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STOCHASTIC ANALYSIS ON SUB-RIEMANNIAN MANIFOLDS WITH TRANSVERSE SYMMETRIES

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Dedicated, with admiration, to Donald Burkholder

We prove a geometrically meaningful stochastic representation of the derivative of the heat semigroup on sub-Riemannian manifolds with tranverse symmetries. This representation is obtained from the study of Bochner–Weitzenböck type formulas for sub-Laplacians on 1-forms. As a consequence, we prove new hypoelliptic heat semigroup gradient bounds under natural global geometric conditions. The results are new even in the case of the Heisenberg group which is the simplest example of a sub-Riemannian manifold with transverse symmetries.

1. Introduction. As shown in the monographs by Hsu [22], Stroock [34] and Wang [37], stochastic analysis provides a set of powerful tools to study the geometry of manifolds. However, as of today, most of the applications are restricted to Riemannian geometry. The goal of the present work is to introduce some stochastic analysis tools in sub-Riemannian geometry. We will, in particular, focus on the special class of sub-Riemannian manifolds with transverse symmetries that was introduced in [7].

A sub-Riemannian manifold is a smooth manifold $\mathbb{M}$ equipped with a nonholonomic, or bracket generating, subbundle $\mathcal{H} \subset T\mathbb{M}$ and a fiber inner product $g_{\mathcal{H}}$. This means that if we denote by $L(\mathcal{H})$ the Lie algebra of the vector fields generated by the global C^∞ sections of $\mathcal{H}$, then $\operatorname{span}\{X(x) | X \in L(\mathcal{H})\} = T_x(\mathbb{M})$ for every $x \in \mathbb{M}$. We note that when $\mathcal{H} = T\mathbb{M}$, a sub-Riemannian manifold is simply a Riemannian one, and thus sub-Riemannian manifolds encompass Riemannian ones. However, some aspects of the geometry of sub-Riemannian manifolds are considerably less regular than their Riemannian ancestors. Some of the major differences between the two geometries are the following:

1. The Hausdorff dimension is usually greater than the manifold dimension.
2. The sub-Riemannian distance to a point x is in general not smooth on any pointed neighborhood of x.

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3. The exponential map defined by the geodesics is in general not a local diffeomorphism in a neighborhood of the point at which it is based (see [28]).

4. The space of horizontal paths joining two fixed points may have singularities (the so-called abnormal geodesics, see [27]).

5. The sub-Riemannian Brownian motion $(X_t)_{t\geq 0}$ does not fill the space in an isotropic way as the Riemannian Brownian does. Intuitively, for small times t the process X_t will move at a speed $\sqrt{t}$ in the direction of the vector fields in $\mathcal{H}$, t in the direction of the vector fields in $[\mathcal{H},\mathcal{H}]$, $t^{3/2}$ in the direction of the vector fields in $[\mathcal{H},[\mathcal{H},\mathcal{H}]]$, and so on (see [3]).

6. The sub-Laplacian is only subelliptic and not elliptic, that is, the diffusion matrix at a point x is in general not invertible.

Sub-Riemannian geometry takes its roots in very old problems related to isoperimetry, but was internationally brought to the attention of mathematicians by E. Cartan's pioneering address [12] at the Bologna International Congress of Mathematicians in 1928. Since then, it has been the focus of numerous studies by geometers. In particular, one should consult the monographs by Agrachev [1], Bellaïche [8], Gromov [21] and Montgomery [27] and the references therein. For the last four decades, sub-Riemannian geometry has also been a center of interest for analysts because it is the natural geometry associated to subelliptic partial differential equations (see [29, 31]). Perhaps more surprisingly, sub-Riemannian geometry has also been widely studied by probabilists since the breakthrough [26] by Malliavin in 1976, where stochastic analysis and hypoellipticity theory merged together. The paper gave birth to what nowadays is called Malliavin calculus. This calculus has then been successfully applied to the study of hypoelliptic heat kernels. We mention in particular the works by Ben Arous [9] and Kusuoka and Stroock [23]. One may also consult the monograph [3] for further connections between probability theory and sub-Riemannian geometry.

Despite being an object of intensive studies, partly due to the above obstructions most of the developments in sub-Riemannian geometry to date are of a local nature, that is, are restricted to compact manifolds only. As a consequence, the theory presently lacks a body of results which, similar to the case of non compact manifolds, connect global properties of solutions of the relevant partial differential equations, or of the relevant stochastic processes to curvature properties of the ambient manifold.

However, in some special sub-Riemannian structures, a notion of *Ricci lower bound* has been made precise in several recent papers [5–7]. Numerous new hypoelliptic functional inequalities were then obtained as a consequence. We mention in particular the subelliptic Li–Yau inequalities (see [7]), the subelliptic parabolic Harnack inequalities (see [7]), the Poincaré inequalities on balls (see [6]) and the log-Sobolev inequalities (see [5]).

In the present paper, by using probabilistic methods, we reprove and actually greatly improve under weaker conditions several inequalities that were obtained in [5] by using purely analytic methods. We also get new hypoelliptic inequalities which seem difficult to prove directly by analytical tools. We mention, that in our opinion, the probability methods are in a sense more direct and overall simpler than the analytical methods that were developed in [7]. More precisely, the results in Sections 3 and 4 in [7] which were used to prove Hypothesis 1.4 in [7] may now be omitted, since this Hypothesis 1.4 is a straightforward consequence of Corollary 4.9 that we prove in this paper.

We now describe our main results. The paper is divided into two parts, a geometric part and a probabilistic part. The geometric part is devoted to the study of Bochner–Weitzenböck type formulas on sub-Riemannian manifolds with transverse symmetries (see Section 2 for the definitions). More precisely, our goal will be to introduce a natural family $\square_\varepsilon$, $\varepsilon > 0$, of sub-Laplacians on one-forms that satisfy the intertwining

$$dL = \square_\varepsilon d, \tag{1.1}$$

where L is the sub-Laplacian and d the exterior derivative. The operator $\square_\varepsilon$ is self-adjoint with respect to a Riemannian metric extension that contracts in the sense of Strichartz [33] to the sub-Riemannian metric when $\varepsilon \to \infty$. Our main geometric result is then Theorem 3.3 where we prove that

$$\square_\varepsilon = -(\nabla_{\mathcal{H}} - \mathfrak{T}^\varepsilon_{\mathcal{H}})^*(\nabla_{\mathcal{H}} - \mathfrak{T}^\varepsilon_{\mathcal{H}}) + \frac{1}{2\varepsilon} J^* J - \mathfrak{Ric}_{\mathcal{H}},$$

and that for any smooth one-form η,

$$\frac{1}{2} L \|\eta\|^2_{2\varepsilon} - \langle \square_\varepsilon \eta, \eta \rangle_{2\varepsilon} = \sum_{i=1}^d \|\nabla_{X_i}\eta - \mathfrak{T}^\varepsilon_{X_i}\eta\|^2_{2\varepsilon} + \left\langle \left(\mathfrak{Ric}_{\mathcal{H}} - \frac{1}{2\varepsilon} J^* J \right)\eta, \eta \right\rangle_{2\varepsilon}.$$

The quantities $\mathfrak{T}^\varepsilon$, J^*J and $\mathfrak{Ric}_{\mathcal{H}}$ are tensors that will be introduced in the text. We should mention that, to our knowledge, this Bochner–Weitzenböck formula is new even in the case of the Heisenberg group and it implies in a straightforward way the horizontal and the vertical Bochner's identities proved in [7].

In the second part of the paper, we exploit the commutation (1.1) to give a probabilistic representation of the derivative dP_t where P_t is the semigroup generated by the sub-Laplacian L. The representation actually follows from (1.1) by adapting in our case classical ideas by Bismut [10], Driver and Thalmaier [15], Elworthy [16, 17] and Thalmaier [35]. We deduce from this representation an integration by part formula in the spirit of Driver [13]. Several new hypoelliptic heat semigroup gradient estimates are then obtained as a consequence.

We point out that these two parts are largely independent and that the more probability inclined reader may skip Section 3 since the main results proved in this section are summarized in Proposition 4.1.

To conclude, we should mention that the inequalities we obtain, in the spirit of [5], involve a vertical gradient. This, of course, does not mean that they are not geometrically meaningful, because we can see for instance that the gradient bound in Corollary 4.9 is actually equivalent to a global lower bound on the horizontal Ricci tensor of the sub-Riemannian connection. These hypoelliptic inequalities with vertical gradient have also been successfully been used in geometry, where real geometric theorems were proved as a consequence, like the subelliptic Bonnet–Myers [7] and in analysis where convergence to equilibrium for hypoelliptic kinetic Fokker–Planck equations were established [4, 36]. On the other hand, we have to say that the Driver–Melcher inequality [14] (see also [2, 24]) in the Heisenberg group, which only involves the horizontal gradient still remains a little mysterious for us, and that it would of course be extremely interesting to connect those type of inequalities to natural geometric quantities.

2. Sub-Riemannian manifolds with transverse symmetries. The notion of sub-Riemannian manifold with transverse symmetries was introduced in [7]. We recall here the main geometric quantities and operators related to this structure that will be needed in the sequel and we refer to [7] for further details. We also introduce some new geometric invariants that shall later be needed.

Let $\mathbb{M}$ be a smooth, connected manifold with dimension $d + \mathfrak{h}$. We assume that $\mathbb{M}$ is equipped with a bracket generating distribution $\mathcal{H}$ of dimension d and a fiberwise inner product $g_{\mathcal{H}}$ on that distribution. The distribution $\mathcal{H}$ is referred to as the set of *horizontal directions*. Sub-Riemannian geometry is the study of the geometry which is intrinsically associated to $(\mathcal{H}, g_{\mathcal{H}})$ (see [33]). In general, there is no canonical vertical complement of $\mathcal{H}$ in the tangent bundle $T\mathbb{M}$, but in some cases the fiberwise inner product $g_{\mathcal{H}}$ determines one.

DEFINITION 2.1. It is said that $\mathbb{M}$ is a sub-Riemannian manifold with transverse symmetries if there exists a $\mathfrak{h}$-dimensional Lie algebra $\mathcal{V}$ of sub-Riemannian Killing vector fields such that for every $x \in \mathbb{M}$,

$$T_x\mathbb{M} = \mathcal{H}(x) \oplus \mathcal{V}(x),$$

where

$$\mathcal{V}(x) = \{Z(x), Z \in \mathcal{V}(x)\}.$$

We recall that a vector field Z on $\mathbb{M}$ is called a sub-Riemannian Killing field if:

- The flow generated by Z infinitesimally preserves $\mathcal{H}$, that is for every horizontal vector field X (i.e., a smooth section of $\mathcal{H}$), the vector field $[Z, X]$ is horizontal.
- The flow generated by Z infinitesimally preserves the metric $g_{\mathcal{H}}$, that is $\mathcal{L}_Z g_{\mathcal{H}} = 0$, where $\mathcal{L}_Z$ denotes the Lie derivative in the direction of Z.

Some of the most interesting examples of sub-Riemannian manifolds with transverse symmetries come from a principal fiber bundle projection $\pi : \mathbb{M} \to \mathbb{N}$ with totally geodesic fibers isomorphic to the structure group. The sub-Riemannian objects of $\mathbb{M}$ we are interested in are then the lifts of the Riemannian objects of $\mathbb{N}$: The sub-Laplacian on $\mathbb{M}$ is the lift of the Laplace–Beltrami operator on $\mathbb{N}$ and the horizontal Brownian motion on $\mathbb{M}$ which is our main object of interest is the lift of the Brownian motion on $\mathbb{N}$. The study of the horizontal diffusion processes associated to this type of submersions has already attracted a lot of attention in the past, mostly in connection with skew-product type decomposition theorems (see Elworthy and Kendall [19] or Liao [25]). In our work, we are more interested in developing an intrinsic horizontal stochastic calculus rather than skew-product considerations. Though a sub-Riemannian manifold with transverse symmetries may not be globally associated with a submersion, it is always locally. More precisely, as recently observed by Elworthy in [18], a sub-Riemannian structure with transverse symmetries induces on $\mathbb{M}$ a Riemannian foliation with totally geodesic leaves and bundle-like metric. We refer to the monograph by Elworthy, Le Jan and Li [20] for a discussion of diffusions on foliated manifolds.

From now on in the sequel of the paper, we assume that $\mathbb{M}$ is a sub-Riemannian manifold with transverse symmetries.

The distribution $\mathcal{V}$ is referred to as the set of *vertical directions*. The choice of an inner product $g_{\mathcal{V}}$ on the Lie algebra $\mathcal{V}$ naturally endows $\mathbb{M}$ with a one-parameter family of Riemannian metrics that makes the decomposition $\mathcal{H} \oplus \mathcal{V}$ orthogonal:

$$g_\varepsilon = g_{\mathcal{H}} \oplus \frac{1}{\varepsilon} g_{\mathcal{V}}, \qquad \varepsilon > 0.$$

For notational convenience, we will often use the notation $\langle \cdot, \cdot \rangle_\varepsilon$, respectively $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, respectively $\langle \cdot, \cdot \rangle_{\mathcal{V}}$, instead of g_ε, respectively $g_{\mathcal{H}}$, respectively $g_{\mathcal{V}}$. We can extend $g_{\mathcal{H}}$ on $T_x\mathbb{M} \times T_x\mathbb{M}$ by the requirement that $g_{\mathcal{H}}(u, v) = 0$ whenever u or v is in $\mathcal{V}(x)$. We similarly extend $g_{\mathcal{V}}$. Hence, for any $u \in T_x\mathbb{M}$,

$$\|u\|_\varepsilon^2 = \|u\|_{\mathcal{H}}^2 + \frac{1}{\varepsilon} \|u\|_{\mathcal{V}}^2.$$

Although g_ε will be useful for the purpose of computations, the geometric objects that we are eventually interested in, like the sub-Laplacian L and its associated semigroup will of course not depend on ε.

The Riemannian volume measure of $(\mathbb{M}, g_\varepsilon)$ is always a multiple of the Riemannian volume measure of $(\mathbb{M}, g_1)$, therefore, we will always use the Riemannian volume measure of $(\mathbb{M}, g_1)$ which we will denote μ.

At every point $x \in \mathbb{M}$, we can find a local frame of vector fields $\{X_1, \dots, X_d, Z_1, \dots, Z_{\mathfrak{h}}\}$ such that on a neighborhood of x:

(a) $\{X_1, \dots, X_d\}$ is a $g_{\mathcal{H}}$-orthonormal basis of $\mathcal{H}$;
(b) $\{Z_1, \dots, Z_{\mathfrak{h}}\}$ is a $g_{\mathcal{V}}$-orthonormal basis of $\mathcal{V}$.

We observe that the following commutation relations hold:

$$[X_i, X_j] = \sum_{\ell=1}^{d} \omega_{ij}^{\ell} X_\ell + \sum_{m=1}^{\mathfrak{h}} \gamma_{ij}^{m} Z_m, \tag{2.1}$$

$$[X_i, Z_m] = \sum_{\ell=1}^{d} \delta_{im}^{\ell} X_\ell, \tag{2.2}$$

for smooth functions ω_{ij}^{ℓ}, γ_{ij}^{m} and δ_{im}^{ℓ} such that

$$\delta_{im}^{\ell} = -\delta_{\ell m}^{i}, \qquad i, \ell = 1, \dots, d \text{ and } m = 1, \dots, \mathfrak{h}. \tag{2.3}$$

Property (2.3) follows from the property of Z_m being a sub-Riemannian Killing field. By convention, $\omega_{ij}^{\ell} = -\omega_{ji}^{\ell}$, $\gamma_{ij}^{m} = -\gamma_{ji}^{m}$ and $\delta_{im}^{\ell} = -\delta_{mi}^{\ell}$.

We define the horizontal gradient $\nabla_{\mathcal{H}} f$ of a function f as the projection of the Riemannian gradient of f on the horizontal bundle. Similarly, we define the vertical gradient $\nabla_{\mathcal{V}} f$ of a function f as the projection of the Riemannian gradient of f on the vertical bundle. In a local adapted frame, we have

$$\nabla_{\mathcal{H}} f = \sum_{i=1}^{d} (X_i f) X_i$$

and

$$\nabla_{\mathcal{V}} f = \sum_{m=1}^{\mathfrak{h}} (Z_m f) Z_m.$$

The canonical sub-Laplacian in a sub-Riemannian manifold with transverse symmetries is the generator of the symmetric Dirichlet form

$$\mathcal{E}_{\mathcal{H}}(f, g) = \int_{\mathbb{M}} \langle \nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} g \rangle_{\mathcal{H}} \, d\mu.$$

It is a diffusion operator L on $\mathbb{M}$ which is symmetric on $C_0^\infty(\mathbb{M})$ with respect to the measure μ.

Actually, it is readily seen that in an adapted frame, one has

$$L = -\sum_{i=1}^{d} X_i^* X_i,$$

where X_i^* is the formal adjoint of X_i. From the commutation relations in an adapted frame, we see that

$$X_i^* = -X_i + \sum_{k=1}^{d} \omega_{ik}^{k},$$

so that

$$L = \sum_{i=1}^{d} X_i^2 + X_0, \tag{2.4}$$

with

$$X_0 = -\sum_{i,k=1}^{d} \omega_{ik}^k X_i. \tag{2.5}$$

On sub-Riemannian manifolds with transverse symmetries, there is a canonical connection.

PROPOSITION 2.2 (See [7]). *There exists a unique connection ∇ on $\mathbb{M}$ satisfying the following properties*:

(i) $\nabla g_\varepsilon = 0$, *for all* $\varepsilon > 0$;
(ii) *If X and Y are horizontal vector fields*, $\nabla_X Y$ *is horizontal*;
(iii) *If* $Z \in \mathcal{V}$, $\nabla Z = 0$;
(iv) *If X, Y are horizontal vector fields and $Z \in \mathcal{V}$, the torsion vector field $T(X,Y)$ is vertical and $T(X,Z) = 0$.*

Intuitively ∇ is the connection which coincides with the Levi–Civita connection of the Riemannian metric g_1 on the horizontal bundle $\mathcal{H}$ and that parallelizes the Lie algebra $\mathcal{V}$. We stress that this connection does not depend on ε and straightforward computations show that one has in a local adapted frame:

$$\nabla_{X_i} X_j = \sum_{k=1}^{d} \frac{1}{2}(\omega_{ij}^k + \omega_{ki}^j + \omega_{kj}^i) X_k, \tag{2.6}$$

$$\nabla_{Z_m} X_i = -\sum_{\ell=1}^{d} \delta_{im}^\ell X_\ell, \tag{2.7}$$

$$\nabla Z_m = 0 \tag{2.8}$$

and

$$T(X_i, X_j) = -\sum_{m=1}^{\mathfrak{h}} \gamma_{ij}^m Z_m.$$

We observe that, thanks to (2.4) and (2.5), in a local adapted frame we have

$$L = \sum_{i=1}^{d} X_i^2 - \nabla_{X_i} X_i.$$

To establish Bochner–Weitzenböck formulas, it will expedient to work in normal frames.

LEMMA 2.3. *Let $x \in \mathbb{M}$. There exists a local adapted frame of vector fields*

$$\{X_1, \dots, X_d, Z_1, \dots, Z_{\mathfrak{h}}\}$$

around x, such that, at x,

$$\nabla_{X_i} X_j(x) = 0.$$

Such frame will be called an adapted normal frame around x.

PROOF. Since ∇ coincides with a Levi–Civita connection on the horizontal bundle, the result essentially boils down to the existence of normal frames in Riemannian geometry. □

Observe that in a normal adapted frame, we have $\omega_{ij}^k = 0$ at the center of the frame. We now introduce some maps that will play an important role in the sequel. For $Z \in \mathcal{V}$, there is a unique skew-symmetric map J_Z defined on the horizontal bundle $\mathcal{H}$ such that for all horizontal vector fields X and Y,

$$g_{\mathcal{H}}\big(J_Z(X), Y\big) = g_{\mathcal{V}}\big(Z, T(X, Y)\big). \tag{2.9}$$

In a local adapted frame, we have

$$J_{Z_m}(X_i) = -\sum_{j=1}^{d} \gamma_{ij}^m X_j.$$

We then extend J_{Z_m} to be 0 on the vertical bundle $\mathcal{V}$.

We finally recall the following definition that was introduced in [7].

DEFINITION 2.4. The sub-Riemannian manifold $\mathbb{M}$ is said to be of Yang–Mills type, if for every horizontal vector field X, and any adapted local frame $\{X_1, \dots, X_d, Z_1, \dots, Z_{\mathfrak{h}}\}$

$$\sum_{\ell=1}^{d} (\nabla_{X_\ell} T)(X_\ell, X) = 0.$$

A quick computation shows that $\mathbb{M}$ is of Yang–Mills type if and only if for every $x \in \mathbb{M}$ and any adapted normal frame $\{X_1, \dots, X_d, Z_1, \dots, Z_{\mathfrak{h}}\}$ around x, we have at x,

$$\sum_{i=1}^{d} X_i \gamma_{ij}^m = 0, \qquad 1 \le j \le d, 1 \le m \le \mathfrak{h}.$$

We conclude the section with simple examples of sub-Riemannian manifolds with transverse symmetries: The 3-dimensional model spaces in K-contact geometry.

Given a number $\rho \in \mathbb{R}$, suppose that $\mathbb{G}(\rho)$ is a simply connected three-dimensional Lie group whose Lie algebra $\mathfrak{g}$ has a basis $\{X, Y, Z\}$ satisfying:

(i) $[X, Y] = Z$,
(ii) $[X, Z] = -\rho Y$,
(iii) $[Y, Z] = \rho X$.

For instance, for $\rho = 0$, $\mathbb{G}(\rho)$ is the Heisenberg group. For $\rho = 1$, $\mathbb{G}(\rho)$ is $\mathbf{SU}(2)$ and for $\rho = -1$, $\mathbb{G}(\rho)$ is $\mathbf{SL}(2)$. It is easy to see that if we consider the left-invariant distribution $\mathcal{H}$ generated by $\{X, Y\}$ and chose for $g_{\mathcal{H}}$ the left-invariant metric that makes $\{X, Y\}$ orthonormal then $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a Yang–Mills sub-Riemannian manifold with transverse symmetry Z.

The sub-Laplacian on $\mathbb{G}(\rho)$ is the left-invariant, second-order differential operator

$$L = X^2 + Y^2$$

and the connection ∇ is given by

$$\nabla_X Y = \nabla_Y X = \nabla_X Z = \nabla_Y Z = 0$$

and

$$\nabla_Z X = -\rho Y, \qquad \nabla_Z Y = \rho X.$$

3. Bochner–Weitzenböck formulas for sub-Laplacians on one-forms. The purpose of the section is to establish the Bochner–Weitzenböck formula for the sub-Laplacian. This formula is the key to the stochastic representation of the heat semigroup on one-forms. The reader only interested in the probabilistic consequences of the formula may directly jump to Section 4 and admit Proposition 4.1 which summarizes the results proved in this section.

From now on, in all the paper we consider a Yang–Mills sub-Riemannian manifold $\mathbb{M}$ with transverse symmetries and adopt the notation of the previous section. In particular L denotes the sub-Laplacian on $\mathbb{M}$.

Obviously, there exist infinitely many second-order differential operators $\mathcal{L}$ defined on one-forms such that for every smooth function f,

$$dLf = \mathcal{L}df,$$

where d is the exterior derivative. In Riemannian geometry, a canonical $\mathcal{L}$ that satisfies the above commutation is the Hodge–de Rham Laplacian. On sub-Riemannian manifolds, even contact manifolds, there is no such canonical sub-Laplacian (see [30]) on one-forms. However, in our case, we will see in this section that there is a distinguished one-parameter family of sub-Laplacians on one-forms which are optimal when interested in Bochner–Weitzenböck's type formulas and that satisfy the above commutation.

We start with some general preliminaries about one-forms. By declaring a one-form horizontal (resp., vertical) if it vanishes on the vertical bundle $\mathcal{V}$ (resp., on the horizontal bundle $\mathcal{H}$), the splitting of the tangent space

$$T_x\mathbb{M} = \mathcal{H}(x) \oplus \mathcal{V}(x)$$

gives a splitting of the cotangent space

$$T_x^*\mathbb{M} = \mathcal{H}^*(x) \oplus \mathcal{V}^*(x).$$

If $\{X_1, \dots, X_d, Z_1, \dots, Z_{\mathfrak{h}}\}$ is a local adapted frame, the dual frame will be denoted $\{\theta_1, \dots, \theta_d, \nu_1, \dots, \nu_{\mathfrak{h}}\}$ and referred to as a local adapted coframe. With a slight abuse of notation, for $\varepsilon > 0$, the metric on $T_x^*\mathbb{M}$ that makes $\{\theta_1, \dots, \theta_d, \frac{1}{\sqrt{\varepsilon}}\nu_1, \dots, \frac{1}{\sqrt{\varepsilon}}\nu_{\mathfrak{h}}\}$ orthonormal will still be denoted g_ε or $\langle \cdot, \cdot \rangle_\varepsilon$. This metric on the cotangent bundle can thus be written

$$g_\varepsilon = g_{\mathcal{H}} \oplus \varepsilon g_{\mathcal{V}}, \qquad \varepsilon > 0, \tag{3.1}$$

where $g_{\mathcal{H}}$ (resp., $g_{\mathcal{V}}$) is the metric on $\mathcal{H}^*$ (resp., $\mathcal{V}^*$) that makes $\{\theta_1, \dots, \theta_d\}$ (resp., $\{\nu_1, \dots, \nu_{\mathfrak{h}}\}$) orthonormal. We use similar notation and conventions as before so that for every η in $T_x^*\mathbb{M}$,

$$\|\eta\|_\varepsilon^2 = \|\eta\|_{\mathcal{H}}^2 + \varepsilon \|\eta\|_{\mathcal{V}}^2.$$

We will denote by $\mathfrak{L}$ the covariant extension on one-forms of the sub-Laplacian. In a local adapted frame, we have thus

$$\mathfrak{L} = \sum_{i=1}^d \nabla_{X_i} \nabla_{X_i} - \nabla_{\nabla_{X_i} X_i}. \tag{3.2}$$

We define then $\mathfrak{Ric}_{\mathcal{H}}$ as the fiberwise symmetric linear map on one forms such that for every smooth functions f, g,

$$\langle \mathfrak{Ric}_{\mathcal{H}}(df), dg \rangle_\varepsilon = \mathbf{Ricci}(\nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} g),$$

where **Ricci** is the Ricci curvature of the connection ∇. Of course, $\mathfrak{Ric}_{\mathcal{H}}$ does not depend on ε because the above definition implies that $\mathfrak{Ric}_{\mathcal{H}}$ is horizontal, that is, it transforms any one-form into a horizontal form. Actually, a computation shows that in a normal adapted frame around x, we have at x,

$$\mathfrak{Ric}_{\mathcal{H}}(\eta) = \sum_{k,\ell=1}^d \frac{1}{2}(\rho_{k\ell} + \rho_{\ell k}) f_k \theta_\ell,$$

where $\eta = \sum_{i=1}^d f_i \theta_i + \sum_{m=1}^{\mathfrak{h}} g_m \nu_m$ and

$$\rho_{k\ell} = \sum_{j=1}^d \sum_{m=1}^{\mathfrak{h}} \gamma_{kj}^m \delta_{jm}^\ell + \sum_{j=1}^d X_\ell \omega_{kj}^j - X_j \omega_{\ell j}^k.$$

Finally, we consider the first-order differential operator $\mathfrak{J}$ defined in a local adapted frame by

$$\mathfrak{J}(\eta) = \sum_{i,j=1}^d \sum_{m=1}^{\mathfrak{h}} \gamma_{ij}^m (X_j g_m) \theta_i,$$

where, again, $\eta = \sum_{i=1}^d f_i\theta_i + \sum_{m=1}^{\mathfrak{h}} g_m\nu_m$. By defining J_{Z_m} on one-forms using the duality

$$J_{Z_m}(\theta_i) = J_{Z_m}(X_i), \qquad J_{Z_m}(\nu_i) = 0,$$

we can write more intrinsically

$$\mathfrak{J} = \sum_{m=1}^{\mathfrak{h}} J_{Z_m}(d\iota_{Z_m}),$$

where ι is the interior product. This last expression shows that $\mathfrak{J}$ does not depend on the choice on the local frame, and is therefore a globally defined first-order differential operator on one-forms.

We are now in a position to prove our first commutation result.

PROPOSITION 3.1. *Let*

$$\Box_\infty = \mathfrak{L} + 2\mathfrak{J} - \mathfrak{Ric}_{\mathcal{H}}.$$

Then, we have for every smooth function f,

$$dLf = \Box_\infty df. \tag{3.3}$$

PROOF. Let $x \in \mathbb{M}$. It is enough to prove this commutation at x in a local adapted normal frame $\{X_1, \dots, X_d, Z_1, \dots, Z_{\mathfrak{h}}\}$ around x. Observing that L and Z_m commute (see [7]), we have at x:

$$\begin{aligned} dLf &= \sum_{i=1}^d (X_i Lf)\theta_i + \sum_{m=1}^{\mathfrak{h}} (Z_m Lf)\nu_m \\ &= \sum_{i=1}^d (LX_i f)\theta_i + \sum_{m=1}^{\mathfrak{h}} (LZ_m f)\nu_m + \sum_{i=1}^d \big([X_i, L]f\big)\theta_i \\ &= \mathfrak{L}df - \sum_{i=1}^d (X_i f)\mathfrak{L}\theta_i + \sum_{i=1}^d \big([X_i, L]f\big)\theta_i. \end{aligned}$$

Keeping in mind that at the center of the frame $\omega_{ij}^k = 0$, and thanks to the Yang–Mills assumption

$$\sum_{i=1}^d X_i\gamma_{ij}^m = 0,$$

we now compute:

$$\begin{aligned} &\sum_{i=1}^d \big([X_i, L]f\big)\theta_i \\ &\quad = \sum_{i,j=1}^d \big([X_i, X_j^2]f\big)\theta_i + \sum_{i=1}^d \big([X_i, X_0]f\big)\theta_i \end{aligned}$$

$$= \sum_{i=1}^{d} \left([X_i, X_j] X_j f + X_j [X_i, X_j] f - \sum_{j,k=1}^{d} [X_i, \omega_{jk}^k X_j] f \right) \theta_i$$

$$= \sum_{i=1}^{d} \left(\sum_{j=1}^{d} \sum_{m=1}^{\mathfrak{h}} \gamma_{ij}^m (Z_m X_j f + X_j Z_m f) + \sum_{j,k=1}^{d} (X_j \omega_{ij}^k - X_i \omega_{kj}^j) X_k f \right) \theta_i$$

$$= \sum_{i=1}^{d} \left(2 \sum_{j=1}^{d} \sum_{m=1}^{\mathfrak{h}} \gamma_{ij}^m (X_j Z_m f) \right.$$

$$\left. - \sum_{j,k=1}^{d} \sum_{m=1}^{\mathfrak{h}} \gamma_{ij}^m \delta_{jm}^k X_k f + \sum_{j,k=1}^{d} (X_j \omega_{ij}^k - X_i \omega_{jk}^k) X_k f \right) \theta_i.$$

It is now elementary to identify the terms in the above equality by using the formula

$$\mathfrak{L} \theta_i = \sum_{j,k=1}^{n} (-X_j \Gamma_{jk}^i) \theta_k,$$

where the Γ_{ij}^k's are the Cristofell symbols of the connection. $\square$

Obviously, $\square_\infty$ is not the only operator that satisfies (3.3). Actually, since $d^2 = 0$, if Λ is any fiberwise linear map from the space of two-forms into the space of one-forms, then we have

$$dLf = (\square_\infty + \Lambda \circ d) df.$$

This raises the question of an *optimal* choice of Λ. The following proposition answers this question if optimality is understood in the sense of a corresponding Bochner–Weitzenböck's formula.

PROPOSITION 3.2. *For any fiberwise linear map Λ from the space of two-forms into the space of one-forms, and any $x \in \mathbb{M}$, we have*

$$\inf_{\eta, \|\eta(x)\|_\varepsilon = 1} \left(\frac{1}{2} (L \|\eta\|_\varepsilon^2)(x) - \langle (\square_\infty + \Lambda \circ d) \eta(x), \eta(x) \rangle_\varepsilon \right)$$

$$\leq \inf_{\eta, \|\eta(x)\|_\varepsilon = 1} \left(\frac{1}{2} (L \|\eta\|_\varepsilon^2)(x) - \left\langle \left(\square_\infty - \frac{1}{\varepsilon} T \circ d \right) \eta(x), \eta(x) \right\rangle_\varepsilon \right),$$

where in the above notation, the torsion tensor T is interpreted, by duality, as a fiberwise linear map from the space of two-forms into the space of one-forms.

PROOF. Let $x \in \mathbb{M}$ and consider a normal adapted frame around x. The following computations are done at the center x of the frame. Let us consider a smooth one-form

$$\eta = \sum_{i=1}^{d} f_i \theta_i + \sum_{m=1}^{\mathfrak{h}} g_m \nu_m .$$

We have

$$\begin{aligned}
&\frac{1}{2}\big(L\|\eta\|_\varepsilon^2\big) - \big\langle(\square_\infty + \Lambda \circ d)\eta, \eta\big\rangle_\varepsilon \\
&\quad = \sum_{i=1}^{d} \|\nabla_{\mathcal{H}} f_i\|_{\mathcal{H}}^2 + \varepsilon \sum_{m=1}^{\mathfrak{h}} \|\nabla_{\mathcal{V}} g_m\|_{\mathcal{V}}^2 - 2 \sum_{i,j=1}^{d} \sum_{m=1}^{\mathfrak{h}} \gamma_{ij}^m (X_j g_m) f_i \\
&\qquad - \big\langle \Lambda(d\eta), \eta\big\rangle_\varepsilon + \langle \mathfrak{Ric}_{\mathcal{H}} \eta, \eta\rangle_{\mathcal{H}}.
\end{aligned} \tag{3.4}$$

On the other hand, the exterior derivative can be computed as follows:

$$\begin{aligned}
d\eta = &\sum_{i,j=1}^{d} \left(X_i f_j - \frac{1}{2} \sum_{m=1}^{\mathfrak{h}} \gamma_{ij}^m g_m \right) \theta_i \wedge \theta_j \\
&+ \sum_{j=1}^{d} \sum_{m=1}^{\mathfrak{h}} \left(X_j g_m - Z_m f_j - \sum_{i=1}^{d} \delta_{jm}^i f_i \right) \theta_j \wedge \nu_m \\
&+ \sum_{m,\ell=1}^{\mathfrak{h}} \alpha_{m,\ell} \nu_\ell \wedge \nu_m ,
\end{aligned}$$

where $\alpha_{m,\ell}$ are coefficients which are unimportant to compute explicitly. Because of the vertical derivatives $Z_m f_i$ and $Z_\ell g_m$ that do not appear in (3.4), the quantity

$$\inf_{\eta, \|\eta(x)\|_\varepsilon = 1} \left(\frac{1}{2}\big(L\|\eta\|_\varepsilon^2\big)(x) - \big\langle(\square_\infty + \Lambda \circ d)\eta(x), \eta(x)\big\rangle_\varepsilon \right) \tag{3.5}$$

is then finite if and only if $\Lambda(\nu_\ell \wedge \nu_m) = \Lambda(\theta_i \wedge \nu_m) = 0$, which we assume from now on. Also, clearly, every nonzero term $\langle \Lambda(\theta_i \wedge \theta_j), \theta_k\rangle_{\mathcal{H}}$ would decrease (3.5), so we can assume $\langle \Lambda(\theta_i \wedge \theta_j), \theta_k\rangle_{\mathcal{H}} = 0$. Completing the squares in (3.4), we see then that the quantity to be maximized is

$$\begin{aligned}
\inf_{\eta, \|\eta(x)\|_\varepsilon = 1} \Bigg(&-\frac{1}{4}\varepsilon^2 \sum_{i,j=1}^{d} \left(\sum_{\ell=1}^{\mathfrak{h}} g_\ell \big\langle \Lambda(\theta_i \wedge \theta_j), \nu_\ell\big\rangle_{\mathcal{V}} \right)^2 \\
&+ \frac{1}{2}\varepsilon \sum_{i,j=1}^{d} \sum_{m,\ell=1}^{\mathfrak{h}} \gamma_{ij}^m g_m g_\ell \big\langle \Lambda(\theta_i \wedge \theta_j), \nu_\ell\big\rangle_{\mathcal{V}} \Bigg).
\end{aligned}$$

We then easily see that the optimal choice of $\langle \Lambda(\theta_i \wedge \theta_j), \nu_\ell\rangle_{\mathcal{V}}$ is given by

$$\big\langle \Lambda(\theta_i \wedge \theta_j), \nu_\ell\big\rangle_{\mathcal{V}} = \frac{1}{\varepsilon}\gamma_{ij}^l . \qquad \square$$

In the sequel, we shall denote

$$\square_\varepsilon = \square_\infty - \frac{1}{\varepsilon} T \circ d.$$

For our purpose, we will need to rewrite $\square_\varepsilon$ in a sum of squares form, from which we will be able to deduce a stochastic representation of the semigroup $e^{(1/2)t\square_\varepsilon}$.

If V is a horizontal vector field, we consider the fiberwise linear map from the space of one-forms into itself which is given by in a local adapted frame by

$$\mathfrak{T}_V^\varepsilon \eta = -\sum_{j=1}^{d} \eta\big(T(V, X_j)\big)\theta_j + \frac{1}{2\varepsilon} \sum_{m=1}^{\mathfrak{h}} \eta(J_{Z_m} V)\nu_m.$$

We see that $\mathfrak{T}_V^\varepsilon$ does not depend of the choice of the local adapted frame and thus, is a globally well-defined, smooth section. In a local adapted frame, if $\eta = \sum_{i=1}^{d} f_i\theta_i + \sum_{m=1}^{\mathfrak{h}} g_m\nu_m$, then we have

$$\mathfrak{T}_{X_i}^\varepsilon \eta = \sum_{j=1}^{d} \sum_{\ell=1}^{\mathfrak{h}} \gamma_{ij}^\ell g_\ell \theta_j - \frac{1}{2\varepsilon} \sum_{j=1}^{d} \sum_{m=1}^{\mathfrak{h}} \gamma_{ij}^m f_j \nu_m.$$

THEOREM 3.3. *In a local adapted frame, we have*

$$\square_\varepsilon = \sum_{i=1}^{d} (\nabla_{X_i} - \mathfrak{T}_{X_i}^\varepsilon)^2 - (\nabla_{\nabla_{X_i} X_i} - \mathfrak{T}_{\nabla_{X_i} X_i}^\varepsilon) + \frac{1}{2\varepsilon} \sum_{m=1}^{\mathfrak{h}} J_{Z_m}^* J_{Z_m} - \mathfrak{Ric}_{\mathcal{H}},$$

and for any smooth one-form η,

$$\begin{aligned} &\frac{1}{2} L \|\eta\|_{2\varepsilon}^2 - \langle \square_\varepsilon \eta, \eta \rangle_{2\varepsilon} \\ &\quad = \sum_{i=1}^{d} \|\nabla_{X_i}\eta - \mathfrak{T}_{X_i}^\varepsilon \eta\|_{2\varepsilon}^2 + \left\langle \left(\mathfrak{Ric}_{\mathcal{H}} - \frac{1}{2\varepsilon} \sum_{m=1}^{\mathfrak{h}} J_{Z_m}^* J_{Z_m} \right)\eta, \eta \right\rangle_{2\varepsilon}. \end{aligned}$$

PROOF. It is enough to prove the two identities at the center of an adapted normal frame. From the definition of $\square_\varepsilon$, at the center of the frame, we have for $\eta = \sum_{i=1}^{d} f_i\theta_i + \sum_{m=1}^{\mathfrak{h}} g_m\nu_m$,

$$\begin{aligned} \square_\varepsilon = {} & \sum_{i=1}^{d} \nabla_{X_i}^2 \eta + 2 \sum_{i,j=1}^{d} \sum_{m=1}^{\mathfrak{h}} \gamma_{ij}^m (X_j g_m)\theta_i \\ & + \frac{1}{\varepsilon} \sum_{i,j=1}^{d} \left(X_i f_j - \frac{1}{2} \sum_{m=1}^{\mathfrak{h}} \gamma_{ij}^m g_m \right) \left(\sum_{m=1}^{\mathfrak{h}} \gamma_{ij}^m \nu_m \right) - \mathfrak{Ric}_{\mathcal{H}}. \end{aligned}$$

On the other hand, still at the center of the frame, we compute

$$(\nabla_{X_i} - \mathfrak{T}^{\varepsilon}_{X_i})\eta = \sum_{j=1}^{d}\left(X_i f_j - \sum_{\ell=1}^{m}\gamma_{ij}^{\ell} g_l\right)\theta_j + \sum_{m=1}^{\mathfrak{h}}\left(X_i g_m + \frac{1}{2\varepsilon}\sum_{j=1}^{d}\gamma_{ij}^{m} f_j\right)\nu_m.$$

Keeping in mind that in a local adapted frame, we have

$$J_{Z_m}(X_i) = -\sum_{j=1}^{d}\gamma_{ij}^{m} X_j,$$

it is now an elementary exercise to check that

$$\square_{\varepsilon} = \sum_{i=1}^{d}(\nabla_{X_i} - \mathfrak{T}^{\varepsilon}_{X_i})^2 + \frac{1}{2\varepsilon}\sum_{m=1}^{\mathfrak{h}} J^*_{Z_m} J_{Z_m} - \mathfrak{Ric}_{\mathcal{H}}.$$

The proof of the second identity follows the same lines as in the proof of Proposition 3.2. The details are let to the reader. □

If V is a horizontal vector field, $\mathfrak{T}^{\varepsilon}_V$ is a skew-symmetric operator for the Riemannian metric $g_{2\varepsilon}$, as a consequence, $\square_{\varepsilon}$ is a symmetric operator for the metric $g_{2\varepsilon}$ on the space of smooth and compactly supported one-forms. It is interesting that $\square_{\varepsilon}$ is symmetric with respect to the metric $g_{2\varepsilon}$ but not g_{ε} which is the one that was used to construct $\square_{\varepsilon}$.

The operator $\sum_{m=1}^{\mathfrak{h}} J^*_{Z_m} J_{Z_m}$ does not depend on the choice of the frame and shall concisely be denoted by J^*J. We can note that in the case where $\mathbb{M}$ is a Sasakian manifold, like the Heisenberg group for instance, J^*J is the identity map on the horizontal distribution.

Similarly, the operator $\sum_{i=1}^{d}(\nabla_{X_i} - \mathfrak{T}^{\varepsilon}_{X_i})^2 - (\nabla_{\nabla_{X_i}X_i} - \mathfrak{T}^{\varepsilon}_{\nabla_{X_i}X_i})$ does not depend on the choice of the frame and can be more intrinsically described as follows.

If η is a one-form, we define the horizontal gradient in a local adapted frame of η as the $(0,2)$ tensor

$$\nabla_{\mathcal{H}}\eta = \sum_{i=1}^{d}\nabla_{X_i}\eta\otimes\theta_i.$$

Similarly, we will use the notation

$$\mathfrak{T}^{\varepsilon}_{\mathcal{H}}\eta = \sum_{i=1}^{d}\mathfrak{T}^{\varepsilon}_{X_i}\eta\otimes\theta_i.$$

It is then easily seen that, in a local adapted frame,

$$-(\nabla_{\mathcal{H}} - \mathfrak{T}^{\varepsilon}_{\mathcal{H}})^*(\nabla_{\mathcal{H}} - \mathfrak{T}^{\varepsilon}_{\mathcal{H}}) = \sum_{i=1}^{d}(\nabla_{X_i} - \mathfrak{T}^{\varepsilon}_{X_i})^2 - (\nabla_{\nabla_{X_i}X_i} - \mathfrak{T}^{\varepsilon}_{\nabla_{X_i}X_i}),$$

where the adjoint is of course understood with respect to the metric $g_{2\varepsilon}$. We therefore globally have

$$\Box_\varepsilon = -(\nabla_{\mathcal{H}} - \mathfrak{T}^\varepsilon_{\mathcal{H}})^*(\nabla_{\mathcal{H}} - \mathfrak{T}^\varepsilon_{\mathcal{H}}) + \frac{1}{2\varepsilon} J^* J - \mathfrak{Ric}_{\mathcal{H}}.$$

To finish the section, we illustrate our formulas in the case of the model space $\mathbb{G}(\rho)$ that was introduced in Section 2. In that case, we have a basis of left invariant vector fields $\{X, Y, Z\}$ satisfying: $[X, Y] = Z$, $[X, Z] = -\rho Y$, and $[Y, Z] = \rho X$ and the sub-Laplacian is given by

$$L = X^2 + Y^2.$$

Every one-form can be written as $\eta = f_1\theta_1 + f_2\theta_2 + g\nu$ where $\{\theta_1, \theta_2, \nu\}$ is the dual basis of $\{X, Y, Z\}$. We identify η with the column vector

$$\eta = \begin{pmatrix} f_1 \\ f_2 \\ g \end{pmatrix}.$$

Elementary computations show then that

$$\mathfrak{Ric}_{\mathcal{H}} = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Box_\varepsilon = \begin{pmatrix} L-\rho & 0 & 2Y \\ 0 & L-\rho & -2X \\ -\frac{1}{\varepsilon}Y & \frac{1}{\varepsilon}X & L-\frac{1}{\varepsilon} \end{pmatrix},$$

$$\mathfrak{T}_X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{2\varepsilon} & 0 \end{pmatrix},$$

$$\mathfrak{T}_Y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ \frac{1}{2\varepsilon} & 0 & 0 \end{pmatrix}$$

and

$$J^*J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

4. Gradient formulas and bounds for the heat semigroup. Throughout the section, we work under the same assumptions as the previous section, and we moreover assume that for every horizontal one-form η,

$$\langle \mathfrak{Ric}_{\mathcal{H}}(\eta), \eta \rangle_{\mathcal{H}} \geq -K \|\eta\|^2_{\mathcal{H}}, \qquad \langle J^*J\eta, \eta \rangle_{\mathcal{H}} \leq \kappa \|\eta\|^2_{\mathcal{H}},$$

with $K, \kappa \geq 0$. We also assume that the manifold $\mathbb{M}$ is metrically complete with respect to the sub-Riemannian distance. Under these assumptions, it was proved in [7] that the sub-Laplacian L is essentially self-adjoint on $C_0^\infty(\mathbb{M})$ and that the semigroup $P_t = e^{(1/2)tL}$ is stochastically complete.

The following result was proved in the previous section.

PROPOSITION 4.1. *Consider the operator defined on one-forms by the formula*

$$\Box_\varepsilon = -(\nabla_{\mathcal{H}} - \mathfrak{T}^\varepsilon_{\mathcal{H}})^*(\nabla_{\mathcal{H}} - \mathfrak{T}^\varepsilon_{\mathcal{H}}) + \frac{1}{2\varepsilon} J^* J - \mathfrak{Ric}_{\mathcal{H}},$$

then for any smooth function f,

$$dLf = \Box_\varepsilon df$$

and for any smooth one-form η

$$\begin{aligned} \frac{1}{2} L\|\eta\|^2_{2\varepsilon} - \langle \Box_\varepsilon \eta, \eta\rangle_{2\varepsilon} &= \sum_{i=1}^d \|\nabla_{X_i}\eta - \mathfrak{T}^\varepsilon_{X_i}\eta\|^2_{2\varepsilon} + \left\langle \left(\mathfrak{Ric}_{\mathcal{H}} - \frac{1}{2\varepsilon} J^* J\right)\eta, \eta \right\rangle_{2\varepsilon} \\ &\geq \left(\rho - \frac{\kappa}{2\varepsilon}\right)\|\eta\|^2_{\mathcal{H}}. \end{aligned}$$

REMARK 4.2. We note again that the operator $\Box_\varepsilon$ depends on ε, but since $dLf = \Box_\varepsilon df$, $\Box_\varepsilon \eta$ does not depend on ε when η is an exact one-form.

4.1. *Heat semigroup on one-forms*. We are interested in a stochastic representation of the semigroup on one-forms which is generated by $\Box_\varepsilon$. This semigroup is well-defined by using the spectral theorem thanks to the following lemma.

LEMMA 4.3. *The operator $\Box_\varepsilon$ is essentially self-adjoint on the space of smooth and compactly supported one-forms for the Riemannian metric $g_{2\varepsilon}$.*

PROOF. Since we assume $\mathbb{M}$ to be metrically complete for the sub-Riemannian distance, it is also complete for the Riemannian distance associated to $g_{2\varepsilon}$, because $g_{2\varepsilon}$ is a Riemannian extension of $g_{\mathcal{H}}$. From [32], there exists therefore a sequence $h_n \in C_0^\infty(\mathbb{M})$, such that $0 \leq h_n \leq 1$ and $\|\nabla_{\mathcal{H}} h_n\|^2_\infty + 2\varepsilon\|\nabla_{\mathcal{V}} h_n\|^2_\infty \to 0$. In particular, note that we have $\|\nabla_{\mathcal{H}} h_n\|_\infty \to 0$.

To prove that $\Box_\varepsilon$ is essentially self-adjoint it is enough (see [32]) to prove that for some $\lambda > 0$, $\Box_\varepsilon \eta = \lambda\eta$ with $\eta \in L^2$ implies $\eta = 0$. So, let $\lambda > 0$ and $\eta \in L^2$ such that $\Box_\varepsilon \eta = \lambda\eta$. We have then

$$\begin{aligned} &\lambda \int_{\mathbb{M}} h_n^2 \|\eta\|^2_{2\varepsilon} \\ &\quad = \int_{\mathbb{M}} \langle h_n^2 \eta, \Box_\varepsilon \eta\rangle_{2\varepsilon} \end{aligned}$$

$$
\begin{aligned}
&= -\int_{\mathbb{M}} \langle \nabla_{\mathcal{H}}(h_n^2\eta) - \mathfrak{T}_{\mathcal{H}}^{\varepsilon}(h_n^2\eta), \nabla_{\mathcal{H}}\eta - \mathfrak{T}_{\mathcal{H}}^{\varepsilon}\eta\rangle_{2\varepsilon} \\
&\quad + \int_{\mathbb{M}} h_n^2 \left\langle \left(\frac{1}{2\varepsilon} J^* J - \mathfrak{Ric}_{\mathcal{H}}\right)(\eta), \eta \right\rangle_{2\varepsilon} \\
&= -\int_{\mathbb{M}} h_n^2 \|\nabla_{\mathcal{H}}\eta - \mathfrak{T}_{\mathcal{H}}^{\varepsilon}\eta\|_{2\varepsilon}^2 - 2\int_{\mathbb{M}} h_n \langle \eta, \nabla_{\nabla_{\mathcal{H}} h_n}\eta\rangle_{2\varepsilon} \\
&\quad + \int_{\mathbb{M}} h_n^2 \left\langle \left(\frac{1}{2\varepsilon} J^* J - \mathfrak{Ric}_{\mathcal{H}}\right)(\eta), \eta \right\rangle_{2\varepsilon}.
\end{aligned}
$$

From our assumptions, the symmetric tensor $\frac{1}{2\varepsilon} J^* J - \mathfrak{Ric}_{\mathcal{H}}$ is bounded from above, thus by choosing λ big enough, we have

$$
\int_{\mathbb{M}} h_n^2 \|\nabla\eta - \mathfrak{T}^{\varepsilon}\eta\|_{2\varepsilon}^2 + 2\int_{\mathbb{M}} h_n \langle \eta, \nabla_{\nabla_{\mathcal{H}} h_n}\eta\rangle_{2\varepsilon} \le 0.
$$

By letting $n \to \infty$, we easily deduce that $\|\nabla_{\mathcal{H}}\eta - \mathfrak{T}_{\mathcal{H}}^{\varepsilon}\eta\|_{2\varepsilon}^2 = 0$ which implies $\nabla_{\mathcal{H}}\eta - \mathfrak{T}_{\mathcal{H}}^{\varepsilon}\eta = 0$. If we come back to the equation $\square_{\varepsilon}\eta = \lambda\eta$ and the expression of $\square_{\varepsilon}$, we see that it implies that:

$$
\left(\frac{1}{2\varepsilon} J^* J - \mathfrak{Ric}_{\mathcal{H}}\right)(\eta) = \lambda\eta.
$$

Our choice of λ forces then $\eta = 0$. $\square$

Since $\frac{1}{2}\square_{\varepsilon}$ is essentially self-adjoint, it admits a unique self-adjoint extension which generates through the spectral theorem a semigroup $Q_t^{\varepsilon} = e^{(1/2)t\square_{\varepsilon}}$. As already mentioned, we will denote by $P_t = e^{(1/2)tL}$ the semigroup generated by $\frac{1}{2}L$. We have the following commutation property.

LEMMA 4.4. *If $f \in C_0^{\infty}(\mathbb{M})$, then for every $t \ge 0$,*

$$
dP_t f = Q_t^{\varepsilon} df.
$$

PROOF. Let $\eta_t = Q_t^{\varepsilon} df$. It is the unique solution in L^2 of the heat equation

$$
\frac{\partial\eta}{\partial t} = \frac{1}{2}\square_{\varepsilon}\eta,
$$

with initial condition $\eta_0 = df$. From [7], we have that $\alpha_t = dP_t f$ is in L^2, and from the fact that

$$
dL = \square_{\varepsilon} d,
$$

we see that α solves the heat equation

$$
\frac{\partial\alpha}{\partial t} = \frac{1}{2}\square_{\varepsilon}\alpha
$$

with the same initial condition $\alpha_0 = df$. We conclude thus $\alpha = \eta$. $\square$

REMARK 4.5. As a consequence, we point out that though Q_t^ε depends on ε, the previous lemma implies that the operator $Q_t^\varepsilon d$ does not depend on ε.

4.2. *Stochastic representation of the semigroup on one-forms.* We now turn to the stochastic representation of Q_t^ε. We denote by $(X_t)_{t\geq 0}$ the symmetric diffusion process generated by $\frac{1}{2}L$. Since P_t is stochastically complete, $(X_t)_{t\geq 0}$ has an infinite lifetime.

Consider the process $\tau_t^\varepsilon : T^*_{X_t}\mathbb{M} \to T^*_{X_0}\mathbb{M}$ to be the solution of the following covariant Stratonovich stochastic differential equation:

$$d[\tau_t^\varepsilon \alpha(X_t)] = \tau_t^\varepsilon \left(\nabla_{\circ dX_t} - \mathfrak{T}^\varepsilon_{\circ dX_t} + \frac{1}{2}\left(\frac{1}{2\varepsilon}J^*J - \mathfrak{Ric}_{\mathcal{H}}\right) dt\right)\alpha(X_t), \tag{4.1}$$
$$\tau_0^\varepsilon = \mathbf{Id},$$

where α is any smooth one-form. We have the following key estimate.

LEMMA 4.6. *For every $t \geq 0$, we have almost surely*

$$\|\tau_t^\varepsilon \alpha(X_t)\|_{2\varepsilon} \leq e^{1/2(K+\kappa/(2\varepsilon))t}\|\alpha(X_0)\|_{2\varepsilon}.$$

PROOF. The estimate stems from the fact that $\mathfrak{T}^\varepsilon$ is skew-symmetric for the Riemannian metric $g_{2\varepsilon}$, which implies that the connection $\nabla - \mathfrak{T}^\varepsilon$ is metric. The deterministic upper bound on τ^ε is therefore a consequence of the pointwise lower bound on $\mathfrak{Ric}_{\mathcal{H}} - \frac{1}{2\varepsilon}J^*J$ and Gronwall's lemma.

More precisely, consider the process $\Theta_t^\varepsilon : T^*_{X_t}\mathbb{M} \to T^*_{X_0}\mathbb{M}$ to be the solution of the following covariant Stratonovich stochastic differential equation:

$$d[\Theta_t^\varepsilon \alpha(X_t)] = \Theta_t^\varepsilon(\nabla_{\circ dX_t} - \mathfrak{T}^\varepsilon_{\circ dX_t})\alpha(X_t), \qquad \tau_0^\varepsilon = \mathbf{Id}, \tag{4.2}$$

where α is any smooth one-form. Since $\mathfrak{T}^\varepsilon$ is skew-symmetric, Θ_t^ε is an isometry for the Riemannian metric $g_{2\varepsilon}$. Consider now the multiplicative functional $(\mathcal{M}_t^\varepsilon)_{t\geq 0}$, solution of the equation

$$\frac{d\mathcal{M}_t^\varepsilon}{dt} = \frac{1}{2}\mathcal{M}_t\Theta_t^\varepsilon\left(\frac{1}{2\varepsilon}J^*J - \mathfrak{Ric}_{\mathcal{H}}\right)(\Theta_t^\varepsilon)^{-1}, \qquad \mathcal{M}_0^\varepsilon = \mathbf{Id}.$$

With the previous notation, we of course have $\tau_t^\varepsilon = \mathcal{M}_t^\varepsilon\Theta_t^\varepsilon$. Thus, the upper bound on τ^ε boils down to an upper bound on $\mathcal{M}^\varepsilon$ which is obtained as a consequence of Gronwall's inequality. $\square$

THEOREM 4.7. *Let η be a smooth and compactly supported one-form. Then for every $t \geq 0$, and $x \in \mathbb{M}$,*

$$(Q_t^\varepsilon \eta)(x) = \mathbb{E}_x(\tau_t^\varepsilon \eta(X_t)).$$

PROOF. It is basically a consequence of the definition of τ_ε and Itô's formula which implies that for every $t \geq 0$ the process

$$N_s = \tau_s^\varepsilon \big(Q_{t-s}^\varepsilon \eta\big)(X_s),$$

is a martingale. □

Combining Lemma 4.4 with Theorem 4.7, we get therefore the following representation for the derivative of the semigroup.

COROLLARY 4.8. *Let $f \in C_0^\infty(\mathbb{M})$. Then for every $t \geq 0$, and $x \in \mathbb{M}$,*

$$dP_t f(x) = \mathbb{E}_x\big(\tau_t^\varepsilon \, df(X_t)\big).$$

This eventually leads to a neat gradient bound for the semigroup P_t.

COROLLARY 4.9. *For every $f \in C_0^\infty(\mathbb{M})$, $\varepsilon > 0$, $t \geq 0$,*

$$\sqrt{\|\nabla_\mathcal{H} P_t f\|_\mathcal{H}^2 + 2\varepsilon \|\nabla_\mathcal{V} P_t f\|_\mathcal{V}^2} \leq e^{1/2(K+\kappa/(2\varepsilon))t} P_t\Big(\sqrt{\|\nabla_\mathcal{H} f\|_\mathcal{H}^2 + 2\varepsilon\|\nabla_\mathcal{V} f\|_\mathcal{V}^2}\Big).$$

We remark that this gradient bound is new in our framework and is stronger than similar gradient bounds in [5]. It also immediately implies that Hypothesis 1.4 of [7] is satisfied on Yang–Mills sub-Riemannian manifolds with transverse symmetries.

4.3. *Integration by parts formula.* As before, we denote by $(X_t)_{t\geq 0}$ the L-diffusion process. The stochastic parallel transport for the connection ∇ along the paths of $(X_t)_{t\geq 0}$ will be denoted by $/\!/_{0,t}$. Since the connection ∇ is horizontal, the map $/\!/_{0,t} : T_{X_0}\mathbb{M} \to T_{X_t}\mathbb{M}$ is an isometry that preserves the horizontal bundle, that is, if $u \in \mathcal{H}_{X_0}$, then $/\!/_{0,t} u \in \mathcal{H}_{X_t}$. We see then that the antidevelopment of $(X_t)_{t\geq 0}$,

$$B_t = \int_0^t /\!/_{0,s}^{-1} \circ dX_s,$$

is a Brownian motion in the horizontal space $\mathcal{H}_{X_0}$. The following integration by parts formula will play an important role in the sequel.

PROPOSITION 4.10. *Let $x \in \mathbb{M}$. For any C^1 adapted process $\gamma : \mathbb{R}_{\geq 0} \to \mathcal{H}_x$ such that $\mathbb{E}_x(\int_0^{+\infty} \|\gamma'(s)\|_\mathcal{H}^2 \, ds) < +\infty$ and any $f \in C_0^\infty(\mathbb{M})$, $t \geq 0$,*

$$\mathbb{E}_x\bigg(f(X_t)\int_0^t \langle \gamma'(s), dB_s\rangle_\mathcal{H}\bigg) = \mathbb{E}_x\bigg(\Big\langle \tau_t^\varepsilon df(X_t), \int_0^t \big(\tau_s^{\varepsilon,*}\big)^{-1} /\!/_{0,s}\gamma'(s)\, ds\Big\rangle_{2\varepsilon}\bigg).$$

PROOF. We fix $t \geq 0$ and denote

$$N_s = \tau_s^\varepsilon (dP_{t-s} f)(X_s).$$

It is a martingale process. We have then for $f \in C_0^\infty(\mathbb{M})$,

$$\begin{aligned}
&\mathbb{E}_x\left(f(X_t) \int_0^t \langle \gamma'(s), dB_s \rangle_{\mathcal{H}} \right) \\
&\quad = \mathbb{E}_x\left(f(X_t) \int_0^t \langle /\!/_{0,s} \gamma'(s), /\!/_{0,s} \, dB_s \rangle_{\mathcal{H}} \right) \\
&\quad = \mathbb{E}_x\left(\big(f(X_t) - \mathbb{E}_x\big(f(X_t)\big)\big) \int_0^t \langle /\!/_{0,s} \gamma'(s), /\!/_{0,s} \, dB_s \rangle_{\mathcal{H}} \right) \\
&\quad = \mathbb{E}_x\left(\int_0^t \langle dP_{t-s} f(X_s), /\!/_{0,s} \, dB_s \rangle_{\mathcal{H}} \int_0^t \langle /\!/_{0,s} \gamma'(s), /\!/_{0,s} \, dB_s \rangle_{\mathcal{H}} \right) \\
&\quad = \mathbb{E}_x\left(\int_0^t \langle dP_{t-s} f(X_s), /\!/_{0,s} \gamma'(s) \rangle_{\mathcal{H}} \, ds \right) \\
&\quad = \mathbb{E}_x\left(\int_0^t \langle \tau_s^\varepsilon dP_{t-s} f(X_s), \big(\tau_s^{\varepsilon,*}\big)^{-1} /\!/_{0,s} \gamma'(s) \rangle_{2\varepsilon} \, ds \right) \\
&\quad = \mathbb{E}_x\left(\int_0^t \langle N_s, \big(\tau_s^{\varepsilon,*}\big)^{-1} /\!/_{0,s} \gamma'(s) \rangle_{2\varepsilon} \, ds \right) \\
&\quad = \mathbb{E}_x\left(\left\langle N_t, \int_0^t \big(\tau_s^{\varepsilon,*}\big)^{-1} /\!/_{0,s} \gamma'(s) \, ds \right\rangle_{2\varepsilon} \right),
\end{aligned}$$

where we integrated by parts in the last equality. □

Let us observe that we can reinterpret the integration by parts formula of Proposition 4.10 in a slightly different way.

COROLLARY 4.11. *Let $x \in \mathbb{M}$. For any C^1 adapted process $\gamma : \mathbb{R}_{\geq 0} \to \mathcal{H}_x$ such that $\mathbb{E}_x(\int_0^{+\infty} \|\gamma'(s)\|_{\mathcal{H}}^2 \, ds) < +\infty$ and any $f \in C_0^\infty(\mathbb{M})$, $t \geq 0$,*

$$\mathbb{E}_x\big(\langle df(X_t), /\!/_{0,t} v(t) \rangle_{2\varepsilon}\big) = \mathbb{E}_x\left(f(X_t) \int_0^t \langle \gamma'(s), dB_s \rangle_{\mathcal{H}} \right),$$

where v is the solution of the Stratonovitch stochastic differential equation in $T_x\mathbb{M}$:

$$\begin{cases} dv(t) = /\!/_{0,t}^{-1} \left(\mathfrak{T}^\varepsilon_{\circ dX_t} + \dfrac{1}{2} \left(\dfrac{1}{2\varepsilon} J^* J - \mathfrak{Ric}_{\mathcal{H}} \right) dt \right) /\!/_{0,t} v(t) + \gamma'(t) \, dt, \\ v(0) = 0. \end{cases}$$

PROOF. It is a consequence of Itô's formula that

$$v(t) = /\!/_{0,t}^{-1} \tau_t^{\varepsilon,*} \int_0^t \big(\tau_s^{\varepsilon,*}\big)^{-1} /\!/_{0,s} \gamma'(s) \, ds$$

is the solution of the above stochastic differential equation. We conclude then with Proposition 4.10. □

As an immediate consequence of the integration by parts formula, we obtain the following Clark–Ocone type representation.

PROPOSITION 4.12. *Let $X_0 = x \in \mathbb{M}$. For every $f \in C_0^\infty(\mathbb{M})$, and every $t \geq 0$,*

$$f(X_t) = P_t f(x) + \int_0^t \langle \mathbb{E}_x\big((\tau_s^\varepsilon)^{-1}\tau_t^\varepsilon\, df(X_t)|\mathcal{F}_s\big), /\!\!/_{0,s}\, dB_s\rangle_{\mathcal{H}},$$

where $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration of $(B_t)_{t\geq 0}$.

PROOF. Let $t \geq 0$. From Itô's integral representation theorem, we can write

$$f(X_t) = P_t f(x) + \int_0^t \langle a_s, dB_s\rangle_{\mathcal{H}},$$

for some adapted and square integrable $(a_s)_{0\leq s\leq t}$. Using the Proposition 4.10, we obtain therefore,

$$\mathbb{E}_x\left(\int_0^t \langle \gamma'(s), a_s\rangle_{\mathcal{H}}\, ds\right) = \mathbb{E}_x\left(\left\langle \tau_t^\varepsilon\, df(X_t), \int_0^t (\tau_s^{\varepsilon,*})^{-1} /\!\!/_{0,s}\gamma'(s)\, ds\right\rangle_{2\varepsilon}\right).$$

Since γ' is arbitrary, we obtain that

$$a_s = \mathbb{E}_x\big(/\!\!/_{0,s}^{-1}(\tau_s^\varepsilon)^{-1}\tau_t^\varepsilon\, df(X_t)|\mathcal{F}_s\big). \qquad \square$$

We deduce first the following Poincaré inequality for the heat kernel measure.

PROPOSITION 4.13. *For every $f \in C_0^\infty(\mathbb{M})$, $t \geq 0$, $x \in \mathbb{M}$, $\varepsilon > 0$,*

$$P_t(f^2)(x) - (P_t f)^2(x) \leq \frac{e^{(K+\kappa/(2\varepsilon))t} - 1}{K + \kappa/(2\varepsilon)}\big[P_t(\|\nabla_{\mathcal{H}} f\|^2)(x) + 2\varepsilon P_t(\|\nabla_{\mathcal{V}} f\|^2)(x)\big].$$

PROOF. From the previous proposition and Lemma 4.6, we have

$$\mathbb{E}_x\big((f(X_t) - P_t f(x))^2\big) \leq \int_0^t e^{(K+\kappa/(2\varepsilon))(t-s)}\, ds\, P_t(\|df\|_{2\varepsilon}^2)(x). \qquad \square$$

We also get the log-Sobolev inequality for the heat kernel measure.

PROPOSITION 4.14. *For every $f \in C_0^\infty(\mathbb{M})$, $t \geq 0$, $x \in \mathbb{M}$, $\varepsilon > 0$,*

$$\begin{aligned} &P_t(f^2 \ln f^2)(x) - P_t(f^2)(x)\ln P_t(f^2)(x) \\ &\quad \leq 2\frac{e^{(K+\kappa/(2\varepsilon))t} - 1}{K + \kappa/(2\varepsilon)}\big[P_t(\|\nabla_{\mathcal{H}} f\|^2)(x) + 2\varepsilon P_t(\|\nabla_{\mathcal{V}} f\|^2)(x)\big]. \end{aligned}$$

PROOF. The method for proving the log-Sobolev inequality from a representation theorem like Proposition 4.12 is due to [11] and the argument is easy to reproduce in our setting. Denote $G = f(X_t)^2$ and consider the martingale $N_s = \mathbb{E}(G|\mathcal{F}_s)$. Applying now Itô's formula to $N_s \ln N_s$ and taking expectation yields

$$\mathbb{E}_x(N_t \ln N_t) - \mathbb{E}_x(N_0 \ln N_0) = \frac{1}{2}\mathbb{E}_x\left(\int_0^t \frac{d[N]_s}{N_s}\right),$$

where $[N]$ is the quadratic variation of N. From Proposition 4.12 applied with f^2, we have

$$dN_s = 2\langle \mathbb{E}\big(f(X_t)\big(\tau_s^\varepsilon\big)^{-1}\tau_t^\varepsilon\, df(X_t)|\mathcal{F}_s\big), /\!/_{0,s}\, dB_s\rangle_{\mathcal{H}}.$$

Thus, we have from Cauchy–Schwarz inequality

$$\begin{aligned}\mathbb{E}_x(N_t \ln N_t) - \mathbb{E}_x(N_0 \ln N_0) &\le 2\mathbb{E}_x\left(\int_0^t \frac{\|\mathbb{E}(f(X_t)(\tau_s^\varepsilon)^{-1}\tau_t^\varepsilon\, df(X_t)|\mathcal{F}_s)\|_{2\varepsilon}^2}{N_s}\, ds\right)\\ &\le 2\int_0^t e^{(K+\kappa/(2\varepsilon))(t-s)}\, ds\, P_t\big(\|df\|_{2\varepsilon}^2\big)(x). \qquad \square\end{aligned}$$

4.4. *Positive curvature and convergence to equilibrium.* In this final section, we prove that if the tensor $\mathfrak{Ric}_{\mathcal{H}}$ is bounded from below by a positive constant on the horizontal bundle, then by exploiting a further geometric quantity we can prove convergence of the semigroup when $t \to +\infty$ and get sharp quantitative estimates in the form of a Poincaré and a log-Sobolev inequality with an exponential decay for the heat kernel measure.

So, we assume throughout the section that for every horizontal one-form η,

$$\langle \mathfrak{Ric}_{\mathcal{H}}(\eta), \eta\rangle_{\mathcal{H}} \ge \rho_1 \|\eta\|_{\mathcal{H}}^2, \qquad \langle J^* J\eta, \eta\rangle_{\mathcal{H}} \le \kappa \|\eta\|_{\mathcal{H}}^2,$$

and that for every vertical one-form η, and any horizontal coframe $\{\theta_1, \dots, \theta_d\}$,

$$\frac{1}{4}\sum_{\ell,j=1}^d \langle T(\theta_\ell, \theta_j), \eta\rangle_{\mathcal{V}}^2 \ge \rho_2 \|\eta\|_{\mathcal{V}}^2,$$

where $\rho_1, \rho_2 > 0$ and $\kappa \ge 0$. As proved in Section 3, this implies that for every one-form η,

$$\frac{1}{2}L\|\eta\|_\varepsilon^2 - \langle \square_\varepsilon \eta, \eta\rangle_\varepsilon \ge \left(\rho_1 - \frac{\kappa}{\varepsilon}\right)\|\eta_{\mathcal{H}}\|_{\mathcal{H}}^2 + \rho_2 \|\eta_{\mathcal{V}}\|_{\mathcal{V}}^2.$$

This implies of course

$$\frac{1}{2}L\|\eta\|_\varepsilon^2 - \langle \square_\varepsilon \eta, \eta\rangle_\varepsilon \ge \inf\left(\rho_1 - \frac{\kappa}{\varepsilon}, \frac{\rho_2}{\varepsilon}\right)\|\eta\|_\varepsilon^2.$$

The constant $\inf(\rho_1 - \frac{\kappa}{\varepsilon}, \frac{\rho_2}{\varepsilon})$ is maximal when $\rho_1 - \frac{\kappa}{\varepsilon} = \frac{\rho_2}{\varepsilon}$, that is $\varepsilon = \frac{\kappa+\rho_2}{\rho_1}$. For this choice of ε, we have then

$$\inf\left(\rho_1 - \frac{\kappa}{\varepsilon}, \frac{\rho_2}{\varepsilon}\right) = \frac{\rho_1\rho_2}{\kappa+\rho_2}.$$

We have then following estimate which is obtained by analyzing the Itô–Stratonovitch correction term in the stochastic differential equation (4.2).

LEMMA 4.15. *Let $\varepsilon = \frac{\kappa+\rho_2}{\rho_1}$. For every $t \geq 0$,*

$$\mathbb{E}\left(\|\tau_t^\varepsilon \alpha(X_t)\|_\varepsilon^2\right) \leq e^{-((\rho_1\rho_2)/(\kappa+\rho_2))t}\mathbb{E}\left(\|\alpha(X_0)\|_\varepsilon^2\right).$$

Arguing then as before, we obtain the following Bakry–Émery, Poincaré and log-Sobolev inequalities.

PROPOSITION 4.16. *For every $f \in C_0^\infty(\mathbb{M})$, $t \geq 0$, $x \in \mathbb{M}$,*

$$\begin{aligned}
&\|\nabla_{\mathcal{H}} P_t f\|^2 + \frac{\kappa+\rho_2}{\rho_1}\|\nabla_{\mathcal{V}} P_t f\|^2 \\
&\quad\leq e^{-((\rho_1\rho_2)/(\kappa+\rho_2))t}\left[P_t\left(\|\nabla_{\mathcal{H}} f\|^2\right)(x) + \frac{\kappa+\rho_2}{\rho_1}P_t\left(\|\nabla_{\mathcal{V}} f\|^2\right)(x)\right], \\
&P_t\left(f^2\right)(x) - (P_t f)^2(x) \\
&\quad\leq \frac{\kappa+\rho_2}{\rho_1\rho_2}\left(1 - e^{-((\rho_1\rho_2)/(\kappa+\rho_2))t}\right) \\
&\qquad\times\left[P_t\left(\|\nabla_{\mathcal{H}} f\|^2\right)(x) + \frac{\kappa+\rho_2}{\rho_1}P_t\left(\|\nabla_{\mathcal{V}} f\|^2\right)(x)\right]
\end{aligned}$$

and

$$\begin{aligned}
&P_t\left(f^2 \ln f^2\right)(x) - P_t\left(f^2\right)(x)\ln P_t\left(f^2\right)(x) \\
&\quad\leq 2\frac{\kappa+\rho_2}{\rho_1\rho_2}\left(1 - e^{-((\rho_1\rho_2)/(\kappa+\rho_2))t}\right) \\
&\qquad\times\left[P_t\left(\|\nabla_{\mathcal{H}} f\|^2\right)(x) + \frac{\kappa+\rho_2}{\rho_1}P_t\left(\|\nabla_{\mathcal{V}} f\|^2\right)(x)\right].
\end{aligned}$$

The first of the above inequality was already proved in [5] by completely different methods and implies $\mu(\mathbb{M}) < +\infty$ and also that when $t \to +\infty$, in L^2, $P_t f \to \frac{1}{\mu(\mathbb{M})}$. It is worth pointing out that in the present framework, the two above Poincaré and log-Sobolev inequalities are new but, by taking the limit when $t \to \infty$ we get the two inequalities

$$\int_{\mathbb{M}} f^2\,d\mu - \left(\int_{\mathbb{M}} f\,d\mu\right)^2 \leq \frac{\kappa+\rho_2}{\rho_1\rho_2}\left[\int_{\mathbb{M}} \|\nabla_{\mathcal{H}} f\|^2\,d\mu + \frac{\kappa+\rho_2}{\rho_1}\int_{\mathbb{M}} \|\nabla_{\mathcal{V}} f\|^2\,d\mu\right]$$

and

$$\int_{\mathbb{M}} f^2 \ln f^2 \, d\mu - \int_{\mathbb{M}} f^2 \, d\mu \ln \int_{\mathbb{M}} f^2 \, d\mu$$
$$\leq \frac{2(\kappa + \rho_2)}{\rho_1 \rho_2} \left[\int_{\mathbb{M}} \|\nabla_{\mathcal{H}} f\|^2 \, d\mu + \frac{\kappa + \rho_2}{\rho_1} \int_{\mathbb{M}} \|\nabla_{\mathcal{V}} f\|^2 \, d\mu \right],$$

which were also already proved in [5] with the very same constants.

REFERENCES

[1] AGRACHEV, A. A. and SACHKOV, Y. L. (2004). *Control Theory from the Geometric Viewpoint. Control Theory and Optimization, II. Encyclopaedia of Mathematical Sciences* **87**. Springer, Berlin. MR2062547

[2] BAKRY, D., BAUDOIN, F., BONNEFONT, M. and CHAFAÏ, D. (2008). On gradient bounds for the heat kernel on the Heisenberg group. *J. Funct. Anal.* **255** 1905–1938. MR2462581

[3] BAUDOIN, F. (2004). *An Introduction to the Geometry of Stochastic Flows*. Imperial College Press, London. MR2154760

[4] BAUDOIN, F. (2013). Bakry–Émery meet Villani. Available at arXiv:1308.4938.

[5] BAUDOIN, F. and BONNEFONT, M. (2012). Log-Sobolev inequalities for subelliptic operators satisfying a generalized curvature dimension inequality. *J. Funct. Anal.* **262** 2646–2676. MR2885961

[6] BAUDOIN, F., BONNEFONT, M. and GAROFALO, N. (2014). A sub-Riemannian curvature-dimension inequality, volume doubling property and the Poincaré inequality. *Math. Ann.* **358** 833–860. MR3175142

[7] BAUDOIN, F. and GAROFALO, N. Curvature-dimension inequalities and Ricci lower bounds for sub-Riemannian manifolds with transverse symmetries. Preprint. Available at arXiv:1101.3590.

[8] BELLAÏCHE, A. and RISLER, J.-J. (1996). *Sub-Riemannian Geometry. Progress in Mathematics* **144**. Birkhäuser, Basel. MR1421821

[9] BEN AROUS, G. (1989). Développement asymptotique du noyau de la chaleur hypoelliptique sur la diagonale. *Ann. Inst. Fourier* (*Grenoble*) **39** 73–99. MR1011978

[10] BISMUT, J.-M. (1984). *Large Deviations and the Malliavin Calculus. Progress in Mathematics* **45**. Birkhäuser, Boston, MA. MR0755001

[11] CAPITAINE, M., HSU, E. P. and LEDOUX, M. (1997). Martingale representation and a simple proof of logarithmic Sobolev inequalities on path spaces. *Electron. Commun. Probab.* **2** 71–81 (electronic). MR1484557

[12] CARTAN, E. (1928). Sur la représentation géométrique des systèmes matériels non holonomes. *Proc. Internat. Congress Math.* **4** 253–261.

[13] DRIVER, B. K. (1992). A Cameron–Martin type quasi-invariance theorem for Brownian motion on a compact Riemannian manifold. *J. Funct. Anal.* **110** 272–376. MR1194990

[14] DRIVER, B. K. and MELCHER, T. (2005). Hypoelliptic heat kernel inequalities on the Heisenberg group. *J. Funct. Anal.* **221** 340–365. MR2124868

[15] DRIVER, B. K. and THALMAIER, A. (2001). Heat equation derivative formulas for vector bundles. *J. Funct. Anal.* **183** 42–108. MR1837533

[16] ELWORTHY, D. (1988). Geometric aspects of diffusions on manifolds. In *École D'Été de Probabilités de Saint-Flour XV–XVII*, 1985–1987. *Lecture Notes in Math.* **1362** 277–425. Springer, Berlin. MR0983375

[17] ELWORTHY, K. D. (1982). *Stochastic Differential Equations on Manifolds. London Mathematical Society Lecture Note Series* **70**. Cambridge Univ. Press, Cambridge. MR0675100

[18] ELWORTHY, K. D. (2014). Decompositions of diffusion operators and related couplings. Preprint.

[19] ELWORTHY, K. D. and KENDALL, W. S. (1986). Factorization of harmonic maps and Brownian motions. In *From Local Times to Global Geometry*, *Control and Physics* (*Coventry*, 1984/85). *Pitman Res*. *Notes Math*. *Ser.* **150** 75–83. Longman, Harlow. MR0894524

[20] ELWORTHY, K. D., LE JAN, Y. and LI, X.-M. (2010). *The Geometry of Filtering*. *Frontiers in Mathematics*. Birkhäuser, Basel. MR2731662

[21] GROMOV, M. (1999). *Metric Structures for Riemannian and Non-Riemannian Spaces*. *Progress in Mathematics* **152**. Birkhäuser, Boston, MA. MR1699320

[22] HSU, E. P. (2002). *Stochastic Analysis on Manifolds*. *Graduate Studies in Mathematics* **38**. Amer. Math. Soc., Providence, RI. MR1882015

[23] KUSUOKA, S. and STROOCK, D. (1987). Applications of the Malliavin calculus. III. *J*. *Fac*. *Sci*. *Univ*. *Tokyo Sect*. *IA Math*. **34** 391–442. MR0914028

[24] LI, H.-Q. (2006). Estimation optimale du gradient du semi-groupe de la chaleur sur le groupe de Heisenberg. *J*. *Funct*. *Anal*. **236** 369–394. MR2240167

[25] LIAO, M. (1989). Factorization of diffusions on fibre bundles. *Trans*. *Amer. Math*. *Soc*. **311** 813–827. MR0929666

[26] MALLIAVIN, P. (1978). Stochastic calculus of variation and hypoelliptic operators. In *Proceedings of the International Symposium on Stochastic Differential Equations* (*Res*. *Inst*. *Math*. *Sci*., *Kyoto Univ*., *Kyoto*, 1976) 195–263. Wiley, New York. MR0536013

[27] MONTGOMERY, R. (2002). *A Tour of Subriemannian Geometries*, *Their Geodesics and Applications*. *Mathematical Surveys and Monographs* **91**. Amer. Math. Soc., Providence, RI. MR1867362

[28] RAYNER, C. B. (1967/1968). The exponential map for the Lagrange problem on differentiable manifolds. *Philos*. *Trans*. *R*. *Soc*. *Lond*. *Ser. A Math*. *Phys*. *Eng*. *Sci*. **262** 299–344. MR0247552

[29] ROTHSCHILD, L. P. and STEIN, E. M. (1976). Hypoelliptic differential operators and nilpotent groups. *Acta Math*. **137** 247–320. MR0436223

[30] RUMIN, M. (2000). Sub-Riemannian limit of the differential form spectrum of contact manifolds. *Geom*. *Funct*. *Anal*. **10** 407–452. MR1771424

[31] SÁNCHEZ-CALLE, A. (1984). Fundamental solutions and geometry of the sum of squares of vector fields. *Invent*. *Math*. **78** 143–160. MR0762360

[32] STRICHARTZ, R. S. (1983). Analysis of the Laplacian on the complete Riemannian manifold. *J*. *Funct*. *Anal*. **52** 48–79. MR0705991

[33] STRICHARTZ, R. S. (1986). Sub-Riemannian geometry. *J*. *Differential Geom*. **24** 221–263. MR0862049

[34] STROOCK, D. W. (2000). *An Introduction to the Analysis of Paths on a Riemannian Manifold*. *Mathematical Surveys and Monographs* **74**. Amer. Math. Soc., Providence, RI. MR1715265

[35] THALMAIER, A. (1998). Some remarks on the heat flow for functions and forms. *Electron*. *Commun*. *Probab*. **3** 43–49 (electronic). MR1637977

[36] VILLANI, C. (2009). Hypocoercivity. *Mem*. *Amer. Math*. *Soc*. **202** iv+141. MR2562709

[37] WANG, F.-Y. (2014). *Analysis for Diffusion Processes on Riemannian Manifolds*. *Advanced Series on Statistical Science & Applied Probability* **18**. World Scientific, Hackensack, NJ. MR3154951

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