

THE MEASURABLE KESTEN THEOREM

BY MIKLÓS ABÉRT¹, YAIR GLASNER² AND BÁLINT VIRÁG³

*Alfréd Rényi Institute of Mathematics, Ben-Gurion University of the Negev
and University of Toronto*

We give an explicit bound on the spectral radius in terms of the densities of short cycles in finite d -regular graphs. It follows that a finite d -regular Ramanujan graph G contains a negligible number of cycles of size less than $c \log \log |G|$.

We prove that infinite d -regular Ramanujan unimodular random graphs are trees. Through Benjamini–Schramm convergence this leads to the following rigidity result. If most eigenvalues of a d -regular finite graph G fall in the Alon–Boppana region, then the eigenvalue distribution of G is close to the spectral measure of the d -regular tree. In particular, G contains few short cycles.

In contrast, we show that d -regular unimodular random graphs with maximal growth are not necessarily trees.

1. Introduction. Let G be a d -regular, finite or infinite connected undirected graph. Let M be the Markov averaging operator on $\ell^2(G)$. When G is infinite, we define the *spectral radius* of G , denoted $\rho(G)$, to be the norm of M . When G is finite, we want to exclude the trivial eigenvalues, and thus define $\rho(G)$ to be the second largest element in the set of absolute values of eigenvalues of M . For an infinite graph G , we have $\rho(G) \geq \rho(T_d) = 2\sqrt{d-1}/d$ where T_d denotes the d -regular tree. For finite graphs, the Alon–Boppana theorem [21] says that $\liminf \rho(G_n) \geq \rho(T_d)$ for any infinite sequence (G_n) of finite connected d -regular graphs with $|G_n| \rightarrow \infty$.

We call G a *Ramanujan graph*, if $\rho(G) \leq \rho(T_d)$. Lubotzky, Philips and Sarnak [15], Margulis [17] and Morgenstern [20] have constructed sequences of d -regular Ramanujan graphs for $d = p^\alpha + 1$. Also, Friedman [9] showed that random d -regular graphs are close to being Ramanujan.

All the Ramanujan graph families above have *large girth*, that is, the minimal size of a cycle tends to infinity with the size of the graph. However, the reason for

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that is group theoretic and not spectral, and a priori, Ramanujan graphs could have many short cycles.

In this paper, we investigate the connection between the densities of short cycles, the spectral radius and the spectral measure for d -regular graphs. We apply our methods to give explicit estimate these invariants, then we pass to graph limits and prove limiting results.

1.1. *Explicit estimates.* A cycle (or k -cycle) in a graph is a walk of length k that starts and ends at the same vertex. It is called *nontrivial* if either for some directed non-loop edge e , the number of times the cycle passes through e differs from the number of times it passes through the reversal of e , or $k = 1$ (see Definition 23). For a finite graph G let $\gamma_k(G)$ denote the number of nontrivial k -cycles in G divided by the number of vertices $|G|$ of G .

THEOREM 1. *Let G be a finite d -regular graph with $|G| \geq d^7$. Then for any $k \geq 1$ we have*

$$\frac{\rho(G)}{\rho(\mathbb{T}_d)} \geq 1 + \frac{\gamma_k(G)}{v_k} - \frac{(3/2) \log \log_{d-1} |G| + 6}{\log_{d-1} |G|},$$

where $v_k = 2 \cdot 10^{11} 2^{4k} (d - 1)^{3k} k$.

Applying this to finite Ramanujan graphs yields that they have few cycles of length $o(\log \log |G|)$.

THEOREM 2. *Let $d \geq 3$ and $\beta = (30 \log(d - 1))^{-1}$. Then for any d -regular finite Ramanujan graph G , the proportion of vertices in G whose $\beta \log \log |G|$ -neighborhood is a d -regular tree is at least $1 - c(\log |G|)^{-\beta}$.*

This answers a question of Lubotzky ([14], Question 10.7.1), who asked for a clarification on the connection between eigenvalues and girth. Note that until now, it was not even known whether a finite Ramanujan graph cannot have a positive density of short cycles.

It is easy to see that infinite Ramanujan graphs can have arbitrarily many short cycles. In fact, every connected, infinite d -regular graph can be embedded as a subgraph of a Ramanujan graph with degree at most d^2 (see Corollary 35). However, it turns out that cycles of bounded size must be sparse in a Ramanujan graph.

THEOREM 3. *Let G be an infinite d -regular graph such that every vertex in G has distance at most R from a k -cycle. Then*

$$\rho(G) \geq \rho(T_d) + \frac{d - 2}{d(d - 1)^{2\lceil R+k/2+1 \rceil}}.$$

1.2. *Graph limits and spectral measure.* The spectral measure μ_{T_d} of the Markov operator on T_d , also known as the Plancherel measure of T_d or the Kesten–McKay measure, has density

$$\frac{d}{2\pi} \frac{\sqrt{\rho^2(T_d) - t^2}}{1 - t^2}.$$

Let (G_n) be a sequence of finite d -regular graphs. We say that (G_n) has *essentially large girth*, if for all k the density of nontrivial cycles satisfies

$$\lim_{n \rightarrow \infty} \gamma_k(G_n) = 0.$$

For a finite graph G , let μ_G denote the eigenvalue distribution of the Markov operator on G . Then the following are equivalent (see Proposition 14):

1. (G_n) has essentially large girth;
2. (G_n) converges to T_d in Benjamini–Schramm convergence;
3. μ_{G_n} weakly converges to μ_{T_d} .

A sequence (G_n) of finite d -regular graphs is *weakly Ramanujan* if

$$\lim_{n \rightarrow \infty} \mu_{G_n}([- \rho(T_d), \rho(T_d)]) = 1,$$

that is, if most eigenvalues of G_n fall in the minimal possible supporting region. Note that a weakly Ramanujan sequence is not necessarily an expander sequence. In fact, the graphs G_n do not even have to be connected.

From (1) \implies (3) and the fact that μ_{T_d} is continuous, it follows immediately that every graph sequence of essentially large girth is weakly Ramanujan (in contrast, ρ is only lower semicontinuous with respect to Benjamini–Schramm convergence of graphs). We show that the converse also holds.

THEOREM 4. *Let (G_n) be a weakly Ramanujan sequence of finite d -regular graphs. Then (G_n) has essentially large girth.*

Theorem 4 can also be looked at as a rigidity result, as it says that if we force most of the eigenvalues of the Markov operator of a large finite graph inside the Alon–Boppana bound, then their distribution will be close to μ_{T_d} .

In the proof of Theorem 4, it is the use of Benjamini–Schramm convergence that allows us to get rid of the bad eigenvalues and clear up the picture. Limit objects with respect to this convergence are random rooted graphs (G, o) called unimodular random graphs. We will sometimes drop the root o from the notation. The notion has been introduced in [2]: for the definition, see Section 2. Unimodular random graphs tend to behave like vertex transitive graphs in many senses. Theorem 4 now follows from the following.

THEOREM 5. *Let (G, o) be a d -regular unimodular random graph that is infinite and Ramanujan a.s. Then $G = T_d$ a.s.*

This is Kesten’s theorem for vertex transitive graphs ([12] and [23]). We give the following two quantitative versions of Theorem 5. For infinite d -regular unimodular random graphs,

$$(1) \quad \mathbf{E} \log \rho(G) - \log \rho(\mathbb{T}_d) \geq \begin{cases} \frac{1}{v_k} \mathbf{E} \gamma_k(G, o), \\ -\frac{1}{k} \mathbf{E} \log \kappa_k^*(G, o). \end{cases}$$

Here, $\gamma_k(G, o)$ denotes the number of nontrivial k -cycles starting at o , and v_k is a constant defined in Theorem 1. Note that for a fixed finite graph G the density $\gamma_k(G)$ equals the expected value of $\gamma_k(G, o)$ over a uniformly chosen root o of G .

To define $\kappa_k^*(G, o)$, consider all paths of length k from o to a vertex v . After attaching a fixed path from v to o , these can be used as generators for a random walk on the fundamental group of G . Then $\kappa_k^*(G, o)$ is the geometric average of the spectral radii of these random walks when v is chosen randomly as the position of the infinite nullcycle (defined in Corollary 20) at time k [see (19), (21) for more details].

Note that if our unimodular random graph G is not a tree, then for k large enough, with positive probability, the Cayley graph of the subgroup of the fundamental group given by above loops as generators has spectral radius less than one. Thus, the second bound clearly implies Theorem 5.

The first bound in (1) is proved in Section 5 as Theorem 28; the proof uses results from Sections 3 and 4. It is just the infinite version of Theorem 1. The advantage of this approach is the linear lower estimate on how the spectral radius grows compared to the tree: we believe this to be sharp. The major advantage of the second bound in (1) (proved in Section 6) is that it is sharp in limit as $k \rightarrow \infty$; however, κ_k^* seems to be hard to compute.

Theorem 4 is related to a paper of Serre [24] (see also [19]) that studies asymptotic properties of graph sequences. Let $d_k(G)$ denote the number of primitive, cyclically reduced cycles of length k in the graph G . Recall that a cycle is primitive if it is not a proper power of another cycle.

THEOREM 6 (Serre). *Let (G_n) be a sequence of finite d -regular graphs, such that the limit*

$$\gamma'_k = \lim_{n \rightarrow \infty} d_k(G_n)/|G_n|$$

exists for every k . Then the measures μ_G weakly converge. If the series

$$\sum_{k=1}^{\infty} \gamma'_k (d-1)^{-k/2}$$

converges, then the sequence of graphs is weakly Ramanujan and the limiting measure is absolutely continuous with respect to the Lebesgue measure on $[-\rho(T_d), \rho(T_d)]$.

Theorem 4 now immediately implies the following.

COROLLARY 7. *If the series*

$$\sum_{k=1}^{\infty} \gamma'_k (d - 1)^{-k/2}$$

converges, then $\gamma'_k = 0$ for all k and the limiting measure of μ_{G_n} equals μ_{T_d} .

It is natural to ask whether a version of Theorem 5 holds for growth instead of spectral radius. In Section 9, we show that the answer is negative.

THEOREM 8. *There exists an infinite d -regular unimodular random graph with the same growth as T_d but not equal to T_d .*

We obtain our example by considering the universal cover of the infinite cluster in supercritical percolation over \mathbb{Z}^2 .

1.3. *The basic method.* There is a common method in the proofs of Theorems 1 and 5 which we can summarize as follows.

The central tool of our analysis is a nullcycle. Recall that a cycle is a walk of finite length that starts and ends at the same vertex.

DEFINITION 9. A *nullcycle* is a cycle in a graph G so that if we keep deleting backtrackings (steps that are immediately reversed), we get a cycle of length 0.

This property does not depend on the order of erasing backtrackings. Equivalently, a nullcycle is a cycle whose lift in the universal covering tree of the graph is also a cycle. In other words, the walk corresponds to a trivial element in the fundamental group of the graph G . The number of nullcycles in a d -regular graph starting at a fixed vertex v equals the number of cycles in the d -regular tree at a fixed vertex.

To bound the spectral radius, we have to count cycles of a given length. In order to bound the spectral radius away from that of the tree, we need to show that there are exponentially more cycles than nullcycles. Consider the set of cycles of length nk starting at a vertex v in a d -regular graph G . We say that w' is a *rewiring* of w if they are at the same place at times that are multiples of k . This definition is used in Section 6; in Section 4.2 we use a slight variant of this.

Consider the equivalence class $[w]$ of a typical nullcycle w under the rewiring equivalence relation. The essence of our argument is to show that for a typical w , the probability that a random element C of $[w]$ is a nullcycle is exponentially small. Essentially, in every segment $[jk, (j + 1)k]$, if there are short cycles around in the graph, there is a positive probability that the rewiring C will use them, and this is likely to stop C from being null-homotopic.

In order to show that C is null-homotopic with exponentially small probability, we need to find a linear number of j so that G has short cycles around $w(jk)$. Fortunately, the random nullcycle w samples the graph G in a homogeneous manner. In particular, if $w(0)$ is a uniformly chosen vertex, then so will be $w(j)$ for every j . This is an advantage of using random nullcycles over random cycles. For infinite graphs, the proof of this step uses unimodularity.

A crucial property that we use to get explicit bounds is one that random nullcycles share with simple random walks. Let G be a d -regular rooted graph and let W be a uniform random nullcycle of length $o(\sqrt{|G|})$, starting at the root. Then the expected number of visits of W at any vertex of G can be bounded above in terms of $\rho(G)$ (without referring to the length of the cycle). In particular, for a good expander graph, the expected number of returns of a random nullcycle is bounded. We need this property to show that a typical rewiring will not use the same short cycles over and over again. This is a technically difficult point that we tackle in Section 4.

Putting all these together, we get that if there are many short cycles, then a typical nullcycle will get close to short cycles at linearly many different times. Thus, a random rewiring will be a nullcycle only with exponentially small probability. In other words, there are exponentially more cycles than nullcycles, which implies that the spectral radius of G is greater than that of the tree \mathbb{T}_d .

1.4. Open problems. It is not clear whether the $\log \log |G|$ is optimal in Theorem 2. For all the known examples of graphs that are close to being Ramanujan, the shortest cycles with positive density are actually logarithmic.

PROBLEM 10. *Is there a constant $c = c(d) > 0$ such that for any d -regular Ramanujan graph sequence (G_n) , the probability that the $c \log |G_n|$ -neighborhood of a uniform random vertex in G_n is a tree converges to 1?*

A standard ergodicity argument says that for an ergodic unimodular random graph G , the weak limit of the random walk neighborhood sampling of G gives back the distribution of G a.s. [6]. This suggests the following possible generalization of Theorem 5.

PROBLEM 11. *Let G be an infinite d -regular rooted Ramanujan graph and let $k > 0$. Let p_n denote the probability that the random walk of length n on G ends on a k -cycle. Is it true that p_n converges to 0?*

That is, is it true that the random walk neighborhood sampling of G converges to T_d ? The answer does not follow from Theorem 5, even when the random walk sampling converges, as the limit is only a stationary distribution on rooted graphs and is not necessarily unimodular. It would also be interesting to see whether Theorem 5 holds for stationary random graphs. The recent paper [11] solves Problem 11

affirmatively in the case when the so-called co-growth of G , the exponent of the probability of return for a non-backtracking random walk, is less than $1/\sqrt{d-1}$. However, when the co-growth equals $1/\sqrt{d-1}$, the graph is still Ramanujan but the answer seems unclear. We thank Tatiana Smirnova-Nagnibeda for communicating this with us. After the first preprint version of this paper appeared, Lyons and Peres, [16] generalized our results and in particular gave a positive answer to Problem 11.

The linear lower estimate in the spectral radius in Theorem 1 seems to be sharp, but we have not been able to settle this with a suitable family of examples. The same is true for unimodular random graphs [see the first bound of (1)].

PROBLEM 12. *Does there exist $C > 0$ such that for every $r > 0$ there exists an infinite d -regular unimodular random graph G with*

$$\rho(G) \leq \rho(T_d) + Cr$$

such that the density of loops in G is at least r ?

One natural idea would be to use a modified universal cover of a finite d -regular graph of size n with a loop, where we never open the loop in the cover. It looks reasonable that this cover (which is a finitely supported random rooted tree with loops) should have spectral radius around $\rho(T_d) + C/n$.

The paper is organized as follows. Section 2 contains the basic definitions and we prove some lemmas that will be used throughout the paper. In Sections 3 and 4, we use properties of cycles in trees to study nullcycles, which are needed for Theorem 1. In Section 5, we prove Theorem 28, a more general version of Theorems 1 and 5. We also show Theorem 29, a more general version of Theorem 2. Finally, in this section we also prove Theorem 4.

Section 6 contains a sharp bound on the spectral radius in terms of random walks on the fundamental group. Section 7 contains the proof of Theorem 3. This section is independent of the rest.

In Section 8, we give example of Ramanujan graphs with loops, and Section 9 we prove Theorem 8.

Note that one can read Sections 5 and 6 independently, after reading Section 2, but reading any of these two will give help when reading the other.

An earlier version of this paper contained a generalization of Kesten's theorem on groups. As the readership of this result is expected to be different from that of the current paper (and the current paper is already long), we decided to publish it in a separate article; see [1].

2. Preliminaries. In this section, we define the notions and state some basic results used in the paper.

We follow Serre's notation for a graph, with a modification on how to define loops. A graph G consists of two sets, a set of vertices denoted by $V(G)$ and a

set of edges denoted $E(G)$. For every edge $e \in E(G)$, there are vertices e^- (the initial vertex) and e^+ (the terminal vertex). We allow $e^- = e^+$: such edge is called a *loop*. For every edge e , there is a *reverse edge* $\bar{e} \in E(G)$ such that $\bar{e}^+ = e^-$ and $\bar{e}^- = e^+$. For a loop e , we allow $\bar{e} = e$; these are called *half-loops*. The degree of a vertex v is

$$\text{deg } v = |\{e \in E(G) | e^- = v\}|.$$

So half-loops contribute 1 to the degree, but loops together with their distinct inverse contribute 2. For spectral and random walk questions, each (non-half) loop can be replaced by two half-loops. So, in this paper, we will assume that all loops are half-loops.

A graph is d -regular, if all vertices have degree d .

A *walk* of length n is a sequence of directed edges $w = (w_1, w_2, \dots, w_n)$ such that $w_{i-1}^+ = w_i^-$ ($2 \leq i \leq n$). The walk is a *cycle* if $w_1^- = w_n^+$. The *vertices* of the walk are defined by $w(i-1) = w_i^-$ and $w(n) = w_n^+$ is the *end of the walk*. The *inverse* of a walk w is defined by $w^{-1} = (\bar{w}_n, \bar{w}_{n-1}, \dots, \bar{w}_1)$. A cycle is a *nullcycle* if its lift to the universal cover of G stays a cycle. That is the same as saying that if we keep erasing backtracks from the cycle, we get to the empty walk. For a rooted graph (G, o) , we will denote the set of nullcycles of length n by \mathcal{N}_n .

For a graph G and $x, y \in V(G)$, let $W_n(x, y)$ denote the set of walks of length n starting at x and ending at y . A *random walk of length n starting at x* is a uniform random walk starting at x . Let $p_n(x, y)$ denote the probability that a random walk of length n started at x ends at y . We call $p_n(x, x)$ the *n -step return probability*.

Let G be a d -regular, connected undirected graph. Let $\ell^2(G)$ be the Hilbert space of all square summable functions on the vertex set of G . Let us define the Markov operator $M : \ell^2 \rightarrow \ell^2$ as follows:

$$(Mf)(x) = \frac{1}{d} \sum_{e \in E(G), e^- = x} f(e^+).$$

When G is infinite, we define the *spectral radius* of G , denoted $\rho(G)$, to be the norm of M . When G is finite, we want to exclude the trivial eigenvalues, and thus define $\rho(G)$ to be the second largest element in the set of absolute values of eigenvalues of M . Note that when the connected graph G is bipartite, then $-d$ is an eigenvalue with multiplicity one; this is not counted in the definition of $\rho(G)$.

In the case when G is infinite and connected, one can express the spectral radius of G from the return probabilities as follows:

$$\rho(G) = \lim_{n \rightarrow \infty} p_{2n}(x, x)^{1/2n},$$

where x is an arbitrary vertex of G .

The Markov operator M is self-adjoint, so we can consider its spectral measure. This is a projection valued measure P such that $P(O) : \ell^2(G) \rightarrow \ell^2(G)$ is a projection for every Borel set $O \subset [-1, 1]$. For every $f \in \ell^2(G)$ with $\|f\|_2 = 1$, the

expression

$$\mu_f(O) = \langle P(O)f, f \rangle$$

defines a Borel probability measure on $[-1, 1]$.

For graph G rooted at o , let the *spectral measure of G* be

$$\mu_{G,o} = \mu_{\delta_o},$$

where $\delta_o \in l^2(G)$ is the indicator function of o . The best way to visualize this measure is to look at its moments, that satisfy the following equality:

$$\int_{[-1,1]} x^k d\mu_{G,o} = p_k(o, o)$$

for all integers $k \geq 0$.

Unimodular random graphs. Heuristically, a unimodular random graph is a probability distribution on rooted graphs that stays invariant under moving the root to any direction. However, one has to be careful with this intuition, as direction is not well-defined and indeed, there exist vertex transitive graphs that we want to exclude from the definition. We follow [3], Section 5.2, in our definition restricted to the d -regular case where it is somewhat simpler.

A *flagged graph* is a graph with a distinguished root and a directed edge starting at the root. One can *invert the flag* by moving the root to the other end of the flag and switching the direction of the flag.

DEFINITION 13. Let G be a probability distribution on rooted d -regular graphs. Pick a uniform random edge from the root and put a flag on it. This gives a probability distribution \tilde{G} on flagged d -regular graphs. We say that G is a *unimodular random graph*, if the distribution \tilde{G} stays invariant under inverting the flag.

That is, if some of the flagged lifts of a given rooted graph are isomorphic, we count it with multiplicity. Note that vertex transitivity in itself does not imply unimodularity. A simple example is the so-called grandmother graph. This can be obtained by taking a 3-regular tree and directing it towards a boundary point, then connecting every vertex to the ascendant of its ascendant (its grandmother) and then erasing directions (see Figure 1).

If one does not mind working with edge directed graphs, it is easier to see the lack of unimodularity in the oriented 3-regular tree itself. There is only one type of rooted graph here that obviously appears with probability 1. The corresponding measure on flagged graphs puts the flag on an outgoing edge with probability $1/3$, but after an inversion we see an outgoing edge with probability $2/3$. See [2] for more about unimodularity.

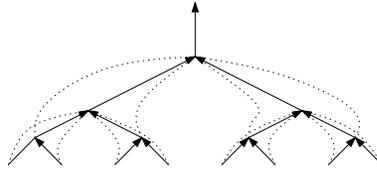


FIG. 1. *The grandmother graph.*

Mass transport principle. The most useful property about unimodular random graphs (that can also be used to define them) is the Mass Transport Principle which is as follows. Let f be a non-negative real-valued function on triples (G, x, y) where G is a d -regular rooted graph and $x, y \in G$ such that f does not depend on the location of the root. Then the expectations

$$\mathbf{E} \left[\sum_{y \in G} f(G, o, y) \right] = \mathbf{E} \left[\sum_{x \in G} f(G, x, o) \right],$$

where o is the root of G . The picture is that if one sets up a paying scheme on the random graph G that is invariant under moving the root, then the expected payout of the root equals its expected income.

Benjamini–Schramm convergence. A d -regular graph sequence (G_n) is defined as a sequence of finite d -regular graphs with size tending to infinity. By a *pattern of radius r* , we mean a rooted graph where every vertex has distance at most r from the root. For a finite graph G and a pattern α of radius r , let the sampling probability $p(G, \alpha)$ be the probability that the r -ball around a uniform random vertex of G is isomorphic to α . We say that a graph sequence (G_n) is Benjamini–Schramm convergent, if $p(G_n, \alpha)$ is convergent for every pattern α . It is easy to see that every graph sequence has a convergent subsequence.

What is a natural limit object of a convergent graph sequence? One can also take pattern densities of a unimodular random graph G ; there $p(G, \alpha)$ denotes the probability that the r -ball around the root of G is isomorphic to α . We say that a graph sequence (G_n) converges to G if

$$\lim_{n \rightarrow \infty} p(G_n, \alpha) = p(G, \alpha) \quad \text{for all patterns } \alpha.$$

Every Benjamini–Schramm convergent graph sequence has a unique limit unimodular random graph (see [3], Section 2.4).

For a finite d -regular graph G let μ_G denote the eigenvalue distribution of the Markov operator on G . Note that for a uniform random vertex o we have $\mu_G = \mathbf{E} \mu_{G,o}$. For an infinite unimodular random graph G , we can also define $\mu_G = \mathbf{E} \mu_{G,o}$.

PROPOSITION 14. *Let (G_n) be a sequence of finite d -regular graphs. Then the following are equivalent:*

- (1) (G_n) has essentially large girth;
- (2) (G_n) converges to T_d in Benjamini–Schramm convergence;
- (3) μ_{G_n} weakly converges to μ_{T_d} .

PROOF. The equivalence of (1) and (2) is immediate from the definition of Benjamini–Schramm convergence.

Assume that (G_n) converges to the unimodular random graph G . We claim that μ_{G_n} weakly converges to the expected spectral measure $\mu_G = \mathbf{E}\mu_{G,o}$. To check this, we can look at the k th moment

$$\int x^k d\mu_G = \mathbf{E}[p_k^G(o, o)].$$

Recall that $p_k^G(o, o)$ denotes the probability of return of the random walk on G starting at o . But for any graph G and vertex v of G , the return probability $p_k^G(v, v)$ only depends on the $k/2$ -ball around o . Since there are only finitely many patterns of a given radius, this implies

$$\mathbf{E}[p_k^G(o, o)] = \sum_{\alpha \text{ is a pattern of radius } \lfloor k/2 \rfloor} p(G, \alpha) p_k^\alpha(v, v),$$

where v is the root of α . Now (G_n) converges to G , so

$$\begin{aligned} \mathbf{E}[p_k^G(o, o)] &= \lim_{n \rightarrow \infty} \sum_{\alpha \text{ is a pattern of radius } \lfloor k/2 \rfloor} p(G_n, \alpha) p_k^\alpha(v, v) \\ &= \lim_{n \rightarrow \infty} \mathbf{E}[p_k^{G_n}(u, u)] = \lim_{n \rightarrow \infty} \int x^k d\mu_{G_n}, \end{aligned}$$

where u is a uniform random vertex in G_n . So, μ_{G_n} weakly converges to μ_G as claimed. Hence, (2) implies (3) follows immediately.

Assume that (1) does not hold, that is, (G_n) is a graph sequence that does not have essentially large girth. Then there exists $k, \varepsilon > 0$ such that the density of k -cycles in G_n is at least ε for infinitely many of the G_n . This implies that for these n ,

$$\int x^k d\mu_{G_n} = \mathbf{E}[p_k^{G_n}(u, u)] \geq p_k^{T_d}(o, o) + \frac{\varepsilon}{d^k} = \int x^k d\mu_{T_d} + \frac{\varepsilon}{d^k},$$

which implies that μ_{G_n} does not converge weakly to μ_{T_d} . Hence, (3) does not hold. We proved the required equivalences. \square

Fundamental group. Let G be a graph rooted at o . We call two cycle starting at o homotopic, if one can get one from the other by inserting and erasing backtracks, that is, walks of type $s\bar{s}$ where s is an edge of G . Then the set of equivalence classes forms a group under concatenation, called the *fundamental group* $\pi_1(G)$. It is well known that the fundamental group of a graph without half-loops is a free group [18], Theorem 5.1. Every half-loop adds a cyclic group of order 2 as a free product. The most important general property of fundamental groups we shall use in this paper is that if H is a subgraph of G , then the induced homomorphism from $\pi_1(H)$ to $\pi_1(G)$ is injective.

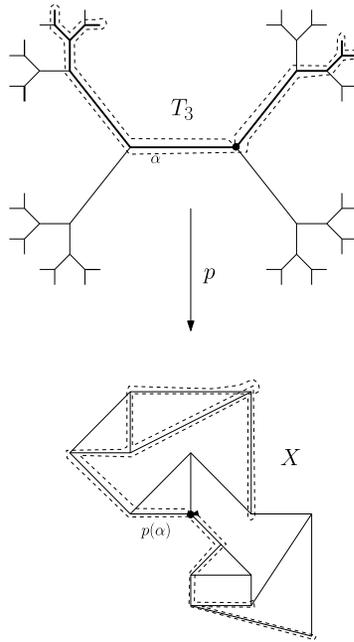


FIG. 2. A cycle in the 3-regular tree.

3. Cycles in \mathbb{T}_d . This section establishes some basic properties of $\mathcal{N}_n = \mathcal{N}_n(d)$, the set of n -cycles in \mathbb{T}_d . Such a cycle $\alpha \in \mathcal{N}_n$ in the 3-regular tree is depicted in Figure 2. Given any covering map $p : \mathbb{T}_3 \rightarrow X$ to a 3-regular graph X , the projection of the cycle $p(\alpha)$ is referred to as a nullcycle in the graph X .

3.1. *Explicit return probability bounds.* We start by estimating the size of \mathcal{N}_n .

LEMMA 15 (Return probabilities of SRW on \mathbb{T}_d). *Let $\rho = \rho(\mathbb{T}_d) = 2\sqrt{d-1}/d$. The n -step return probability $r_n = d^{-n}|\mathcal{N}_n(d)|$ for simple random walk in \mathbb{T}_d for even $n > 0$ satisfies*

$$\frac{2}{3} \frac{\rho^n}{n^{3/2}} < r_n < 10 \frac{\rho^n}{n^{3/2}}.$$

PROOF. Return probabilities are moments of the spectral measure. The spectral measure in \mathbb{T}_d is supported on $[-\rho, \rho]$ with density given by

$$\frac{d}{2\pi} \frac{\sqrt{\rho^2 - t^2}}{1 - t^2},$$

see [25], formula (19.27). So for even n , by symmetry, we may write

$$r_n = \frac{d}{\pi} \int_0^\rho t^n \frac{\sqrt{\rho^2 - t^2}}{1 - t^2} dt = \frac{d}{2\pi} \int_0^{\rho^2} s^{(n-1)/2} \frac{\sqrt{\rho^2 - s}}{1 - s} ds.$$

Then, with

$$a = \rho^{-2} \int_0^{\rho^2} s^{(n-1)/2} \sqrt{\rho^2 - s} \, ds = \frac{\sqrt{\pi}}{2} \rho^n \frac{\Gamma(n/2 + 1/2)}{\Gamma(n/2 + 2)}$$

we have

$$\frac{d\rho^2}{2\pi} a < r_n < \frac{d\rho^2}{2\pi(1 - \rho^2)} a.$$

A small computation shows that for $d \geq 3$ we have

$$\frac{8}{3} \leq 4 - \frac{4}{d} = d\rho^2, \quad \frac{d\rho^2}{1 - \rho^2} = \frac{4d^2 - 4d}{(d - 2)^2} \leq 24.$$

Now for $n \geq 4$ we have

$$\kappa n^{-3/2} \leq \frac{\Gamma(n/2 + 1/2)}{\Gamma(n/2 + 2)} \leq 2^{3/2} n^{-3/2}, \quad \kappa = 4^{3/2} \frac{\Gamma(2.5)}{\Gamma(4)}.$$

The upper bound also holds for $n = 2$. (We manually check that the lower bound of the lemma holds for $r_2 = 1/d$.) To complete the proof, we bound the lower and upper constants factors

$$\frac{8}{3} \frac{1}{2\pi} \frac{\sqrt{\pi}}{2} \kappa = \frac{2}{3}, \quad 9.57 \sim 24 \frac{1}{2\pi} \frac{\sqrt{\pi}}{2} 2^{3/2} = 12\sqrt{2/\pi} < 10. \quad \square$$

Our next goal is to study the expected number of visits for random cycles in \mathbb{T}_d . This will be based on the same question for random walk excursions on \mathbb{Z} . Recall that an *excursion* of length n on \mathbb{Z} is a walk that stays positive except for time 0 and n , when it is zero.

3.2. Visits of cycles.

LEMMA 16 (Counting excursions). *Let $w_{n,k}$ be the number of walks of length $n \geq 1$ from 0 to $k \geq 0$ in \mathbb{Z} . Then*

$$w_{n,k} < \sqrt{2/\pi} \frac{2^n}{\sqrt{n}} e^{-k^2/(2n)}.$$

Let $w_{n,k}^+$ be the number of such paths that stay positive after time 0. Then for $k > 0$ we have

$$w_{n,k}^+ < \sqrt{2/\pi} \frac{2^n k}{n^{3/2}} e^{-k^2/(2n)}.$$

PROOF. We may assume that n and k are the same parity. Then

$$w_{n,k} = \binom{n}{\frac{n+k}{2}}.$$

We use the inequality

$$\binom{n}{\lfloor n/2 \rfloor} < \sqrt{2/\pi} \frac{2^n}{\sqrt{n}},$$

which holds since the ratio of the two sides is increasing along even (resp., odd) n and converges to 1. For n even, we now write

$$\binom{\frac{n}{2} + k}{\frac{n}{2}} \binom{n}{n/2}^{-1} = \frac{((n-k)/2 + 1) \cdots (n/2)}{(n/2 + 1) \cdots ((n+k)/2)} \leq \left(\frac{n-k}{n}\right)^{k/2} \leq e^{-k^2/(2n)},$$

and the odd case follows similarly.

By the Ballot theorem (see Section 2.7.1 in [13]), we have

$$w_{n,k}^+ = \frac{k}{n} w_{n,k} \leq \sqrt{2/\pi} 2^n \frac{k}{n^{3/2}} e^{-k^2/(2n)}. \quad \square$$

Recall that a *simple random walk excursion* of length n on \mathbb{Z} is a uniform choice from all excursions of length n . In other words, it is the simple random walk conditioned to stay positive except for time 0 and n , when it is zero. Now we are ready to bound the expected number of visits for simple random walk excursions on \mathbb{Z} .

LEMMA 17 (Visits of SRW excursions on \mathbb{Z}). *The expected number of visits $v_{k,n}$ to level $k > 0$ for the simple random walk excursion of length n on \mathbb{Z} satisfies $v_{k,n} \leq 64k$.*

PROOF. Let $w_{n,k}^+$ denote the number of walks of length n starting at 0 and ending at $k \geq 0$ that stay positive except perhaps at time 0 and n . If X_m is a random walk excursion of length n , then

$$\begin{aligned} v_{k,n} &= \mathbf{E} \sum_{m=1}^{n-1} \mathbf{1}(X_m = k) = \sum_{m=1}^{n-1} \mathbf{P}(X_m = k) \\ &= \frac{1}{w_{n,0}^+} \sum_{m=1}^{n-1} w_{m,k}^+ w_{n-m,k}^+ \leq \frac{2}{w_{n,0}^+} \sum_{m=1}^{n/2} w_{m,k}^+ w_{n-m,k}^+. \end{aligned}$$

For $n = 2$, the claim is easy to check. For $n \geq 4$, even we have the lower bound using the Catalan number formula (for general estimates on Catalan numbers see [8])

$$w_{n,0}^+ = \frac{2w_{n-2,0}}{n} \geq \frac{1}{\sqrt{2\pi}} \frac{2^n}{n^{3/2}},$$

where the last inequality holds since the ratio of the two sides is decreasing and converges to 1. Together with Lemma 16 this gives the bound

$$v_{k,n} \leq 2 \cdot \frac{2}{\pi} k^2 \sqrt{2\pi} n^{3/2} \sum_{m=1}^{n/2} \frac{e^{-k^2/(2m)}}{m^{3/2} (n-m)^{3/2}} \leq \frac{2}{\pi} \cdot 2 \cdot \sqrt{2\pi} \cdot 2^{3/2} k^2 \sum_{m=1}^{n/2} \frac{e^{-k^2/(2m)}}{m^{3/2}}.$$

Let a_m denote the last summand, even for non-integer m . Then for all $m \geq 1$ and $\delta \in [0, 1]$ we have $a_{m+\delta} \geq 2^{-3/2}a_m$. Thus, we can bound the sum by

$$2^{3/2} \int_1^\infty \frac{e^{-k^2/(2x)}}{x^{3/2}} dx < 2^{3/2} \int_0^\infty \frac{e^{-k^2/(2x)}}{x^{3/2}} dx = \frac{4\sqrt{\pi}}{k}. \quad \square$$

A *random cycle* is a cycle chosen from uniform measure from the set of cycles with the same starting point.

LEMMA 18 (Visits of cycles in \mathbb{T}_d). *The expected amount of time a random cycle of even length n in \mathbb{T}_d spends at distance $k > 0$ from its starting point is at most $2 \cdot 10^4 k$. For $k = 0$, it is at most 301.*

PROOF. Consider a random cycle of length n in \mathbb{T}_d from the root o . Let R_j be the distance of the walk from o at time j . The following is well known; see Section 2 of [7].

Let $0 = T_0 < T_1 < \dots < T_M = n$ be the (random) times when R_j is zero. Given the values of T_i and M , the sections of R_j in between are independent simple random walk excursions on \mathbb{Z} . In particular, given this information, Lemma 17 implies that the conditional expectation of the number of visits of R_j to k is bounded above by $64kM$. So, by Lemma 17, it suffices to show that \mathbf{EM} is bounded by a constant independent of n .

Let r_n be the probability that the simple random walk on \mathbb{T}_d visits its starting point at time n . By the Markov property, we have

$$\begin{aligned} \mathbf{EM} &= 1 + \sum_{k=1}^{n/2-1} P(R_{2k} = 0) = 1 + \frac{1}{r_n} \sum_{k=1}^{n/2-1} r_{2k} r_{n-2k} \\ &\leq 1 + \frac{3}{2} \cdot 10^2 \sum_{k=1}^{n/2-1} \frac{n^{3/2}}{(2k)^{3/2}(n-2k)^{3/2}}, \end{aligned}$$

where the last inequality follows from Lemma 15. Since the summand is convex as a function of k , the k term is bounded above by

$$\int_{k-1/2}^{k+1/2} \frac{n^{3/2}}{(2x)^{3/2}(n-2x)^{3/2}} dx$$

and the entire sum is at most

$$\int_{1/2}^{n/2-1/2} \frac{n^{3/2}}{(2x)^{3/2}(n-2x)^{3/2}} dx = \frac{2(n-2)}{\sqrt{(n-1)n}} < 2.$$

This gives $\mathbf{EM} < 301$. \square

Finally, we consider the limiting process of the random cycle starting at o in \mathbb{T}_d .

PROPOSITION 19 (The infinite cycle in \mathbb{T}_d). *Let $(X_k^n, k = 0, \dots, n)$ be the random cycle of even length n from o to o in \mathbb{T}_d . Then as $n \rightarrow \infty$ the process $(X_k^n, k = 0, \dots, n)$ converges in distribution to a process $(X_k, k \geq 0)$ called the infinite cycle, a time-homogeneous Markov process with transition probabilities (2).*

PROOF. The random cycle of length n is a time-inhomogeneous Markov process. Let $p_k^n(x, y)$ be denote its transition probabilities from x to y at time k . It suffices to show that the ratios of $p_k^n(x, x_+)/p_k^n(x, x_-)$ converge, (where x_+, x_- denotes a child or the parent of x , resp.) as any probability of the form

$$\mathbf{P}(X_1^n = x_1, \dots, X_k^n = x_k)$$

can be written as an expression containing finitely many of these probabilities. With $p_n(x, y)$ denoting the simple random walk transition probabilities in \mathbb{T}_d , the standard path counting argument gives

$$\frac{p_k^n(x, x_+)}{p_k^n(x, x_-)} = \frac{p_{n-k-1}(x_+, o)}{p_{n-k-1}(x_-, o)}.$$

We now use Theorem 19.30 in [25] which for x fixed and $n \rightarrow \infty$ gives

$$p_n(o, x) = (c + o(1)) \left(1 + \frac{d-2}{d}|x| \right) (d-1)^{-|x|/2} \rho(\mathbb{T}_d)^n n^{-3/2},$$

where $|x|$ is the graph distance of x from o , to get

$$(2) \quad \lim_{n \rightarrow \infty} \frac{p_{n-k-1}(x_+, o)}{p_{n-k-1}(x_-, o)} = \frac{1}{d-1} \frac{d + (d-2)(|x| + 1)}{d + (d-2)(|x| - 1)} =: \frac{p^*(x, x_+)}{p^*(x, x_-)}.$$

So, $(X_k, g \geq 0)$ is a time-homogeneous Markov process with transition probabilities p^* [which are determined by (2) since they sum over the neighbors of x to 1]. Clearly, $|X_n|$ is also a time-homogeneous Markov process, which has up/down transition probability ratio from $x \in \mathbb{Z}_+$ given by

$$\frac{d + (d-2)(x + 1)}{d + (d-2)(x - 1)}.$$

Note that when $d = 2$ we get the reflected simple random walk, as expected. \square

COROLLARY 20 (The infinite nullcycle). *Let G be a d -regular graph, and $(\bar{X}_k^n, k = 0, \dots, n)$ be the k th step of a uniformly chosen random nullcycle from a vertex o to o . Then \bar{X}_k^n converges in distribution as $n \rightarrow \infty$ to a limiting process $(\bar{X}_k, k \geq 0)$ called the infinite nullcycle. In particular, the fixed-time distributions converge.*

PROOF. Note that \bar{X}_k^n is just the image under the universal cover map from \mathbb{T}_d to G of the random cycle in \mathbb{T}_d . So, the claim follows from Proposition 19. \square

4. Properties of nullcycles. This section establishes some important properties of random nullcycles in graphs. But first we need a simple well-known lemma.

LEMMA 21 (Spectral radius and hitting probabilities). *Let G be a connected d -regular graph and let o be a vertex. Let $p_n(o, A)$ denote the probability that a random walk of length n starting at o ends in the finite vertex set A . Then with the spectral radius $\rho(G)$ we have*

$$p_n(o, A) \leq \sqrt{|A|}\rho(G)^n + \frac{2|A|}{|G|}.$$

PROOF. We prove the claim for finite graphs, the infinite case is similar but simpler. Let $m = |G|$, the number of vertices of G . Let v_0 denote the function on G that takes value $1/\sqrt{m}$ everywhere. Then $v_0M = v_0$. When G is not bipartite, let $l_*^2(G)$ denote the orthogonal subspace of v_0 in $l^2(G)$. When G is bipartite, let \mathcal{I} be an independent subset of G of size $m/2$ containing o and let v_1 be the function on G that takes values $1/\sqrt{m}$ on \mathcal{I} and $-1/\sqrt{m}$ otherwise. Then $v_1M^n = (-1)^n v_1$. Let $l_*^2(G)$ denote the subspace orthogonal to v_0 and v_1 in $l^2(G)$.

Now $\rho(G)$ equals the norm of M on $l_*^2(G)$. Let δ_A denote the indicator function of the vertex set A . Let v be a projection of δ_o onto $l_*^2(G)$, and let $v_* = \delta_o - v$. Then $\|v\| \leq 1$. For G bipartite, we can write $v_* = a(v_0 + v_1)$, with $a = 1/\sqrt{m}$. We have

$$(3) \quad \langle v_*M^n, \delta_A \rangle = \langle a(v_0 + v_1)M^n, \delta_A \rangle = \langle a(v_0 + (-1)^n)v_1, \delta_A \rangle.$$

Since v_0 and v_1 are orthonormal, writing δ_A in the orthonormal basis we see that (3) is bounded above by

$$a\langle v_0, \delta_A \rangle + a|\langle v_1, \delta_A \rangle| \leq 2|A|/m.$$

Similarly, in the non-bipartite case, $\langle v_*M^n, \delta_A \rangle = |A|/m$. We now have

$$\begin{aligned} p_n(o, A) &= \langle \delta_oM^n, \delta_A \rangle = \langle v_*M^n, \delta_A \rangle + \langle vM^n, \delta_A \rangle \\ &\leq 2|A|/|G| + \|v\| \cdot \rho(G)^n \cdot \|\delta_A\|. \end{aligned}$$

Here, $\|\delta_A\| = \sqrt{|A|}$. The claim follows. \square

4.1. Visits of nullcycles.

PROPOSITION 22 (Visits of nullcycles). *For any infinite d -regular rooted connected graph (G, o) with $\rho(G) < 1$ the number of visits V_A to a finite vertex set A of a random nullcycle of length n starting at o satisfies*

$$\mathbf{E}V_A \leq 2 \cdot 10^4 \frac{|A|}{(1 - \rho(G))^2}.$$

This is at most $10^7|A|$ if $\rho(G) \leq 19/20$. Note that $19/20 > \rho(\mathbb{T}_d)$ for $d \geq 3$.

For any finite d -regular graph G , we also have

$$\mathbf{E}V_A \leq 4 \cdot 10^4 |A| \left(\frac{1}{(1 - \rho(G))^2} + \frac{72n^2}{|G|} \right).$$

This is at most $2 \cdot 10^7 |A|$ if $\rho(G) \leq 19/20$ and $n^2 \leq |G|$.

PROOF. Let X_j be a random cycle in the d -regular tree \mathbb{T}_d started at the root o , and let \bar{X}_j be its projection to the graph G . Then we have

$$\mathbf{E}V_A = \mathbf{E} \sum_{j=0}^n \mathbf{1}(\bar{X}_j \in A) = \sum_{j=0}^n \mathbf{P}(\bar{X}_j \in A).$$

Condition on $|X_j|$, the distance from the root, and then sum over all possible options to get

$$\mathbf{E}V_A = \sum_{j=0}^n \sum_{k=0}^n P(|X_j| = k) P(\bar{X}_j \in A | |X_j| = k).$$

Note that given $|X_j| = k$, the distribution of X_j is uniform on the k -sphere about o in the tree. Thus, the distribution on \bar{X}_j in the graph G is that of the k th step of a non-backtracking random walk. So let p_k denote the probability that the k th step of the non-backtracking walk is in A .

Switching the order of summation, we get

$$\mathbf{E}V_A = \sum_{k=0}^n p_k \sum_{j=0}^n P(|X_j| = k) \leq 500 p_0 + \sum_{k=1}^n 2 \cdot 10^4 k p_k,$$

where the last inequality is based on the fact that the j -sum gives the expected number of visits to distance k for the random cycle in \mathbb{T}_d , and the result of Lemma 18. Note that $p_0 = \mathbf{1}(o \in A)$. The above can be bounded by Green function techniques as follows. Define

$$\mathcal{C}(z) = \sum_{k=0}^{\infty} p_k z^k,$$

the generating function for the proportion of non-backtracking paths that start from o and end in A . For any $z \in (0, 1]$, we have

$$\sum_{k=0}^n k p_k \leq z^{1-n} \sum_{k=1}^{\infty} k p_k z^{k-1} = z^{1-n} \mathcal{C}'(z).$$

The right-hand side is a power series with non-negative coefficients, so it always makes sense but may equal $+\infty$. Rewriting our bound in terms of \mathcal{C} , we get

$$\mathbf{E}V_A \leq 2 \cdot 10^4 z^{1-n} \mathcal{C}'(z) + 500 \cdot \mathbf{1}(o \in A).$$

Let $\mathcal{G}(z)$ be the analogous generating function for simple random walk. It was shown in [5] (see formula (2.3) in [22]), that for any d -regular graph we have

$$\mathcal{C}(z) = \frac{\mathbf{1}(o \in A)}{d} + \frac{(d-1)^2 - z^2}{d(d-1+z^2)} \mathcal{G}\left(\frac{dz}{d-1+z^2}\right).$$

Now with $x = dz/(d-1+z^2)$ we compute

$$\mathcal{C}'(z) = a_0 \mathcal{G}(x) + a_1 \mathcal{G}'(x),$$

where

$$a_0 = -\frac{2(d-1)z}{(d+z^2-1)^2} \leq 0,$$

$$a_1 = \frac{d^3 - d^2(z^2+3) + d(z^2+3) + z^4 - 1}{(d+z^2-1)^3} \leq 1,$$

for our range of parameters $d \geq 2$ and $z \in (0, 1]$. We now consider two cases.

1. For G infinite with $\rho = \rho(G) < 1$, we use the case $z = 1$, noting that the radius of convergence of \mathcal{G} is $1/\rho > 1$. Since \mathcal{G} and its derivative are non-negative, we get the upper bound

$$\frac{1}{2} 10^{-4} \mathbf{E}V_A \leq |A| + \mathcal{G}'(1) \leq |A| + \bar{\mathcal{G}}'(1), \quad \bar{\mathcal{G}}(z) = \frac{|A|}{1-z\rho}.$$

The last inequality uses the fact that the probability that simple random walk at time k is in A is bounded above by $|A|\rho^k$, so we can replace $\mathcal{G}'(z)$ by $\bar{\mathcal{G}}'(z)$. Finally, we have

$$|A| + \bar{\mathcal{G}}'(1) = |A| \frac{1-\rho+\rho^2}{(1-\rho)^2} \leq \frac{|A|}{(1-\rho)^2}.$$

2. For G finite, we use the case $z < 1$. Since \mathcal{G} and its derivatives are non-negative, we get the upper bound

$$\mathcal{C}'(z) \leq \mathcal{G}'(x) \leq \bar{\mathcal{G}}'(x).$$

For the last inequality, we use $\rho = \rho(G)$,

$$\bar{\mathcal{G}}(x) = |A| \sum_{k=0}^{\infty} x^k (\rho^k + 2/|G|) = \frac{2}{|G|} \frac{|A|}{1-x} + \frac{|A|}{1-x\rho}$$

and use Lemma 21 to bound the return probabilities. This gives

$$|A| + \bar{\mathcal{G}}'(x) = \frac{2|A|}{|G|} \frac{1}{(1-x)^2} + |A| \frac{\rho + (1-\rho x)^2}{(1-\rho x)^2} \leq \frac{2}{|G|} \frac{|A|}{(1-x)^2} + \frac{|A|}{(1-\rho)^2}.$$

We now have

$$\frac{1}{1-x} = \frac{d-1+z^2}{(d-1-z)(1-z)} \leq \frac{d}{d-2} \frac{1}{1-z} \leq \frac{3}{1-z}$$

and set $z = 1 - 1/(2n)$ to get

$$\begin{aligned} \mathbf{E}V_A &\leq 2 \cdot 10^4 z^{-n} (C'(z) + |A|) \\ &\leq 2 \cdot 10^4 (1 - 1/(2n))^{-n} |A| \left(\frac{2 \cdot 3^2 \cdot 2^2 n^2}{|G|} + \frac{1}{(1 - \rho)^2} \right) \end{aligned}$$

since for $n \geq 1$ the $(1 - 1/(2n))^{-n} \leq 2$, and the claim follows. \square

4.2. *Cycles and nullcycles.* We now turn to the connection between ordinary cycles and nullcycles. We recall the definition of nontrivial cycles.

DEFINITION 23. Call a cycle of length k in a graph a *nontrivial cycle* if either:

- for some directed non-loop edge e , the number of times the cycle passes through e differs from the number of times it passes through the reversal of e
- or $k = 1$.

This definition differs slightly from “vanishing in homology”, but is precisely what we need in our proof (briefly we use \mathbb{Z} -homology for $k \geq 2$, and \mathbb{Z}_2 -homology for $k = 1$). Our goal there is to take a nullcycle and make it non-null-homotopic. We do this by swapping the direction of nontrivial sub-cycles of length $k \geq 2$. For loops this does not work (they do not have direction), so we have to have a separate argument for $k = 1$: we add or erase them.

Cycles not covered by this definition are called *trivial*. For example, nullcycles are trivial and simple cycles are nontrivial.

The following theorem is another main ingredient in the proof of Theorem 1. Let \mathcal{N}_n denote the set of nullcycles starting at o in the rooted graph (G, o) .

THEOREM 24 (Cycles and nullcycles). *Let (G, o) be a d -regular rooted graph, and let $n, k, \ell > 0$.*

For a nullcycle $w \in \mathcal{N}_{nk}$ let $\chi(w, a, k) = \chi_\ell(w, a, k)$ denote the indicator function that the path segment w_a, \dots, w_{a+k} is a nontrivial k -cycle and that the vertex w_a is visited at most ℓ times by w . Let

$$(4) \quad \chi_w = \sum_{j=0}^{n-1} \chi(w, jk, k).$$

Then with $c_1 = 1/16$ and $c_k = (d - 1)^{-k} / 2$ (for $k \geq 2$) we have

$$|W_{nk}(o, o)| \geq \frac{1}{14} \sum_{w \in \mathcal{N}_{nk}} \exp(c_k \chi_w / \ell),$$

where $W_{nk}(o, o)$ is the set of cycles of length nk starting at o .

The quantity χ_ω will be estimated in terms of the parameters $\gamma_k(G)$. Heuristically, if it is large, it means that there are many different places in ω where rewiring is possible. The point in limiting the number of visits by ℓ is a convenient way to make sure that if there are many possible rewiring times, then they happen also at many different locations.

PROOF OF THEOREM 24. Let us denote $W = W_{nk}(o, o)$, and $\mathcal{N} = \mathcal{N}_{nk}$, the subset of nullcycles. We first break W into equivalence classes, called *rewiring classes*. A loop is called *single* if its vertex has no other loops. Otherwise, we call it a *multiple loop*.

When $k = 1$ we break up the sum on the right of (4) into a sum over single loops and a sum over multiple loops, counted as $\chi_{1w} + \chi_{2w} = \chi_w$. We choose k (and for $k = 1$ we choose single or multiple loops), and consider rewiring classes depending on our choice.

Case $k = 1$, single loops. Given a path w , let \bar{w} denote the path in which all self-loops whose vertex is visited at most ℓ times (not counting consecutive visits) have been erased. Let $w \equiv w'$ if $\bar{w} = \bar{w}'$. (“Not counting consecutive visits” means that visits to v that are at consecutive times count as a single visit.)

Case $k = 1$, multiple loops. Two paths are equivalent if for all times i the vertices satisfy $w_i = w'_i$, and w and w' agree except at times when they traverse multiple self-loops.

Case $k \geq 2$. The paths w and w' are equivalent if for all $0 \leq j \leq n - 1$ the following holds:

- If $w_{jk} \neq w_{jk+k}$ then the path segment between these times of w and w' is equal.
- If $w_{jk} = w_{jk+k}$ and the path segment between these times of w is trivial, then it equals the corresponding path segment in w' .
- If $w_{jk} = w_{jk+k}$ and the path segment between these times of w is nontrivial, then it either equals the corresponding path segment in w or is the time-reversal of that. We call jk a *proper cycle time* of w , and the corresponding path segment a *proper cycle* of w .

This is illustrated in the example depicted in Figure 3.

For $w \in W$, let $[w]$ denote the equivalence class of w , called *rewiring class*. Note that the rewiring defined here is more complex than the one in Section 1.3. For $w \in \mathcal{N}$, let $p(w)$ denote the probability that a uniform random element of $[w]$ is null-homotopic.

Then we have

$$|W| = \sum_{A \text{ is a rewiring class}} |A| \geq \sum_{\substack{A \text{ is a rewiring} \\ \text{class, } A \cap \mathcal{N} \neq \emptyset}} |A| = \sum_{w \in \mathcal{N}} \frac{|[w]|}{|[w] \cap \mathcal{N}|} = \sum_{w \in \mathcal{N}} p(w)^{-1}.$$

This path has one independent proper cycle.

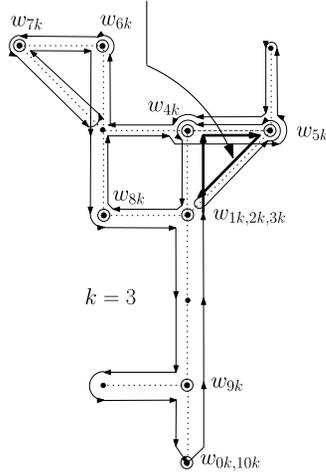


FIG. 3. A nullcycle $\omega \in \mathcal{N}_{30}$ with two proper cycles of length $k = 3$. These are opposite to each other and in particular dependent. Changing the direction in one of them gives raise to a nontrivial cycle equivalent to ω .

What remains is to show that for all $w \in \mathcal{N}$ we have

$$p(w) \leq 14 \exp(-c_k \chi_w / \ell).$$

We will do this case by case.

Case $k = 1$, single loops. We call a vertex with a single loop (and its loop) *reclusive* for w , if w visits it at most ℓ times (not counting consecutive visits). Whether a vertex is reclusive or not depends only on $[w]$.

Let $\tau_i, i = 1, \dots, \kappa$ denote the times when \bar{w} visits a reclusive vertex, and let ϕ be the number of loops erased from w to get \bar{w} . Then an element of $[w]$ is determined by X_1, \dots, X_κ , the number of loops inserted into \bar{w} at times $\tau_1, \dots, \tau_\kappa$. A uniform random element of $[w]$ corresponds to a uniform random choice of the X_i so that their sum is ϕ . Let $\text{tr } w$ denote the function that assigns to every reclusive loop of $[w]$ the number of times modulo 2 that w passes through it. Then

$$p(w) \leq \mathbf{P}(\text{tr } w = 0),$$

where the right-hand side refers to a random element of $[w]$. This is exactly the probability that for each reclusive vertex the sum of the X_i corresponding to that vertex is even. By Lemma 25, this is at most $14 \exp(-\min(m, \phi/\ell)/14)$, where m is the number of different reclusive vertices visited. Note that $m \geq \chi_{1w}/\ell$ and $\phi \geq \chi_{1w}$, so we get the bound $14 \exp(-\frac{\chi_{1w}}{14\ell})$.

Case $k = 1$, multiple loops. We call a vertex *important* if it has a loop traversed by w . Further, we call a loop *important* if its vertex is important (even if not traversed by w). Note that the set of important loops (or vertices) only depends on the equivalence class of w .

For a path, let tr denote the function that assigns to each important loop the number of times modulo 2 that it is traversed. Consider a random element w of $[w]$. For each important vertex v with k_v loops, let $\bar{X}_v = (X_{v,1}, \dots, X_{v,k_v})$ record the number of times w visits its loops. Note \bar{X}_v are independent as v varies, and each have a multinomial distribution with probabilities $1/k_v$ for each option; each traversal is assigned to one of the loops uniformly at random.

Conditioning on the assignment of all traversals except for the last one, we see that the probability that $X_{v,1}, \dots, X_{v,k_v}$ are all even numbers is at most $1/k_v \leq 1/2$. So, if i is the number of important vertices visited at most ℓ times, then we have $i \geq \chi_{2w}/\ell$ and

$$p(w) \leq \mathbf{P}(\text{tr } w = 0) \leq 2^{-i} \leq 2^{-\chi_{2w}/\ell}.$$

Case $k \geq 2$. For a path, let tr denote the antisymmetric edge function that sums 1 over all forward steps of a path and -1 over all backward steps (here ignoring self-loops). Note that the trace of a random element w in $[w]$ can be written as

$$(5) \quad \text{tr } w = \text{tr } \bar{w} + \sum_{\text{proper cycles } c \text{ of } w} X_c \text{tr } c,$$

where the X_c are independent random variables uniform on $\{-1, 1\}$, and \bar{w} denotes w with all its proper cycles removed. We claim that

$$p(w) \leq \mathbf{P}(\text{tr } w = 0) \leq 2^{-|w|_o},$$

where $|w|_o$ is the maximum size of a subset of linearly independent proper cycles of w . Indeed, consider such a set \mathcal{C} , and complete it to a basis for antisymmetric edge functions. Fix all values of X_c for $c \notin \mathcal{C}$. Then for $c \in \mathcal{C}$, looking at the a c -coordinate of equation (5), we see that it can hold only if X_c equals some fixed value, which has probability $1/2$ or 0 , independently over the coordinates. The claim follows.

Our next step is to bound the number of independent cycles. Fix a j_0 , and we consider the set J of indices j so that the $\chi(w, jk, k) = \chi(w, j_0k, k) = 1$, and the cycles of w at jk and j_0k share an edge. For a vertex v , let $J(v)$ denote the number of $j \in J$ so that $w_{jk} = v$. Since for $j \in J$ the vertex w_{jk} is visited at most ℓ times, we have $J(v) \leq \ell$. If two k -cycles share an edge, then a vertex on one and a vertex on the other are of distance at most $k - 1$ from each other. Thus, we have

$$|J| = \sum_{v \in B(w_{j_0k}, k-1)} J(v) \leq \ell |B(w_{j_0k}, k-1)| \leq \ell d(d-1)^{k-2},$$

where $B(v, r)$ is the ball of radius r about v . This means that the dependency graph of cycles has degree at most $d(d-1)^{k-2}\ell$ and size χ_w and, therefore, contains an independent set of size $\chi_w / (d(d-1)^{k-2}\ell + 1)$. So, we get $p(w) \leq 2^{-\chi_w / (d(d-1)^{k-2}\ell + 1)} \leq e^{-\chi_w / (2(d-1)^k\ell)}$.

Now we have either $\chi_{1w} \geq \frac{7}{8}\chi_w$ or $\chi_{2w} \geq \frac{1}{8}\chi_w$. In either case, we get

$$p(w) \leq 14 \exp(-\chi_w / (16\ell)).$$

Together with the $k \geq 2$ case, this completes the proof. \square

The following simple probabilistic lemma was used in the proof of Theorem 24.

LEMMA 25. *Let $X = (X_1, \dots, X_k)$ be a uniform random variable on the set of k -tuples of non-negative integers with even sum $n \geq 2$.*

(a) *For any integer k -vector x with $k \geq 2$, we have*

$$(6) \quad \mathbf{P}(X \equiv x \pmod 2) \leq \frac{\binom{n/2+k-1}{k-1}}{\binom{n+k-1}{k-1}} \leq \exp\left(-\frac{1}{4/k + 2/n}\right),$$

with equality at the first location if $x = 0$.

(b) *Consider a partition of $\{1, \dots, k\}$ into m non-empty parts so that $k \leq m\ell$ for some $\ell \geq 2$. Then with \wedge denoting minimum, we have*

$$\mathbf{P}\left(\sum_{i \in p} X_i \text{ is even for each part } p\right) \leq 14 \exp\left(-\frac{m \wedge (n/\ell)}{14}\right).$$

PROOF. (a) (We thank P. Csikváry for this simplification of our previous proof.) To count the number of k tuples that are equal to $x \pmod 2$, we subtract 1 from each odd entry and divide each resulting entry by 2. We get a bijection between such k -tuples and the number of k -tuples with entry sum $(n - o)/2$, where o is the number of odd entries of x . Thus,

$$\mathbf{P}(X \equiv x \pmod 2) = \frac{\binom{(n-o)/2+k-1}{k-1}}{\binom{n+k-1}{k-1}} \leq \frac{\binom{n/2+k-1}{k-1}}{\binom{n+k-1}{k-1}}.$$

This shows the first inequality. For the second, note that the right-hand side equals

$$\frac{n/2 + 1}{n + 1} \frac{n/2 + 2}{n + 2} \dots \frac{n/2 + k - 1}{n + k - 1}.$$

Each factor is at most $1 - \frac{n/2}{n+k-1}$, giving a bound of

$$\exp\left(-\frac{(k-1)n/2}{n+k-1}\right) \leq \exp\left(-\frac{1/2}{2/k + 1/n}\right).$$

The last inequality holds for $k \geq 2$.

(b) Let \bar{X} denote the vector formed by the sums of the entries of X over the parts of our partition. Let $M \subset \{1, \dots, k\}$ be a subset of indices, one in each part, and let M' be its complement. Let $S = \sum_{i \in M} X_i$. Then

$$\mathbf{E}S = \sum_{i \in M} \mathbf{E}X_i = m \frac{n}{k} \geq \frac{n}{\ell}.$$

We first bound the probability that S is exceptionally small, namely that it is at most $(k \wedge n)/(4\ell)$. S has a discrete beta distribution. By a standard construction, $S + m$ can be realized as the time of the m th black sample when sampling without replacement from n white and $k - 1$ black balls. From this, we get

$$p_s = P(S = s) = \frac{\binom{s+m-1}{m-1} \binom{(n-s)+(k-m)-1}{(k-m)-1}}{\binom{n+k-1}{k-1}}.$$

We compute the ratio of these probabilities for two consecutive values of s

$$(7) \quad \begin{aligned} \frac{p_s}{p_{s+1}} &= \frac{(s+1)(k-m+n-s-1)}{(m+s)(n-s)} \\ &\leq \frac{s(k-m+n-s)}{(m+s)(n-s)} + \frac{k-m+n-s}{(m+s)(n-s)}. \end{aligned}$$

Assume that $s \leq s_0 = (m/2) \wedge (n/(2\ell))$. We first bound the second term in (7), which equals

$$\frac{k-m}{m+s} \frac{1}{n-s} + \frac{1}{m+s} \leq \ell \frac{1}{n/2} + \frac{1}{m} \leq \frac{3}{m \wedge (n/\ell)}$$

since $n/2 \leq n-s$ and $k \leq m\ell$. The first term in (7) is increasing in k so we substitute the smallest possible value $k = m\ell$ to get the upper bound

$$1 - \frac{m(n-\ell s)}{(n-s)(m+s)} \leq 1 - \frac{mn/2}{n(m+m/2)} = \frac{2}{3}.$$

Thus, when

$$(8) \quad 3/(m \wedge (n/\ell)) \leq 1/12$$

the whole expression in (7) is bounded above by $3/4$. Now note that from $s = s_0$ down the probability of $S = s$ decreases by at least a factor of $3/4$. So,

$$\begin{aligned} \mathbf{P}(S \leq (k \wedge n)/4\ell) &= \sum_{s=0}^{s_0/2} p_s \leq \sum_{i=s_0/2}^{\infty} p_{(k \wedge n)/2\ell} \left(\frac{3}{4}\right)^i \\ &\leq 4 \left(\frac{3}{4}\right)^{s_0/2} \leq 4 \exp\left(-\frac{m \wedge (n/\ell)}{14}\right). \end{aligned}$$

Condition on the random variables in $X'_M = (X_i, i \in M')$. Given this information the random variable $X_M = (X_i, i \in M)$ is uniform on the set of k -tuples of non-negative integers with sum S .

$$\mathbf{P}(\bar{X} = 0 \text{ mod } 2) = \mathbf{E}[\mathbf{P}(\bar{X}_M = \bar{X}_{M'} \text{ mod } 2 | X_{M'})] \leq \mathbf{E}[\mathbf{P}(\bar{X}_M = 0 \text{ mod } 2 | X_{M'})].$$

The inequality follows from part (a). The last conditional probability depends only on the value of S . Using part (a), we can break the expression up with $s = s_0/2$ as

$$\begin{aligned} &\mathbf{P}(S < s) + \mathbf{E}[\mathbf{E}(\mathbf{1}(X_m = 0 \text{ mod } 2)\mathbf{1}(S \geq s) | S)] \\ &\leq 4 \exp\left(-\frac{m \wedge (n/\ell)}{14}\right) + \exp\left(-\frac{1}{4/k + 2/s}\right) \leq 5 \exp\left(-\frac{m \wedge (n/\ell)}{14}\right). \end{aligned}$$

We increase the prefactor 5 to 14 in order to get a trivial bound when (8) fails. \square

5. Explicit bounds on the spectral radius.

5.1. *A preliminary bound on the return probability.*

PROPOSITION 26. *Let G be a d -regular unimodular random graph and let $k, \ell > 0$. Then with*

$$c_k = \begin{cases} 1/16, & \text{for } k = 1, \\ (d - 1)^{-k}/2, & \text{for } k \geq 2, \end{cases}$$

we have

$$\mathbf{E} \log |W_{nk}| \geq \log |\mathcal{N}_{nk}| - 3 + \frac{c_k n}{\ell |\mathcal{N}_{nk}|} \sum_{w \in \mathcal{N}_{nk}} \mathbf{E} \chi_\ell(w, 0, k).$$

PROOF. By Theorem 24 and the inequality of arithmetic and geometric means, we have

$$\begin{aligned} |W_{nk}| &\geq e^{-3} \sum_{w \in \mathcal{N}_{nk}} \exp\left(c_k \sum_{j=0}^{n-1} \chi(w, jk, k) / \ell\right) \\ &\geq e^{-3} |\mathcal{N}_{nk}| \left(\prod_{w \in \mathcal{N}_{nk}} \prod_{j=0}^{n-1} \exp(c_k \chi(w, jk, k) / \ell) \right)^{1/|\mathcal{N}_{nk}|}. \end{aligned}$$

Taking logarithm of both sides gives us

$$\log |W_{nk}| - \log |\mathcal{N}_{nk}| \geq -3 + \frac{c_k}{\ell |\mathcal{N}_{nk}|} \sum_{w \in \mathcal{N}_{nk}} \sum_{j=0}^{n-1} \chi(w, jk, k).$$

Taking expected value of both sides over the random graph, we get

$$\mathbf{E} \log |W_{nk}| - \log |\mathcal{N}_{nk}| \geq -3 + \frac{c_k}{\ell |\mathcal{N}_{nk}|} \sum_{w \in \mathcal{N}_{nk}} \sum_{j=0}^{n-1} \mathbf{E} \chi(w, jk, k).$$

We will use the mass transport principle to show that the expression

$$(9) \quad \sum_{w \in \mathcal{N}_{nk}} \mathbf{E} \chi(w, jk, k),$$

does not depend on the position j . Let the mass transport be defined as

$$\begin{aligned} f(G, x, y) &= \sum_{w \in \mathcal{N}_{nk}(x)} \mathbf{1}(w_{(n-j)k} = y) \chi(w, 0, k) \\ &= \sum_{w \in \mathcal{N}_{nk}(y)} \mathbf{1}(w_{jk} = x) \chi(w, jk, k). \end{aligned}$$

That is, for every null-homotopic path w starting at x , x sends mass $\chi(w, 0, k)$ to the $(n - j)$ th position of w . The second equality follows by rooting the path at y instead of x . Trivially, the mass transport does not depend on the root of G , so the mass transport principle gives us

$$\sum_{y \in V(G)} \mathbf{E}f(G, o, y) = \sum_{x \in V(G)} \mathbf{E}f(G, x, o)$$

that is, the expected mass sent from the root equals the expected mass received by the root. Plugging in the corresponding equations, we get

$$\sum_{w \in \mathcal{N}_{nk}(o)} \mathbf{E}\chi(w, 0, k) = \sum_{w \in \mathcal{N}_{nk}(o)} \mathbf{E}\chi(w, jk, k)$$

and we get that the expression (9) does not depend on j . This proves the theorem. □

LEMMA 27. *Let (G, o) be a d -regular rooted graph with $\rho(G) \leq 19/20$. Let $\gamma_k(G, o)$ be the number of nontrivial cycles of length k starting at the root o . Let n satisfy $2k \leq n \leq \sqrt{|G|}$ and let $\ell = 6 \cdot 10^8(4d - 4)^k$. Then we have*

$$\frac{1}{|\mathcal{N}_n|} \sum_{w \in \mathcal{N}_n} \chi_\ell(w, 0, k) \geq \frac{\gamma_k(G, o)}{30(4d - 4)^k}.$$

PROOF. We may assume $\gamma_k(G, o) \geq 1$, otherwise the claim is trivial. In this lemma, G is fixed, so the probabilistic language for nullcycles will not cause confusion. So, let w be a uniform random element of \mathcal{N}_n .

The probability that a random cycle of length n in \mathbb{T}_d traverses a specific path for its first k steps can be bounded below easily by requiring the path to retrace its steps in the following k times. If r_n is the return probability of simple random walk in \mathbb{T}_d , then the total number of paths that do this is given by $r_{n-2k}d^{n-2k}$, so the probability is at least

$$\frac{r_{n-2k}d^{n-2k}}{r_n d^n} \geq \frac{1}{15} (d\rho(\mathbb{T}_d))^{-2k} = \frac{1}{15} (4d - 4)^{-k} =: p,$$

and the inequality uses both sides of Lemma 15 (but one side in the special case $n = 2k$). So, if G has $\gamma_k(G, o)$ cycles of length k at o , then the event A that w passes through one of them in the first k steps satisfies $\mathbf{P}A \geq p\gamma_k(G, o)$. Let V_o be the number of times the random nullcycle w traverses o . By Proposition 22, we have

$$\mathbf{E}V_o \leq 2 \cdot 10^7 = c, \quad \mathbf{E}(V_o|A) \leq \frac{\mathbf{E}V_o}{\mathbf{P}A} \leq \frac{c}{p\gamma_k(G, o)}.$$

By Markov's inequality with $\ell = 2c/p$,

$$\mathbf{P}(V_o \geq \ell|A) \leq \frac{\mathbf{E}(V_o|A)}{\ell} \leq \frac{1}{2\gamma_k(G, o)} \leq \frac{1}{2}.$$

This implies (using probabilistic notation for averaging over \mathcal{N}_n)

$$\begin{aligned} \mathbf{E}\chi(w, 0, k) &= \mathbf{P}(A, V_o \leq \ell) = \mathbf{P}(A) - \mathbf{P}(V_o > \ell|A)\mathbf{P}(A) \\ &\geq \frac{\mathbf{P}(A)}{2} \geq \frac{p\gamma_k(G, o)}{2} \end{aligned}$$

as claimed. \square

5.2. *Main bounds on spectral radius.* The following theorem implies Theorems 1 and 5.

THEOREM 28 (Main results). *Let (G, o) be a d -regular unimodular random graph and let $k \geq 1$. Let $\gamma_k(G, o)$ be the number of nontrivial cycles of length k starting at o . Let*

$$v_k = 2 \cdot 10^{11} 2^{4k} (d - 1)^{3k} k.$$

For G infinite a.s., we have

$$(10) \quad \mathbf{E} \log \rho(G) \geq \log \rho(\mathbb{T}_d) + \frac{1}{v_k} \mathbf{E} \gamma_k(G, o).$$

For G infinite and ergodic, we have $\rho(G) \geq \rho(\mathbb{T}_d) e^{\mathbf{E} \gamma_k(G, o)/v_k}$.

Let G be a finite connected d -regular graph with $|G| \geq d^7$. Then for the root o chosen uniformly at random we have

$$(11) \quad \frac{\rho(G)}{\rho(\mathbb{T}_d)} \geq 1 + \frac{1}{v_k} \mathbf{E} \gamma_k(G, o) - \frac{(3/2) \log \log_{d-1} |G| + 6}{\log_{d-1} |G|}.$$

In particular, for finite Ramanujan graphs with $|G| \geq d^7$ we have

$$(12) \quad \mathbf{E} \gamma_k(G, o) \leq v_k \frac{(3/2) \log \log_{d-1} |G| + 6}{\log_{d-1} |G|}.$$

PROOF. Let nk be even and $n \geq 1$. First, assume that G which may be finite or infinite satisfies $\mathbf{P}(|G| \geq (nk)^2) = 1$. We will use Lemma 27, which requires $\rho(G) \leq 19/20$. We first take care of the other case. For (10) and (11), we need to show for every such G we have

$$\log \rho(G) \geq \log \rho(\mathbb{T}_d) + \gamma_k(G)/v_k.$$

Since $\gamma_k(G) \leq d^k$, this inequality follows from

$$\rho(G) \geq 19/20, \quad \rho(\mathbb{T}_d) \leq \rho(\mathbb{T}_3) = 2\sqrt{2}/3, \quad \gamma_k(G)/v_k \leq \frac{1}{2 \cdot 10^{11}}.$$

Now assume $\rho(G) \leq 19/20$. By Proposition 26 and Lemma 27 for $\ell = 6 \cdot 10^8 (4d - 4)^k$, $n \geq 1$ with $c_1 = 1/16x$ and $c_k = (d - 1)^{-k}/2$ for $k \geq 2$ we have

$$(13) \quad \mathbf{E} \log |W_{nk}| \geq \log |\mathcal{N}_{nk}| - 3 + \frac{c_k}{\ell} n \frac{\mathbf{E} \gamma_k(G, o)}{30(4d - 4)^k},$$

where

$$\frac{c_k}{30(4d - 4)^k \ell_k} \geq \frac{1}{v_k}.$$

For the first claim (11), we divide (13) by nk and use the bounded convergence theorem. The second claim (10) follows from the fact that for G ergodic $\rho(G)$ is constant.

The bound on \mathcal{N}_n of Lemma 15 now shows that

$$(14) \quad \mathbf{E} \log p_{nk}(o, o) \geq nk \log \rho(\mathbb{T}_d) - \frac{3}{2} \log(nk) - 4 + \frac{nk}{v_k} \mathbf{E} \gamma_k(G, o).$$

For G finite and $d \geq 3$, we have

$$(15) \quad |G| \geq d^7 \implies \rho(G) \geq 1/(d - 1)^{5/6},$$

which follows from $\rho(G)^2 + 2/|G| \geq p_2(o, o) \geq 1/d$, a consequence of Lemma 21. Note that a lower bound on $|G|$ is needed for (15) since the complete graph with loops has $|G| = d$ and $\rho(G) = 0$.

Assume $|G| \geq d^7$, and $\log_{d-1} |G| \geq 10k$. Set $n = 2 \lceil \frac{1}{2k} \log_{d-1} |G| \rceil$ so that

$$\rho(G)^{nk} \geq (d - 1)^{-(5/6)nk} \geq (d - 1)^{-(5/6)(\log_{d-1} |G| + 2k)} \geq 1/|G|.$$

[Here, the power $5/6$ from (15) is used to offset the effect of $\lceil \cdot \rceil$, and thus yield a cleaner final bound.] By Lemmas 15 and 21, the left-hand side of (14) is at most

$$\begin{aligned} \log(\rho(G)^{nk} + 2/|G|) &= nk \log \rho(G) + \log\left(1 + \frac{2}{|G| \rho(G)^{nk}}\right) \\ &\leq nk \log \rho(G) + \log 3. \end{aligned}$$

We use this and divide (14) by nk and get the lower bound

$$(16) \quad \begin{aligned} \log \rho(G) &\geq \log \rho(\mathbb{T}_d) + \frac{1}{v_k} \mathbf{E} \gamma_k(G, o) \\ &\quad - \frac{(3/2) \log \log_{d-1} |G| + 4 + \log 3}{\log_{d-1} |G|}. \end{aligned}$$

This proves (11) for the case $\log_{d-1} |G| \geq 10k$.

The rest of the proof is standard and can be skipped. Its goal is to remove the restriction $\log_{d-1} |G| \geq 10k$. The same argument as above, using the trivial comparison with \mathbb{T}_d for walks of length $2 \lceil \frac{1}{2} \log_{d-1} |G| \rceil$ gives the (suboptimal) Alon–Boppana-type bound

$$(17) \quad \log \rho(G) \geq \log \rho(\mathbb{T}_d) + \frac{(3/2) \log \log_{d-1} |G| + 4 + \log 3}{\log_{d-1} |G|}$$

as long as $\log_{d-1} |G| \geq 10$. For $d \geq 4$, (15) can be improved to

$$(18) \quad |G| \geq d^7 \implies \rho(G) \geq 1/(d - 1)^{3/4},$$

and this yields that (17) holds as long as $\log_{d-1} |G| \geq 6$. So, both for $d = 3$ and $d \geq 4$ we get that (17) holds as long as $|G| \geq d^7$. Equation (17) implies (11) if

$$\frac{\gamma_k(G, o)}{\nu_k} \leq \frac{2 - \log 3}{\log_{d-1} |G|}.$$

With the trivial bound $\gamma_k(G, o) \leq d(d - 1)^{k-1}$, in the case $\log_{d-1} |G| \leq 10k$ this is implied by

$$\frac{d(d - 1)^{k-1}}{\nu_k} \leq \frac{2 - \log 3}{10k},$$

which holds trivially. \square

5.3. *Bounds for graphs close to the Ramanujan threshold.* The following theorem implies Theorem 2.

THEOREM 29 (Short cycles in Ramanujan graphs). *Let $\alpha > 0$, $d \geq 3$ and consider finite, connected d -regular graphs G that are close to Ramanujan in the sense that*

$$\rho(G) \leq \rho(\mathbb{T}_d) + \frac{1}{(\log |G|)^\alpha}.$$

Fix $\beta, \varepsilon > 0$ so that $\beta + \varepsilon < \frac{\alpha \wedge 1}{6 \log(d-1) + 8 \log 2}$ [for example $\beta = \frac{\alpha \wedge 1}{16 \log(d-1)}$]. Then as $|G| \rightarrow \infty$, the proportion of vertices in G whose $\beta \log \log |G|$ -neighborhood is not a d -regular tree is $o((\log |G|)^{-\varepsilon})$.

PROOF. Note that if the $k = \beta \log \log |G|$ neighborhood of a vertex v is not a tree, then v is contained in a nontrivial cycle of length $2k$, or its k -neighborhood contains a vertex with a loop. We rule out these two cases separately.

By Theorem 28, (11), we have

$$\mathbf{E} \gamma_k(G, o) \leq c(d - 1)^{3k} 2^{4k} k^3 \left(\frac{\log \log |G|}{\log |G|} + \frac{1}{(\log |G|)^\alpha} \right)$$

so if $k = 2\beta \log \log |G|$, then the dominant factor is

$$(\log |G|)^{-\alpha \wedge 1 + 4\beta \log(4d-4)}$$

and this is $o(\log |G|^{-\varepsilon'})$ for some $\varepsilon' > \varepsilon$ since

$$\beta + \varepsilon < \frac{\alpha \wedge 1}{\log((d - 1)^3 2^4)}.$$

The inequality also holds uniformly for all smaller k [with a uniform constant in the $o(\cdot)$ term], and summing over all such we get that the expected number of

nontrivial cycles at o of length at most $2k$ is $o(\log |G|^{-\varepsilon} \log \log |G|) \rightarrow 0$. This rules out the first option.

For the second option, we use a simple mass transport argument (see the proof of Theorem 24 for the formal setup). Let each vertex with a loop send mass k to all elements in its k -neighborhood. Then the expected amount of mass sent is at most $d(d - 1)^{k-1} \mathbf{E}\gamma_1(G)$. The amount of mass received is the number of vertices with loops in the k -neighborhood; lets call this N . So, we have

$$\mathbf{E}N \leq c(d - 1)^k \left(\frac{\log \log |G|}{\log |G|} + \frac{1}{(\log |G|)^\alpha} \right).$$

By the same argument as before, this is $o(\log |G|^{-\varepsilon})$ with the above choice of β . □

5.4. *Weakly Ramanujan sequences.* We are ready to prove that a d -regular weakly Ramanujan sequence of finite graphs converges to the d -regular tree.

PROOF OF THEOREM 4. Let (G_n) be a weakly Ramanujan sequence of finite d -regular graphs. Assume by contradiction, that it does not have essentially large girth. Then, by passing to a suitable subsequence, there exists $c > 0$ and $L > 0$ such that the cycle densities $\gamma_L(G_n) > c$.

By passing to a subsequence, we can also assume that (G_n) is Benjamini–Schramm convergent. Let G be the limit of (G_n) .

We claim that G is infinite a.s. Assume this is not the case, then there exists $R > 0$ such that G has size R with probability $p > 0$. This means, that with probability at least p , the $R + 1$ -ball around the root has the same size as the R -ball. So, for large enough n , the same holds for all G_n with $p/2$. That is, at least $|G_n|p/2$ vertices lie in a connected component of size at most R' , where R' is the size of the R -ball in the d -regular tree. This implies that the number of connected components of G_n is at least $|G_n|p/2R'$. Hence,

$$\mu_{G_n}(1) \geq \frac{p}{2R'}.$$

This contradicts the assumption that (G_n) is weakly Ramanujan. So, our claim holds.

We claim that G is Ramanujan a.s. By the proof of Proposition 14, μ_{G_n} weakly converges to the expected spectral measure μ_G , which yields $\mu_G([- \rho(T_d), \rho(T_d)]) = 1$, and this implies that $\mu_{G,o}([- \rho(T_d), \rho(T_d)]) = 1$ a.s. Since the spectral radius equals the radius of the support of the spectral measure $\mu_{G,o}$ for any rooted connected graph G (see [12], Lemma 2.1), this implies that $\rho(G) \leq \rho(T_d)$ a.s. and our claim holds.

Now using Theorem 5, $G = T_d$ a.s., that is, (G_n) converges to T_d and so by Proposition 14, it has essentially large girth, a contradiction. Our theorem holds. □

6. Spectral radius and the fundamental group—A sharp bound.

6.1. *Relations in deterministic graphs.* In this section, we analyze the spectral radius of a fixed rooted d -regular infinite graph using random walks on its fundamental group.

For a graph G rooted at $o \in V(G)$ and an arbitrary finite multiset N of cycles in G starting at o , we will also use N to denote the corresponding Markov operator on the fundamental group $\pi_1(G, o)$ (which is a free product of copies of \mathbb{Z} and the group of order 2), where the step distribution is the uniform measure on N . Let $\|N\|$ denote the operator norm. The adjoint of the operator N is the operator corresponding to the multi-set $N^{-1} = \{w^{-1} | w \in N\}$. The multi-set N may not be closed to taking inverses, so the Markov operator need not be self-adjoint. We will use the property

$$\|N\| = \sqrt{\|NN^{-1}\|}.$$

Let G be a graph, let vertices $x, y \in V(G)$ and $k > 0$ let $W = W_k(x, y)$ denote the set of walks of length k in G starting at x and ending at y . Let $o \in V(G)$, let u be a walk from o to x and let v be a walk from y to o . When W is non-empty, let

$$N = \{uvw | w \in W\} \subseteq \pi_1(G, o)$$

and let

$$(19) \quad \kappa_k(x, y) = \|N\|.$$

Now $\kappa_k(x, y)$ does not depend on the choice of o, u and v , because the multi-set

$$NN^{-1} = \{uw'w^{-1}u^{-1} | w, w' \in W\}$$

(defined with multiplicities), so the corresponding Markov operator is the conjugate of the operator belonging to WW^{-1} by the fixed element u .

Note that the norm κ_k satisfies

$$\kappa_k(x, y)^2 = \rho(\text{Cay}(\pi_1(G, x), WW^{-1})) \in [0, 1].$$

Let \mathcal{N}_k denote the set of nullcycles of length k starting at o (see Definition 9). The following lemma relates $|\mathcal{N}_k|/|W_k(o, o)|$, the probability that a random cycle of length k is a nullcycle to the spectral radius κ_k . This relation can be established also with respect to paths connecting two vertices.

LEMMA 30. *Let G be a d -regular graph rooted at o and let $k > 0$. Let x be a vertex in G , and let w be a path of length $|w|$ from x to o . Then*

$$|(W_k(o, x)w) \cap \mathcal{N}_{k+|w|}| \leq |W_k(o, x)|\kappa_k(o, x) \leq (d\rho(T_d))^{k+|w|}.$$

In particular, with $x = o$ and w trivial we have

$$|\mathcal{N}_k| \leq |W_k(o, o)|\kappa_k(o, o) \leq (d\rho(T_d))^k.$$

PROOF. We have

$$|W_k(o, x)w \cap \mathcal{N}_{k+|w|}| = |W_k(o, x)w| \frac{|W_k(o, x)w \cap \mathcal{N}_{k+|w|}|}{|W_k(o, x)w|}$$

and the second factor on the right-hand side equals the one step return probability of the random walk on $\pi_1(G, o)$ with uniform step distribution on $W_k(o, x)w$; hence, it is at most the spectral radius of the corresponding Markov operator. This proves the left inequality in the lemma.

Now consider

$$\begin{aligned} |W_k(o, x)w|^n &= |(W_k(o, x)w)^n \cap \mathcal{N}_{n(k+|w|)}| \frac{|(W_k(o, x)w)^n|}{|(W_k(o, x)w)^n \cap \mathcal{N}_{n(k+|w|)}|} \\ &\leq |\mathcal{N}_{n(k+|w|)}| \frac{|(W_k(o, x)w)^n|}{|(W_k(o, x)w)^n \cap \mathcal{N}_{n(k+|w|)}|}. \end{aligned}$$

The second factor on the right-hand side equals the inverse of the n -step return probability of the same random walk as above. Taking n th roots and the limit as n goes to infinity gives us the right-hand side inequality of the lemma. \square

THEOREM 31. Let G be a d -regular graph rooted at o and let $n, k > 0$. Then

$$|W_{nk}(o, o)| \geq \sum_{w \in \mathcal{N}_{nk}} \prod_{j=0}^{n-1} \kappa_k(w_{jk}, w_{(j+1)k})^{-1} \geq \frac{|\mathcal{N}_k|^n}{(d\rho(T_d))^{nk}} |W_k(o, o)|^n.$$

This implies

$$d\rho(G) \geq \left(\sum_{w \in \mathcal{N}_{nk}} \prod_{j=0}^{n-1} \kappa_k(w_{jk}, w_{(j+1)k})^{-1} \right)^{1/nk}.$$

Moreover, when we take the limit of the right-hand side as $k \rightarrow \infty$ (and n changing arbitrarily) we get equality.

PROOF. Let us denote $W = W_{nk}(o, o)$ and $\mathcal{N} = \mathcal{N}_{nk}$. We say that $w' \in W$ is a rewiring of $w \in W$ if $w'_{jk} = w_{jk}$ for $0 \leq j \leq n - 1$.

As an example following the line of proof below, all possible rewirings of a nullcycle of length $35 = 5 \cdot 7$ are shown in Figure 4 below. It should be helpful to refer to that figure while reading the proof. Rewiring is an equivalence relation, and for $w \in W$ let $[w]$ denote the equivalence class of w . For $w \in \mathcal{N}$ let $p(w)$ denote the probability that a uniform random element of $[w]$ is null-homotopic.

Then we have

$$|W| = \sum_{A \text{ is a rewiring class}} |A| \geq \sum_{\substack{A \text{ is a rewiring} \\ \text{class, } A \cap \mathcal{N} \neq \emptyset}} |A| = \sum_{w \in \mathcal{N}} \frac{|[w]|}{|[w] \cap \mathcal{N}|} = \sum_{w \in \mathcal{N}} p(w)^{-1}.$$

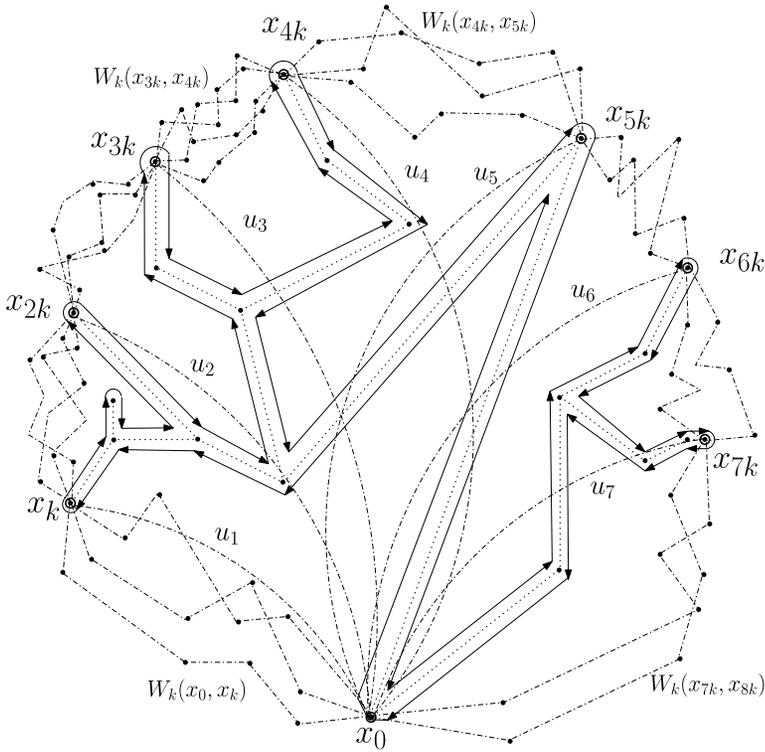


FIG. 4. Here, we see how all possible rewirings of a path $\omega \in \mathcal{N}_{35}$ are obtained upon replacing segments of length $k = 5$ by other possible replacements. The essential part of the proof is giving an estimate to the probability that such a rewiring w' chosen at random will again be a nullcycle. Namely that the path w' represents the trivial element of the fundamental group $\pi_1(G, x_0)$.

We claim that for all $w \in \mathcal{N}$ we have

$$p(w) \leq \prod_{j=0}^{n-1} \kappa_k(w_{jk}, w_{(j+1)k}).$$

To prove this, for $0 \leq j \leq n$ let u_j be a path from o to w_{jk} . Assume that u_0 and u_n are the empty paths. For $0 \leq j \leq n - 1$, let

$$N_j = \{u_j w u_{j+1}^{-1} \mid w \in W_k(w_{jk}, w_{(j+1)k})\} \subseteq \pi_1(G, o)$$

and let v_j be a uniform random element of N_j . Let N_j also denote the Markov operator corresponding to the multi-set N_j . Then $\|N_j\| = \kappa_k(w_{jk}, w_{(j+1)k})$ by definition.

Now the random element $v = v_0 \cdots v_{n-1}$ and the uniform random element of $[w]$ have the same distribution as elements on the fundamental group. Indeed, they are related by adding or deleting the nullcycles $u_j^{-1} u_j$. That is, $p(w)$ equals

the probability that v is null-homotopic. Let e be the characteristic vector of the identity element in $\pi_1(G, o)$. Using the Cauchy–Schwarz inequality, this gives

$$\begin{aligned}
 p(w) &= \left\langle e, e \prod_{j=0}^{n-1} N_j \right\rangle \leq \left\langle e \prod_{j=0}^{n-1} N_j, e \prod_{j=0}^{n-1} N_j \right\rangle^{1/2} \\
 &\leq \left\| \prod_{j=0}^{n-1} N_j \right\| \leq \prod_{j=0}^{n-1} \|N_j\| = \prod_{j=0}^{n-1} \kappa_k(w_{jk}, w_{(j+1)k})
 \end{aligned}$$

and our claim holds.

Together with our first estimate on $|W|$ this completes the proof of the first inequality of the theorem. For the second claim, note that restricting the sum to nullcycles that return to o at every time kj we get the lower bound

$$\begin{aligned}
 \sum_{w \in \mathcal{N}_{nk}} \prod_{j=0}^{n-1} \kappa_k(w_{jk}, w_{(j+1)k})^{-1} &\geq |\mathcal{N}_k|^n \kappa_k(o, o)^{-n} \\
 &\geq \frac{|\mathcal{N}_k|^n}{(d\rho(T_d))^{nk}} |W_k(o, o)|^n.
 \end{aligned}$$

Here, the last inequality follows from Lemma 30. \square

Let G be a d -regular graph rooted at o . We define a new distribution on the vertices of G as follows. For $k, n > 0$ where n is even and $x \in V(G)$, let $p(k, n, x)$ denote the probability that a uniform random null-homotopic walk of length n starting at o is at x at time k . Let

$$(20) \quad p_k(x) = \lim_{n \rightarrow \infty} p(k, n, x),$$

which, for each k that describes where the first k -segment of a the infinite bride of large length ends. The fact that this limit exists is a consequence of Corollary 20.

Let

$$(21) \quad \kappa_k^*(G, o) = \prod_{x \in V(G)} \kappa_k(o, x)^{p_k(x)}$$

that is, the geometric mean of the $\kappa_k(o, x)$ averaged over the vertices x with respect to the distribution p_k .

LEMMA 32. *For any connected d -regular infinite graph G , we have*

$$\rho(G) = \rho(\mathbb{T}_d) \lim_{k \rightarrow \infty} \kappa_k^*(G, o)^{-1/k}.$$

Moreover, the terms $\kappa_k^*(G, o)^{-1/k}$ are bounded above by a constant depending on d only.

PROOF. For any vertex x , Lemma 30 gives the lower bound

$$|W_k(o, x)|^{-1} |\overline{W}_k(o, \bar{x})| \leq \kappa_k(o, x) \leq |W_k(o, x)|^{-1} (d\rho(T_d))^{k+|x|},$$

where \overline{W} is the function W for the covering tree and \bar{x} is a lift of x corresponding to w in that lemma. Using the simplest lower bounds for the number of paths, we get

$$\rho(G)^{-k} |\overline{W}_{k-|x|}(o, o)| \leq \kappa_k(o, x) \leq |W_{k-|x|}(o, o)|^{-1} (d\rho(T_d))^{k+|x|}.$$

Note that $\rho(k, \cdot)$ assigns probability q_k tending to 1 to vertices x with $|x| \leq k^{2/3}$.

$$\begin{aligned} \rho(G)^{-k} |\overline{W}_{k-k^{2/3}}(o, o)|^{q_k} &\leq \kappa_k^*(G, o) \\ &\leq |W_{k-k^{2/3}}(o, o)|^{-q_k} (\rho(T_d))^{(k+k^{2/3})q_k} d^{k+k^{2/3}}. \end{aligned}$$

The second claim follows by taking k th roots; the first follows by letting $k \rightarrow \infty$ and noting that the left- and right-hand sides both converge to $\rho(T_d)/\rho(G)$. \square

6.2. An asymptotically sharp bound.

THEOREM 33. Let G be a d -regular infinite unimodular random graph. Then for any $k > 0$, we have

$$\mathbf{E} \log \rho(G) \geq \log \rho(T_d) - \frac{1}{k} \mathbf{E} \log \kappa_k^*(G, o)$$

and these bounds are sharp in the sense that

$$\mathbf{E} \log \rho(G) = \log \rho(T_d) - \lim_{k \rightarrow \infty} \frac{1}{k} \mathbf{E} \log \kappa_k^*(G, o).$$

PROOF. By Theorem 31 and the inequality of arithmetic and geometric means, we have

$$\begin{aligned} |W_{nk}| &\geq \sum_{w \in \mathcal{N}_{nk}} \prod_{j=0}^{n-1} \kappa_k(w_{jk}, w_{(j+1)k})^{-1} \\ &\geq |\mathcal{N}_{nk}| \left(\prod_{w \in \mathcal{N}_{nk}} \prod_{j=0}^{n-1} \kappa_k(w_{jk}, w_{(j+1)k})^{-1} \right)^{1/|\mathcal{N}_{nk}|}. \end{aligned}$$

Taking logarithm of both sides gives us

$$\log |W_{nk}| - \log |\mathcal{N}_{nk}| \geq \frac{-1}{|\mathcal{N}_{nk}|} \sum_{w \in \mathcal{N}_{nk}} \sum_{j=0}^{n-1} \log \kappa_k(w_{jk}, w_{(j+1)k}).$$

Taking expected value of both sides over the random graph, we get

$$\mathbf{E} \log |W_{nk}| - \log |\mathcal{N}_{nk}| \geq - \sum_{j=0}^{n-1} \frac{1}{|\mathcal{N}_{nk}|} \sum_{w \in \mathcal{N}_{nk}} \mathbf{E} \log \kappa_k(w_{jk}, w_{(j+1)k}).$$

We will use the mass transport principle to show that the expression

$$(22) \quad \sum_{w \in \mathcal{N}_{nk}} \mathbf{E} \log \kappa_k(w_{jk}, w_{(j+1)k})$$

does not depend on the position j . Let the mass transport be defined as

$$\begin{aligned} f(G, x, y) &= \sum_{w \in \mathcal{N}_{nk}(x)} \mathbf{1}(w_{(n-j)k} = y) \log \kappa_k(w_0, w_k) \\ &= \sum_{w \in \mathcal{N}_{nk}(y)} \mathbf{1}(w_{jk} = x) \log \kappa_k(w_{jk}, w_{(j+1)k}). \end{aligned}$$

That is, for every null-homotopic path w starting at x , x sends mass $\log \kappa_k(w_0, w_k)$ to the $(n - j)k$ th position of w . The second equality follows by rooting the path at y instead of x . Trivially, the mass transport does not depend on the root of G , so the mass transport principle gives us

$$\sum_{y \in V(G)} \mathbf{E} f(G, o, y) = \sum_{x \in V(G)} \mathbf{E} f(G, x, o)$$

that is, the expected mass sent from the root equals the expected mass received by the root. Plugging in the corresponding equations, we get

$$\sum_{w \in \mathcal{N}_{nk}(o)} \mathbf{E} \log \kappa_k(w_0, w_k) = \sum_{w \in \mathcal{N}_{nk}(o)} \mathbf{E} \log \kappa_k(w_{jk}, w_{(j+1)k})$$

and we get that the expression (22) does not depend on j .

This gives

$$\frac{\mathbf{E} \log |W_{nk}| - \log |\mathcal{N}_{nk}|}{nk} \geq \frac{-1}{k|\mathcal{N}_{nk}|} \sum_{w \in \mathcal{N}_{nk}(o)} \mathbf{E} \log \kappa_k(w_0, w_k).$$

The right-hand side now equals

$$-\frac{1}{k} \mathbf{E} \sum_{x \in V(G)} p(k, nk, x) \log \kappa_k(o, x)$$

with p defined in (20). For G, k fixed, the right-hand side is an average of a bounded function $\log \kappa_k(o, x)$ on the vertices x of G with respect to the distribution $p(k, nk, \cdot)$. As $n \rightarrow \infty$, this distribution converges to the distribution $p_k(\cdot)$ by Corollary 20, and so does the corresponding average by the bounded convergence theorem. Since each average is a bounded function of G , applying the bounded convergence theorem again, now for the expectation over G , we get the limiting inequality

$$\begin{aligned} \mathbf{E} \log \rho(G) - \log \rho(T_d) &\geq -\frac{1}{k} \mathbf{E} \sum_{x \in V(G)} p(k, x) \log \kappa_k(o, x) \\ &= -\frac{1}{k} \mathbf{E} \log \kappa_k^*(G, o). \end{aligned}$$

This completes the proof of the first claim of the theorem. To prove the second claim, take expectation of the logarithm of the result of Lemma 32 and use the bounded convergence theorem. \square

7. Graphs with uniformly dense short cycles. In this section, we prove Theorem 3. This part of the paper is independent of the rest as it does not use any of the results in the rest and vice versa. Theorem 3 immediately implies that vertex transitive Ramanujan graphs are trees; the proof for that [23] is to first show that every vertex transitive graph that is not a tree can be covered by a Cayley graph that is also not a tree, and then use the original Kesten’s theorem. The proof presented here is purely combinatorial. It seems tempting to try to prove Theorem 28 using this method, but we did not manage to do so.

PROOF OF THEOREM 3. Let G be an infinite d -regular graph such that every vertex in G has distance at most R from a k -cycle. For a vertex $x \in G$, let $N(x)$ be the list of endpoints of edges starting at x . For $n \geq 0$, let

$$g(n) = \frac{d + (d - 2)n}{d\sqrt{d - 1}^n}.$$

Then $g(0) = 1$ and for $n > 0$ we have

$$\frac{1}{d}(g(n - 1) + (d - 1)g(n + 1)) = \frac{2\sqrt{d - 1}}{d}g(n).$$

Also, for $n \geq 0$ the function is monotonically decreasing, as

$$(23) \quad \frac{1}{\sqrt{d - 1}} < \frac{g(n + 1)}{g(n)} \leq \frac{2\sqrt{d - 1}}{d} = \frac{g(1)}{g(0)} < 1.$$

This is the spherical function that demonstrates $\rho(T_d) \geq 2\sqrt{d - 1}/d$.

Fix $o \in G$ forever. For $r \geq 0$, let

$$S_r = \{x \in G \mid d(o, x) = r\}$$

and for abbreviation let us denote $g_r = g(r)$.

For $x \in S_r$, let

$$\begin{aligned} \deg^+(x) &= |N(x) \cap S_{r+1}|, & \deg^0(x) &= |N(x) \cap S_r| \quad \text{and} \\ \deg^-(x) &= |N(x) \cap S_{r-1}|. \end{aligned}$$

Let the set of *return points* be defined as

$$A = \{x \in G \mid \deg^-(x) + \deg^0(x) \geq 2\}.$$

Let $k' = \lfloor R + k/2 + 1 \rfloor$. By the assumption of the theorem, the k' -neighborhood of A equals the whole G .

Let $R > 0$ (this will tend to infinity later). Let us define $f_R : G \rightarrow \mathbb{R}$ as follows:

$$f_R(x) = \begin{cases} g(d(o, x)), & \text{if } d(o, x) \leq R, \\ 0, & \text{otherwise.} \end{cases}$$

Then $f_R \in l^2(G)$ and we have $\langle f_R, f_R \rangle = \sum_{r=0}^R |S_r|g_r^2$.

Let $x \in G$ and let $r = d(o, x)$.

If $r < R$ and $x \notin A$, then

$$Mf_R(x) = \frac{1}{d}(g_{r-1} + (d-1)g_{r+1}) = \frac{2\sqrt{d-1}}{d}g_r$$

otherwise

$$\begin{aligned} Mf_R(x) &= \frac{1}{d}(\text{deg}^-(x)g_{r-1} + \text{deg}^0(x)g_r + \text{deg}^+(x)g_{r+1}) \\ &\geq \frac{2\sqrt{d-1}}{d}g_r + \frac{1}{d}(g_r - g_{r+1}). \end{aligned}$$

If $r = R$, then

$$Mf_R(x) \geq \frac{1}{d}g_{R-1} \geq \frac{1}{d}g_R.$$

Using

$$g_r - g_{r+1} \geq g_r \left(1 - \frac{2\sqrt{d-1}}{d}\right) = \frac{d - 2\sqrt{d-1}}{d}g_r$$

this gives us

$$\begin{aligned} \langle Mf_R, f_R \rangle &\geq \frac{2\sqrt{d-1}}{d} \sum_{r=0}^{R-1} |S_r|g_r^2 \\ &\quad + \frac{d - 2\sqrt{d-1}}{d^2} \sum_{r=0}^{R-1} |S_r \cap A|g_r^2 + \frac{1}{d}|S_R|g_R^2 \\ &= \frac{2\sqrt{d-1}}{d} \sum_{r=0}^R |S_r|g_r^2 + \frac{d - 2\sqrt{d-1}}{d^2} \sum_{r=0}^{R-1} |S_r \cap A|g_r^2 \\ &\quad - \frac{2\sqrt{d-1} - 1}{d}|S_R|g_R^2. \end{aligned}$$

For each $x \in G$, let $a(x) \in A$ be a closest vertex in A . Then $d(x, a(x)) \leq k'$ and so evenly distributing the weight $g^2(d(o, a))$ on a to all $x \in G$ with $a(x) = a$, we get

$$\begin{aligned} \sum_{r=0}^{R-1} |S_r \cap A|g_r^2 &= \sum_{x \in A, d(o,x) \leq R-1} g^2(d(o, x)) \\ &\geq \frac{1}{B} \sum_{x \in G, d(o,x) \leq R-(k'+1)} g^2(d(o, a(x))), \end{aligned}$$

where $B = d((d - 1)^{k'} - 1)/(d - 2)$ is the size of the k' -ball in T_d . On the other hand, (23) implies

$$\frac{g^2(d(o, a(x)))}{g^2(d(o, x))} > \frac{1}{(d - 1)^{d(x, a(x))}} \geq \frac{1}{(d - 1)^{k'}}$$

and so we get

$$\sum_{r=0}^{R-1} |S_r \cap A| g_r^2 > \frac{1}{B(d - 1)^{k'}} \sum_{r=0}^{R-(k'+1)} |S_r| g_r^2.$$

Putting together and trivially estimating B , we get

$$\begin{aligned} \frac{\langle Mf_R, f_R \rangle}{\langle f_R, f_R \rangle} &> \left(\frac{2\sqrt{d-1}}{d} + \frac{d-2}{d(d-1)^{2k'}} \right) \\ &\quad - \frac{C \sum_{r=R-k'}^R |S_r| g_r^2}{\sum_{r=0}^R |S_r| g_r^2}, \end{aligned}$$

where C is an absolute constant. We get the required estimate if we show that

$$\lim_{R \rightarrow \infty} \frac{|S_R| g_R^2}{\sum_{r=0}^R |S_r| g_r^2} = 0.$$

For $r \geq 0$ let $s_r = |S_r|/(d - 1)^r$. Then trivially $s_r \geq s_{r+1}$ and

$$|S_r| g_r^2 = \frac{1}{d^2} s_r (d + (d - 2)r)^2$$

thus we get

$$\sum_{r=0}^R |S_r| g_r^2 \geq \frac{1}{d^2} s_R \sum_{r=0}^R (d + (d - 2)r)^2.$$

This gives us

$$\frac{\sum_{r=0}^R |S_r| g_r^2}{|S_R| g_R^2} \geq \frac{\sum_{r=0}^R (d + (d - 2)r)^2}{(d + (d - 2)R)^2},$$

which tends to infinity with R . The theorem is proved. \square

8. Examples of Ramanujan graphs.

8.1. *Tolerance of loops in Ramanujan graphs.* In this section, we build examples of finite and infinite Ramanujan graphs with some loops. It turns out that for infinite trees, there is a tolerance phenomenon; the tree lets us insert some loops before giving up being Ramanujan.

Recall that a Cayley graph of a group G together with a finite set of generators $S = S^{-1}$ is the graph with vertex set G and edge set $\{\{v, vs\}, s \in S\}$. Our first result shows that every Cayley graph sequence that is Ramanujan gives rise to another Ramanujan sequence with loops. The first examples for sequences of Ramanujan graphs with loops were constructed in [10].

THEOREM 34. *Let G_n be an expander sequence of finite d -regular Cayley graphs with $|G_n| \rightarrow \infty$. Then there exists H_n with $|H_n| \rightarrow \infty$ such that for all n , H_n contains a loop and G_n covers H_n . In particular, $\rho(H_n) \leq \rho(G_n)$.*

PROOF. Let F be the free group with the alphabet S and let K_n be the normal subgroup in F such that $G_n = \text{Cay}(K_n \setminus F, S)$. Let $s \in S$ and let $F_n = \langle K_n, s \rangle$ be the subgroup generated by K_n and s . Let $H_n = \text{Sch}(F_n \setminus F, S)$. Then the map between coset spaces $K_n g \mapsto F_n g$ is a covering map from G_n to H_n , since F_n contains K_n . Every eigenvector of H_n can be pulled back to be an eigenvector of G_n , which implies $\rho(H_n) \leq \rho(G_n)$. Also, $F_n s = F_n$, so H_n contains a loop.

Assume now that when passing to a subsequence, H_n has bounded size. Let N be the intersection of the K_n . Since F has only finitely many subgroups of a given index, $N \langle s \rangle$ has finite index in F . Thus, $N \setminus F$ has a cyclic subgroup of finite index; hence, it is amenable. Now a subsequence of the G_n locally converges to an infinite Cayley graph G and G is a quotient of $\text{Cay}(N \setminus F, S)$; hence, it is amenable as well. But then G has a Følner sequence, which then can be also found in the finite sequence. This implies that G_n is not an expander family, a contradiction. So, $|H_n| \rightarrow \infty$ as claimed. \square

Note that this proof only guarantees *one* loop in H_n . The known Lubotzky–Philips–Sarnak construction does not allow us to create two loops by factoring out with two generators. For infinite graphs, the picture is very different.

8.2. Infinite Ramanujan graphs are abundant. Unlike finite Ramanujan graphs, which are notoriously difficult to construct, infinite Ramanujan graphs are abundant. In fact, let G be any graph whose degrees are bounded by m . There is a unique way of embedding G into an m -regular graph $Y := \text{Tree}_m(G)$ in such a way that the embedding $\iota : G \rightarrow Y$ induces an isomorphism on fundamental groups. In fact, the graph Y is constructed by “gluing trees at every vertex” in the unique possible way that would make the resulting graph m -regular.

Now fix a base vertex $o \in G \subset Y$ and let $W_n^Y(o, o)$ [resp., $V_n^Y(o, o)$] be the sets of n -cycles (resp., non-backtracking cycles) on the graph Y . The asymptotic of these are governed by the spectral radius $\rho(Y) = \frac{1}{m} \limsup_{n \rightarrow \infty} |W_n^Y(o, o)|^{1/n}$ and the co-growth $\alpha = \alpha(Y) = \limsup_{n \rightarrow \infty} |V_n^Y(o, o)|^{1/n}$. Now Grigorchuk’s famous

co-growth formula relates these two numbers by the following formula:

$$\rho = \begin{cases} \frac{\sqrt{m-1}}{m} \left(\frac{\alpha}{\sqrt{m-1}} + \frac{\sqrt{m-1}}{\alpha} \right), & \text{if } \alpha > \sqrt{m-1}, \\ \frac{2\sqrt{m-1}}{m}, & \text{otherwise.} \end{cases}$$

This formula is obtained by comparing the radii of convergence of the generating functions corresponding to these two types of random walks; see [22], equation (2.3). This equation also plays a central role in our proof of Proposition 22.

COROLLARY 35. *Let G be a graph with maximal degree bounded by m . Then $\text{Tree}_m(G)$ is Ramanujan if and only if $m \geq \alpha^2(G) + 1$. In particular, if G is d -regular then $\text{Tree}_m(G)$ is Ramanujan whenever $m \geq d^2 - 2d + 2$.*

PROOF. Clearly, $\alpha(G) = \alpha(Y)$. The first statement follows, since by definition the graph $Y = \text{Tree}_m(G)$ is Ramanujan if and only if it falls into the second clause of the above formula. The second statement follows since $\alpha(G) \leq d - 1$ for any d -regular graph. \square

An open question of Itai Benjamini (private communication) asks whether there exist infinite Ramanujan graphs where all bounded harmonic functions are constant. This calls for different examples.

9. A unimodular random graph of maximal growth. For a rooted graph G , let S_n denote the vertices at distance n from the root. Let

$$\underline{\text{gr}} G = \liminf_{n \rightarrow \infty} |S_n|^{1/n}.$$

Clearly, for every d -regular graph $\underline{\text{gr}} G \leq d - 1$, and $\underline{\text{gr}} \mathbb{T}_d = d - 1$. The goal of this section is to prove Theorem 8 from the Introduction, namely to exhibit a d -regular unimodular random graph G different from \mathbb{T}_d where $\underline{\text{gr}} G = \underline{\text{gr}} \mathbb{T}_d = d - 1$ almost surely.

For this, we consider site percolation on \mathbb{Z}^2 , namely a random induced subgraph where every vertex is present with probability p and absent with probability $1 - p$, independently. For p large, the connected component of the origin is infinite with positive probability. Let $\overline{\mathcal{C}}$ denote the distribution of the universal cover of the cluster given that it is infinite; this is a tree with degree bounded by 4, but is not 4-regular. It can be made 4-regular by adding loops.

THEOREM 36. *The rooted random graph $\overline{\mathcal{C}}$ is a unimodular random graph satisfying $\underline{\text{gr}} \overline{\mathcal{C}} = 3$ with probability 1.*

The following lemma follows from the definition of unimodular random graphs.

LEMMA 37. *The universal cover of a unimodular random graph is a unimodular random graph.*

Let \mathcal{C} be a connected, induced subgraph of \mathbb{Z}^2 , and let b_r be the size of the largest square fully contained in \mathcal{C} whose center is at distance at most r in \mathcal{C} from a fixed vertex. Fix $a > 0$, and consider the following property of \mathcal{C} :

$$(24) \quad \liminf_{r \rightarrow \infty} \frac{b_r}{\log r} \geq a.$$

It is clear that this property does not depend on the fixed vertex. Whether the infinite cluster in supercritical percolation has this property is a tail event, so it has probability 0 or 1, although we will not use this. We will argue for the latter.

LEMMA 38. *There is $a = a(p)$ so that the supercritical percolation cluster \mathcal{C} satisfies property (24) with probability 1.*

PROOF. The fact that the set of open vertices in a percolation cluster with $p > 0$ satisfies this property (with distance in \mathbb{Z}^2 instead of distance in \mathcal{C}) is a simple exercise using independence and the Borel–Cantelli lemmas.

We now use the two-round exposure technique, namely the following construction of the set of open vertices of supercritical percolation at parameter p . Take the union of open vertices in a supercritical percolation with parameter $p' < p$, and an independent site percolation with parameter p'' where $p = p' + p'' - p'p''$.

Consider the percolation at p' . Note that its infinite cluster \mathcal{C}' is unique and dense in \mathbb{Z}^2 . Dense here means that the root (and so every vertex) has a positive probability of being contained in this cluster. Moreover, by the standard Antral–Pisztora result [4], there is a constant η so that the set of vertices \mathcal{C}^+ in \mathcal{C}' whose distance in \mathcal{C} is at most η times their \mathbb{Z}^2 distance from the vertex in \mathcal{C} closest to 0 is also dense.

Given this dense set of vertices \mathcal{C}^+ , we can use the independent percolation at p'' to add squares of size $c \log r$ at distance r that are connected to \mathcal{C}^+ . It follows that the infinite open cluster in the union of the two site percolations has the desired properties. \square

LEMMA 39. *Let \mathcal{C} be a connected subgraph of \mathbb{Z}^2 satisfying property (24). Then the probability that simple random walk exits from \mathcal{C} in r steps decays slower than exponentially in r .*

PROOF. Note that the probability that the random walk on \mathbb{Z}^2 starting at the center of a square of volume v in \mathbb{Z}^2 , stays there for time at least t is bounded below by $q^{t/v}$ for some $q < 1$.

So, the probability that the random walk moves in \mathcal{C} on a geodesic to a square of size $c \log r$ at distance r , and there for time $r \log r$, is at least $e^{-c'r}$. The claim follows. \square

LEMMA 40. *Let \mathcal{C} be a subgraph of a d -regular graph so that the probability that the random walk stays in \mathcal{C} for n steps decays slower than exponentially in n . Then the universal cover of \mathcal{C} has lower growth $d - 1$.*

PROOF. Let A_n denote the event that random walk stays in \mathcal{C} for n steps. Let s_n be the size of the sphere in the universal cover. Then the probability of the event B_n that non-backtracking random walk on the base graph stays in \mathcal{C} until time n is given by

$$P(B_n) = \frac{s_n}{d(d-1)^{n-1}}.$$

Note also that running ordinary random walk until time n and deleting the back-trackings, we get non-backtracking random walk run until a random time $N_n \leq n$. Indeed, erasing the backtrackings just means taking the geodesic from the starting point to the current vertex in the universal cover tree.

Standard arguments show that $N_n/n \rightarrow 1 - 2/d$ and the event that $N_n/n < \alpha$ for $\alpha < 1 - 2/d$ fixed has probability that is exponentially small in n . Thus, we have

$$\begin{aligned} P(A_n) &\leq \sum_{k=0}^n P(N_n = k)P(B_k) \leq P(N_n < \alpha n) + \sum_{k=\alpha n}^n P(N_n = k)P(B_{\alpha n}) \\ &\leq P(N_n < \alpha n) + P(B_{\alpha n}) \end{aligned}$$

and, therefore,

$$\frac{s_n}{d(d-1)^{n-1}} = P(B_n) \geq P(A_{n/\alpha}) - P(N_{n/\alpha} < n),$$

where the first probability decays slower than exponentially, and the second exponentially. The claim follows. \square

PROOF OF THEOREM 36. The component of the origin in the supercritical percolation in \mathbb{Z}^2 is unimodular, so it must be one even when conditioned to be infinite. In this case, it satisfies property (24). Then its universal cover is a unimodular random graph with lower growth $d - 1$. \square

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M. ABÉRT
ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS
REÁLTANODA UTCA 13-15, H-1053
BUDAPEST
HUNGARY
E-MAIL: abert@renyi.hu

Y. GLASNER
DEPARTMENT OF MATHEMATICS
BEN-GURION UNIVERSITY OF THE NEGEV
P.O.B. 653
BE'ER SHEVA 84105
ISRAEL
E-MAIL: yairgl@math.bgu.ac.il

B. VIRÁG
DEPARTMENTS OF MATHEMATICS AND STATISTICS
UNIVERSITY OF TORONTO
TORONTO, ONTARIO
M5S 2E4
CANADA
E-MAIL: balint@math.toronto.edu