (5.16) that for all $(s, x, w) \in D$, and $x \geq \frac{M-p}{r}$,

$$V(s, x, w) - V(s + h, x, w + h) \leq 4\eta \quad \text{whenever } h < \delta.$$

Combining this with the case $x < \frac{M-p}{r}$ argued previously, we see that

$$\liminf_{h \downarrow 0} [V(s + h, x, w + h) - V(s, x, w)] \geq 0 \quad \text{uniformly in } (s, x, w) \in D,$$

proving the opposite inequality of (5.3), whence the proposition. \square

Combining Theorems 3.3 and 5.1, we have proved the following theorem.

THEOREM 5.3. *Assume that Assumption 2.1 is in force. Then the value function $V(s, x, w)$ is uniformly continuous in w , uniformly on $(s, x, w) \in D$.*

Next, we observe from definition (6.2) that the law of \tilde{X}^h on $[s - h_2, T - h_2]$ and that of \hat{X}^h on $[s, T]$ are identical. We have

$$\begin{aligned}
 J_2 &= J(s - h_2, x, w - h_2, \tilde{\pi}^h) - J(s, x, w - h_2, \hat{\pi}^h) \\
 &= \mathbb{E}_{(s-h_2)x(w-h_2)} \left[\int_{s-h_2}^{\tau^{\tilde{\pi}^h} \wedge T} e^{-c(t-s+h_2)} \tilde{a}_t^h dt \right] \\
 &\quad - \mathbb{E}_{s x(w-h_2)} \left[\int_s^{\tau^{\hat{\pi}^h} \wedge T} e^{-c(t-s)} \hat{a}_t^h dt \right] \\
 (6.4) \quad &= e^{-ch_2} \mathbb{E}_{(s-h_2)x(w-h_2)} \left[\int_{s-h_2}^{\tau^{\tilde{\pi}^h} \wedge (T-h_2)} e^{-c(t-s)} \tilde{a}_t^h dt \right] \\
 &\quad - \mathbb{E}_{s x(w-h_2)} \left[\int_s^{\tau^{\hat{\pi}^h} \wedge T} e^{-c(t-s)} \hat{a}_t^h dt \right] \\
 &\quad + \mathbb{E}_{(s-h_2)x(w-h_2)} \left[\int_{\tau^{\tilde{\pi}^h} \wedge (T-h_2)}^{\tau^{\tilde{\pi}^h} \wedge T} e^{-c(t-s+h_2)} \tilde{a}_t^h dt \right] \\
 &:= e^{-ch_2} J_2^1 - J_2^2 + J_2^3,
 \end{aligned}$$

where $J_2^i, i = 1, 2, 3$ are the three expectations on the right-hand side, respectively. Note that by definition of the $\hat{\pi}^h$ and $\tilde{\pi}^h$, it is easy to check that $J_2^1 = J_2^2$. Thus, (6.4) becomes

$$(6.5) \quad J_2 \leq J_2^3 = \mathbb{E}_{(s-h_2)x(w-h_2)} \left[\int_{\tau^{\tilde{\pi}^h} \wedge (T-h_2)}^{\tau^{\tilde{\pi}^h} \wedge T} e^{-c(t-s+h_2)} \tilde{a}_t^h dt \right] \leq Mh_2.$$

Finally, from the proofs of Theorem 4.2 and Lemma 4.1, we see that the mapping $x \mapsto J(s, x, w, \pi)$ is continuous in x , uniformly for $(s, x, w) \in D$ and $\pi \in \mathcal{U}_{ad}[s, T]$. Therefore, for any $\varepsilon > 0$, we can find $\delta > 0$, depending only on ε , such that, for $0 < h_1 < \delta$, it holds that

$$J_3 = J(s, x, w - h_2, \hat{\pi}^h) - J(s, x - h_1, w - h_2, \hat{\pi}^h) < \varepsilon/3 \quad \forall h_2 \in (0, w).$$

We can then assume that δ is small enough, so that for $h_2 < \delta$, it holds that $J_1 < \varepsilon/3, J_2 < \varepsilon/3$, uniformly in $(s, x, w) \in D$ and $\pi \in \mathcal{U}_{ad}[s, T]$, thanks to (6.3) and (6.5). Consequently, we have

$$J(s, x, w, \pi) - J(s, x - h_1, w - h_2, \hat{\pi}^h) \leq J_1 + J_2 + J_3 < \varepsilon,$$

proving (6.1), whence the lemma. \square

We are now ready to prove the first main result of this paper: the Bellman principle of optimality or Dynamic Programming Principle (DPP). Recall that for a given $\pi \in \mathcal{U}_{ad}[s, T]$ and $(s, x, w) \in D$, we denote $R_t^\pi = R_t^{\pi, s, x, w} = (t, X_t^{\pi, s, x, w}, W_t^{s, w})$, $t \in [s, T]$.

THEOREM 6.2. *Assume that Assumption 2.1 is in force. Then, for any $(s, x, w) \in D$ and any stopping time $\tau \in [s, T]$, it holds that*

$$(6.6) \quad \begin{aligned} &V(s, x, w) \\ &= \sup_{\pi \in \mathcal{U}_{ad}[s, T]} \mathbb{E}_{s,x,w} \left[\int_s^{\tau \wedge \tau^\pi} e^{-c(t-s)} a_t dt + e^{-c(\tau \wedge \tau^\pi - s)} V(R_{\tau \wedge \tau^\pi}^\pi) \right]. \end{aligned}$$

PROOF. The idea of the proof is more or less standard. We shall first argue that (6.6) holds for deterministic $\tau = s + h$, for $h \in (0, T - s)$. That is, denoting

$$\begin{aligned} &v(s, x, w; s + h) \\ &:= \sup_{\pi \in \mathcal{U}_{ad}[s, T]} \mathbb{E}_{s,x,w} \left[\int_s^{(s+h) \wedge \tau^\pi} e^{-c(t-s)} a_t dt \right. \\ &\quad \left. + e^{-c((s+h) \wedge \tau^\pi - s)} V(R_{(s+h) \wedge \tau^\pi}^\pi) \right], \end{aligned}$$

we now show that $V(s, x, w) = v(s, x, w; s + h)$. To this end, let $\pi = (\gamma, a) \in \mathcal{U}_{ad}[s, T]$, and write

$$(6.7) \quad \begin{aligned} J(s, x, w; \pi) &= \mathbb{E}_{s,x,w} \left[\int_s^{(s+h) \wedge \tau^\pi} e^{-c(t-s)} a_t dt \right] \\ &\quad + \mathbb{E}_{s,x,w} \left[\int_{s+h}^{\tau^\pi} e^{-c(t-s)} a_t dt : \tau^\pi > s + h \right]. \end{aligned}$$

Now applying Lemma 2.4 we see that the second term on the right-hand side of (6.7) becomes

$$\begin{aligned} &\mathbb{E}_{s,x,w} \left[\int_{s+h}^{\tau^\pi} e^{-c(t-s)} a_t dt : \tau^\pi > s + h \right] \\ &= e^{-ch} \mathbb{E}_{s,x,w} \left[\mathbb{E} \left[\int_{s+h}^{\tau^\pi} e^{-c(t-(s+h))} a_t dt \middle| \mathcal{F}_{s+h}^s \right] : \tau^\pi > s + h \right] \\ &= e^{-ch} \mathbb{E}_{s,x,w} [J^\pi(s + h, X_{s+h}^\pi, W_{s+h}^\pi) : \tau^\pi > s + h] \\ &\leq e^{-ch} \mathbb{E}_{s,x,w} [V(R_{s+h}^\pi) : \tau^\pi > s + h] \\ &\leq \mathbb{E}_{s,x,w} [e^{-c((s+h) \wedge \tau^\pi - s)} V(R_{(s+h) \wedge \tau^\pi}^\pi)]. \end{aligned}$$

Plugging this into (6.7) and taking supremum, we obtain that $V(s, x, w) \leq v(s, x, w; s + h)$.

The proof of the reversed inequality is slightly more involved, as usual. To begin with, we recall Lemma 6.1. For any $\varepsilon > 0$, let $\delta > 0$ be the constant in Lemma 6.1. Next, let $0 = x_0 < x_1 < \dots$ and $0 = w_0 < w_1 < \dots < w_n = T$ be a partition of $[0, \infty) \times [0, T]$, so that $x_{i+1} - x_i < \delta$ $w_{j+1} - w_j < \delta$. Denote $D_{ij} := [x_{i-1}, x_i) \times$

$[w_{j-1}, w_j], i, j \in \mathbb{N}$. For $0 \leq s < s + h < T, i \in \mathbb{N}$ and $0 \leq j \leq n$, we choose $\pi^{ij} \in \mathcal{W}_{ad}^{s+h, w_j}[s + h, T]$ such that $J(s + h, x_i, w_j; \pi^{ij}) > V(s + h, x_i, w_j) - \varepsilon$.

Now applying Lemma 6.1, for each $(x, w) \in D_{ij}$ and $\pi^{ij} \in \mathcal{W}_{ad}^{s+h, w_j}[s + h, T]$, we can define the strategy $\hat{\pi}^{ij} = \hat{\pi}^{ij}(x, w) \in \mathcal{W}_{ad}^{s+h, w}[s + h, T]$, such that

$$(6.8) \quad \begin{aligned} J(s + h, x, w; \hat{\pi}^{ij}) &\geq J(s + h, x_i, w_j; \pi^{ij}) - \varepsilon \\ &\geq V(s + h, x_i, w_j) - 2\varepsilon \geq V(s + h, x, w) - 3\varepsilon. \end{aligned}$$

In the above, the last inequality is due to the uniform continuity of V on the variables (x, w) .

Now for any $\pi \in \mathcal{W}_{ad}^{s, w}[s, T]$, we define a new strategy π^* as follows:

$$\pi_t^* = \pi_t \mathbf{1}_{[s, s+h)}(t) + \sum_{i=0}^{\infty} \sum_{j=0}^{n-1} \hat{\pi}_t^{ij}(X_{s+h}^\pi, W_{s+h}) \mathbf{1}_{D_{ij}}(X_{s+h}^\pi, W_{s+h}) \mathbf{1}_{[s+h, T)}(t).$$

Then $\pi^* \in \mathcal{W}_{ad}^{s, w}[s, T]$, and $\{\tau^{\pi^*} \leq s + h\} = \{\tau^\pi \leq s + h\}$. Furthermore, when $\tau^{\pi^*} > s + h$ we have

$$(6.9) \quad \begin{aligned} J(s + h, X_{s+h}^\pi, W_{s+h}; \pi^*) &\geq V(s + h, X_{s+h}^\pi, W_{s+h}) - 3\varepsilon, \\ &\mathbb{P}\text{-a.s. on } \{\tau^{\pi^*} > s + h\}, \end{aligned}$$

thanks to (6.8). Consequently, similar to (6.7) we have

$$(6.10) \quad \begin{aligned} &V(s, x, w) \\ &\geq J(s, x, w; \pi^*) \\ &= \mathbb{E}_{s, x, w} \left[\int_s^{(s+h) \wedge \tau^\pi} e^{-c(t-s)} a_t dt \right. \\ &\quad \left. + \mathbf{1}_{\{\tau^\pi > s+h\}} e^{-ch} \int_{s+h}^{\tau^{\pi^*} \wedge T} e^{-c(t-(s+h))} a_t^* dt \right] \\ &\geq \mathbb{E}_{s, x, w} \left[\int_s^{(s+h) \wedge \tau^\pi} e^{-c(t-s)} a_t dt + e^{-c((s+h) \wedge \tau^\pi - s)} V(R_{(s+h) \wedge \tau^\pi}^\pi) \right] - 3\varepsilon. \end{aligned}$$

Here, in the last inequality we used the fact that $\mathbf{1}_{\{\tau^\pi \leq s+h\}} V(R_{(s+h) \wedge \tau^\pi}^\pi) = \mathbf{1}_{\{\tau^\pi \leq s+h\}} V(R_{\tau^\pi}^\pi) = 0$. Since π is arbitrary, (6.10) implies $V(s, x, w) \geq v(s, x, w; s + h) - 3\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain that $V(s, x, w) \geq v(s, x, w; s + h)$, proving (6.6) for $\tau = s + h$.

We now consider the general case when $s < \tau < T$ is a stopping time. Let $s = t_0 < t_1 < \dots < t_n = T$ be a partition of $[s, T]$. We assume that $t_k := s + \frac{k}{n}(T - s), k = 0, 1, \dots, n$. Define $\tau_n := \sum_{k=0}^{n-1} t_k \mathbf{1}_{[t_k, t_{k+1})}(\tau)$. Clearly, τ_n takes only a finite

number of values and $\tau_n \rightarrow \tau$, \mathbb{P} -a.s. It is easy to check, using the same argument above when τ is deterministic to each subinterval $[s, T]$, that $V(s, x, w) \leq v(s, x, w; \tau_n)$. We shall prove by induction (on n) that

$$(6.11) \quad V(s, x, w) \geq v(s, x, w; \tau_n) \quad \forall n \geq 1.$$

Indeed, for $n = 1$, we have $\tau_1 \equiv s$, so there is nothing to prove. Now suppose that (6.11) holds for τ_{n-1} , and $n \geq 2$. We shall argue that (6.11) holds for τ_n as well. For any $\pi \in \mathcal{U}_{ad}^{s,w}[s, T]$, we have

$$(6.12) \quad \begin{aligned} & \mathbb{E}_{s,x,w} \left\{ \int_s^{\tau_n \wedge \tau^\pi} e^{-c(t-s)} a_t dt + e^{-c(\tau_n \wedge \tau^\pi - s)} V(R_{\tau_n \wedge \tau^\pi}^\pi) \right\} \\ &= \mathbb{E}_{s,x,w} \left\{ \mathbf{1}_{\{\tau^\pi \leq t_1\}} \int_s^{\tau^\pi} e^{-c(t-s)} a_t dt \right\} \\ &+ \mathbb{E}_{s,x,w} \left\{ \left[\int_s^{\tau_n \wedge \tau^\pi} e^{-c(t-s)} a_t dt \right. \right. \\ &+ \left. \left. e^{-c(\tau_n \wedge \tau^\pi - s)} V(R_{\tau_n \wedge \tau^\pi}^\pi) \right] \mathbf{1}_{\{\tau_n > t_1\}} \mathbf{1}_{\{\tau^\pi > t_1\}} \right. \\ &+ \left. \left[\int_s^{t_1} e^{-c(t-s)} a_t dt + e^{-c(t_1-s)} V(R_{t_1}^\pi) \right] \mathbf{1}_{\{\tau_n = t_1\}} \mathbf{1}_{\{\tau^\pi > t_1\}} \right\}. \end{aligned}$$

Note that on the set $\{\tau_n > t_1\}$, τ_n takes only $n - 1$ values, by inductual hypothesis, we have

$$\begin{aligned} & \mathbb{E}_{s,x,w} \left\{ \left[\int_{t_1}^{\tau_n \wedge \tau^\pi} e^{-c(t-s)} a_t dt + e^{-c(\tau_n \wedge \tau^\pi - s)} V(R_{\tau_n \wedge \tau^\pi}^\pi) \right] \mathbf{1}_{\{\tau_n > t_1\}} \mathbf{1}_{\{\tau^\pi > t_1\}} \right\} \\ & \leq \mathbb{E}_{s,x,w} \left\{ e^{-c(t_1-s)} v(t_1, X_{t_1}^\pi, W_{t_1}; \tau_n) \mathbf{1}_{\{\tau_n > t_1\}} \mathbf{1}_{\{\tau^\pi > t_1\}} \right\} \\ & \leq \mathbb{E}_{s,x,w} \left\{ e^{-c(t_1-s)} V(R_{t_1}^\pi) \mathbf{1}_{\{\tau_n > t_1\}} \mathbf{1}_{\{\tau^\pi > t_1\}} \right\}. \end{aligned}$$

Plugging this into (6.12) we obtain

$$(6.13) \quad \begin{aligned} & \mathbb{E}_{s,x,w} \left\{ \int_s^{\tau_n \wedge \tau^\pi} e^{-c(t-s)} a_t dt + e^{-c(\tau_n \wedge \tau^\pi - s)} V(R_{\tau_n \wedge \tau^\pi}^\pi) \right\} \\ & \leq \mathbb{E}_{s,x,w} \left\{ \mathbf{1}_{\{\tau^\pi \leq t_1\}} \int_s^{\tau^\pi} e^{-c(t-s)} a_t dt \right\} \\ & + \mathbb{E}_{s,x,w} \left\{ \left[\int_s^{t_1} e^{-c(t-s)} a_t dt + e^{-c(t_1-s)} V(R_{t_1}^\pi) \right] \mathbf{1}_{\{\tau_n > t_1\}} \mathbf{1}_{\{\tau^\pi > t_1\}} \right. \\ & + \left. \left[\int_s^{t_1} e^{-c(t-s)} a_t dt + e^{-c(t_1-s)} V(R_{t_1}^\pi) \right] \mathbf{1}_{\{\tau_n = t_1\}} \mathbf{1}_{\{\tau^\pi > t_1\}} \right\} \\ & = \mathbb{E}_{s,x,w} \left\{ \mathbf{1}_{\{\tau^\pi \leq t_1\}} \int_s^{\tau^\pi} e^{-c(t-s)} a_t dt \right\} \end{aligned}$$

$$\begin{aligned}
 & + \mathbb{E}_{s,x,w} \left\{ \mathbf{1}_{\{\tau^\pi > t_1\}} \left[e^{-c(t_1-s)} V(R_{t_1}^\pi) + \int_s^{t_1} e^{-c(t-s)} a_t dt \right] \right\} \\
 & = \mathbb{E}_{s,x,w} \left\{ \int_s^{t_1 \wedge \tau^\pi} e^{-c(t-s)} a_t dt + e^{-c(t_1 \wedge \tau^\pi - s)} V(R_{t_1 \wedge \tau^\pi}^\pi) \right\} \leq V(s, x, w).
 \end{aligned}$$

In the above, we again used the fact $V(R_{t_1}^\pi) = 0$, and the last inequality is due to (6.6) for fixed time $t_1 = s + h$. Consequently, we obtain $v(s, x, w; \tau_n) \leq V(s, x, w)$, whence $v(s, x, w; \tau_n) = V(s, x, w)$. A simple application of dominated convergence theorem, together with the uniform continuity of the value function, will then lead to the general form of (6.6). The proof is now complete. \square

7. The Hamilton–Jacobi–Bellman equation. We are now ready to investigate the main subject of the paper: the Hamilton–Jacobi–Bellman (HJB) equation associated to our optimization problem (2.7). We note that such a PDE characterization of the value function is only possible after the clock process W is brought into the picture. Recall the sets $\mathcal{D} \subset \mathcal{D}^* \subset D$ defined in (2.8).

Next, we denote $\mathbb{C}_0^{1,2,1}(D)$ to be the set of all functions $\varphi \in \mathbb{C}^{1,2,1}(\mathcal{D})$ such that for $\eta = \varphi, \varphi_t, \varphi_x, \varphi_{xx}, \varphi_w$, it holds that $\lim_{(t,y,v) \rightarrow (s,x,w)} \eta(t, y, v) = \eta(s, x, w)$, for all $(s, x, w) \in D$; and $\varphi(s, x, w) = 0$, for $(s, x, w) \notin D$. We note that while a function $\varphi \in \mathbb{C}_0^{1,2,1}(D)$ is well defined on D , it is not necessarily continuous on the boundaries $\{(s, x, w) : x = 0 \text{ or } w = 0 \text{ or } w = s\}$.

Next, we define the following function:

$$\begin{aligned}
 (7.1) \quad & H(s, x, w, u, \xi, A, z, \gamma, a) \\
 & := \frac{\sigma^2}{2} \gamma^2 x^2 A + [p + (r + (\mu - r)\gamma)x - a] \xi^1 \\
 & \quad + \xi^2 + \lambda(w)z + (a - cu),
 \end{aligned}$$

where $\xi = (\xi^1, \xi^2) \in \mathbb{R}^2$, $u, A, z \in \mathbb{R}$, and $(\gamma, a) \in [0, 1] \times [0, M]$. For $\varphi \in \mathbb{C}_0^{1,2,1}(D)$, we define

$$\begin{aligned}
 (7.2) \quad & \mathcal{H}(s, x, w, \varphi, \varphi_x, \varphi_w, \varphi_{xx}, \gamma, a) \\
 & := H(s, x, w, \varphi, \nabla\varphi, \varphi_{xx}, I(\varphi), \gamma, a),
 \end{aligned}$$

where $\nabla\varphi := (\varphi_x, \varphi_w)$ and $I[\varphi]$ is the integral operator defined by

$$\begin{aligned}
 (7.3) \quad & I[\varphi] := \int_0^\infty [\varphi(s, x - u, 0) - \varphi(s, x, w)] dG(u) \\
 & = \int_0^x \varphi(s, x - u, 0) dG(u) - \varphi(s, x, w).
 \end{aligned}$$

Here, the last equality is due to the fact that $\varphi(s, x, w) = 0$ for $x < 0$. The main purpose of this section is to show that the value function V is a viscosity solution of the following HJB equation:

$$(7.4) \quad \{V_s + \mathcal{L}[V]\}(s, x, w) = 0; \quad (s, x, w) \in \mathcal{D}; \quad V(T, x, w) = 0,$$

where $\mathcal{L}[\cdot]$ is the second-order partial integro-differential operator: for $\varphi \in \mathbb{C}_0^{1,2,1}(D)$,

$$(7.5) \quad \mathcal{L}[\varphi](s, x, w) := \sup_{\gamma \in [0, 1], a \in [0, M]} \mathcal{H}(s, x, w, \varphi, \varphi_x, \varphi_w, \varphi_{xx}, \gamma, a).$$

REMARK 7.1. (i) We note that even a classical solution to (7.4) may have discontinuity on the boundary $\{x = 0\} \cup \{w = 0\} \cup \{w = s\}$, and (7.4) only specifies the boundary value at $s = T$.

(ii) To guarantee the well-posedness we shall consider the *constrained* viscosity solutions (cf., e.g., [40]), for which the following observation is crucial. Let $V \in \mathbb{C}_0^{1,2,1}(D)$ be a classical solution so that (7.4) holds on \mathcal{D}^* . Consider the point $(s, 0, w) \in \partial\mathcal{D}^*$. Let $\varphi \in \mathbb{C}_0^{1,2,1}(D)$ be such that $0 = [V - \varphi](s, 0, w) = \max_{(t, y, v) \in \mathcal{D}^*} [V - \varphi](t, y, v)$. Then one must have $(\partial_t, \nabla)(V - \varphi)(s, 0, w) = \alpha v$ for some $\alpha > 0$, where $\nabla = (\partial_x, \partial_w)$ and v is the outward normal vector of \mathcal{D}^* at the boundary $\{x = 0\}$ [i.e., $v = (0, -1, 0)$], and $I[V - \varphi](s, 0, w) = -[V - \varphi](s, 0, w) = 0$ since $[V - \varphi](s, y, w) = 0$ for $y \leq 0$. Thus, for any $(\gamma, a) \in [0, 1] \times [0, M]$ we obtain that

$$(7.6) \quad \begin{aligned} & [\varphi_s + \mathcal{H}(\cdot, \varphi, \varphi_x, \varphi_w, \varphi_{xx}, \gamma, a)](s, 0, w) \\ &= [\varphi_s + ((p - a, 1), \nabla\varphi) + \lambda I[\varphi] + (a - c\varphi)](s, 0, w) \\ &= [V_s + \mathcal{H}(\cdot, V, \nabla V, V_{xx}, I(V), \gamma, a)](s, 0, w) + \alpha(p - a). \end{aligned}$$

Consequently, assuming $a \leq p$ (which is natural in the case $x = 0$!) we have

$$(7.7) \quad \{\varphi_s + \mathcal{L}[\varphi]\}(s, 0, w) \geq \{V_s + \mathcal{L}[V]\}(s, 0, w) = 0.$$

For the other two boundaries $\{w = 0\}$ and $\{w = s\}$, we note that $[V_{xx} - \varphi_{xx}] \leq 0$ and the corresponding outward normal vectors are $v = (0, 0, -1)$ and $(-1, 0, 1)$, respectively. Therefore, a similar calculation as (7.6), noting that $((1, p + rx - a, 1), v) = -1, 0$, respectively, would lead to (7.7) in both cases. In other words, we can extend the “subsolution property” of (7.4) to \mathcal{D}^* .

We are now ready to give the definition of the so-called *constrained* viscosity solution.

DEFINITION 7.2. Let $\mathcal{O} \subseteq \mathcal{D}^*$ be a subset such that $\partial_T \mathcal{O} := \{(T, y, v) \in \partial\mathcal{O}\} \neq \emptyset$, and let $v \in \mathbb{C}(\mathcal{O})$. We say that v is a viscosity subsolution (resp., supersolution) of (7.4) on \mathcal{O} , if $v(T, y, v) \leq 0$ (resp., ≥ 0) for $(T, y, v) \in \partial_T \mathcal{O}$; and for $(s, x, w) \in \mathcal{O}$ and $\varphi \in \mathbb{C}_0^{1,2,1}(\mathcal{O})$ such that $0 = [v - \varphi](s, x, w) = \max_{(t, y, v) \in \mathcal{O}} [v - \varphi](t, y, v)$ (resp., $0 = [v - \psi](s, x, w) = \min_{(t, y, v) \in \mathcal{O}} [v - \psi](t, y, v)$), it holds that

$$(7.8) \quad \varphi_s(s, x, w) + \mathcal{L}[\varphi](s, x, w) \geq 0 \quad (\text{resp. } \leq 0).$$

We say that $v \in \mathbb{C}(D)$ is a “constrained viscosity solution” of (7.4) on \mathcal{D}^* if it is both a viscosity subsolution of (7.4) on \mathcal{D}^* and a viscosity supersolution of (7.4) on \mathcal{D} .

REMARK 7.3. (i) We note that the main feature of the constrained viscosity solution is that its subsolution is defined on \mathcal{D}^* , which is justified in Remark 7.1(ii). This turns out to be essential for the comparison theorem, whence the uniqueness.

(ii) The inequalities in (7.8) are opposite than the usual sub and supersolutions, due to the fact that the HJB equation (7.4) is a terminal value problem.

As in the viscosity theory, it is often convenient to study viscosity solution in terms of the sub(super) differentials [or parabolic sub(super)jets], which we now define.

DEFINITION 7.4. Let $\mathcal{O} \subseteq \mathcal{D}^*$, $u \in \mathbb{C}(\mathcal{O})$, and $(s, x, w) \in \mathcal{O}$. The set of parabolic superjets of u at (s, x, w) , denoted by $\mathcal{P}_O^{+(1,2,1)}u(s, x, w)$, is defined as the set of all $(q, \xi, A) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$ such that for all $(s, X) := (s, x, w)$, $(t, Y) := (t, y, v) \in \mathcal{O}$, it holds that

$$(7.9) \quad \begin{aligned} u(t, Y) \leq & u(s, X) + q(t - s) + (\xi, Y - X) + \frac{1}{2}A(x - y)^2 \\ & + o(|t - s| + |w - v| + |y - x|^2). \end{aligned}$$

The set of parabolic subjets of u at $(s, x, w) \in \mathcal{O}$, denoted by $\mathcal{P}_O^{-(1,2,1)}u(s, x, w)$, is the set of all $(q, p, A) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$ such that (7.9) holds with “ \leq ” being replaced by “ \geq .”

The closure of $\mathcal{P}_O^{+(1,2,1)}u(s, x, w)$ [resp., $\mathcal{P}_O^{-(1,2,1)}u(s, x, w)$], denoted by $\bar{\mathcal{P}}_O^{+(1,2,1)}u(s, x, w)$ [resp., $\bar{\mathcal{P}}_O^{-(1,2,1)}u(s, x, w)$], is defined as the set of all $(q, \xi, A) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$ such that there exist $(s_n, x_n, w_n) \in \mathcal{O}$ and $(q_n, \xi_n, A_n) \in \mathcal{P}_O^{+(1,2,1)}u(s_n, x_n, w_n)$ [resp., $\mathcal{P}_O^{-(1,2,1)}u(s_n, x_n, w_n)$], and that $((s_n, x_n, w_n), u(s_n, x_n, w_n), q_n, \xi_n, A_n) \rightarrow ((s, x, w), u(s, x, w), q, \xi, A)$, as $n \rightarrow \infty$.

We now define the constrained viscosity solution in terms of the parabolic jets. The equivalence between the two definitions in such a setting can be found in, for example, [5, 7].

DEFINITION 7.5. Let $\mathcal{O} \subseteq \mathcal{D}^*$, $u \in \mathbb{C}(\mathcal{O})$. We say that \underline{u} (resp., \bar{u}) $\in \mathbb{C}(\mathcal{O})$ is a viscosity subsolution (resp., supersolution) of (7.4) on \mathcal{O} if for any $(s, x, w) \in \mathcal{O}$, it holds that

$$\begin{aligned} q + \sup_{\gamma \in [0, 1], a \in [0, M]} H(s, x, w, \underline{u}, \xi, A, I[\underline{u}], \gamma, a) &\geq 0, \\ \left(\text{resp. } q + \sup_{\gamma \in [0, 1], a \in [0, M]} H(s, x, w, \bar{u}, \xi, A, I[\bar{u}], \gamma, a) \right) &\leq 0, \end{aligned}$$

for all $(q, (p^1, p^2), A) \in \mathcal{P}_O^{+(1,2,1)}\underline{u}(s, x, w)$ [resp., $\mathcal{P}_O^{-(1,2,1)}\bar{u}(s, x, w)$].

In particular, we say that u is a “constrained viscosity solution” of (7.4) on \mathcal{D}^* if it is both a viscosity subsolution on \mathcal{D}^* , and a viscosity supersolution on \mathcal{D} .

In the rest of the paper, we shall assume that all solutions of (7.4) satisfy $u(s, x, w) = 0$, for $(s, x, w) \notin D$. We now give the main result of this section.

THEOREM 7.6. *Assume that Assumption 2.1 is in force. Then the value function V of problem (2.7) is a constrained viscosity solution of (7.4) on \mathcal{D}^* .*

PROOF. *Supersolution.* Given $(s, x, w) \in \mathcal{D}$. Let $\varphi \in \mathbb{C}_0^{1,2,1}(D)$ be such that $V - \varphi$ attains its minimum at (s, x, w) with $\varphi(s, x, w) = V(s, x, w)$. For any $h > 0$ such that $s \leq s + h < T$, let us denote $\tau_s^h := s + h \wedge T_1^{s,w}$, and $\tilde{U}_1 = \Delta Q_{T_1^{s,w}}^{s,w}$. For any $(\gamma_0, a_0) \in [0, 1] \times [0, M]$, we consider the following ‘‘feedback’’ strategy: $\pi_t^0 = (\gamma_0, a_0 \mathbf{1}_{\{t < \tau_0\}} + p \mathbf{1}_{\{t \geq \tau_0\}})$, $t \in [s, T]$, where $\tau_0 = \inf\{t > s, X_t^{\pi^0} = 0\}$. Then $\pi^0 \in \mathcal{W}_{ad}[s, T]$, and it is readily seen from (3.1) that ruin can only happen at a jump time, that is, $T_1^{s,w} \leq \tau_s^{\pi^0}$, and $R_t^0 := (t, X_t^{\pi^0, s, w, x}, W_t^{s,w}) \in \mathcal{D}$, for $t \in [s, \tau_s^h]$.

Next, by DPP (Theorem 6.2) and the properties of φ we have

$$\begin{aligned}
 0 &\geq \mathbb{E}_{s,x,w} \left[\int_s^{\tau_s^h} e^{-c(t-s)} (a_0 \mathbf{1}_{\{t < \tau_0\}} + p \mathbf{1}_{\{t \geq \tau_0\}}) dt + e^{-c(\tau_s^h-s)} V(R_{\tau_s^h}^0) \right] \\
 &\quad - V(s, x, w) \\
 (7.10) \quad &\geq \mathbb{E}_{s,x,w} \left[\int_s^{\tau_s^h} e^{-c(t-s)} a_0 dt \mathbf{1}_{\{\tau_s^h < \tau_0\}} + e^{-c(\tau_s^h-s)} \varphi(R_{\tau_s^h}^0) \right] - \varphi(s, x, w) \\
 &= \mathbb{E}_{s,x,w} \left[\int_s^{\tau_s^h} e^{-c(t-s)} a_0 dt \mathbf{1}_{\{\tau_s^h < \tau_0\}} \right] \\
 &\quad + \mathbb{E}_{s,x,w} [e^{-c(\tau_s^h-s)} [\varphi(R_{\tau_s^h}^0) - \varphi(R_{\tau_s^h-}^0)] \mathbf{1}_{\{T_1^{s,w} < h\}}] \\
 &\quad + \mathbb{E}_{s,x,w} [e^{-c(\tau_s^h-s)} \varphi(R_{\tau_s^h-}^0) - \varphi(s, x, w)] := I_1 + I_2 + I_3,
 \end{aligned}$$

where $I_i, i = 1, 2, 3$ are the three terms on the right-hand side above. Clearly, we have

$$\begin{aligned}
 (7.11) \quad I_1 &= \frac{a_0}{c} \left\{ [1 - e^{-ch}] \mathbb{P}(\tau_0 > s + h, T_1^{s,w} > h) \right. \\
 &\quad \left. + \int_0^h [1 - e^{-ct}] \mathbb{P}(\tau_0 > s + t) dF_{T_1^{s,w}}(t) \right\}.
 \end{aligned}$$

Since $\tau_s^h = s + T_1^{s,w}$ on $\{T_1^{s,w} < h\}$, we have

$$\begin{aligned}
 (7.12) \quad I_2 &= \mathbb{E}_{s,x,w} [e^{-cT_1^{s,w}} [\varphi(R_{s+T_1^{s,w}}^0) - \varphi(R_{(s+T_1^{s,w})-}^0)] \mathbf{1}_{\{T_1^{s,w} < h\}}] \\
 &= \mathbb{E}_{s,x,w} \left[\int_0^\infty \int_0^h e^{-ct} [\varphi(s + t, X_{(s+t)-}^{\pi^0} - u, 0) \right. \\
 &\quad \left. - \varphi(t, X_{(s+t)-}^{\pi^0}, W_{(s+t)-}^{s,w})] dF_{T_1^{s,w}}(t) dG(u) \right].
 \end{aligned}$$

Since there is no jumps on $[s, \tau_s^h)$, by Itô's formula [denoting $\theta(x) := (r + (\mu - r)\gamma_0)x + p$] we get

$$\begin{aligned}
 I_3 &= \mathbb{E}_{s,x,w} \left[\int_s^{\tau_s^h} e^{-c(t-s)} \left\{ -c\varphi + \varphi_t + ((\theta(X_t^{\pi^0}) - a_0, 1), \nabla\varphi) \right. \right. \\
 &\quad \left. \left. + \frac{(\sigma\gamma_0 X_t^{\pi^0})^2}{2} \varphi_{xx}^2 \right\} (R_t^0) dt \right] \\
 (7.13) \quad &= \mathbb{E}_{s,x,w} \left[\int_0^h \mathbf{1}_{\{T_1^{s,w} \geq t\}} e^{-ct} \left\{ -c\varphi + \varphi_t + ((\theta(X_{s+t}^{\pi^0}) - a_0, 1), \nabla\varphi) \right. \right. \\
 &\quad \left. \left. + \frac{(\sigma\gamma_0 X_{s+t}^{\pi^0})^2}{2} \varphi_{xx}^2 \right\} (R_{s+t}^0) dt \right] \\
 &= \mathbb{E}_{s,x,w} \left[\int_0^h \bar{F}_{T_1^{s,w}}(t) e^{-ct} \left\{ -c\varphi + \varphi_t + ((\theta(X_{s+t}^{\pi^0}) - a_0, 1), \nabla\varphi) \right. \right. \\
 &\quad \left. \left. + \frac{(\sigma\gamma_0 X_{s+t}^{\pi^0})^2}{2} \varphi_{xx}^2 \right\} (R_{s+t}^0) dt \right].
 \end{aligned}$$

Recall that $dF_{T_1^{s,w}}(t) = \lambda(w)\bar{F}_{T_1^{s,w}}(t) dt = \lambda(w)e^{-\int_w^{w+t} \lambda(u) du} dt$, and $\bar{F}_{T_1^{s,w}}(0) = 1$, dividing both sides of (7.10) by h and then sending h to 0 we obtain, in light of (7.11)–(7.13),

$$(7.14) \quad 0 \geq \{ \varphi_t + \mathcal{H}(\cdot, \varphi, \varphi_x, \varphi_w, \varphi_{xx}, \gamma_0, a_0) \}(s, x, w).$$

Since (γ_0, a_0) is arbitrary, we conclude that V is a viscosity supersolution on \mathcal{D} .

Subsolution. We shall now argue that V is a viscosity subsolution on \mathcal{D}^* . Suppose not, then we shall first show that there exist $(s, x, w) \in \mathcal{D}^*$, $\psi \in \mathbb{C}_0^{1,2,1}(D)$, and constants $\varepsilon > 0$, $\rho > 0$, such that $0 = [V - \psi](s, x, w) = \max_{(t,y,v) \in \mathcal{D}^*} [V - \psi](t, y, v)$, but

$$(7.15) \quad \begin{cases} \{ \psi_s + \mathcal{L}[\psi] \}(t, y, v) \leq -\varepsilon c, \\ \quad (t, y, v) \in \overline{B_\rho(s, x, w)} \cap \mathcal{D}^* \setminus \{t = T\}; \\ V(t, y, v) \leq \psi(t, y, v) - \varepsilon, \\ \quad (t, y, v) \in \partial B_\rho(s, x, w) \cap \mathcal{D}^*, \end{cases}$$

where $B_\rho(s, x, w)$ is the open ball centered at (s, x, w) with radius ρ . To see this, we note that if V is not a viscosity subsolution on \mathcal{D}^* , then there must exist $(s, x, w) \in \mathcal{D}^*$ and $\psi^0 \in \mathbb{C}_0^{1,2,1}(D)$, such that $0 = [V - \psi^0](s, x, w) = \max_{(t,y,v) \in \mathcal{D}^*} [V - \psi^0](t, y, v)$, but

$$(7.16) \quad \{ \psi_s^0 + \mathcal{L}[\psi^0] \}(s, x, w) = -2\eta < 0 \quad \text{for some } \eta > 0.$$

We shall consider two cases.

Case 1. $x > 0$. In this case, we introduce the function

$$(7.17) \quad \psi(t, y, v) := \psi^0(t, y, v) + \frac{\eta[(t-s)^2 + (y-x)^2 + (v-w)^2]^2}{\lambda(w)(x^2 + w^2)^2},$$

$$(t, y, v) \in D.$$

Clearly, $\psi \in \mathbb{C}_0^{1,2,1}(D)$, $\psi(s, x, w) = \psi^0(s, x, w) = V(s, x, w)$, and $\psi(t, y, v) > V(t, y, v)$, for all $(t, y, v) \in D \setminus (s, x, w)$. Furthermore, it is easy to check that $(\psi_s, \nabla\psi)(s, x, w) = (\psi_s^0, \nabla\psi^0)(s, x, w)$, $\psi_{yy}(s, x, w) = \psi_{yy}^0(s, x, w)$, and

$$\lambda(w) \int_0^x \psi(s, x-u, 0) dG(u) \leq \lambda(w) \int_0^x \psi^0(s, x-u, 0) dG(u) + \eta.$$

Consequently, we see that

$$\{\psi_s + \mathcal{L}[\psi]\}(s, x, w) \leq \{\psi_s^0 + \mathcal{L}[\psi^0]\}(s, x, w) + \eta = -\eta < 0.$$

By continuity of $\psi_s + \mathcal{L}[\psi]$, we can then find $\rho > 0$ such that

$$(7.18) \quad \{\psi_t + \mathcal{L}[\psi]\}(t, y, v) < -\eta/2$$

$$\text{for } (t, y, v) \in \overline{B_\rho(s, x, w)} \cap \mathcal{D}^* \setminus \{t = T\}.$$

Note also that for $(t, y, v) \in \partial B_\rho(s, x, w) \cap \mathcal{D}^*$, one has

$$(7.19) \quad V(t, y, v) \leq \psi(t, y, v) - \frac{\eta\rho^4}{\lambda(w)(x^2 + w^2)^2}.$$

Thus, if we choose $\varepsilon = \min\{\frac{\eta}{2c}, \frac{\eta\rho^4}{\lambda(w)(x^2+w^2)^2}\}$, then (7.18) and (7.19) become (7.15).

Case 2. $x = 0$. In this case, we have

$$\psi_s^0 - \mathcal{L}[\psi^0](s, 0, w)$$

$$= \sup_{a \in [0, M]} [((1, p - a, 1), (\psi_s^0, \nabla\psi^0))(s, 0, w) - (c + \lambda(w))\psi^0(s, 0, w) + a].$$

If we define $\psi(t, y, v) = \psi^0(t, y, v) + \eta[(t-s)^2 + y^2 + (v-w)^2]$, for $(t, y, v) \in D$, and $\varepsilon = \min\{\frac{\eta}{2c}, \rho^2\}$, then a similar calculation as before shows that (7.15) still holds, proving the claim.

We now argue that this will lead to a contradiction. Fix any $\pi = (\gamma, a) \in \mathcal{U}_{ad}^{s,w}[s, T]$, and let $R_t^{s,x,w} = (t, X_t^{s,x,w}, W_t^{s,w})$. Define $\tau_\rho := \inf\{t > s : R_t \notin \overline{B_\rho(s, x, w)} \cap \mathcal{D}^*\}$, $\tau := \tau_\rho \wedge T_1^{s,w}$, and denote $R_t = R_t^{s,x,w}$ for simplicity. Applying Itô's formula to $e^{-c(t-s)}\psi(R_t)$ from s to τ , we have

$$\int_s^\tau e^{-c(t-s)} a_t dt + e^{-c(\tau-s)} V(R_\tau)$$

$$= \int_s^\tau e^{-c(t-s)} a_t dt + e^{-c(\tau-s)} [\psi(R_\tau) + (V(R_\tau) - \psi(R_\tau))]$$

$$\begin{aligned}
 &= e^{-c(\tau-s)} [V(R_\tau) - \psi(R_\tau)] + \psi(s, x, w) \\
 (7.20) \quad &+ \int_s^\tau e^{-c(t-s)} \left\{ a_t - c\psi + \psi_t + \psi_w \right. \\
 &+ [(r + (\mu - r)\gamma_t)X_t + p - a_t]\psi_x + \frac{1}{2}X_t^2\sigma^2\gamma_t^2\psi_{xx} \left. \right\} (R_t) dt \\
 &+ \int_s^\tau e^{-c(t-s)} \psi_x(R_t)\sigma\gamma_t X_t dW_t \\
 &+ \sum_{s \leq t \leq \tau} e^{-c(t-s)} (\psi(R_t) - \psi(R_{t-})).
 \end{aligned}$$

Then, on the set $\{\tau_\rho \geq T_1^{s,w}\}$, we have $\tau = T_1^{s,w}$. Since the ruin only happens at the claim arrival times, we have $\tau^\pi \geq T_1^{s,w}$. In the case that $\tau^\pi = T_1^{s,w}$, $X_{T_1^{s,w}} < 0$ and $V(R_{T_1^{s,w}}) = \psi(R_{T_1^{s,w}}) = 0$; whereas in the case $\tau^\pi > T_1^{s,w}$, we have $R_{T_1^{s,w}} \in D$, and $V(R_{T_1^{s,w}}) \leq \psi(R_{T_1^{s,w}})$.

On the other hand, we note that on the set $\{\tau_\rho < T_1^{s,w}\}$, $\tau = \tau_\rho$, and since $(\tau_\rho, X_{\tau_\rho}, W_{\tau_\rho}) \in \partial B_\rho(s, x, w) \cap \mathcal{D}^*$, we derive from (7.15) that $[V(R_{\tau_\rho}) - \psi(R_{\tau_\rho})] \leq -\varepsilon$. Thus, noting that $W_{T_1^{s,w}} = 0$, and that both ψ_x and γ are bounded, we deduce from (7.20) that

$$\begin{aligned}
 &\mathbb{E}_{s,x,w} \left[\int_s^\tau e^{-c(t-s)} a_t dt + e^{-c(\tau-s)} V(\tau, X_\tau, W_\tau) \right] \\
 &\leq \mathbb{E} \left[\psi(s, x, w) - \varepsilon e^{-c(\tau_\rho-s)} \mathbf{1}_{\{\tau_\rho < T_1^{s,w}\}} \right. \\
 &\quad \left. + \int_s^\tau e^{-c(t-s)} [\psi_t + \mathcal{H}(\dots, \gamma_t, a_t)(R_t)] dt \right] \\
 (7.21) \quad &\leq \psi(s, x, w) - \varepsilon \mathbb{E}_{s,x,w} [e^{-c(\tau-s)} \mathbf{1}_{\{\tau_\rho < T_1^{s,w}\}} + (1 - e^{-c(\tau-s)})] \\
 &= V(s, x, w) - \varepsilon \mathbb{E}_{s,x,w} [(1 - e^{-c(T_1^{s,w}-s)}) \mathbf{1}_{\{\tau_\rho \geq T_1^{s,w}\}}] \\
 &\leq V(s, x, w) - \varepsilon \mathbb{E}_{s,x,w} (1 - e^{-c(T_1^{s,w}-s)}).
 \end{aligned}$$

Since $\mathbb{P}\{T_1^{s,w} > s\} = 1$, we see that (7.21) contradicts the DPP (6.6). \square

8. Comparison principle and uniqueness. In this section, we present a comparison theorem that would imply the uniqueness among a certain class of the constrained viscosity solutions of (7.4) to which the value function belong. To be more precise, we introduce to following subset of $\mathbb{C}(D)$.

DEFINITION 8.1. We say that a function $u \in \mathbb{C}(D)$ is of class (L) if:

- (i) $u(s, x, w) \geq 0$, $(s, x, w) \in D$, and u is uniformly continuous on D ;

- (ii) the mapping $x \mapsto u(s, x, w)$ is increasing, and $\lim_{x \rightarrow \infty} u(s, x, w) = \frac{M}{c}[1 - e^{-c(T-s)}]$;
- (iii) $u(T, y, v) = 0$ for any $(y, v) \in [0, \infty) \times [0, T]$.

Clearly, the value function V of (2.7) is of class (L), thanks to Propositions 3.2, 3.3, Theorem 4.2 and Corollary 5.3. Our goal is to show the following *comparison principle*.

THEOREM 8.2 (Comparison principle). *Assume that Assumption 2.1 is in force. Let \underline{u} be a viscosity subsolution of (7.4) on \mathcal{D}^* and \bar{u} be a viscosity supersolution of (7.4) on \mathcal{D} . If both \bar{u} and \underline{u} are of class (L), then $\underline{u} \leq \bar{u}$ on D .*

Consequently, there is at most one constrained viscosity solution of class (L) to (7.4) on D .

PROOF. We first perturb the supersolution slightly so that all the inequalities involved become strict. Define, for $\rho > 1, \theta, \zeta > 0$,

$$\bar{u}^{\rho, \theta, \zeta}(t, y, v) = \rho \bar{u}(t, y, v) + \theta \frac{T - t + \zeta}{t}.$$

Then it is straightforward to check that $\bar{u}^{\rho, \theta, \zeta}(t, y, v)$ is also a supersolution of (7.4) on \mathcal{D} . In fact, it is easy to see that $\rho \bar{u}$ is a supersolution of (7.4) in \mathcal{D} as $\rho > 1$, and for any $(s, x, w) \in \mathcal{D}$ and $\varphi \in C_0^{1,2,1}(D)$ such that $0 = [\bar{u}^{\rho, \theta, \zeta} - \varphi](s, x, w) = \min_{(t, y, v) \in \mathcal{D}} [\bar{u}^{\rho, \theta, \zeta} - \varphi](t, y, v)$, it holds that

$$\begin{aligned} & \left[\varphi_t + \sup_{\gamma, a} \mathcal{H}(\cdot, \bar{u}^{\rho, \theta, \zeta}, \varphi_x, \varphi_w, \varphi_{xx}, \gamma, \alpha) \right](s, x, w) \\ & \leq \left[\varphi_t + \sup_{\gamma, a} \mathcal{H}(\cdot, \rho \bar{u}, \tilde{\varphi}_x, \tilde{\varphi}_w, \tilde{\varphi}_{xx}, \gamma, a) \right](s, x, w) \leq 0, \end{aligned}$$

where $\tilde{\varphi}(t, y, v) := \varphi(t, y, v) - \theta(T - t + \zeta)/t$, that is, $\bar{u}^{\rho, \theta, \zeta}$ is a viscosity supersolution on \mathcal{D} . We shall argue that $\underline{u} \leq \bar{u}^{\rho, \theta, \zeta}$, which will lead to the desired comparison result as $\lim_{\rho \downarrow 0, \theta \downarrow 0, \zeta \downarrow 0} \bar{u}^{\rho, \theta, \zeta} = \bar{u}$.

To this end, we first note that $\lim_{t \rightarrow 0} \bar{u}^{\rho, \theta, \zeta}(t, y, v) = +\infty$. Thus, we need only show that $\underline{u} \leq \bar{u}^{\rho, \theta, \zeta}$ on $\mathcal{D}^* \setminus \{t = 0\}$. Next, note that both \underline{u} and \bar{u} are of class (L), we have (recall Definition 8.1)

$$\begin{aligned} (8.1) \quad & \lim_{y \rightarrow \infty} (\underline{u}(t, y, v) - \bar{u}^{\rho, \theta, \zeta}(t, y, v)) \\ & = (1 - \rho) \frac{M}{c} [1 - e^{-c(T-t)}] - \frac{\theta(T - t + \zeta)}{t} \leq -\frac{\theta \zeta}{T} < 0, \end{aligned}$$

for all $0 < t \leq T$. Thus, by Dini's theorem, the convergence in (8.1) is uniform in (t, y, v) , and we can choose $b > 0$ so that $\underline{u}(t, y, v) < \bar{u}^{\rho, \theta, \zeta}(t, y, v)$ for $y \geq b$,

$0 < t < T$, and $0 \leq v \leq t$. Consequently, it suffices to show that

$$(8.2) \quad \begin{aligned} \underline{u}(t, y, v) &\leq \bar{u}^{\rho, \theta, \varsigma}(t, y, v) \\ &\text{on } \mathcal{D}_b = \{(t, y, v) : 0 < t < T, 0 \leq y < b, 0 \leq v \leq t\}. \end{aligned}$$

Suppose (8.2) is not true, then there exists $(t^*, y^*, v^*) \in \bar{\mathcal{D}}_b$ such that

$$(8.3) \quad \begin{aligned} M_b &:= \sup_{\mathcal{D}_b} (\underline{u}(t, y, v) - \bar{u}^{\rho, \theta, \varsigma}(t, y, v)) \\ &= \underline{u}(t^*, y^*, v^*) - \bar{u}^{\rho, \theta, \varsigma}(t^*, y^*, v^*) > 0. \end{aligned}$$

Next, we denote $\mathcal{D}_b^0 := \text{int} \mathcal{D}_b$, and

$$(8.4) \quad \mathcal{D}_b^1 := \partial \mathcal{D}_b \cap \mathcal{D}_b = \partial \mathcal{D}_b \setminus [\{t = 0\} \cup \{t = T\} \cup \{y = b\}].$$

We note that $\underline{u}(t, y, v) - \bar{u}^{\rho, \theta, \varsigma}(t, y, v) \leq 0$, for $t = 0, T$ or $y = b$; thus (t^*, y^*, v^*) can only happen on $\mathcal{D}_b^0 \cup \mathcal{D}_b^1$. We shall consider the following two cases separately.

Case 1. We assume that $(t^*, y^*, v^*) \in \mathcal{D}_b^0$, but

$$(8.5) \quad \underline{u}(t, y, v) - \bar{u}^{\rho, \theta, \varsigma}(t, y, v) < M_b, \quad (t, y, v) \in \mathcal{D}_b^1.$$

In this case, we follow a more or less standard argument. For $\varepsilon > 0$, we define an auxiliary function:

$$(8.6) \quad \begin{aligned} \Sigma_\varepsilon^b(t, x, w, y, v) &= \underline{u}(t, x, w) - \bar{u}^{\rho, \theta, \varsigma}(t, y, v) - \frac{1}{2\varepsilon}(x - y)^2 - \frac{1}{2\varepsilon}(w - v)^2, \end{aligned}$$

for $(t, x, w, y, v) \in \mathcal{C}_b := \{(t, x, w, y, v) : t \in [0, T], x, y \in [0, b], w, v \in [0, t]\}$. Since \mathcal{C}_b is compact, there exist $\{(t_\varepsilon, x_\varepsilon, w_\varepsilon, y_\varepsilon, v_\varepsilon)\}_{\varepsilon > 0} \subset \mathcal{C}_b$, such that

$$(8.7) \quad M_{\varepsilon, b} := \max_{\mathcal{C}_b} \Sigma_\varepsilon^b(t, x, w, y, v) = \Sigma_\varepsilon^b(t_\varepsilon, x_\varepsilon, w_\varepsilon, y_\varepsilon, v_\varepsilon).$$

We claim that for some $\varepsilon_0 > 0$, $(t_\varepsilon, x_\varepsilon, w_\varepsilon, y_\varepsilon, v_\varepsilon) \in \text{int} \mathcal{C}_b$, whenever $0 < \varepsilon < \varepsilon_0$.

Indeed, suppose not, then there is a sequence $\varepsilon_n \downarrow 0$, such that $(t_{\varepsilon_n}, x_{\varepsilon_n}, w_{\varepsilon_n}, y_{\varepsilon_n}, v_{\varepsilon_n}) \in \partial \mathcal{C}_b$, the boundary of \mathcal{C}_b , and that (8.7) holds for each n . Now since $\partial \mathcal{C}_b$ is compact, we can find a subsequence, may assume $(t_{\varepsilon_n}, x_{\varepsilon_n}, w_{\varepsilon_n}, y_{\varepsilon_n}, v_{\varepsilon_n})$ itself, such that $(t_{\varepsilon_n}, x_{\varepsilon_n}, w_{\varepsilon_n}, y_{\varepsilon_n}, v_{\varepsilon_n}) \rightarrow (\hat{t}, \hat{x}, \hat{w}, \hat{y}, \hat{v}) \in \partial \mathcal{C}_b$.

Note that the function \underline{u} is continuous and bounded on D , and

$$(8.8) \quad \Sigma_{\varepsilon_n}^b(t_{\varepsilon_n}, x_{\varepsilon_n}, w_{\varepsilon_n}, y_{\varepsilon_n}, v_{\varepsilon_n}) = M_{\varepsilon_n, b} \geq \Sigma_{\varepsilon_n}^b(t^*, y^*, v^*; y^*, v^*) = M_b > 0,$$

it follows from (8.6) and (8.8) that

$$\frac{(x_{\varepsilon_n} - y_{\varepsilon_n})^2}{2\varepsilon_n} + \frac{(w_{\varepsilon_n} - v_{\varepsilon_n})^2}{2\varepsilon_n} \leq \underline{u}(t_{\varepsilon_n}, x_{\varepsilon_n}, w_{\varepsilon_n}) \leq \frac{M}{c}.$$

Letting $n \rightarrow \infty$, we obtain that $\hat{x} = \hat{y}$, $\hat{w} = \hat{v}$, which implies, by (8.8),

$$(8.9) \quad \begin{aligned} \underline{u}(\hat{t}, \hat{x}, \hat{w}) - \bar{u}^{\rho, \theta, \varsigma}(\hat{t}, \hat{x}, \hat{w}) &= \Sigma_\varepsilon^b(\hat{t}, \hat{x}, \hat{w}, \hat{x}, \hat{w}) \\ &= \lim_{n \rightarrow \infty} \Sigma_\varepsilon^b(t_{\varepsilon_n}, x_{\varepsilon_n}, w_{\varepsilon_n}, y_{\varepsilon_n}, v_{\varepsilon_n}) \geq M_b > 0. \end{aligned}$$

But as before we note that $\underline{u}(t, y, v) - \bar{u}^{\rho, \theta, \varsigma}(t, y, v) \leq 0$ for $t = 0$, $t = T$ and $y = b$, we conclude that $\hat{t} \neq 0, T$ and $\hat{x} < b$. In other words, $(\hat{t}, \hat{x}, \hat{w}) \in \partial \mathcal{D}_b^0 \setminus (\{t = 0\} \cup \{t = T\} \cup \{y = b\}) = \mathcal{D}_b^1$. This, together with (8.9), contradicts the assumption (8.5).

In what follows, we shall assume that $(t_\varepsilon, x_\varepsilon, w_\varepsilon, y_\varepsilon, v_\varepsilon) \in \text{int} \mathcal{C}_b, \forall \varepsilon > 0$. Applying [16], Theorem 8.3, one shows that for any $\delta > 0$, there exist $q = \hat{q} \in \mathbb{R}$ and $A, B \in \mathcal{S}^2$ such that

$$\begin{cases} (q, ((x_\varepsilon - y_\varepsilon)/\varepsilon, (w_\varepsilon - v_\varepsilon)/\varepsilon), A) \in \bar{\mathcal{P}}_{\mathcal{D}_b^0}^{1,2,+} \underline{u}(t_\varepsilon, x_\varepsilon, w_\varepsilon), \\ (\hat{q}, ((x_\varepsilon - y_\varepsilon)/\varepsilon, (w_\varepsilon - v_\varepsilon)/\varepsilon), B) \in \bar{\mathcal{P}}_{\mathcal{D}_b^0}^{1,2,-} \bar{u}^{\rho, \theta, \varsigma}(t_\varepsilon, y_\varepsilon, v_\varepsilon), \end{cases}$$

where $\bar{\mathcal{P}}_{\mathcal{D}_b^0}^{1,2,+} u(t, x, w)$ and $\bar{\mathcal{P}}_{\mathcal{D}_b^0}^{1,2,-} u(t, y, v)$ are the closures of the usual parabolic super(sub)jets of the function u at $(t, x, w), (t, y, v) \in \mathcal{D}_b^0$, respectively (see [16]), such that

$$(8.10) \quad \frac{1}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \frac{2\delta}{\varepsilon^2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \geq \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix}$$

where I is the 2×2 identity matrix. Taking $\delta = \varepsilon$, we have

$$(8.11) \quad \frac{3}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \geq \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix}.$$

Note that if we denote $A = [A_{ij}]_{i,j=1}^2$ and $B = [B_{ij}]_{i,j=1}^2$ and $\xi_\varepsilon := ((x_\varepsilon - y_\varepsilon)/\varepsilon, (w_\varepsilon - v_\varepsilon)/\varepsilon)$, then $(q, \xi_\varepsilon, A) \in \bar{\mathcal{P}}_{\mathcal{D}_b^0}^{1,2,+} \underline{u}(t_\varepsilon, x_\varepsilon, w_\varepsilon)$, [resp., $(\hat{q}, \xi_\varepsilon, B) \in \bar{\mathcal{P}}_{\mathcal{D}_b^0}^{1,2,-} \bar{u}^{\rho, \theta, \varsigma}(t_\varepsilon, y_\varepsilon, v_\varepsilon)$] implies that $(q, \xi_\varepsilon, A_{11}) \in \bar{\mathcal{P}}_{\mathcal{D}^*}^{+(1,2,1)} \underline{u}(t_\varepsilon, x_\varepsilon, w_\varepsilon)$ [resp., $(\hat{q}, \xi_\varepsilon, B_{11}) \in \bar{\mathcal{P}}_{\mathcal{D}}^{-(1,2,1)} \bar{u}^{\rho, \theta, \varsigma}(t_\varepsilon, y_\varepsilon, v_\varepsilon)$]. Since the functions $\underline{u}, \bar{u}^{\rho, \theta, \varsigma}$, and H are all continuous in all variables, we may assume without loss of generality that $(q, \xi_\varepsilon, A_{11}) \in \mathcal{P}_{\mathcal{D}^*}^{+(1,2,1)} \underline{u}(t_\varepsilon, x_\varepsilon, w_\varepsilon)$ [resp., $(\hat{q}, \xi_\varepsilon, B_{11}) \in \mathcal{P}_{\mathcal{D}}^{-(1,2,1)} \bar{u}^{\rho, \theta, \varsigma}(t_\varepsilon, y_\varepsilon, v_\varepsilon)$] and, by Definition 7.5,

$$\begin{cases} q + \sup_{\gamma \in [0,1], a \in [0,M]} H(t_\varepsilon, x_\varepsilon, w_\varepsilon, \underline{u}, \xi_\varepsilon, A_{11}, I[\underline{u}], \gamma, a) \geq 0, \\ q + \sup_{\gamma \in [0,1], a \in [0,M]} H(t_\varepsilon, y_\varepsilon, v_\varepsilon, \bar{u}^{\rho, \theta, \varsigma}, \xi_\varepsilon, B_{11}, I[\bar{u}^{\rho, \theta, \varsigma}], \gamma, a) \leq 0. \end{cases}$$

Furthermore, we note that (8.11) in particular implies that

$$(8.12) \quad A_{11}x_\varepsilon^2 - B_{11}y_\varepsilon^2 \leq \frac{3}{\varepsilon}(x_\varepsilon - y_\varepsilon)^2.$$

Thus, if we choose $(\gamma_\varepsilon, a_\varepsilon) \in \operatorname{argmax}_{(\gamma, a) \in [0, 1] \times [0, M]} H(t_\varepsilon, y_\varepsilon, v_\varepsilon, \underline{u}, \xi_\varepsilon, A_{11}, I[\underline{u}], \gamma, a)$, then we have

$$H(t_\varepsilon, x_\varepsilon, w_\varepsilon, \underline{u}, \xi_\varepsilon, A_{11}, \gamma_\varepsilon, a_\varepsilon) - H(t_\varepsilon, y_\varepsilon, v_\varepsilon, \bar{u}^{\rho, \theta, \varsigma}, \xi_\varepsilon, B_{11}, \gamma_\varepsilon, a_\varepsilon) \geq 0.$$

Therefore, by definition (7.2) we can easily deduce that

$$\begin{aligned} & c(\underline{u}(t_\varepsilon, x_\varepsilon, w_\varepsilon) - \bar{u}^{\rho, \theta, \varsigma}(t_\varepsilon, y_\varepsilon, v_\varepsilon)) + \lambda(w_\varepsilon)\underline{u}(t_\varepsilon, x_\varepsilon, w_\varepsilon) \\ & - \lambda(v_\varepsilon)\bar{u}^{\rho, \theta, \varsigma}(t_\varepsilon, y_\varepsilon, v_\varepsilon) \\ & \leq \frac{1}{2}\sigma^2\gamma_\varepsilon^2(A_{11}x_\varepsilon^2 - B_{11}y_\varepsilon^2) + [r + (\mu - r)\gamma_\varepsilon]\frac{(x_\varepsilon - y_\varepsilon)^2}{\varepsilon} \\ & + \lambda(w_\varepsilon)\int_0^{x_\varepsilon} \underline{u}(t_\varepsilon, x_\varepsilon - u, 0) dG(u) \\ (8.13) \quad & - \lambda(v_\varepsilon)\int_0^{y_\varepsilon} \bar{u}^{\rho, \theta, \varsigma}(t_\varepsilon, y_\varepsilon - u, 0) dG(u) \\ & \leq \left(\frac{3\sigma^2}{2} + \mu\right)\frac{(x_\varepsilon - y_\varepsilon)^2}{\varepsilon} \\ & + \lambda(w_\varepsilon)\int_0^{x_\varepsilon} \underline{u}(t_\varepsilon, x_\varepsilon - u, 0) dG(u) \\ & - \lambda(v_\varepsilon)\int_0^{y_\varepsilon} \bar{u}^{\rho, \theta, \varsigma}(t_\varepsilon, y_\varepsilon - u, 0) dG(u). \end{aligned}$$

Now, again, since $(t_\varepsilon, x_\varepsilon, w_\varepsilon, y_\varepsilon, v_\varepsilon) \in \mathcal{C}_b \subset \bar{\mathcal{C}}_b$ which is compact, there exists a sequence $\varepsilon_m \rightarrow 0$ such that $(t_{\varepsilon_m}, x_{\varepsilon_m}, w_{\varepsilon_m}, y_{\varepsilon_m}, v_{\varepsilon_m}) \rightarrow (\bar{t}, \bar{x}, \bar{w}, \bar{y}, \bar{v}) \in \bar{\mathcal{C}}_b$. By repeating the arguments, before one shows that $\bar{t} \in (0, T)$, $\bar{x} = \bar{y} \in [0, b)$, $\bar{w} = \bar{v} \in [0, t]$, that is, and

$$\underline{u}(\bar{t}, \bar{x}, \bar{w}) - \bar{u}^{\rho, \theta, \varsigma}(\bar{t}, \bar{x}, \bar{w}) = \lim_{\varepsilon_m \rightarrow 0} M_{\varepsilon_m, b} \geq M_b,$$

we obtain that $(\bar{t}, \bar{x}, \bar{w}) \in \mathcal{D}_b^0$. But on the other hand, replacing ε by ε_m and letting $m \rightarrow \infty$ in (8.13) we have

$$(c + \lambda(\bar{w}))M_b \leq \lambda(\bar{w})\int_0^{\bar{x}} [\underline{u}(\bar{t}, \bar{x} - u, 0) - \bar{u}^{\rho, \theta, \varsigma}(\bar{t}, \bar{x} - u, 0)] dG(u) \leq \lambda(\bar{w})M_b.$$

This is a contradiction as $c > 0$ and $M_b > 0$.

Case 2. We now consider the case $(t^*, y^*, v^*) \in \mathcal{D}_b^1$. We shall first move the point away from the boundary \mathcal{D}_b^1 into the interior \mathcal{D}_b^0 and then argue as Case 1. To this end, we borrow some arguments from [13, 26] and [40]. First, since (t^*, y^*, v^*) is on the boundary of a simple polyhedron and $0 < t^* < T$, it is not hard to see that there exist $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$, and $a > 0$ such that for any $(t, x, w) \in B_a^3(t^*, y^*, v^*) \cap \mathcal{D}_b^0$, $0 < \delta \leq 1$, it holds that

$$(8.14) \quad (t, y, v) \subset \mathcal{D}_b^0 \quad \text{whenever } (y, v) \in B_{\delta a}^2(x + \delta\eta_1, w + \delta\eta_2).$$

Here, $B_\rho^n(\xi)$ denotes the ball centered at $\xi \in \mathbb{R}^n$ with radius ρ . For any $\varepsilon > 0$ and $0 < \beta < 1$, define the auxiliary functions: for $(t, x, w, y, v) \in \mathcal{C}_b$,

$$\begin{aligned} \phi_{\varepsilon,\beta}(t, x, w, y, v) := & \left(\frac{x-y}{\sqrt{2\varepsilon}} + \beta\eta_1\right)^2 + \left(\frac{w-v}{\sqrt{2\varepsilon}} + \beta\eta_2\right)^2 \\ & + \beta[(t-t^*)^2 + (x-y^*)^2 + (w-v^*)^2]. \end{aligned}$$

$\Sigma_{\varepsilon,\beta}(t, x, w, y, v) := \underline{u}(t, x, w) - \bar{u}^{\rho,\theta,\varsigma}(t, y, v) - \phi_{\varepsilon,\beta}(t, x, w, y, v)$. Again, we have

$$\begin{aligned} M_{\varepsilon,\beta,b} := & \sup_{\mathcal{C}_b} \Sigma_{\varepsilon,\beta}(t, x, w, y, v) \geq \Sigma_{\varepsilon,\beta}(t^*, y^*, v^*, y^*, v^*) \\ (8.15) \quad & = M_b - \beta^2|\eta|^2 > 0, \end{aligned}$$

for any $\varepsilon > 0$ and $\beta < \beta_0$, for some $\beta_0 > 0$. Now we fix $\beta \in (0, \beta_0)$ and denote, for simplicity, $(t_\varepsilon, x_\varepsilon, w_\varepsilon, y_\varepsilon, v_\varepsilon) \in \operatorname{argmax}_{\mathcal{C}_b} \Sigma_{\varepsilon,\beta}$. We have

$$\begin{aligned} (8.16) \quad \Sigma_{\varepsilon,\beta}(t_\varepsilon, x_\varepsilon, w_\varepsilon, y_\varepsilon, v_\varepsilon) & \geq \Sigma_{\varepsilon,\beta}(t^*, y^*, v^*, y^* + \beta\sqrt{2\varepsilon}\eta_1, v^* + \beta\sqrt{2\varepsilon}\eta_2), \end{aligned}$$

which implies that

$$\begin{aligned} (8.17) \quad & \left(\frac{x_\varepsilon - y_\varepsilon}{\sqrt{2\varepsilon}} + \beta\eta_2\right)^2 + \left(\frac{w_\varepsilon - v_\varepsilon}{\sqrt{2\varepsilon}} + \beta\eta_3\right)^2 \\ & + \beta[(t_\varepsilon - t^*)^2 + (x_\varepsilon - y^*)^2 + (w_\varepsilon - v^*)^2] \\ & \leq \underline{u}(t_\varepsilon, x_\varepsilon, w_\varepsilon) - \bar{u}^{\rho,\theta,\varsigma}(t_\varepsilon, y_\varepsilon, v_\varepsilon) - \underline{u}(t^*, y^*, v^*) \\ & \quad + \bar{u}^{\rho,\theta,\varsigma}(t^*, y^* + \beta\sqrt{2\varepsilon}\eta_1, v^* + \beta\sqrt{2\varepsilon}\eta_2) \\ & \leq \frac{2M(1+\rho)}{c} + \frac{\theta(T-t^*+\varsigma)}{t^*}. \end{aligned}$$

It follows that $[(x_\varepsilon - y_\varepsilon)^2 + (w_\varepsilon - v_\varepsilon)^2]/\varepsilon \leq C_\beta$ for some constant $C_\beta > 0$. Thus, possibly along a subsequence, we have $\lim_{\varepsilon \rightarrow 0} [(x_\varepsilon - y_\varepsilon)^2 + (v_\varepsilon - w_\varepsilon)^2] = 0$. By the continuity of the functions \underline{u} and $\bar{u}^{\rho,\theta,\varsigma}$ and the definition of (t^*, y^*, v^*) , we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} [\underline{u}(t_\varepsilon, x_\varepsilon, w_\varepsilon) - \bar{u}^{\rho,\theta,\varsigma}(t_\varepsilon, y_\varepsilon, v_\varepsilon)] \\ & \leq M_b = \lim_{\varepsilon \rightarrow 0} [\underline{u}(t^*, y^*, v^*) - \bar{u}^{\rho,\theta,\varsigma}(t^*, y^* + \beta\sqrt{2\varepsilon}\eta_1, v^* + \beta\sqrt{2\varepsilon}\eta_2)]. \end{aligned}$$

Therefore, sending $\varepsilon \rightarrow 0$ in (8.17) we obtain that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \left[\left(\frac{x_\varepsilon - y_\varepsilon}{\sqrt{2\varepsilon}} + \beta\eta_1\right)^2 + \left(\frac{w_\varepsilon - v_\varepsilon}{\sqrt{2\varepsilon}} + \beta\eta_2\right)^2 \right. \\ & \quad \left. + \beta[(t_\varepsilon - t^*)^2 + (x_\varepsilon - y^*)^2 + (w_\varepsilon - v^*)^2] \right] \leq 0. \end{aligned}$$

Consequently, we conclude that

$$(8.18) \quad \begin{cases} \lim_{\varepsilon \rightarrow 0} (t_\varepsilon, x_\varepsilon, w_\varepsilon) = \lim_{\varepsilon \rightarrow 0} (t_\varepsilon, y_\varepsilon, v_\varepsilon) = (t^*, y^*, v^*), \\ \lim_{\varepsilon \rightarrow 0} \left(\frac{1}{\sqrt{2\varepsilon}}(x_\varepsilon - y_\varepsilon) + \beta\eta_1 \right)^2 + \left(\frac{1}{\sqrt{2\varepsilon}}(w_\varepsilon - v_\varepsilon) + \beta\eta_2 \right)^2 = 0. \end{cases}$$

In other words, we have shown that $y_\varepsilon = x_\varepsilon + \beta\sqrt{2\varepsilon}\eta_1 + o(\sqrt{2\varepsilon})$, $v_\varepsilon = w_\varepsilon + \beta\sqrt{2\varepsilon}\eta_2 + o(\sqrt{2\varepsilon})$. It then follows from (8.14) that $(t_\varepsilon, y_\varepsilon, v_\varepsilon) \in \mathcal{D}_b^0$ for $\varepsilon > 0$ small enough. Namely, we have now returned to the situation of Case 1, with a slightly different penalty function $\phi_{\varepsilon, \beta}$. The rest of the proof follows a similar line of arguments; we present it briefly for completeness. First, we apply [16], Theorem 8.3, again to assert that for any $\delta > 0$, there exist $q, \hat{q} \in \mathbb{R}$ and $A, B \in \mathcal{S}^2$ such that

$$(8.19) \quad \begin{cases} (q, (\xi_\varepsilon^1 + 2\beta(x_\varepsilon - y^*), \xi_\varepsilon^2 + 2\beta(w_\varepsilon - v^*)), A) \in \bar{\mathcal{P}}_D^{1,2,+} \underline{u}(t_\varepsilon, x_\varepsilon, w_\varepsilon) \\ (\hat{q}, (\xi^1, \xi^2), B) \in \bar{\mathcal{P}}_D^{1,2,-} \bar{u}^{\rho, \theta, \varsigma}(t_\varepsilon, y_\varepsilon, v_\varepsilon), \end{cases}$$

where $q - \hat{q} = 2\beta(t_\varepsilon - t^*)$, $\xi_\varepsilon^1 := (x_\varepsilon - y_\varepsilon)/\varepsilon + 2\beta\eta_1/\sqrt{2\varepsilon}$, $\xi_\varepsilon^2 := (w_\varepsilon - v_\varepsilon)/\varepsilon + 2\beta\eta_2/\sqrt{2\varepsilon}$, and

$$(8.20) \quad \begin{aligned} & \begin{pmatrix} \left(\left(2\beta + \frac{1}{\varepsilon} \right) I & -\frac{1}{\varepsilon} I \right) & \\ & -\frac{1}{\varepsilon} I & \frac{1}{\varepsilon} I \end{pmatrix} + \delta \begin{pmatrix} \left(\frac{2}{\varepsilon^2} + 4\beta^2 + \frac{4\beta}{\varepsilon} \right) I & -\left(\frac{2}{\varepsilon^2} + \frac{2\beta}{\varepsilon} \right) I \\ & -\left(\frac{2}{\varepsilon^2} + \frac{2\beta}{\varepsilon} \right) I & \frac{2}{\varepsilon^2} I \end{pmatrix} \\ & \geq \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix}. \end{aligned}$$

Now, setting $\delta = \varepsilon$ we have

$$(8.21) \quad \frac{3}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \begin{pmatrix} (6\beta + 4\beta^2\varepsilon)I & -2\beta I \\ -2\beta I & 0 \end{pmatrix} \geq \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix},$$

which implies, in particular,

$$(8.22) \quad A_{11}x_\varepsilon^2 - B_{11}y_\varepsilon^2 \leq \frac{3}{\varepsilon}(x_\varepsilon - y_\varepsilon)^2 + (6\beta + 4\beta^2\varepsilon)x_\varepsilon^2 - 4\beta x_\varepsilon y_\varepsilon.$$

Again, as in Case 1 we can easily argue that, without loss of generality, one may assume that $(q, (\xi_\varepsilon^1 + 2\beta(x_\varepsilon - y^*), \xi_\varepsilon^2 + 2\beta(w_\varepsilon - v^*)), A_{11}) \in \mathcal{P}_{\mathcal{D}^*}^{+(1,2,1)} \underline{u}(t_\varepsilon, x_\varepsilon, w_\varepsilon)$ and $(\hat{q}, (\xi^1, \xi^2), B_{11}) \in \mathcal{P}_{\mathcal{D}}^{-(1,2,1)} \bar{u}^{\rho, \theta, \varsigma}(t_\varepsilon, x_\varepsilon, w_\varepsilon)$. It is important to notice that, while $(t_\varepsilon, y_\varepsilon, v_\varepsilon) \in \mathcal{D}_b^0$, it is possible that the point $(t_\varepsilon, x_\varepsilon, w_\varepsilon)$ is on the boundary of \mathcal{D}^* . Thus it is crucial that viscosity (subsolution) property is satisfied on \mathcal{D}^* ,

including the boundary points. Thus, by Definition 7.5 we have

$$q + \sup_{\gamma \in [0,1], a \in [0,M]} H(t_\varepsilon, x_\varepsilon, w_\varepsilon, \underline{u}, \xi_\varepsilon^1 + 2\beta(x_\varepsilon - y^*), \xi_\varepsilon^2 + 2\beta(w_\varepsilon - v^*), A_{11}, I[\underline{u}], \gamma, a) \geq 0,$$

$$\hat{q} + \sup_{\gamma \in [0,1], a \in [0,M]} H(t_\varepsilon, y_\varepsilon, v_\varepsilon, \bar{u}^{\rho, \theta, \varsigma}, \xi_\varepsilon^1, \xi_\varepsilon^2, B_{11}, I[\bar{u}^{\rho, \theta, \varsigma}], \gamma, a) \leq 0.$$

Now if we take $(\gamma_\varepsilon, a_\varepsilon) \in \operatorname{argmax} H(t_\varepsilon, x_\varepsilon, w_\varepsilon, \underline{u}, \xi_\varepsilon^1 + 2\beta(x_\varepsilon - y^*), \xi_\varepsilon^2 + 2\beta(w_\varepsilon - v^*), A_{11}, I[\underline{u}], \gamma, a)$, then we have

$$0 \leq (q - \hat{q}) + H(t_\varepsilon, x_\varepsilon, w_\varepsilon, \underline{u}, (\xi_\varepsilon^1 + 2\beta(x_\varepsilon - y^*), \xi_\varepsilon^2 + 2\beta(w_\varepsilon - v^*)), A_{11}, I[\underline{u}], \gamma_\varepsilon, a_\varepsilon) - H(t_\varepsilon, y_\varepsilon, v_\varepsilon, \bar{u}^{\rho, \theta, \varsigma}, (\xi_\varepsilon^1, \xi_\varepsilon^2), B_{11}, I[\bar{u}^{\rho, \theta, \varsigma}], \gamma_\varepsilon, a_\varepsilon),$$

or equivalently, denoting $\Gamma_\varepsilon := r + (\mu - r)\gamma_\varepsilon$,

$$(8.23) \quad \begin{aligned} & (c + \lambda(w_\varepsilon))\underline{u}(t_\varepsilon, x_\varepsilon, w_\varepsilon) - (c + \lambda(v_\varepsilon))\bar{u}^{\rho, \theta, \varsigma}(t_\varepsilon, y_\varepsilon, v_\varepsilon) \\ & \leq \frac{1}{2}\sigma^2\gamma_\varepsilon^2(A^{11}x_\varepsilon^2 - B^{11}y_\varepsilon^2) + \Gamma_\varepsilon(x_\varepsilon - y_\varepsilon)^2/\varepsilon + 2(x_\varepsilon - y_\varepsilon)\Gamma_\varepsilon\beta\eta_1/\sqrt{2\varepsilon} \\ & \quad + 2\beta[(\Gamma_\varepsilon x_\varepsilon + p - a_\varepsilon)(x_\varepsilon - y^*) + (w_\varepsilon - v^*)] + 2\beta(t_\varepsilon - t^*) \\ & \quad + \lambda(w_\varepsilon) \int_0^{x_\varepsilon} \underline{u}(t_\varepsilon, x_\varepsilon - u, 0) dG(u) \\ & \quad - \lambda(v_\varepsilon) \int_0^{y_\varepsilon} \bar{u}^{\rho, \theta, \varsigma}(t_\varepsilon, y_\varepsilon - u, 0) dG(u) \\ & \leq (3\sigma^2\gamma_\varepsilon^2/2 + r)(x_\varepsilon - y_\varepsilon)^2/\varepsilon + 2(x_\varepsilon - y_\varepsilon)\Gamma_\varepsilon\beta\eta_1/\sqrt{2\varepsilon} \\ & \quad + 2\beta[(\Gamma_\varepsilon x_\varepsilon + p - a_\varepsilon)(x_\varepsilon - y^*) + (3 + 2\beta\varepsilon)x_\varepsilon^2 - 2x_\varepsilon y_\varepsilon + (w_\varepsilon - v^*) \\ & \quad + (t_\varepsilon - t^*)] \\ & \quad + \lambda(w_\varepsilon) \int_0^{x_\varepsilon} \underline{u}(t_\varepsilon, x_\varepsilon - u, 0) dG(u) \\ & \quad - \lambda(v_\varepsilon) \int_0^{y_\varepsilon} \bar{u}^{\rho, \theta, \varsigma}(t_\varepsilon, y_\varepsilon - u, 0) dG(u). \end{aligned}$$

First, sending $\varepsilon \rightarrow 0$ then sending $\beta \rightarrow 0$, and noting (8.18), we obtain from (8.23) that

$$(c + \lambda(v^*))M_b \leq \lambda(v^*) \left(\int_0^{y^*} (\underline{u}(t^*, y^* - u, 0) - \bar{u}^{\rho, \theta, \varsigma}(t^*, y^* - u, 0)) dG(u) \right) \leq \lambda(v^*)M_b.$$

Again, this is a contradiction as $c > 0$ and $M_b > 0$. The proof is now complete. \square

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