Electronic Journal of Statistics
Vol. 11 (2017) 78–98
ISSN: 1935-7524
DOI: 10.1214/16-EJS1221

Estimation of the global regularity of a multifractional Brownian motion

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Abstract: This paper presents a new estimator of the global regularity index of a multifractional Brownian motion. Our estimation method is based upon a ratio statistic, which compares the realized global quadratic variation of a multifractional Brownian motion at two different frequencies. We show that a logarithmic transformation of this statistic converges in probability to the minimum of the Hurst functional parameter, which is, under weak assumptions, identical to the global regularity index of the path.

MSC 2010 subject classifications: 60G15, 60G22, 62G05, 62M09, 60G17.
Keywords and phrases: Consistency, Hurst parameter, multifractional Brownian motion, power variation.

Received July 2016.

Contents

1. Introduction

Fractional Brownian motion (fBm) is one of the most prominent Gaussian processes in the probabilistic and statistical literature. Popularized by Mandelbrot and van Ness [MVN68] in 1968, it found various applications in modeling stochastic phenomena in physics, biology, telecommunication and finance among many other fields. Fractional Brownian motion is characterized by its self-similarity property, the stationarity of its increments and by its ability to match any prescribed constant local regularity. Mathematically speaking, for any $H \in (0,1)$, a fBm with Hurst index H, denoted by $B^H = (B^H_t)_{t\geq 0}$, is a zero mean Gaussian process with the covariance function given by

$$\mathbf{E}[B^H_s B^H_t] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

Various representations of fBm can be found in the existing literature; we refer to [Nua06, Mis08, Nou12, LLVH14] and references therein. The Hurst parameter $H \in (0,1)$ determines the path properties of the fBm: (i) The process $(B^H_t)_{t\geq 0}$ is self-similar with index H, i.e. $(a^H B^H_t)_{t\geq 0} = (B^H_{at})_{t\geq 0}$ in distribution for any $a > 0$, (ii) $(B^H_t)_{t\geq 0}$ has Hölder continuous paths of any order strictly smaller than H, (iii) fractional Brownian motion has short memory if and only if $H \in (0,1/2]$. Moreover, fBm exhibits long range dependance if H belongs to $(1/2,1)$. The statistical estimation of the Hurst parameter H in the high frequency setting, i.e. the setting of mesh converging to 0 while the interval length remaining fixed, is often performed by using power variation of B^H. Recall that a standard power variation of an auxiliary process $(Y_t)_{t\geq 0}$ on the interval $[0,T]$ is defined by

$$V(Y,p)^n_T := \sum_{i=0}^{[nT]} \left| Y_{\frac{i+1}{n}} - Y_{\frac{i}{n}} \right|^p.$$

This type of approach has been investigated in numerous papers; we refer to e.g. [GL89, IL97] among many others. The fact that most of the properties of fBm are governed by the single parameter H restricts its application in some situations. In particular, its Hölder exponent remains the same along all its trajectories. This does not seem to be adapted to describe adequately natural terrains as it has been shown in [BELV12], for instance. In addition, long range dependence requires $H > 1/2$, and thus imposes paths smoother than the ones of Brownian motion. Multifractional Brownian motion (mBm) was introduced to overcome these limitations. Several definitions of a multifractional Brownian motion exist. The first ones were proposed in [PLV95] and [BJR97]. A more general approach was introduced in [ST06] while the most recent definition of mBm (which contains all the previous ones) has been given in [LLVH14]. The latter definition is both more flexible and retains the essence of this class of Gaussian processes. Recall first that a fractional Brownian field on $\mathbb{R}_+ \times (0,1)$ denoted by $\mathbf{B} = (\mathbf{B}(t,H))_{(t,H)\in\mathbb{R}_+\times(0,1)}$ is a Gaussian field such that, for any H, the process $(\mathbf{B}(t,H))_{t\in\mathbb{R}_+}$ is a fBm with Hurst parameter H. Define for any $(t,H) \in \mathbb{R}_+ \times (0,1)$

$$\mathbf{B}_1(t,H) := \frac{1}{c_H} \int_{\mathbb{R}} \frac{e^{itu}-1}{|u|^{H+1/2}} \widetilde{\mathbb{W}_1}(du), \tag{1.1}$$

$$\mathbf{B}_2(t,H) := \int_{\mathbb{R}} \left(|t-u|^{H-1/2} - |u|^{H-1/2} \right) \mathbb{W}_2(du),$$

$$\mathbf{B}_3(t,H) := \int_{\mathbb{R}} \big((t-u)_+^{H-1/2} - (-u)_+^{H-1/2}\big)\mathbb{W}_3(du),$$
$$\mathbf{B}_4(t,H) := \int_0^T \mathbb{1}_{\{0\le u<t\le T\}}(t,u)\ K_H(t,u)\ \mathbb{W}_4(du),$$

where

$$c_H := \left(\frac{2\cos(\pi H)\Gamma(2-2H)}{H(1-2H)}\right)^{\frac{1}{2}}, \tag{1.2}$$

Γ denotes the standard gamma function, $d_H := \big(\frac{2H\Gamma(3/2-H)}{\Gamma(1/2+H)\Gamma(2-2H)}\big)^{1/2}$ and

$$K_H(t,s) := d_H\,(t-s)^{H-1/2} + c_H(1/2-H)\int_s^t (u-s)^{H-3/2}\big(1-\big(\tfrac{s}{u}\big)^{1/2-H}\big)du,$$

Here $\mathbb{W}_i$, $i=1,2,3,4$, denotes an independently scattered standard Gaussian measure on $\mathbb{R}$, and $\widetilde{\mathbb{W}_1}$ denotes the complex-valued Gaussian measure which can be associated in a unique way to $\mathbb{W}_1$ (see [ST06, p.203-204] and [ST94, p.325-326] for more details). It is straightforward to check that all Gaussian fields $(\mathbf{B}_i(t,H))_{(t,H)\in\mathbb{R}\times(0,1)}$ are fractional Brownian fields. A multifractional Brownian motion is simply a "path" traced on a fractional Brownian field. More precisely, it has been defined in [LLVH14, Definition 1.2] as follows:

Definition 1.1. *Let $h:\mathbb{R}_+\to(0,1)$ be a deterministic function and $\mathbf{B} := (\mathbf{B}(t,H))_{(t,H)\in\mathbb{R}_+\times(0,1)}$ be a fractional Brownian field. A multifractional Brownian motion (mBm) with functional parameter h is the Gaussian process $B^h = (B^h_t)_{t\in\mathbb{R}_+}$ defined by $B^h_t := \mathbf{B}(t,h(t))$, for all $t\in\mathbb{R}_+$.*

It is easy to verify that the process $B^h := (B^h_t)_{t\in\mathbb{R}_+}$ defined by

$$B^h_t = \frac{1}{c_{h(t)}}\int_{\mathbb{R}} \frac{\exp(itx)-1}{|x|^{h(t)+1/2}}\ \widetilde{W}(dx), \tag{1.3}$$

where $\widetilde{W}$ denotes the complex-valued Gaussian measure is a multifractional Brownian motion with functional parameter h (which lies on the underlying fractional field $\mathbf{B}_1$, defined in (1.1)). It is straightforward to check that any multifractional Brownian motion in the sense of [ST06, Def.1.1] is also a mBm in sense of Definition 1.1. Multifractional Brownian motions $(\mathbf{B}_1(t,h(t)))_{t\in\mathbb{R}_+}$ and $(\mathbf{B}_2(t,h(t)))_{t\in\mathbb{R}_+}$ lead to the so-called harmonizable mBm, first considered in [BJR97]. $(\mathbf{B}_3(t,h(t)))_{t\in\mathbb{R}_+}$ yields the moving average mBm defined in [PLV95]. Both are particular cases of mBms in the sense of [ST06]. Finally, $(\mathbf{B}_4(t,h(t)))_{t\in\mathbb{R}_+}$ corresponds to the Volterra multifractional Gaussian process studied in [BDM10]. This last process is an mBm in our sense.

Intuitively speaking, the multifractional Brownian motion behaves *locally* as fractional Brownian motion, but the functional parameter h is time-varying. Moreover, it remains linked to local regularity of B^h, but in a less simple way than in the case of the fBm. More precisely, if we assume that h belongs to the set $C^\eta([0,1],\mathbb{R})$, for some $\eta>0$, and is such that

$$0 < h_{\min} := \min_{t\in[0,1]} h(t) \leq h_{\max} := \max_{t\in[0,1]} h(t) < \min\{1, \eta\}, \tag{1.4}$$

then $h_{\min}$ is the regularity parameter of B^h (see [ACLV00, Corollaries 1,2 and Proposition 10]). In this setting the functional parameter h needs to be estimated locally in order to get a full understanding of the path properties of the multifractional Brownian motion B^h. Bardet and Surgailis [BS13] have proposed to use a local power variation of higher order filters of increments of B^h to estimate the function h. More specifically, they prove the law of large numbers and a central limit theorem for the local estimator of h (i) based on log-regression of the local quadratic variation, (ii) based on a ratio of local quadratic variations.

In this paper we are aiming at the estimation of the parameter $h_{\min}$, which represents the regularity (or smoothness) of the multifractional Brownian motion $B^h = (B^h_t)_{t\geq 0}$ defined in (1.3). For this particular statistical problem the local estimation approach investigated in [BS13] appears to be rather inconvenient. Instead our method relies on a ratio statistic, which compares the *global* quadratic variation at two different frequencies. We remark that in general it is impossible to find a global rate a_n such that the normalized power variation $a_n V(B^h, p)^n_T$ converges to a non-trivial limit. However, ratios of global power variations can very well be useful for statistical inference. Indeed, we will show that under appropriate conditions on the functional parameter h, the convergence

$$S_n(B^h) := \frac{\sum_{i=0}^{n-1}\left(B^h_{\frac{i+1}{n}} - B^h_{\frac{i}{n}}\right)^2}{\sum_{i=0}^{n-2}\left(B^h_{\frac{i+2}{n}} - B^h_{\frac{i}{n}}\right)^2} \underset{n\to+\infty}{\longrightarrow} 2^{-2h_{\min}}, \text{holds almost surely.}$$

Then a simple log transformation gives a strongly consistent estimator of the global regularity $h_{\min}$ of a mBm.

The paper is structured as follows. Section 2 presents the basic distribution properties of the multifractional Brownian motion, reviews the estimation methods from [BS13] and states the main asymptotic results of the paper. Proofs are given in Section 3.

2. Background and main results

In [BS13] Bardet and Surgailis deal with a little bit more general processes than multifractional Brownian motions. However, in order not to overload the notations we will focus in this paper on the normalized multifractional Brownian motion (i.e. the mBm defined by (1.3)). From now on we will refer to this process as the multifractional Brownian motion and denote it by $B^h = (B^h_t)_{t\geq 0}$.

2.1. Basic properties and local estimation of the functional parameter h

We start with the basic properties of the mBm B^h with functional parameter h. Its covariance function is given by the expression

$$R_h(t,s) := \mathbf{E}[B_t^h B_s^h] = \frac{c_{h_{t,s}}^2}{2c_{h(t)}c_{h(s)}}\left(|t|^{2h_{t,s}} + |s|^{2h_{t,s}} - |t-s|^{2h_{t,s}}\right), \tag{2.1}$$

where $h_{t,s} := \frac{h(t)+h(s)}{2}$ and c_x has been defined in (1.2). It is easy to check that $x \mapsto c_x$ is a $C^\infty((0,1))$-function. The local behaviour of the multifractional Brownian motion is best understood via the relationship

$$\left(u^{-h(t)}(B_{t+us}^h - B_t^h)\right)_{s\geq 0} \xrightarrow{f.d.d.} \left(B_s^{h(t)}\right)_{s\geq 0} \qquad \text{as } u \to 0,$$

where $\xrightarrow{f.d.d.}$ denotes the convergence of finite dimensional distributions. Hence, in the neighbourhood of any t in $(0,1)$, the mBm B^h behaves as fBm with Hurst parameter $h(t)$. This observation is essential for the local estimation of the functional parameter h. In the following we will briefly review the statistical methods of local inference investigated in Bardet and Surgailis [BS13], which is based on high frequency observations $B_0^h, B_{1/n}^h, \ldots, B_{(n-1)/n}^h, B_1^h$. While the original paper is investigating rather general Gaussian models whose tangent process is a fractional Brownian motion, we will specialize their asymptotic results to the framework of multifractional Brownian motion.

Let us introduce the generalized increments of a process $Y = (Y_t)_{t\geq 0}$. Consider a vector of coefficients $a = (a_0, \ldots, a_q) \in \mathbb{R}^{q+1}$ and a natural number $m \geq 1$ such that

$$\sum_{j=0}^q j^k a_j = 0 \quad \text{for } k = 0, \ldots, m-1 \qquad \text{and} \qquad \sum_{j=0}^q j^m a_j \neq 0.$$

In this case the vector $a \in \mathbb{R}^{q+1}$ is called a filter of order m. The generalised increments of Y associated with filter a at stage i/n are defined as

$$\Delta_{i,a}^n Y := \sum_{j=0}^q a_j Y_{\frac{i+j}{n}}.$$

Standard examples are $a^{(1)} = (-1,1)$, $\Delta_{i,a^{(1)}}^n Y = Y_{(i+1)/n} - Y_{i/n}$ (first order differences) and $a^{(2)} = (1,-2,1)$, $\Delta_{i,a^{(2)}}^n Y = Y_{(i+2)/n} - 2Y_{(i+1)/n} + 2Y_{i/n}$ (second order differences). In both cases we have that $q = m$. Now, we set $\psi(x,y) := (|x+y|)/(|x|+|y|)$ and set

$$\Lambda(H) := \mathbf{E}[\psi(\Delta_{0,a}^n B^H, \Delta_{1,a}^n B^H)], \qquad H \in (0,1).$$

The function Λ does not depend on n and is strictly increasing on the interval $(0,1)$. For any $\alpha \in (0,1)$, which determines the local bandwidth, the ratio type estimator of $h(t)$ is defined as

$$\widehat{h}_t^{n,\alpha} := \Lambda^{-1}\left(\frac{\sum_{i\in[\![0,n-q-1]\!]:\ |i/n-t|\leq n^{-\alpha}} \psi(\Delta_{i,a}^n B^h, \Delta_{i+1,a}^n B^h)}{\operatorname{card}\{i \in [\![0,n-q-1]\!] : |i/n-t| \leq n^{-\alpha}\}}\right).$$

Here and throughout the paper we denote $[\![p,q]\!] := \{p, p+1, p+2, \ldots, q\}$ for any $p, q \in \mathbb{N}$ with $p \leq q$. The authors of [BS13] only investigate the estimator $\widehat{h}_t^{n,\alpha}$

relative to the filter $a = a^{(2)}$, which we assume in this subsection from now on. The consistency and asymptotic normality of the estimator $\widehat{h}_t^{n,\alpha}$ is summarized in the following theorem. We remark that the condition for the central limit theorem crucially depends on the interplay between the bandwidth parameter α and the Hölder index η of the function h.

Theorem 2.1. *([BS13, Proposition 3]) Assume that h belongs to $C^\eta([0,1])$ and that condition (1.4) is satisfied.*

(i) For any $t \in (0,1)$ and $\alpha \in (0,1)$ it holds that

$$\widehat{h}_t^{n,\alpha} \xrightarrow{\mathbb{P}} h(t), \qquad \text{as } n \to \infty.$$

(ii) When $\alpha > \max\left(\frac{1}{1+2\min(\eta,2)}, 1 - 4(\min(\eta,2) - \sup_{t\in(0,1)} h(t))\right)$ it holds that

$$\sqrt{2n^{1-\alpha}}\left(\widehat{h}_t^{n,\alpha} - h(t)\right) \xrightarrow{d} \mathcal{N}(0,\tau^2) \qquad \text{as } n \to \infty,$$

where the asymptotic variance τ^2 is defined in [BS13, Eq. (2.17)].

The paper [BS13] contains the asymptotic theory for a variety of other local estimators of $h(t)$. We dispense with the detailed exposition of these estimators, since only $\widehat{h}_t^{n,\alpha}$ is somewhat related to our estimation method.

Remark 2.1. Nowadays, it is a standard procedure to consider higher order filters for Gaussian processes to obtain a central limit theorem for the whole range of Hurst parameters. Let us shortly recall some classical asymptotic results, which are usually referred to as Breuer-Major central limit theorems. We consider the scaled power variation of a fractional Brownian motion B^H with Hurst parameter $H \in (0,1)$ based on first order filter $a^{(1)}$ and second order filter $a^{(2)}$:

$$V(B^H, p; a^{(1)})^n := n^{-1+pH} \sum_{i=0}^{n-1} |\Delta_{i,a^{(1)}}^n B^H|^p$$

and

$$V(B^H, p; a^{(2)})^n := n^{-1+pH} \sum_{i=0}^{n-2} |\Delta_{i,a^{(2)}}^n B^H|^p.$$

It is well known that, after an appropriate normalization, the statistic $V(B^H, p; a^{(1)})^n$ exhibits asymptotic normality for $H \in (0, 3/4]$, while it converges to the Rosenblatt distribution for $H \in (3/4, 1)$. On the other hand, the statistic $V(B^H, p; a^{(2)})^n$ exhibits asymptotic normality for all $H \in (0,1)$. We refer to [BM83, Taq79] for a detailed exposition. □

2.2. Estimation of the global regularity parameter $h_{\min}$

In this section we will construct a consistent estimator of the global regularity parameter $h_{\min}$, which has been defined at (1.4). Our first condition is on the set $h^{-1}(\{h_{\min}\})$, which is necessarily compact since h belongs to $C^\eta([0,1])$. We assume that this set has the following form

$$\mathcal{M}_h := h^{-1}(\{h_{\min}\}) = \left(\bigcup_{i=1}^{q}[a_i,b_i]\right)\bigcup\left(\bigcup_{j=1}^{m}\{x_j\}\right), \qquad (q,m)\in\mathbb{N}^2\setminus(0,0), \tag{2.2}$$

where $\mathbb{N}=\{0,1,2,\ldots\}$ and the intervals $[a_i,b_i]$ are disjoint and such that none of the x_j's belongs to $\bigcup_{i=1}^{q}[a_i,b_i]$. Depending on whether $q\geq 1$ or $q=0$, we will need an additional assumption. Below, we denote by $h_l^{(p)}(x)$ (resp. $h_r^{(p)}(x)$) the pth left (resp. right) derivative of h at point x.

($\mathscr{A}$) There exist positive integers p_j such that function h is p_j times continuously left and right differentiable at point x_j for $j=1,\ldots,m$ such that

$$p_j = \min\{p:\ h_l^{(p)}(x_j)\neq 0\} = \min\{p:\ h_r^{(p)}(x_j)\neq 0\}.$$

We remark that since h reaches its minimum at points x_j, we necessarily have that $h_r^{(p_j)}(x_j)>0$ and that $h_l^{(p_j)}(x_j)>0$ if p_j is even and $h_l^{(p_j)}(x_j)<0$ if p is odd. Now, we proceed with the construction of the consistent estimator of the global regularity parameter $h_{\min}$ based on high frequency observations $B_0^h, B_{1/n}^h,\ldots,B_{(n-1)/n}^h, B_1^h$. First of all, let us remark that considering the estimator $\min_{t\in[0,1]}\widehat{h}_t^{n,\alpha}$, where $\widehat{h}_t^{n,\alpha}$ has been introduced in the previous section, is not a trivial matter since the functional version of Theorem 2.1 is not available. Instead our statistics relies on the global quadratic variation rather than local estimates.

For the mBm $B^h=(B_t^h)_{t\in[0,1]}$, we introduce the notations

$$V(B^h;k)^n := \sum_{i=0}^{n-k}\left(B^h_{\frac{i+k}{n}} - B^h_{\frac{i}{n}}\right)^2, \qquad S_n(B^h) := \frac{V(B^h;1)^n}{V(B^h;2)^n}. \tag{2.3}$$

Our first result determines the limit of $\mathbf{E}[V(B^h;1)^n]/\mathbf{E}[V(B^h;2)^n]$.

Proposition 2.2. *Let $h:[0,1]\to(0,1)$ be a deterministic $C^\eta([0,1])$-function satisfying (1.4) and such that the set $\mathcal{M}_h$ has the form (2.2). If $q=0$ we also assume that condition ($\mathscr{A}$) holds. Define*

$$\mathscr{U}_n^h := \frac{\mathbf{E}[V(B^h;1)^n]}{\mathbf{E}[V(B^h;2)^n]}.$$

Then it holds that

$$\lim_{n\to+\infty}\mathscr{U}_n^h = \left(\frac{1}{2}\right)^{2h_{\min}}. \tag{2.4}$$

The convergence result of Proposition 2.2 is rather intuitive when $q\geq 1$, which means that the minimum of the function h is reached on a set of positive Lebesgue measure. In this setting it is quite obvious that the statistic $V(B^h;k)^n$

is dominated by squared increments $(B^h_{(i+k)/n} - B^h_{i/n})^2$ for $i/n \in \cup_{i=1}^q [a_i, b_i]$. Thus, the estimation problem is similar to the estimation of the Hurst parameter of a fractional Brownian motion $(B^{h_{\min}}_t)_{t\in \cup_{i=1}^q [a_i,b_i]}$ with Hurst parameter h_{min}, for which the convergence at (2.4) is well known. When $q = 0$, and hence $\mathrm{Leb}(\mathcal{M}_h) = 0$, the proof of Proposition 2.2 becomes much more delicate.

Remark 2.2. Assume for illustration purpose that $q = 0$, $m = 1$, $x := x_1$ and $p := p_1$. Condition $(\mathscr{A})$ is crucial to determine the precise asymptotic expansion of the quantity $\mathbf{E}[V(B^h;k)^n]$. As a prototypical example let us consider the simple function

$$h(t) = c + dt^p, \qquad t \in [0,1], \tag{2.5}$$

where $c \in (0,1)$ and $d > 0$ such that $c + d < 1$. In this case $h_{\min} = c$ and $x = 0$. We obtain the following asymptotic decomposition:

$$\mathbf{E}[V(B^h;k)^n] \approx \sum_{i=0}^{n-k} \left(\frac{k}{n}\right)^{2h(i/n)} = \left(\frac{k}{n}\right)^{2c} \sum_{i=0}^{n-k} \exp\left(2d(i/n)^p\{\ln k - \ln n\}\right).$$

Observing that the map $x \mapsto x^p$, $x \in [0,1]$, is monotone increasing, we conclude from the latter

$$\begin{aligned}
\mathbf{E}[V(B^h;k)^n] &\approx n\left(\frac{k}{n}\right)^{2c} \int_0^1 \exp\left(2dx^p\{\ln k - \ln n\}\right) dx \\
&= k^{2c} \frac{n^{1-2c}}{p(\ln(n/k))^{1/p}} \int_0^{\ln(n/k)} y^{-1+1/p} \exp\left(-2dy\right) dy \\
&\approx k^{2c} \frac{n^{1-2c}}{p(\ln(n/k))^{1/p}} \int_0^{\infty} y^{-1+1/p} \exp\left(-2dy\right) dy. \qquad (2.6)
\end{aligned}$$

From this simple example we learn that the constant p from condition $(\mathscr{A})$ determines the leading term of $\mathbf{E}[V(B^h;k)^n]$. Indeed, a similar argumentation and the lower and upper bounds in (3.16) and (3.17) in the proof show that

$$\mathbf{E}[V(B^h;k)^n] = \mathrm{O}\left(\frac{n^{1-2h_{\min}}}{(\ln n)^{1/p}}\right) \quad \text{as } n \to +\infty, \qquad \text{for } k = 1, 2,$$

in the general setting of Proposition 2.2. Furthermore, in the framework of (2.5), we may easily determine the bias associated with convergence at (2.4) using (2.6):

$$\mathscr{U}^h_n - \left(\frac{1}{2}\right)^{2h_{\min}} = \mathrm{O}\left(\frac{1}{\ln n}\right) \quad \text{as } n \to +\infty. \tag{2.7}$$

The condition $\min\{p:\ h_l^{(p)}(x) \neq 0\} = \min\{p:\ h_r^{(p)}(x) \neq 0\}$ of assumption $(\mathscr{A})$ is not essential for the proofs. For instance, when $\min\{p:\ h_l^{(p)}(x) \neq 0\} > \min\{p:\ h_r^{(p)}(x) \neq 0\}$ the expectation $\mathbf{E}[V(B^h;k)^n]$ would be dominated by the terms in the small neighbourhood on the right hand side of x and the statement of Proposition 2.2 can be proved in the same manner. $\square$

Our main result shows strong consistency of the statistic $S_n(B^h)$.

Theorem 2.3. *Assume that $h \in C^2([0,1])$ and the set $\mathcal{M}_h$ has the form (2.2). If $q=0$ we also assume that condition $(\mathscr{A})$ holds. Then we have the following result:*

$$S_n(B^h) \xrightarrow{a.s.} \left(\frac{1}{2}\right)^{2h_{\min}}. \tag{2.8}$$

In particular, the following convergence holds:

$$\widehat{h}_{\min} := -\frac{\ln(S_n(B^h))}{2\ln(2)} \xrightarrow{a.s.} h_{\min}. \tag{2.9}$$

The asymptotic result of Theorem 2.3 can be extended to more general Gaussian processes than the mere multifractional Brownian motion. As it has been discussed in [BS13], when a Gaussian process possesses a tangent process $B^{h(t)}$ at time t, we may expect Theorem 2.3 to hold under certain assumptions on its covariance kernel. We refer to assumptions $(A)_\kappa$ and $(B)_\alpha$ therein for more details on sufficient conditions.

When $q \geq 1$ we obtain the following weak limit theorem.

Theorem 2.4. *Assume that $h \in C^2([0,1])$ and the set $\mathcal{M}_h$ has the form (2.2). If $q \geq 1$ and $\sup_{t\in[0,1]} h(t) < 3/4$ we obtain the central limit theorem*

$$n^{-1/2+2h_{\min}}\left(\sum_{i=0}^{n-k}\left\{\left(B^h_{\frac{i+k}{n}} - B^h_{\frac{i}{n}}\right)^2 - \mathbf{E}\left[\left(B^h_{\frac{i+k}{n}} - B^h_{\frac{i}{n}}\right)^2\right]\right\}\right)_{k=1,2} \xrightarrow{d} \mathcal{N}_2(0,\Sigma), \tag{2.10}$$

where the matrix $\Sigma \in \mathbb{R}^{2\times 2}$ is defined by

$$\Sigma_{11} = 2r\sum_{j\in\mathbb{Z}}\rho_{11}^2(j), \quad \Sigma_{22} = 2^{4H+1}r\sum_{j\in\mathbb{Z}}\rho_{22}^2(j), \quad \Sigma_{12} = \Sigma_{21} = 2^{2H+1}r\sum_{j\in\mathbb{Z}}\rho_{12}^2(j)$$

with $r = \sum_{j=1}^q (b_j - a_j)$ and

$$\begin{aligned}
\rho_{11}(j) &= cov(B_i^{h_{\min}} - B_{i-1}^{h_{\min}}, B_{i+j}^{h_{\min}} - B_{i+j-1}^{h_{\min}}),\\
\rho_{22}(m) &= cov(B_i^{h_{\min}} - B_{i-2}^{h_{\min}}, B_{i+j}^{h_{\min}} - B_{i+j-2}^{h_{\min}}),\\
\rho_{12}(m) &= cov(B_i^{h_{\min}} - B_{i-1}^{h_{\min}}, B_{i+j}^{h_{\min}} - B_{i+j-2}^{h_{\min}}), \qquad j \in \mathbb{Z},
\end{aligned}$$

and $B^{h_{\min}}$ denotes the fractional Brownian motion with Hurst parameter $h_{\min}$.

It is well known that $|\rho_{kk'}(j)| \leq C|j|^{2h_{\min}-2}$ for $k,k' = 1,2$ and thus $\Sigma < \infty$ when $h_{\min} < 3/4$. As stated in Remark 2.1 a central limit theorem can be obtained without the restriction $\sup_{t\in[0,1]} h(t) < 3/4$ when the first order increments are replaced by second order increments. In the setting $q=0$, which implies that $\text{Leb}(\mathcal{M}_h) = 0$, the weak limit theorem seems to be out of reach.

Remark 2.3. The main result (2.10) can be reformulated as follows:

$$\sqrt{n}\left(n^{-1+2h_{\min}}V(B^h;k)^n - E_n(k)\right)_{k=1,2} \xrightarrow{d} \mathcal{N}_2(0,\Sigma),$$

where $E_n(k) := n^{-1+2h_{\min}}\mathbf{E}[V(B^h;k)^n]$.

Following the arguments of Section 3.1.1 we may conclude that

$$\lim_{n\to+\infty} E_n(k) = rk^{2h_{\min}} \qquad k=1,2,$$

where $r=\sum_{j=1}^{q}(b_j-a_j)$. Applying the δ-method to the function $f(x,y)=x/y$, we obtain the central limit theorem

$$\sqrt{n}\left(\frac{V(B^h;1)^n}{V(B^h;2)^n}-\mathscr{U}_n^h\right) \xrightarrow{d} \mathcal{N}\left(0,(r^{-1}2^{-2h_{\min}},-2^{-2h_{\min}})\Sigma(r^{-1}2^{-2h_{\min}},-2^{-2h_{\min}})^\star\right),$$

where $y^\star$ denotes the transpose of y, under conditions of Theorem 2.4. However, the bias associated with Proposition 2.2 has a logarithmic rate. To illustrate this fact we consider a simple example

$$h(t)=c1_{[0,1/2]}(t)+(c+d(t-1/2)^p)1_{(1/2,1]}(t),$$

where $c\in(0,3/4)$, $d>0$ and $c+d/2^p<3/4$ (cf. (2.5)). Following the arguments in (2.6) we deduce the asymptotic expansion

$$\mathbf{E}[V(B^h;k)^n]=\frac{1}{2}n\left(\frac{k}{n}\right)^{2c}\left(1+\frac{C(p,d)}{(\ln(n/k))^{1/p}}\right)+\mathrm{o}\left(n^{1-2c}(\ln n)^{-1/p}\right),$$

where $C(p,d)$ is a constant that depends on p and d. In this framework we obviously obtain that

$$\mathscr{U}_n^h-\left(\frac{1}{2}\right)^{2h_{\min}}=\mathrm{O}\left(\frac{1}{(\ln n)^{1/p}}\right) \quad \text{as } n\to+\infty.$$

Hence, the bias dominates the variance and in this situation the central limit theorem of (2.10) is of little use. □

3. Proofs

Throughout this section we denote all positive constants by C, or C_p if they depend on an external parameter p, although they may change from line to line.

3.1. Proof of Proposition 2.2

For $k=1,2$ we introduce the notation

$$V_n^{(k)} := \sum_{i=0}^{n-k}\left(\frac{k}{n}\right)^{2h(i/n)}, \tag{3.1}$$

which serves as the first order approximation of the quantity $\mathbf{E}[V(B^h;k)^n]$. Applying [BS10, Lemma 1 p.13] we conclude that

$$\left|\mathbf{E}[V(B^h;k)^n]-V_n^{(k)}\right| \le C\frac{\ln n}{n^{\eta\wedge 1}}\sum_{i=0}^{n-k}\left(\frac{i}{n}\right)^{2h(k/n)} \le C\frac{\ln n}{n^{2h_{\min}-1+\eta\wedge 1}} \tag{3.2}$$

for any $(n,k)\in\mathbb{N}\times\{1,2\}$. We have the inequality

$$\left|\mathscr{U}_n^h - \left(\frac{1}{2}\right)^{2h_{\min}}\right| \le \frac{|\mathbf{E}[V(B^h;1)^n] - V_n^{(1)}| + |\mathbf{E}[V(B^h;2)^n] - V_n^{(2)}|}{V_n^{(2)}} + \left|\frac{V_n^{(1)}}{V_n^{(2)}} - \left(\frac{1}{2}\right)^{2h_{\min}}\right| =: \mu_n^{(1)} + \mu_n^{(2)}. \tag{3.3}$$

We first show that $\mu_n^{(1)} \to 0$ as $n \to \infty$. When $h_{\min} = h_{\max}$ we trivially have $\mu_n^{(1)} = 0$. If $h_{\min} < h_{\max}$, we fix $\epsilon \in (0, h_{\max} - h_{\min})$. By Leb$(A)$ we denote the Lebesgue measure of any measurable set A. We have that

$$\text{Leb}\left(h^{-1}([h_{\min}, h_{\min} + \epsilon])\right) > 0.$$

Thus, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ it holds that

$$\text{Card}\{i \in [\![0, n-k]\!];\ h(i/n) \in [h_{\min}, h_{\min}+\epsilon]\} \ge n\,\text{Leb}\left(h^{-1}([h_{\min}, h_{\min} + \epsilon])\right)/2.$$

This implies that

$$V_n^{(2)} \ge \sum_{i\in[\![0,n-k]\!];\ h(i/n)\in[h_{\min},h_{\min}+\epsilon]} \left(\frac{2}{n}\right)^{2h(i/n)} \ge Cn^{1-2(h_{\min}+\epsilon)}.$$

Hence, applying Inequality (3.2), we conclude that:

$$\mu_n^{(1)} \le C \ln n \cdot n^{2\epsilon - \eta\wedge 1},$$

which proves that $\mu_n^{(1)} \underset{n\to+\infty}{\to} 0$, for any ϵ small enough. □

3.1.1. *Convergence of $\mu_n^{(2)}$ in the case $q \ge 1$*

We first prove that $\mu_n^{(2)} \to 0$ in the case $q \ge 1$. Assume again that $h_{\min} < h_{\max}$. First, we observe the lower bound

$$\begin{aligned} V_n^{(k)} &\ge \sum_{l=1}^{q} \sum_{i\in[\![0,n-k]\!];\ i/n\in[a_l,b_l]} \left(\frac{k}{n}\right)^{2h(i/n)} \\ &\ge \left(\frac{k}{n}\right)^{2h_{\min}} \sum_{l=1}^{q} \text{card}\{i \in [\![0, n-k]\!];\ i/n \in [a_l, b_l]\} \\ &\ge n\left(\frac{k}{n}\right)^{2h_{\min}} \sum_{l=1}^{q} \left(b_l - a_l - \tfrac{2}{n}\right). \end{aligned} \tag{3.4}$$

For the upper bound we fix $0 < \epsilon < h_{\max} - h_{\min}$ and consider the decomposition

$$V_n^{(k)} = \sum_{i\in[\![0,n-k]\!];\ h(i/n)\in[h_{\min},h_{\min}+\epsilon]} \left(\frac{k}{n}\right)^{2h(i/n)}$$

$$+ \sum_{i \in [\![0,n-k]\!];\ h(i/n) \notin [h_{\min}, h_{\min}+\epsilon]} \left(\frac{k}{n}\right)^{2h(i/n)}.$$

Setting $\lambda_n(\epsilon) := n^{-1}\mathrm{card}\{i \in [\![0, n-k]\!];\ h(i/n) \in [h_{\min}, h_{\min}+\epsilon]\}$, we deduce the assertions

$$\lambda_n(\epsilon) \to \mathrm{Leb}\left(h^{-1}([h_{\min}, h_{\min}+\epsilon])\right) \quad \text{as } n \to \infty,$$

$$\mathrm{Leb}\left(h^{-1}([h_{\min}, h_{\min}+\epsilon])\right) \to \mathrm{Leb}\left(h^{-1}(\{h_{\min}\})\right) = \sum_{l=1}^{q}(b_l - a_l) > 0 \quad \text{as } \epsilon \to 0.$$

Now, we conclude that

$$V_n^{(k)} \leq n\lambda_n(\epsilon)\left(\frac{k}{n}\right)^{2h_{\min}} + n(1-\lambda_n(\epsilon))\left(\frac{k}{n}\right)^{2(h_{\min}+\epsilon)}. \tag{3.5}$$

Throughout the proofs we write $\underline{\lim}$ for $\liminf$ and $\overline{\lim}$ for $\limsup$. Applying inequalities (3.4) and (3.5), we obtain that

$$\begin{aligned}
&\underline{\lim}_{n\to+\infty} \frac{n\sum_{l=1}^{q}(b_l - a_l - \frac{2}{n})}{n\lambda_n(\epsilon) + n(1-\lambda_n(\epsilon))\left(\frac{2}{n}\right)^{2\epsilon}} \\
&\qquad \leq \underline{\lim}_{n\to+\infty} 2^{2h_{\min}} \frac{V_n^{(1)}}{V_n^{(2)}} \leq \overline{\lim}_{n\to+\infty} 2^{2h_{\min}} \frac{V_n^{(1)}}{V_n^{(2)}} \leq \\
&\qquad\qquad \overline{\lim}_{n\to+\infty} \frac{n\lambda_n(\epsilon) + n(1-\lambda_n(\epsilon))\left(\frac{1}{n}\right)^{2\epsilon}}{n\sum_{l=1}^{q}(b_l - a_l - \frac{2}{n})}.
\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}
&\frac{2^{-2h_{\min}}\mathrm{Leb}\left(h^{-1}(\{h_{\min}\})\right)}{\mathrm{Leb}\left(h^{-1}([h_{\min}, h_{\min}+\epsilon])\right)} \\
&\qquad \leq \underline{\lim}_{n\to+\infty} \frac{V_n^{(1)}}{V_n^{(2)}} \leq \overline{\lim}_{n\to+\infty} \frac{V_n^{(1)}}{V_n^{(2)}} \leq \\
&\qquad\qquad \frac{2^{-2h_{\min}}\mathrm{Leb}\left(h^{-1}([h_{\min}, h_{\min}+\epsilon])\right)}{\mathrm{Leb}\left(h^{-1}(\{h_{\min}\})\right)}.
\end{aligned}$$

By letting ϵ tend to 0, we readily deduce taht $\mu_n^{(2)} \to 0$ as $n \to +\infty$.

3.1.2. Convergence of $\mu_n^{(2)}$ in the case $q = 0$

Without loss of generality we assume that $m = 1$ and $\mathcal{M}_h = h^{-1}(\{h_{\min}\}) = \{x\}$ with $x \in (0,1)$. Recall that in this setting we assume condition $(\mathscr{A})$ with $p := p_1$. We let γ be a positive number such that $\gamma < 2^{-1}\min\{|h_l^{(p)}(x)|, h_r^{(p)}(x)\}$. Now, there exists a $\epsilon = \epsilon(\gamma) > 0$ with $\epsilon < \min\{x, 1-x, \gamma\}$ such that:

$\forall y > x$ with $0 < y - x < \epsilon$:

$$h_{\min} + \frac{1}{p!}(y-x)^p \ (h_r^{(p)}(x) - \gamma) \le h(y) \le h_{\min} + \frac{1}{p!}(y-x)^p \ (h_r^{(p)}(x) + \gamma), \tag{3.6}$$

$\forall y < x$ with $0 < x - y < \epsilon$:

$$h_{\min} + \frac{1}{p!}(y-x)^p \ (h_l^{(p)}(x) - (-1)^p\gamma) \le h(y) \le h_{\min} + \frac{1}{p!}(y-x)^p \ (h_l^{(p)}(x) + (-1)^p\gamma). \tag{3.7}$$

We proceed with the derivation of upper and lower bounds for the quantity $\mu_n^{(2)}$. We start with the decomposition $V_n^{(k)} = \Gamma_{n,k}^{(1)}(\gamma,\epsilon) + \Gamma_{n,k}^{(2)}(\gamma,\epsilon) + \Gamma_{n,k}^{(3)}(\gamma,\epsilon)$ where

$$\Gamma_{n,k}^{(1)}(\gamma,\epsilon) := \sum_{i\in[\![0,n-k]\!];\ i/n\in[x,x+\epsilon]} \left(\frac{k}{n}\right)^{2h(i/n)};$$

$$\Gamma_{n,k}^{(2)}(\gamma,\epsilon) := \sum_{i\in[\![0,n-k]\!];\ i/n\in[x-\epsilon,x)} \left(\frac{k}{n}\right)^{2h(i/n)};$$

$$\Gamma_{n,k}^{(3)}(\gamma,\epsilon) := \sum_{i\in[\![0,n-k]\!];\ i/n\in[x-\epsilon,x+\epsilon]^c} \left(\frac{k}{n}\right)^{2h(i/n)}.$$

It is clear that $\Gamma_{n,k}^{(3)}(\gamma,\epsilon) \le n(k/n)^{2h(y_\epsilon)}$, where we have set

$$y_\epsilon := \operatorname{argmin}\{h(u):\ u \in (x-\epsilon, x+\epsilon)^c \cap [0,1]\}.$$

For the other two quantities, we deduce that $\underline{\Gamma}_{n,k}^{(r)}(\gamma,\epsilon) \le \Gamma_{n,k}^{(r)}(\gamma,\epsilon) \le \overline{\Gamma}_{n,k}^{(r)}(\gamma,\epsilon)$ with

$$\underline{\Gamma}_{n,k}^{(1)}(\gamma,\epsilon) := \left(\frac{k}{n}\right)^{2h_{\min}} \sum_{i\in[\![0,n-k]\!]:\ i/n\in[x,x+\epsilon]} \left(\frac{k}{n}\right)^{2(p!)^{-1}(i/n-x)^p(h_r^{(p)}(x)+\gamma)},$$

$$\underline{\Gamma}_{n,k}^{(2)}(\gamma,\epsilon) := \left(\frac{k}{n}\right)^{2h_{\min}} \sum_{i\in[\![0,n-k]\!]:\ i/n\in[x-\epsilon,x)} \left(\frac{k}{n}\right)^{2(p!)^{-1}(i/n-x)^p(h_l^{(p)}(x)+(-1)^p\gamma)}$$

and $\overline{\Gamma}_{n,k}^{(1)}(\gamma,\epsilon) := \underline{\Gamma}_{n,k}^{(1)}(-\gamma,\epsilon)$ and $\overline{\Gamma}_{n,k}^{(2)}(\gamma,\epsilon) := \underline{\Gamma}_{n,k}^{(2)}(-\gamma,\epsilon)$. Using (3.6) and (3.7), it is easy to see that, for every $(k,n) \in \{1,2\} \times \mathbb{N}$:

$$\underline{\mu}_n^{(2)}(\gamma,\epsilon) \le \frac{V_n^{(1)}}{V_n^{(2)}} \le \overline{\mu}_n^{(2)}(\gamma,\epsilon), \tag{3.8}$$

with

$$\underline{\mu}_n^{(2)}(\gamma,\epsilon) := \frac{\underline{\Gamma}_{n,1}^{(1)}(\gamma,\epsilon) + \underline{\Gamma}_{n,1}^{(2)}(\gamma,\epsilon)}{\overline{\Gamma}_{n,2}^{(1)}(\gamma,\epsilon) + \overline{\Gamma}_{n,2}^{(2)}(\gamma,\epsilon) + n(2/n)^{2h(y_\epsilon)}},$$

$$\overline{\mu}_n^{(2)}(\gamma,\epsilon) := \frac{\overline{\Gamma}_{n,1}^{(1)}(\gamma,\epsilon)+\overline{\Gamma}_{n,1}^{(2)}(\gamma,\epsilon)+n(1/n)^{2h(y_\varepsilon)}}{\underline{\Gamma}_{n,2}^{(1)}(\gamma,\epsilon)+\underline{\Gamma}_{n,2}^{(2)}(\gamma,\epsilon)}.$$

From (3.8) we obtain that

$$0 \le 2^{2h_{\min}}\mu_n^{(2)} \le \left|2^{2h_{\min}}\frac{V_n^{(1)}}{V_n^{(2)}}-1\right| \le U_n(\gamma,\epsilon)+U_n(-\gamma,\epsilon), \tag{3.9}$$

where

$$U_n(\gamma,\epsilon) := |\underline{\Delta}_{n,2}(\gamma,\epsilon)|^{-1}\left(|2^{2h_{\min}}\overline{\Delta}_{n,1}(\gamma,\epsilon)-\underline{\Delta}_{n,2}(\gamma,\epsilon)|+2n^{1-2h(y_\epsilon)}\right), \tag{3.10}$$

$$\underline{\Delta}_{n,k}(\gamma,\epsilon) := \underline{\Gamma}_{n,k}^{(1)}(\gamma,\epsilon)+\underline{\Gamma}_{n,k}^{(2)}(\gamma,\epsilon), \qquad \overline{\Delta}_{n,k}(\gamma,\epsilon) := \underline{\Delta}_{n,k}(-\gamma,\epsilon). \tag{3.11}$$

In view of (3.9) it is sufficient to show that $\overline{\lim}_{\gamma\to 0}\ \overline{\lim}_{n\to+\infty} U_n(\gamma,\epsilon)=0$. Define

$$d_\gamma := 2(p!)^{-1}(h_r^{(p)}(x)+\gamma) \qquad \text{and} \qquad d'_\gamma := 2(p!)^{-1}(h_l^{(p)}(x)+(-1)^p\gamma).$$

For any (a,b) in $\mathbb{R}_+\times(\mathbb{R}\setminus\{0\})$, we also set

$$S_{n,k}(a,\epsilon) := \sum_{i\in[\![0,n-k]\!]:\ i/n\in[x,x+\epsilon]}\left(\frac{k}{n}\right)^{a(i/n-x)^p}, \tag{3.12}$$

$$T_{n,k}(b,\epsilon) := \sum_{i\in[\![0,n-k]\!]:\ i/n\in[x-\epsilon,x)}\left(\frac{k}{n}\right)^{b(i/n-x)^p}. \tag{3.13}$$

We deduce the identities $\underline{\Gamma}_{n,k}^{(1)}(\gamma,\epsilon) = (k/n)^{2h_{\min}}S_{n,k}(d_\gamma,\epsilon)$ and $\underline{\Gamma}_{n,k}^{(2)}(\gamma,\epsilon) = (k/n)^{2h_{\min}}T_{n,k}(d'_\gamma,\epsilon)$. Note moreover that $d'_\gamma>0$ when p is even and $d'_\gamma<0$ when p is odd. We therefore assume from now on that $b>0$ when p is even and that $b<0$ when p is odd. For any $\eta\in\mathbb{R}\setminus\{0\}$, we define

$$f_{n,k}^{(\eta)}(u) := \left(\frac{k}{n}\right)^{\eta(u-x)^p}.$$

Since $i\mapsto f_{n,k}^{(a)}(i/n)$ is decreasing on $[\![[nx]+1,[n(x+\epsilon)]]\!]$ while $i\mapsto f_{n,k}^{(b)}(i/n)$ is increasing if p even (resp. decreasing if p odd) on $[\![[n(x-\epsilon)]+1,[nx]]\!]$, one can use an integral test for convergence, which provides us with the following upper bounds

$$\frac{n\int_{\alpha_n(a)}^{\beta_n(a)} y^{1/p-1}e^{-y}\,dy}{p(a\ln(n/k))^{1/p}} \le S_{n,k}(a,\epsilon) \le \frac{n\int_{\tau_n(a)}^{\mu_n(a)} y^{1/p-1}e^{-y}\,dy}{p(a\ln(n/k))^{1/p}}, \tag{3.14}$$

$$\frac{n\left(\int_{\alpha'_n(b)}^{\beta'_n(b)} y^{1/p-1}e^{-y}\,dy-\rho_{n,k}^{(b)}(\epsilon)\right)}{p((-1)^p b\ln(n/k))^{1/p}} \le T_{n,k}(b,\epsilon)$$

$$\leq \frac{n\left(\int_{\tau_n'(b)}^{\mu_n'(b)} y^{1/p-1}e^{-y}\ dy - \rho_{n,k}^{(b)}(\epsilon)\right)}{p((-1)^p b\ln(n/k))^{1/p}}. \tag{3.15}$$

Here we use the notation

$$\alpha_n(a) := a\ln(n/k)\left(\frac{[nx]+1}{n} - x\right)^p, \quad \beta_n(a) := a\ln(n/k)\left(\frac{[n(x+\epsilon)]+1}{n} - x\right)^p,$$
$$\tau_n(a) := a\ln(n/k)\left(\frac{[nx]}{n} - x\right)^p, \quad \mu_n(a) := a\ln(n/k)\left(\frac{[n(x+\epsilon)]}{n} - x\right)^p$$

and $\rho_{n,k}^{(b)}(\epsilon) := f_{n,k}^{(b)}(\frac{[n(x-\epsilon)]+1}{n}) + f_{n,k}^{(b)}(\frac{[nx]}{n})$. Furthermore,

$$(\alpha_n'(b), \beta_n'(b), \tau_n'(b), \mu_n'(b)) := (z_n^{(1)}(b), z_n^{(2)}(b), z_n^{(3)}(b), z_n^{(4)}(b)) \text{ if } p \text{ is even},$$
$$(\alpha_n'(b), \beta_n'(b), \tau_n'(b), \mu_n'(b)) := (z_n^{(3)}(b), z_n^{(4)}(b), z_n^{(1)}(b), z_n^{(2)}(b)) \text{ if } p \text{ is odd},$$

where we have set

$$z_n^{(1)}(b) := b\ln(n/k)\left(\frac{[nx]-2}{n} - x\right)^p,$$
$$z_n^{(2)}(b) := b\ln(n/k)\left(\frac{[n(x-\epsilon)]+1}{n} - x\right)^p,$$
$$z_n^{(3)}(b) := b\ln(n/k)\left(\frac{[nx]-1}{n} - x\right)^p,$$
$$z_n^{(4)}(b) := b\ln(n/k)\left(\frac{[n(x-\epsilon)]+2}{n} - x\right)^p.$$

In view of the inequalities (3.14) and (3.15), as well as identities (3.12) and (3.13), we then deduce that

$$\frac{n^{1-2h_{\min}}k^{2h_{\min}}u_{n,k,p}(d_\gamma)}{(\ln(n/k))^{1/p}} \cdot \left(\frac{1}{d_\gamma}\right) \leq \underline{\Gamma}_{n,k}^{(1)}(\gamma,\epsilon)$$
$$\leq \frac{n^{1-2h_{\min}}k^{2h_{\min}}v_{n,k,p}(d_\gamma)}{(\ln(n/k))^{1/p}} \cdot \left(\frac{1}{d_\gamma}\right), \tag{3.16}$$
$$\frac{n^{1-2h_{\min}}k^{2h_{\min}}u_{n,k,p}'(d_\gamma')}{(\ln(n/k))^{1/p}} \cdot \left(\frac{1}{|d_\gamma'|}\right) \leq \underline{\Gamma}_{n,k}^{(2)}(\gamma,\epsilon)$$
$$\leq \frac{n^{1-2h_{\min}}k^{2h_{\min}}v_{n,k,p}'(d_\gamma')}{(\ln(n/k))^{1/p}} \cdot \left(\frac{1}{|d_\gamma'|}\right). \tag{3.17}$$

Here we have used the notation

$$u_{n,k,p}(a) := \frac{1}{p}\int_{\alpha_n(a)}^{\beta_n(a)} y^{1/p-1}e^{-y}\ dy, \qquad v_{n,k,p}(a) := \frac{1}{p}\int_{\tau_n(a)}^{\mu_n(a)} y^{1/p-1}e^{-y}\ dy,$$

$$u'_{n,k,p}(b) := \frac{1}{p}\int_{\alpha'_n(b)}^{\beta'_n(b)} y^{1/p-1}e^{-y}\ dy - \frac{\big((-1)^p b\ln(n/k)\big)^{1/p}\rho^{(b)}_{n,k}(\epsilon)}{pn},$$
$$v'_{n,k,p}(b) := \frac{1}{p}\int_{\tau'_n(b)}^{\mu'_n(b)} y^{1/p-1}e^{-y}\ dy - \frac{\big((-1)^p b\ln(n/k)\big)^{1/p}\rho^{(b)}_{n,k}(\epsilon)}{pn}.$$

Since $\underline{\Gamma}^{(r)}_{n,k}(\gamma,\epsilon) = \overline{\Gamma}^{(r)}_{n,k}(-\gamma,\epsilon)$, (3.16) and (3.17) also provide us with upper and lower bounds for $\overline{\Gamma}^{(r)}_{n,k}(\gamma,\epsilon)$. Finally, we obtain the following lower and upper bounds

$$\frac{n^{1-2h_{\min}}k^{2h_{\min}}}{(\ln(n/k))^{1/p}}\cdot\Lambda_{n,k}(\gamma,\epsilon) \le \underline{\Delta}_{n,k}(\gamma,\epsilon) \le \frac{n^{1-2h_{\min}}k^{2h_{\min}}}{(\ln(n/k))^{1/p}}\Lambda'_{n,k}(\gamma,\epsilon), \tag{3.18}$$
$$\frac{n^{1-2h_{\min}}k^{2h_{\min}}}{(\ln(n/k))^{1/p}}\cdot\Lambda_{n,k}(-\gamma,\epsilon) \le \overline{\Delta}_{n,k}(\gamma,\epsilon) \le \frac{n^{1-2h_{\min}}k^{2h_{\min}}}{(\ln(n/k))^{1/p}}\Lambda'_{n,k}(-\gamma,\epsilon), \tag{3.19}$$

where

$$\Lambda_{n,k}(\gamma,\epsilon) := \frac{1}{d_\gamma}\cdot u_{n,k,p}(d_\gamma) + \frac{1}{|d'_\gamma|}\cdot u'_{n,k,p}(d'_\gamma),$$
$$\Lambda'_{n,k}(\gamma,\epsilon) := \frac{1}{d_\gamma}\cdot v_{n,k,p}(d_\gamma) + \frac{1}{|d'_\gamma|}\cdot v'_{n,k,p}(d'_\gamma).$$

Denote $c_p := \int_0^{+\infty} y^{1/p-1}e^{-y}\ dy$. Recalling the definition of the constants d_γ and d'_γ, a straightforward computation shows that, for any $(k,k')\in\{1,2\}^2$ with $k\neq k'$:

$$\lim_{n\to+\infty}\Lambda_{n,k}(\gamma,\epsilon) = \lim_{n\to+\infty}\Lambda'_{n,k}(\gamma,\epsilon) = \frac{c_p}{p}(1/d_\gamma + 1/|d'_\gamma|), \tag{3.20}$$
$$\begin{aligned}\overline{\lim_{n\to+\infty}}|\Lambda'_{n,k'}(\gamma,\epsilon) - \Lambda_{n,k}(-\gamma,\epsilon)| &\le C\ (2|\gamma| + |1/d_{-\gamma} - 1/d_\gamma + 1/|d'_{-\gamma}| - 1/|d'_\gamma||)\\ &\le C|\gamma|.\end{aligned} \tag{3.21}$$

Starting from (3.18), and using (3.20) and (3.21), we see that there exists a positive integer n_0 and $C>0$ such that for all $n\ge n_0$

$$|\underline{\Delta}_{n,k}(\gamma,\epsilon)|^{-1} \le C\frac{(\ln(n/k))^{1/p}}{n^{1-2h_{\min}}k^{2h_{\min}}}. \tag{3.22}$$

Finally, inequalities (3.20), (3.21) and (3.22) imply that there exists a positive integer N such that for all $n\ge N$:

$$U_n(\gamma,\epsilon) \le C\bigg(|\gamma| + \frac{(\ln n)^{1/p}}{n^{2(h(y_\epsilon)-h_{\min})}}\bigg).$$

From the previous inequality, $\overline{\lim_{n\to+\infty}} U_n(\gamma,\epsilon) \le C|\gamma|$ and thus we get $\overline{\lim_{\gamma\to 0}}\ \overline{\lim_{n\to+\infty}} U_n(\gamma,\epsilon) = 0$, which completes the proof. □

Remark 3.1. In the previous proof (in the case $q=0$), using (3.20), one can also see that the bias related to the convergence of $\mu_n^{(2)}$ to 0 is of order $1/\ln n$. □

3.2. Proof of Theorem 2.3

In the first step we will find an upper bound for the covariance function of the increments of B^h. We define

$$r_n(i,j) := \operatorname{cov}\left(B^h_{\frac{i+k}{n}} - B^h_{\frac{i}{n}}, B^h_{\frac{j+k}{n}} - B^h_{\frac{j}{n}}\right), \qquad k = 1,2.$$

Recalling the notation at (2.1), we conclude the identity

$$r_n(i,j) = R_h\left(\frac{i+k}{n}, \frac{j+k}{n}\right) - R_h\left(\frac{i}{n}, \frac{j+k}{n}\right) - R_h\left(\frac{i+k}{n}, \frac{j}{n}\right) + R_h\left(\frac{i}{n}, \frac{j}{n}\right).$$

Since $h \in C^2([0,1])$ and the function c defined at (1.2) is a $C^\infty((0,1))$-function, we deduce by an application of Taylor expansion

$$|r_n(i,j)| \le n^{-2} \sum_{l,l'=1}^{2} |\partial_{ll'} R_h(\psi^n_{ij})| \qquad \text{for } |i-j| > 2, \tag{3.23}$$

where $\partial_{ll'} R_h$ denotes the second order derivative in the direction of x_l and $x_{l'}$, and $\psi^n_{ij} \in (i/n, (i+k)/n) \times (j/n, (j+k)/n)$. Now, we will compute an upper bound for the right side of (3.23) for $i \neq j$. First, we observe that

$$R_h(t,s) = F(t,s)\ G(t,s,h(t)+h(s)),$$

where

$$F(t,s) = \frac{c^2_{h_{t,s}}}{c_{h(t)} c_{h(s)}}, \qquad G(t,s,H) = \frac{1}{2}\left(|t|^H + |s|^H - |t-s|^H\right).$$

We remark that $G(t,s,2H)$ is the covariance kernel of the fractional Brownian motion with Hurst parameter $H \in (0,1)$.

Since $h \in C^2([0,1])$, $c \in C^\infty((0,1))$ and $c_x \neq 0$ for $x \in (0,1)$, we conclude that

$$|\partial_l F(t,s)|,\ |\partial_{ll'} F(t,s)| \le C, \qquad l,l' = 1,2, \quad (t,s) \in [0,1]^2.$$

We concentrate on the second order derivative $\partial_{11} R_h(\psi^n_{ij})$; the estimates for the other second order derivatives are obtained similarly. We have that

$$\begin{aligned}
&\partial_{11} R_h(t,s) \\
&\quad = \partial_{11} F(t,s) \cdot G(t,s,h(t)+h(s)) \\
&\quad + 2\partial_1 F(t,s)\,[\partial_1 G(t,s,h(t)+h(s)) + h'(t) \cdot \partial_3 G(t,s,h(t)+h(s))] \\
&\quad + F(t,s)\,[\partial_{11} G(t,s,h(t)+h(s)) + 2h'(t) \cdot \partial_{13} G(t,s,h(t)+h(s))) \\
&\qquad + h''(t) \cdot \partial_3 G(t,s,h(t)+h(s)) + (h'(t))^2 \cdot \partial_{33} G(t,s,h(t)+h(s)))\Big].
\end{aligned}$$

For the derivatives of the function G, we deduce the following estimates

$$|\partial_1 G(t,s,h(t)+h(s))| \le C\left(t^{h(t)+h(s)-1} + |t-s|^{h(t)+h(s)-1}\right),$$

$$|\partial_3 G(t,s,h(t)+h(s))| \le C\Big(-\ln t \cdot t^{h(t)+h(s)} - \ln s \cdot s^{h(s)+h(s)} - \ln|t-s| \cdot |t-s|^{h(t)+h(s)}\Big)$$

$$|\partial_{11} G(t,s,h(t)+h(s))| \le C\left(t^{h(t)+h(s)-2} + |t-s|^{h(t)+h(s)-2}\right)$$

$$|\partial_{13} G(t,s,h(t)+h(s))| \le C\Big((1-\ln t)\ t^{h(t)+h(s)-1} + (1-\ln|t-s|)|t-s|^{h(t)+h(s)-1}\Big)$$

$$|\partial_{33} G(t,s,h(t)+h(s))| \le C\Big(\ln^2 t \cdot t^{h(t)+h(s)} + \ln^2 s \cdot s^{h(t)+h(s)} + \ln^2|t-s| \cdot |t-s|^{h(t)+h(s)}\Big),$$

which hold for $t,s \in (0,1]$ with $t \neq s$ and the third inequality holds whenever $h(t)+h(s) \neq 1$ (if $h(t)+h(s)=0$ we simply have $\partial_{11}G(t,s,h(t)+h(s))=0$). Similar formulas and bounds are obtained for other second order derivatives of R_h. Using the boundedness of functions F, h and its derivatives, together with the above estimates and (3.23) we obtain the inequality

$$\begin{aligned}|r_n(i,j)| &\le Cn^{-h(i/n)-h(j/n)}\Big(i^{h(i/n)+h(j/n)-2} + j^{h(i/n)+h(j/n)-2} + |i-j|^{h(i/n)+h(j/n)-2}\Big) \qquad (3.24)\\ &\le Cn^{-2h_{\min}}\left(i^{2h_{\min}-2} + j^{2h_{\min}-2} + |i-j|^{2h_{\min}-2}\right), \quad i,j \ge 1, |i-j| > 2.\end{aligned}$$

When $|i-j| \le 2$ we deduce from [BS10, Lemma 1 p.13] that

$$|r_n(i,j)| \le \operatorname{var}\left(B^h_{\frac{i+k}{n}} - B^h_{\frac{i}{n}}\right) + \operatorname{var}\left(B^h_{\frac{j+k}{n}} - B^h_{\frac{j}{n}}\right) \le Cn^{-2h_{\min}}. \qquad (3.25)$$

We recall the identity $\operatorname{cov}(Z_1^2, Z_2^2) = 2\operatorname{cov}(Z_1,Z_2)^2$ for a Gaussian vector (Z_1,Z_2). By (3.24) and (3.25) we immediately conclude that

$$\operatorname{var}(V(B^h;k)^n) \le Cn^{-4h_{\min}+1}\sum_{i=1}^n i^{4h_{\min}-4} \le C\begin{cases} n^{-4h_{\min}+1} & h_{\min} \in (0,3/4)\\ \ln n \cdot n^{-2} & h_{\min} = 3/4\\ n^{-2} & h_{\min} \in (3/4,1)\end{cases} \qquad (3.26)$$

In view of Proposition 2.2 it is sufficient to show that

$$\frac{V(B^h;k)^n)}{\mathbf{E}[V(B^h;k)^n]} \xrightarrow{\text{a.s.}} 1, \qquad k=1,2, \qquad (3.27)$$

to prove Theorem 2.3. We assume again without loss of generality that $q=0$, $m=1$ and $\mathcal{M}_h = h^{-1}\{h_{\min}\} = \{x\}$. Using the notations from the previous subsection together with the inequalities (3.16) and (3.17), we deduce the following lower bound, for n large enough and for ϵ small enough:

$$\mathbf{E}[V(B^h;k)^n] \geq \underline{\Gamma}^{(1)}_{n,k}(\gamma,\epsilon) + \underline{\Gamma}^{(2)}_{n,k}(\gamma,\epsilon) \geq C_\epsilon \frac{n^{1-2h_{\min}}}{(\ln n)^{1/p}}. \tag{3.28}$$

Now, observe that the random variable

$$R_n(k) = \frac{V(B^h;k)^n)}{\mathbf{E}[V(B^h;k)^n]} - 1 = \mathbf{E}[V(B^h;k)^n]^{-1}\left(V(B^h;k)^n) - \mathbf{E}[V(B^h;k)^n]\right)$$

is an element of the second order Wiener chaos. Thus, for any $q \geq 2$ there exists a constant C_q such that

$$\mathbf{E}[|R_n(k)|^q]^{1/q} \leq C_q \mathbf{E}[|R_n(k)|^2]^{1/2}, \tag{3.29}$$

which is due to the hypercontractivity property on a Wiener chaos of a fixed order (see e.g. [NP12, Theorem 2.7.2]). The inequalities (3.26) and (3.28) imply the existence of a constant $r > 0$ with $\mathbf{E}[|R_n(k)|^2]^{1/2} \leq Cn^{-r}$. We conclude that

$$\mathbf{E}[|R_n(k)|^q] \leq C_q n^{-rq}.$$

Choosing q sufficiently large to ensure that $qr > 1$, we deduce that $R_n(k) \xrightarrow{\text{a.s.}} 0$ by Borel-Cantelli lemma. This completes the proof of Theorem 2.3. □

3.3. Proof of Theorem 2.4

We use the following decomposition:

$$\begin{aligned} V(B^h;k)^n &= \sum_{i:\ i/n \in \bigcup_{j=1}^q [a_j,b_j]} \left(B^h_{\frac{i+k}{n}} - B^h_{\frac{i}{n}}\right)^2 + \sum_{i:\ i/n \in [0,1]\setminus \bigcup_{j=1}^q [a_j,b_j]} \left(B^h_{\frac{i+k}{n}} - B^h_{\frac{i}{n}}\right)^2 \\ &=: V_1(B^h;k)^n + V_2(B^h;k)^n, \\ E_1^n(k) &:= \mathbf{E}[V_1(B^h;k)^n], \qquad E_2^n(k) := \mathbf{E}[V_2(B^h;k)^n]. \end{aligned}$$

We recall that $h(x) = h_{\min}$ for all $x \in \bigcup_{j=1}^q [a_j,b_j]$. Applying classical results for fractional Brownian motion with Hurst parameter $h_{\min} \in (0,3/4)$ (see e.g. [IL97]) we obtain the central limit theorem

$$n^{-1/2+2h_{\min}}\left(V_1(B^h;k)^n - E_1^n(k)\right)_{k=1,2} \xrightarrow{d} \mathcal{N}_2(0,\Sigma),$$

where the matrix $\Sigma \in \mathbb{R}^{2\times 2}$ is defined in Theorem 2.4. We introduce the sets $D(\epsilon) := \{x \in [0,1]:\ h(x) \in [h_{\min}, h_{\min}+\epsilon]\} \setminus \bigcup_{j=1}^q [a_j,b_j]$ and $D'(\epsilon) := \{x \in [0,1]:\ h(x) > h_{\min}+\epsilon\}$ for $\epsilon > 0$. Due to condition (2.2) we have that

$$\text{Leb}(D(\epsilon)) \to 0 \qquad \text{as } \epsilon \to 0.$$

Observe the decomposition

$$V_2(B^h;k)^n = \sum_{i:\ i/n \in D(\epsilon)} \left(B^h_{\frac{i+k}{n}} - B^h_{\frac{i}{n}} \right)^2 + \sum_{i:\ i/n \in D'(\epsilon)} \left(B^h_{\frac{i+k}{n}} - B^h_{\frac{i}{n}} \right)^2 .$$

Now, we use the fact that $\sup_{t\in[0,1]} h(t) < 3/4$ and inequality (3.26) to conclude the upper bound

$$\mathrm{var}\left(n^{-1/2+2h_{\min}} V_2(B^h;k)^n \right) \leq C\left(D_n(\epsilon) + n^{-2\epsilon} \right)$$

where we have set $D_n(\epsilon) := n^{-1}\mathrm{Card}\{i \in [\![0, n-k]\!] : \ i/n \in D(\epsilon)\}$. Since $\lim_{n\to+\infty} D_n(\epsilon) = \mathrm{Leb}(D(\epsilon))$, for any $\epsilon > 0$, we deduce that $n^{-1/2+2h_{\min}} \left(V_2(B^h;k)^n - E_2^n(k) \right)_{k=1,2} \xrightarrow{\mathbb{P}} 0$, which completes the proof of Theorem 2.4. □

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