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The speed of a biased walk on a Galton–Watson tree without leaves is monotonic with respect to progeny distributions for high values of bias

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Abstract. Consider biased random walks on two Galton–Watson trees without leaves having progeny distributions P_1 and P_2 ($\mathrm{GW}(P_1)$ and $\mathrm{GW}(P_2)$) where P_1 and P_2 are supported on positive integers and P_1 dominates P_2 stochastically. We prove that the speed of the walk on $\mathrm{GW}(P_1)$ is bigger than the same on $\mathrm{GW}(P_2)$ when the bias is larger than a threshold depending on P_1 and P_2. This partially answers a question raised by Ben Arous, Fribergh and Sidoravicius (*Comm. Pure Appl. Math.* **67** (2014) 519–530).

Résumé. Nous considérons des marches aléatoires biaisées sur deux arbres de Galton–Watson sans feuilles $\mathrm{GW}(P_1)$ et $\mathrm{GW}(P_2)$ ayant des lois de reproduction respectivement P_1 et P_2, deux lois supportées par les entiers positifs telles que P_1 domine stochastiquement P_2. Nous prouvons que la vitesse de la marche sur $\mathrm{GW}(P_1)$ est supérieure ou égale à celle sur $\mathrm{GW}(P_2)$ si le biais est plus grand qu'un seuil dépendant de P_1 et P_2. Ceci répond partiellement à une question posée par Ben Arous, Fribergh et Sidoravicius (*Comm. Pure Appl. Math.* **67** (2014) 519–530).

MSC: 60K37; 60J80; 60G50

Keywords: Random walk in random environment; Galton–Watson tree; Speed; Stochastic domination

1. Introduction and main results

1.1. *Introduction*

Consider a supercritical Galton–Watson tree, i.e., a random rooted tree, where the offspring size of all individuals are i.i.d. copies of an integer random variable Z, which satisfies $P(Z=k)=p_k$, $k=0,1,\dots$ and $\sum_{k\geq 1} kp_k > 1$. The tree has no leaves if $p_0=0$. We shall use $|x|$ to denote the distance of a vertex x from the root. Moreover x_* will denote the ancestor of x for any vertex x different from the root and x_i will denote the ith child of x. Given a tree T and $\beta>0$, we define β-biased random walk $(X_n)_{n\geq 0}$ on T as follows. Transitions to each of the children of the root are equally likely. If the vertex x has k children and x is not the root then the transition probabilities are given by

$$P(X_{n+1}=x_*|X_n=x)=\frac{1}{1+\beta k},$$
$$P(X_{n+1}=x_i|X_n=x)=\frac{\beta}{1+\beta k},\quad i=1,2,\dots,k.$$

We start the walk from the root of the tree and denote by P^ω the law of $(X_n)_{n\geq 0}$ on a tree ω. We define the averaged law as the semi-direct product $\mathbb{P} = \overline{P} \times P^\omega$ where $\overline{P}$ is the Galton–Watson measure (associated with offspring distribution P) on the space of rooted trees conditioned on non-extinction.

Lyons [6] proved that if $\beta > \frac{1}{E[Z]}$, then the random walk is transient, i.e., $\lim_{n\to\infty} |X_n| = \infty$. Lyons, Pemantle and Peres [7] showed that $\mathbb{P}$ almost surely, the speed

$$v(\beta, P) := \lim_{n\to\infty} \frac{|X_n|}{n} \tag{1.1}$$

exists and is a non-random constant. A lot of work has been done on the behavior of the speed as a function of β. It was conjectured in [7] that $v(\beta, P)$ increases in β on $(\frac{1}{E[Z]}, \infty)$ when the tree has no leaves, i.e., $P\{0\} = 0$. The conjecture has been open for a long time until proven recently in [3] for large values of β.

Theorem ([3]). *The speed $v(\beta, P)$ of a β-biased random walk on a Galton–Watson tree without leaves is increasing for $\beta > \beta_c$ for some $\beta_c > 0$ very large when $P\{0\} = 0$.*

Very recently, Aïdékon obtained an expression for the speed v.

Theorem ([1]).

$$v(\beta, P) = \frac{\mathbb{E}[(\beta Z - 1)Y_0/(1 - \beta + \beta \sum_{i=0}^{Z} Y_i)]}{\mathbb{E}[(\beta Z + 1)Y_0/(1 - \beta + \beta \sum_{i=0}^{Z} Y_i)]}, \tag{1.2}$$

where Y_i are i.i.d. copies distributed as $P_x(\tau_{x_*} = \infty)$ and τ_y is the first hitting time of y.

Using his own formula, Aïdékon (private communications) can prove the monotonicity for $\beta \geq 2$ when $P\{0\} = 0$. However, the original conjecture is still open in the sense that it is not known if the monotonicity holds for every $\beta > 1/E[Z]$.

In this paper we shall investigate how the speed changes when one changes the progeny distribution keeping the bias fixed.

The paper is organized as the following. In Section 1.2, we shall introduce our main results. In Section 2, we shall describe in details our coupling method. Finally, in Section 3, we shall provide the proofs of all the results in Section 1.2.

1.2. *Main results*

In [3], the authors raised the following interesting question, if P_1 stochastically dominates P_2, does it imply that $v(\beta, P_1) \geq v(\beta, P_2)$? We show that this is indeed the case at least when the bias is large.

Throughout this paper, when we say P_1 dominates P_2 stochastically, we also mean that $P_1 \neq P_2$. We also recall that if P_1 dominates P_2 stochastically then there is a coupling of the random variables Z_1 and Z_2 having distributions P_1 and P_2 respectively such that $Z_2 \leq Z_1$.

We have the following result.

Theorem 1. *Assume that P_1 and P_2 are two probability measures on positive integers such that P_1 dominates P_2 stochastically. Consider β-biased random walks on $GW(P_1)$ and $GW(P_2)$. Then for every $\delta > 0$, there exists a $\beta_0 := \beta_0(P_1, P_2, \delta) > 0$ such that for any $\beta > \beta_0$, we have $v(\beta, P_1) > v(\beta, P_2)$. The constant β_0 equals $\max\{\beta_1, \frac{23}{4} + \delta\}$ where*

$$\beta_1 := c_\delta \cdot \min\left\{ \frac{E[(1/Z_1 - 1/Z_2)1_{Z_1 < Z_2}]}{E[1/Z_2' - 1/Z_1']}, \frac{E[Z_2'(1/Z_2' - 1/Z_1')]}{E[1/Z_2' - 1/Z_1']} + 1 \right\}, \tag{1.3}$$

and c_δ is a universal constant depending only on δ. Here, Z_1, Z_2 are independent and are distributed according to P_1 and P_2 respectively, Z_1' and Z_2' are jointly distributed so that $Z_1' \geq Z_2'$ almost surely and their marginal distributions are P_1 and P_2.

Remark 2. *There is a universal cut-off $\beta_1 = \beta_1(M)$ which works for all P_2 supported on $\{1, 2, \dots, M\}$ since we have*

$$E\left[Z_2'\left(\frac{1}{Z_2'} - \frac{1}{Z_1'}\right)\right] \le M \cdot E\left[\frac{1}{Z_2'} - \frac{1}{Z_1'}\right].$$

The other expression inside the parentheses in the definition of β_1 in Theorem 1 is more useful when "the distribution of Z_1 is much larger than that of Z_2", we shall illustrate this in Corollary 5.

Remark 3. *Suppose P_1 dominates P_2 and are both supported on positive integers. Then $v(\beta, P_1) \ge v(\beta, P_2)$ follows trivially in the following cases.*

(i) *It is easy to see* (*via a coupling argument*) *that if the maximum of the support of P_2 is not larger than the minimum of the support of P_1, then for any $\beta > 0$, we have $v(\beta, P_1) \ge v(\beta, P_2)$.*
(ii) *We have $v(1, P_1) \ge v(1, P_2)$ just by considering the expression*

$$v(1, P) = E_P\left[\frac{Z-1}{Z+1}\right]$$

obtained in [7].
(iii) *Note that $v(1/E_{P_2}[Z], P_2) = 0$, $v(1/E_{P_2}[Z], P_1) > 0$, and $v(\beta, P_j)$ is continuous in β for $j = 1, 2$. Thus, for some small $\varepsilon > 0$ we have $v(\beta, P_1) \ge v(\beta, P_2)$ for $0 < \beta < \varepsilon + 1/E_{P_2}[Z]$.*

Further (ii) *and* (iii) *hold even when the offspring distributions are supported on non-negative integers as long as we define the speed as in* (1.1) *conditional on non-extinction of the trees.*

We can improve the threshold β_0 of Theorem 1 by making stronger assumptions.

Theorem 4. *Suppose P_1 and P_2 are two probability measures on positive integers such that for some $\ell > 1$, there exists a coupling of $Z_1^{(1)}, Z_1^{(2)}, \dots, Z_1^{(\ell)}$ and $Z_2^{(1)}, Z_2^{(2)}, \dots, Z_2^{(\ell)}$ for which $\min\{Z_1^{(1)}, Z_1^{(2)}, \dots, Z_1^{(\ell)}\} \ge \max\{Z_2^{(1)}, Z_2^{(2)}, \dots, Z_2^{(\ell)}\}$ almost surely, where $Z_j^{(1)}, \dots, Z_j^{(\ell)}$ are i.i.d. distributed according to P_j for $j = 1, 2$. Then for any $\delta > 0$, we have $v(\beta, P_1) \ge v(\beta, P_2)$ for any $\beta > \max\{K \cdot \beta_1^{1/\ell}, \frac{23}{4} + \delta\}$ where the constant K equals $\frac{27}{4} \cdot 3^{5/3}$.*

Corollary 5. *Assume that P_1 and P_2 are two probability measures on positive integers such that P_1 stochastically dominates P_2. Let $m_i := E_{P_i}[Z]$ and $Z_i^{(n)}$ be the number of children in the nth generation in $GW(P_i)$, denote the law of $Z_i^{(n)}$ by $P_i^{(n)}$ for $i = 1, 2$. Assume that there exists some $\theta > 0$ such that $E[e^{\theta Z_1^{(1)}}] < \infty$. Let f be the generating function for P_1 and $\alpha := -\log f'(0)/\log f'(1)$. Further assume that $m_1 > m_2^{\max\{2/\alpha, (1/\alpha)+1\}}$ (if $P_1\{1\} = 0$, then $\alpha = \infty$ and this condition is automatically satisfied).*

Then, for any $\beta > 23/4$, there exists some $k = k(P_1, P_2, \beta)$ such that $v(\beta, P_1^{(k)}) > v(\beta, P_2^{(k)})$. (*We emphasize that $v(\beta, P_i^{(k)})$ is the speed of a β-biased random walk on a Galton–Watson tree having $P_i^{(k)}$ as its offspring distribution.*)

The following corollary is the counterpart to Theorem 1.2 in [3].

Corollary 6. *Assume all the assumptions in Theorem 1 and recall the definition of β_0 from there. Moreover, assume that the minimum degrees of both P_1 and P_2 are bigger than d (for some $d \ge 2$), i.e., $d_i := \min\{k \ge 1, P_i(Z = k) > 0\} \ge d$, for $i \in \{1, 2\}$.*

Then we have, $v(\beta, P_1) > v(\beta, P_2)$ for any $\beta > \beta_0/d$.

2. Constructing the walks

Let us describe precisely the coupling we use. Let U_1 have uniform distribution on $(1/(\beta + 1), 1)$. Let $(U_i)_{i\ge 2}$ be i.i.d. uniformly distributed random variables on $[0, 1]$ independent of U_1. Let $\{(Z'_{1,k}, Z'_{2,k})\}_{k\ge 1}$ be i.i.d. random vectors such that for each k, $Z'_{1,k}$ has the marginal distribution P_1 and $Z'_{2,k}$ has the marginal distribution

P_2 and with probability 1, we have $Z'_{2,k} \le Z'_{1,k}$. Finally let $\{Z_{i,k}\}_{k\ge1}$ be i.i.d. P_i for $i = 1, 2$. The sequences $\{U_i\}_{i\ge1}, \{Z_{1,k}\}_{k\ge1}, \{Z_{2,k}\}_{k\ge1}, \{(Z'_{1,k}, Z'_{2,k})\}_{k\ge1}$ are independent of each other.

In our proof we shall work conditional on an event which ensures that the roots are only visited once, for this reason we only need one copy of U_1. Note that our definition of U_1 is slightly different from the one in [3].

We construct two random walks $X_n^{(1)}$ and $X_n^{(2)}$ (on $GW(P_1)$ and $GW(P_2)$) and another walk Y_n on $\mathbb{Z}_{\ge0}$ in the following way. Define $Y_0 := 0$ and for $n \ge 1$,

$$Y_n := \sum_{i=1}^{n} \{1_{U_i>1/(\beta+1)} - 1_{U_i\le1/(\beta+1)}\}, \quad n \in \mathbb{N}.$$

We start $X^{(1)}$ and $X^{(2)}$ at the roots and grow the trees $GW(P_1)$ and $GW(P_2)$ dynamically. For simplicity we drop the time parameter n and denote the position of $X_n^{(i)}$ by $x^{(i)}$.

Now, if at time $n \ge 0$, $X_n^{(1)}$ and $X_n^{(2)}$ are at two sites $x^{(1)}$ and $x^{(2)}$, neither of them visited before by the corresponding walks, then we assign $Z'_{1,n+1}$ and $Z'_{2,n+1}$ many children to $x^{(1)}$ and $x^{(2)}$ respectively (recall that $Z'_{1,n+1} \ge Z'_{2,n+1}$).

If at time n, one of the walks, say $X^{(1)}$ is at a site $x^{(1)}$ previously visited by the walk while the other walk $X^{(2)}$ is at a new site $x^{(2)}$ then we assign $Z_{2,n+1}$ many children to $x^{(2)}$.

Let us now explain the rules for transition. Denote the number of offsprings of $x^{(i)}$ by Z_i and let $x_k^{(i)}$ be the kth child of $x^{(i)}$ $(i = 1, 2)$.

Define

$$\eta_1 := \frac{\beta}{(\beta+1)Z_1}, \qquad \eta_2 := \left(\frac{\beta}{\beta+1}\right)\left(\frac{1}{Z_2} - \frac{1}{Z_1}\right), \qquad \eta_3 := \left(\frac{1}{\beta+1} - \frac{1}{Z_2\beta+1}\right)\frac{1}{Z_2},$$
$$\eta_4 := \left(\frac{1}{\beta+1} - \frac{1}{Z_1\beta+1}\right)\frac{1}{Z_1}, \qquad \eta_5 := |\eta_3 - \eta_4|.$$

Then whenever $Z_1 \ge Z_2$, we move according to the rule explained below.

When $U_{n+1} \in (1/(\beta+1), 1)$ we have the following cases.

(1) Consider the random walk $X^{(1)}$.

- If $U_{n+1} \in (\frac{1}{\beta+1} + (i-1)\eta_1, \frac{1}{\beta+1} + i\eta_1]$, then $X_{n+1}^{(1)} = x_{Z_1+1-i}^{(1)}$ for $i = 1, 2, \dots, Z_1$.

(2) Consider the random walk $X^{(2)}$.

- If $U_{n+1} \in (\frac{1}{\beta+1} + (i-1)\eta_2, \frac{1}{\beta+1} + i\eta_2]$, then we have $X_{n+1}^{(2)} = x_{Z_2+1-i}^{(2)}$, where $i = 1, 2, \dots, Z_2$.
- If $U_{n+1} \in (\frac{1}{\beta+1} + Z_2\eta_2 + (i-1)\eta_1, \frac{1}{\beta+1} + Z_2\eta_2 + i\eta_1]$, then we have $X_{n+1}^{(2)} = x_{Z_2+1-i}^{(2)}$, where $i = 1, 2, \dots, Z_2$.

When $U_{n+1} \in (0, 1/(\beta+1))$ we have to consider two cases. If $\eta_3 \ge \eta_4$, then we use the following coupling. Figure 1 gives an illustration.

(1) Consider the random walk $X^{(1)}$.

- If $U_{n+1} \in [0, \frac{1}{Z_1\beta+1}]$, then we have $X_{n+1}^{(1)} = x_*^{(1)}$.
- If $U_{n+1} \in (\frac{1}{Z_1\beta+1} + (i-1)\eta_4, \frac{1}{Z_1\beta+1} + i\eta_4]$, then we have $X_{n+1}^{(1)} = x_{Z_1+1-i}^{(1)}$, where $i = 1, 2, \dots, Z_1$.

(2) Consider the random walk $X^{(2)}$.

- If $U_{n+1} \in [0, \frac{1}{Z_2\beta+1}]$, then we have $X_{n+1}^{(2)} = x_*^{(2)}$.
- If $U_{n+1} \in (\frac{1}{Z_2\beta+1} + (i-1)\eta_5, \frac{1}{Z_2\beta+1} + i\eta_5]$, then we have $X_{n+1}^{(2)} = x_{Z_2+1-i}^{(2)}$, where $i = 1, 2, \dots, Z_2$.
- If $U_{n+1} \in (\frac{1}{Z_2\beta+1} + Z_2\eta_5 + (i-1)\eta_4, \frac{1}{Z_2\beta+1} + Z_2\eta_5 + i\eta_4]$, then we have $X_{n+1}^{(2)} = x_{Z_2+1-i}^{(2)}$, where $i = 1, 2, \dots, Z_2$.

If $\eta_3 < \eta_4$, then we use the following coupling. Figure 2 is an illustration of the following coupling.

(1) Consider the random walk $X^{(1)}$.

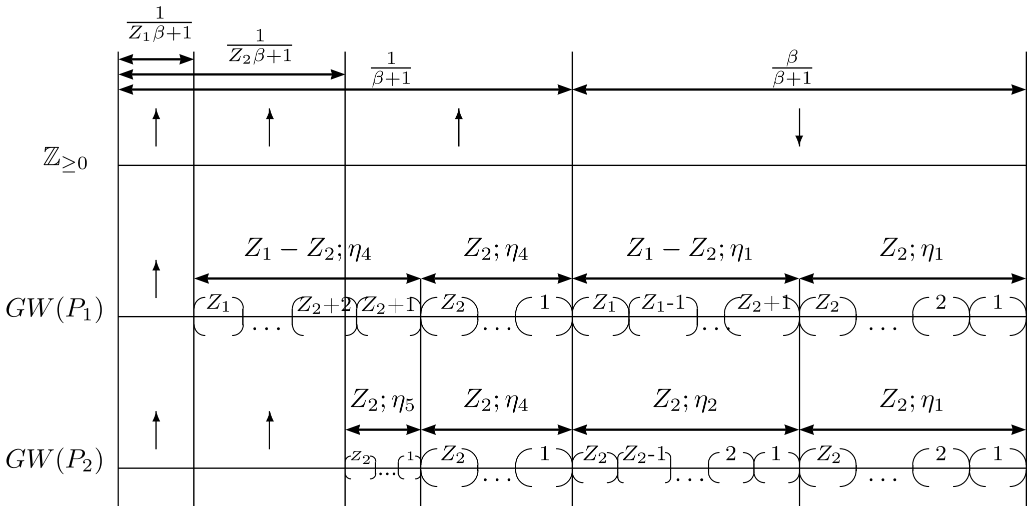

Fig. 1. The coupling for $\eta_3 \geq \eta_4$. In the illustration, we use Z_2; η_4 etc. to denote Z_2 many subintervals with each subinterval of length η_4 etc.

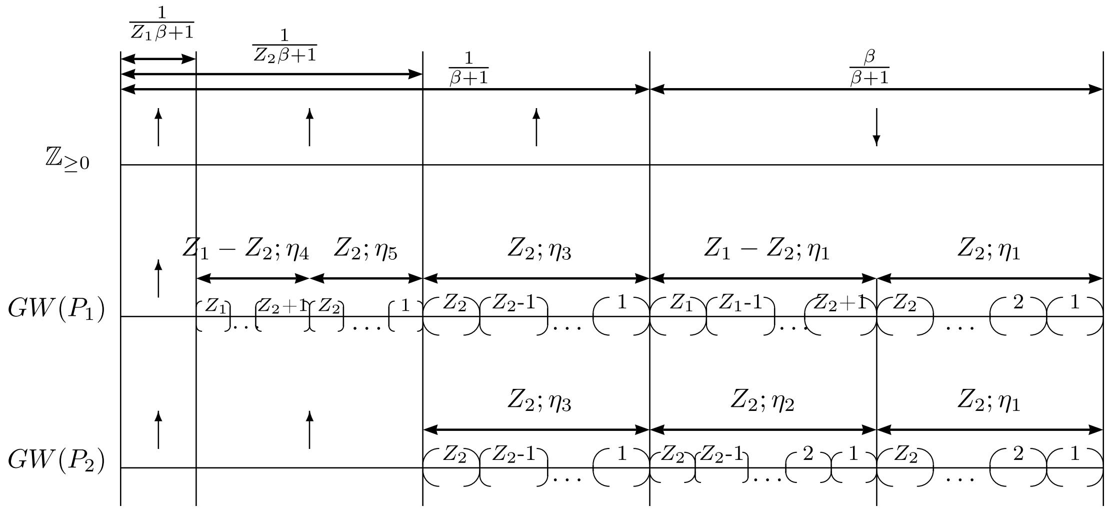

Fig. 2. The coupling for $\eta_4 > \eta_3$.

- If $U_{n+1} \in [0, \frac{1}{Z_1\beta+1}]$, then we have $X^{(1)}_{n+1} = x^{(1)}_*$.
- If $U_{n+1} \in (\frac{1}{Z_1\beta+1} + (i-1)\eta_4, \frac{1}{Z_1\beta+1} + i\eta_4]$, then we have $X^{(1)}_{n+1} = x^{(1)}_{Z_1+1-i}$, where $i = 1, 2, \dots, Z_1 - Z_2$.
- If $U_{n+1} \in (\frac{1}{Z_1\beta+1} + (Z_1 - Z_2)\eta_4 + (i-1)\eta_5, \frac{1}{Z_1\beta+1} + (Z_1 - Z_2)\eta_4 + i\eta_5]$, then $X^{(1)}_{n+1} = x^{(1)}_{Z_2+1-i}$, where $i = 1, 2, \dots, Z_2$.
- If $U_{n+1} \in (\frac{1}{Z_2\beta+1} + (i-1)\eta_3, \frac{1}{Z_2\beta+1} + i\eta_3]$, then we have $X^{(1)}_{n+1} = x^{(1)}_{Z_2+1-i}$, where $i = 1, 2, \dots, Z_2$.

(2) Consider the random walk $X^{(2)}$.

- If $U_{n+1} \in [0, \frac{1}{Z_2\beta+1}]$, then we have $X^{(2)}_{n+1} = x^{(2)}_*$.
- If $U_{n+1} \in (\frac{1}{Z_2\beta+1} + (i-1)\eta_3, \frac{1}{Z_2\beta+1} + i\eta_3]$, then we have $X^{(2)}_{n+1} = x^{(2)}_{Z_2+1-i}$, where $i = 1, 2, \dots, Z_2$.

Finally if $Z_1 < Z_2$ we move according to the following rule.

(1) For $i = 1, 2$:

- If $U_{n+1} \in [0, \frac{1}{Z_i\beta+1}]$, then we have $X^{(i)}_{n+1} = x^{(i)}_*$.
- If $U_{n+1} \in (\frac{1}{Z_i\beta+1} + (j-1)\frac{\beta}{Z_i\beta+1}, \frac{1}{Z_i\beta+1} + j\frac{\beta}{Z_i\beta+1}]$, then we have $X^{(i)}_{n+1} = x^{(i)}_j$, where $j = 1, 2, \dots, Z_i$.

It is routine to check that $X^{(i)}$ is a β-biased random walk on $GW(P_i)$ for $i = 1, 2$.

3. Proofs

The main idea in our proof is to use a technique originally used in [3], to couple the walks on the Galton–Watson trees with a random walk on $\mathbb{Z}$. We shall use a super-regeneration time which is a regeneration time for all the three walks Y, $GW(P_1)$ and $GW(P_2)$. Regeneration time is an often-used technique in the study of random walks in random media. (See, e.g., [10].) Informally, a regeneration time is a maximum of a random walk which is also a minimum of the future of the random walk. A time τ is a regeneration time for the β-biased random walk $(Y_n)_{n\ge 0}$ on $\mathbb{Z}$ if we have

$$Y_\tau > \max_{n<\tau} Y_n \quad \text{and} \quad Y_\tau < \min_{n>\tau} Y_n.$$

Consider the regeneration time for walks on $GW(P_1)$ and $GW(P_2)$ in the sense that is usually defined on trees (see [8]). As in [3] if τ is a regeneration time for $(Y_n)_{n\ge 0}$, then it is also a regeneration time for $GW(P_1)$ and $GW(P_2)$. In this respect, τ is called a super-regeneration time.

Let us consider the event that 0 is a regeneration time for $(Y_n)_{n\ge 0}$. Following the notation in [3], we denote this event by $\{0 - SR\}$. Then, we have

$$p_\infty := P(0 - SR) = \frac{\beta - 1}{\beta}.$$

Let us define the probability measure $\tilde{P}$ as

$$\tilde{P}(\cdot) := P(\cdot | 0 - SR).$$

Under $\tilde{P}$, 0 is a regeneration time and let τ_i be ith non-zero regeneration time.

Then, $(|X_{\tau_{i+1}} - X_{\tau_i}|, \tau_{i+1} - \tau_i)_{i\ge 1}$ is a sequence of i.i.d. random vectors having the same distribution as $(|X_{\tau_1}|, \tau_1)$ under $\tilde{P}$ and as in [3], we have, for any $\beta > 1$,

$$v(\beta, P_1) = \frac{\tilde{E}[|X^{(1)}_{\tau_1}|]}{\tilde{E}[\tau_1]} \quad \text{and} \quad v(\beta, P_2) = \frac{\tilde{E}[|X^{(2)}_{\tau_1}|]}{\tilde{E}[\tau_1]}.$$

Hence, $v(\beta, P_1) > v(\beta, P_2)$ is equivalent to $\tilde{E}[|X^{(1)}_{\tau_1}|] > \tilde{E}[|X^{(2)}_{\tau_1}|]$.

Let us denote by $\mathcal{B}$ the set of times before τ_1 when the random walk on $\mathbb{Z}_{\ge 0}$ takes a step back, i.e., $\mathcal{B} = \{j \le \tau_1 | U_j \le 1/(\beta+1)\}$.

We quote the following lemma from [3].

Lemma 7 (Lemma 4.1, [3]). *If* $\{|\mathcal{B}| = k\}$, *then* $\{\tau_1 \le 3k + 2\}$.

Proof of Theorem 1. Consider $|\mathcal{B}| = k$, i.e., $\mathcal{B} = \{i_1 < \cdots < i_k\}$, where $k \ge 1$ and $\tau_1 = n$. Let us make two simple observations.

(i) $|X^{(1)}_{\tau_1}| - |X^{(2)}_{\tau_1}| = 2$ or 0 when $k = 1$.
(ii) $|X^{(1)}_{\tau_1}| - |X^{(2)}_{\tau_1}| \ge -2(k-1)$ when $k \ge 2$.

We have

$$\begin{aligned}
&\tilde{E}\big[\big|X^{(1)}_{\tau_1}\big| - \big|X^{(2)}_{\tau_1}\big|\big] \\
&\quad = \tilde{E}\big[\big|X^{(1)}_{\tau_1}\big| - \big|X^{(2)}_{\tau_1}\big|; |\mathcal{B}| = 1\big] + \sum_* \tilde{E}\big[\big|X^{(1)}_{\tau_1}\big| - \big|X^{(2)}_{\tau_1}\big|; \mathcal{B} = \{i_1 < \cdots < i_k\}, \tau_1 = n\big]
\end{aligned}$$

$$\geq \tilde{E}\big[\big|X_{\tau_1}^{(1)}\big| - \big|X_{\tau_1}^{(2)}\big|; |\mathcal{B}| = 1\big]$$
$$- \sum_{*} 2(k-1)\tilde{P}\big(\big|X_{\tau_1}^{(1)}\big| - \big|X_{\tau_1}^{(2)}\big| < 0; \mathcal{B} = \{i_1 < \cdots < i_k\}, \tau_1 = n\big),$$

where $\sum_*$ stands for summation over all $n \geq 2, k \geq 2$ and $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ for which the walk Y_k does not come back to the origin.

For the first term, we have

$$\tilde{E}\big[\big|X_{\tau_1}^{(1)}\big| - \big|X_{\tau_1}^{(2)}\big|; |\mathcal{B}| = 1\big] \geq 2\left(\frac{\beta}{\beta+1}\right)^4 E\left[\frac{1}{Z_2\beta+1} - \frac{1}{Z_1\beta+1}\right]. \tag{3.1}$$

Let us explain the inequality in (3.1). Let $\varepsilon_i = \mathbb{I}(U_i \geq 1/(\beta+1)) - \mathbb{I}(U_i < 1/(\beta+1))$. When $|\mathcal{B}| = 1$, $|X_{\tau_1}^{(1)}| - |X_{\tau_1}^{(2)}| = 2$ or 0, hence we have

$$\tilde{E}\big[\big|X_{\tau_1}^{(1)}\big| - \big|X_{\tau_1}^{(2)}\big|; |\mathcal{B}| = 1\big] = \frac{2}{p_\infty} P\big(\big|X_{\tau_1}^{(1)}\big| - \big|X_{\tau_1}^{(2)}\big| = 2; |\mathcal{B}| = 1; 0 - SR\big)$$

and thus we get the lower bound in (3.1) by considering the event

$$\mathcal{A} = \big\{\varepsilon_1 = \varepsilon_2 = 1, \varepsilon_3 = -1 \text{ and } \big|X_3^{(1)}\big| - \big|X_3^{(2)}\big| = 2, \varepsilon_4 = \varepsilon_5 = 1, \tau_1 = 5\big\}.$$

For the second term, we have

$$\tilde{P}\big(\big|X_{\tau_1}^{(1)}\big| - \big|X_{\tau_1}^{(2)}\big| < 0; \mathcal{B} = \{i_1 < \cdots < i_k\}, \tau_1 = n\big)$$
$$\leq \frac{1}{p_\infty} P\big(\big|X_{\tau_1}^{(1)}\big| - \big|X_{\tau_1}^{(2)}\big| < 0; \mathcal{B} = \{i_1 < \cdots < i_k\}, \tau_1 = n\big).$$

On $\{|X_{\tau_1}^{(1)}| - |X_{\tau_1}^{(2)}| < 0\}$, let σ be the first time when the walk on $GW(P_1)$ goes up but the walk on $GW(P_2)$ goes down, necessarily $\sigma \in \mathcal{B}$. (When $\{|X_{\tau_1}^{(1)}| - |X_{\tau_1}^{(2)}| \geq 0\}$, define $\sigma = \infty$.) We introduce some notation here. Given a sequence $\theta = \{\theta_n\}_{n\geq 1}$ where $\theta_n = \pm 1$ we denote by $\tau(\theta)$, the first non-zero regeneration time for the walk $Z_n = \sum_{i=1}^n \theta_i$; e.g., the first non-zero regeneration time for Y_n, τ_1 equals $\tau(\varepsilon)$ where $\varepsilon = \{\varepsilon_n\}_{n\geq 1}$ and $\varepsilon_n = \mathbb{I}(U_n \geq 1/(\beta+1)) - \mathbb{I}(U_n < 1/(\beta+1))$. Define

$$\tau_1^{(j)} = \tau\big(\varepsilon^{(j)}\big) \quad \text{where } \varepsilon^{(j)} := \{\varepsilon_1, \ldots, \varepsilon_{j-1}, -1, \varepsilon_{j+1}, \ldots\}.$$

Let $\mathcal{B}^{(j)} := \{i \leq \tau_1^{(j)}\colon \varepsilon_i^{(j)} = -1\}$. Also define

$$\tau 1_{(j)} = \tau(\varepsilon_{(j)}) \quad \text{where } \varepsilon_{(j)} := \{\varepsilon_1, \ldots, \varepsilon_{j-1}, +1, \varepsilon_{j+1}, \ldots\}.$$

Let $\mathcal{B}_{(j)} := \{i \leq \tau 1_{(j)}\colon \varepsilon_{(j)_i} = -1\}$. Note that, for fixed j, $\varepsilon^{(j)}$ and $\varepsilon_{(j)}$ are functions of $\{U_i\colon i \neq j\}$, and are hence independent of U_j.

Also note that if $|X_{\tau_1}^{(1)}| - |X_{\tau_1}^{(2)}| < 0$ then the event

$$\mathcal{E} = \bigcup_{i,j\leq\tau_1} \{Z_{1,i} < Z'_{2,j}\} \bigcup_{i,j\leq\tau_1} \{Z'_{1,i} < Z_{2,j}\} \bigcup_{i,j\leq\tau_1} \{Z_{1,i} < Z_{2,j}\} \bigcup_{\substack{i\neq j\\ i,j\leq\tau_1}} \{Z'_{1,i} < Z'_{2,j}\}$$

is true. For every $n \geq 2$, let $\mathcal{E}_{i_\ell}^n := \bigcup_{j=1}^4 \mathcal{E}_{j,i_\ell}^n$ where

$$\mathcal{E}_{1,i_\ell}^n := \bigcup_{i,j\leq n}\left[\{Z_{1,i} < Z'_{2,j}\} \cap \left\{U_{i_l} \in \left(\frac{1}{Z'_{2,j}\beta+1}, \frac{1}{Z_{1,i}\beta+1}\right)\right\}\right]$$

and $\mathcal{E}^n_{2,i_\ell}, \mathcal{E}^n_{3,i_\ell}, \mathcal{E}^n_{4,i_\ell}$ are defined similarly (i.e., use the random variables $Z'_{1,i}, Z_{2,j}$ in the definition of $\mathcal{E}^n_{2,i_\ell}$ etc.).

$$\begin{aligned}
&P\big(\big|X^{(1)}_{\tau_1}\big| - \big|X^{(2)}_{\tau_1}\big| < 0; \mathcal{B} = \{i_1 < \cdots < i_k\}, \tau_1 = n\big) \\
&\quad \le \sum_{\ell=1}^k P\big(\mathcal{B} = \{i_1 < \cdots < i_k\}, \tau_1 = n, \sigma = i_\ell\big) \\
&\quad = \sum_{\ell=1}^k P\big(\mathcal{B}^{(i_\ell)} = \{i_1 < \cdots < i_k\}, \tau_1^{(i_\ell)} = n, \sigma = i_\ell\big) \\
&\quad \le \sum_{\ell=1}^k P\big(\big\{\mathcal{B}^{(i_\ell)} = \{i_1 < \cdots < i_k\}, \tau_1^{(i_\ell)} = n\big\}; \mathcal{E}^n_{i_\ell}\big) \\
&\quad \le \sum_{\ell=1}^k 4n^2 P\big(\mathcal{B}^{(i_\ell)} = \{i_1 < \cdots < i_k\}, \tau_1^{(i_\ell)} = n\big) E\left[\frac{1}{Z_1\beta+1} - \frac{1}{Z_2\beta+1}; 1_{Z_1<Z_2}\right],
\end{aligned}$$

where we used independence of $\varepsilon^{(i_\ell)}$ and U_{i_ℓ}. Then, by Lemma 7,

$$\begin{aligned}
&P\big(\big|X^{(1)}_{\tau_1}\big| - \big|X^{(2)}_{\tau_1}\big| < 0; \mathcal{B} = \{i_1 < \cdots < i_k\}, \tau_1 = n\big) \\
&\quad \le 4(3k+2)^2 \sum_{\ell=1}^k P\left(\mathcal{B}^{(i_\ell)} = \{i_1 < \cdots < i_k\}, \tau_1^{(i_\ell)} = n, U_{i_\ell} \le \frac{1}{\beta+1}\right) \\
&\qquad \cdot (\beta+1) \cdot E\left[\frac{1}{Z_1\beta+1} - \frac{1}{Z_2\beta+1}; 1_{Z_1<Z_2}\right] \\
&\quad = 4(\beta+1)(3k+2)^2 E\left[\frac{1}{Z_1\beta+1} - \frac{1}{Z_2\beta+1}; 1_{Z_1<Z_2}\right] \sum_{\ell=1}^k P\big(\mathcal{B} = \{i_1 < \cdots < i_k\}, \tau_1 = n\big) \\
&\quad \le 8\beta k(3k+2)^2 E\left[\frac{(Z_2 - Z_1)\beta}{(Z_1\beta+1)(Z_2\beta+1)} 1_{Z_1<Z_2}\right] P\big(\mathcal{B} = \{i_1 < \cdots < i_k\}, \tau_1 = n\big) \\
&\quad \le 8k(3k+2)^2 E\left[\left(\frac{1}{Z_1} - \frac{1}{Z_2}\right) 1_{Z_1<Z_2}\right] P\big(\mathcal{B} = \{i_1 < \cdots < i_k\}, \tau_1 = n\big).
\end{aligned}$$

Therefore, by using the simple upper bound $P(|\mathcal{B}| = k) \le c(\frac{27}{4(1+\beta)})^k$ (Lemma 6.1 in [3]) for a universal constant c and the fact that $p_\infty = (\beta-1)/\beta$, we get

$$\begin{aligned}
&\tilde{E}\big[\big|X^{(1)}_{\tau_1}\big| - \big|X^{(2)}_{\tau_1}\big|\big] \\
&\quad \ge 2\left(\frac{\beta}{\beta+1}\right)^4 E\left[\frac{1}{Z_2\beta+1} - \frac{1}{Z_1\beta+1}\right] - \sum_{k=2}^\infty \frac{16}{p_\infty} k(k-1)(3k+2)^2 E\left[\left(\frac{1}{Z_1} - \frac{1}{Z_2}\right) 1_{Z_1<Z_2}\right] P\big(|\mathcal{B}| = k\big) \\
&\quad \ge 2\left(\frac{\beta}{\beta+1}\right)^4 E\left[\frac{(Z'_1 - Z'_2)\beta}{(Z'_2\beta+1)(Z'_1\beta+1)}\right] \\
&\qquad - \frac{c}{p_\infty} E\left[\left(\frac{1}{Z_1} - \frac{1}{Z_2}\right) 1_{Z_1<Z_2}\right] \sum_{k=2}^\infty 16k(k-1)(3k+2)^2 \left(\frac{27}{4(1+\beta)}\right)^k \\
&\quad \ge 2\left(\frac{\beta}{\beta+1}\right)^4 E\left[\frac{(Z'_1 - Z'_2)\beta}{4Z'_2 Z'_1 \beta^2}\right] - \frac{c\beta}{(\beta-1)} E\left[\left(\frac{1}{Z_1} - \frac{1}{Z_2}\right) 1_{Z_1<Z_2}\right] \sum_{k=2}^\infty 16k(k-1)(3k+2)^2 \left(\frac{27}{4(1+\beta)}\right)^k
\end{aligned}$$

$$\geq 2\left(\frac{\beta}{\beta+1}\right)^4 E\left[\frac{(Z_1'-Z_2')}{4Z_2'Z_1'\beta}\right] - \frac{c\cdot 27^2\beta}{4^2(\beta-1)(\beta+1)^2} E\left[\left(\frac{1}{Z_1}-\frac{1}{Z_2}\right)1_{Z_1<Z_2}\right]$$
$$\cdot \sum_{k=2}^{\infty} 16k(k-1)(3k+2)^2\left(\frac{27}{4(1+\beta)}\right)^{k-2}, \tag{3.2}$$

where we used the fact that $Z_1' \geq Z_2' \geq 1$ and $\beta > 1$. Hence we conclude that for any $\delta > 0$, $\tilde{E}[|X_{\tau_1}^{(1)}| - |X_{\tau_1}^{(2)}|] > 0$ if we have

$$\beta > \max\left\{c_\delta \cdot \frac{E[((1/Z_1)-(1/Z_2))1_{Z_1<Z_2}]}{E[(1/Z_2')-(1/Z_1')]}, \frac{23}{4}+\delta\right\}, \tag{3.3}$$

for some universal constant $c_\delta > 0$ that depends only on δ.

Now we derive the other lower bound in (1.3). On $\{|X_{\tau_1}^{(1)}| - |X_{\tau_1}^{(2)}| < 0\}$, let us define the events E and F as

$$E := \left\{\text{for some } \sigma_1 \leq \tau_1, \left|X_{\sigma_1+1}^{(1)}\right| \neq \left|X_{\sigma_1+1}^{(2)}\right| \text{ and } X_j^{(1)} = X_j^{(2)} \text{ for any } j \leq \sigma_1\right\},$$
$$F := \left\{\text{for some } \sigma_2 \leq \tau_1, X_j^{(1)} = X_j^{(2)} \text{ for any } j \leq \sigma_2, \text{ and } X_{\sigma_2+1}^{(1)} \neq X_{\sigma_2+1}^{(2)}, \text{but } \left|X_{\sigma_2+1}^{(1)}\right| = \left|X_{\sigma_2+1}^{(2)}\right|\right\}.$$

In other words, E is the event that the first time the walks on $GW(P_1)$ and $GW(P_2)$ decouple, the walk on $GW(P_2)$ goes up and the walk on $GW(P_1)$ goes down. Clearly this happens at time $\sigma_1 \in \mathcal{B}$. F is the event that the first time the walks on $GW(P_1)$ and $GW(P_2)$ decouple, they both go downwards but to different offsprings. This happens at time σ_2 which may or may not be in $\mathcal{B}$.

Next,

$$P\left(\left|X_{\tau_1}^{(1)}\right| - \left|X_{\tau_1}^{(2)}\right| < 0; \mathcal{B} = \{i_1,\dots,i_k\}, \tau_1 = n\right)$$
$$= P\left(\left|X_{\tau_1}^{(1)}\right| - \left|X_{\tau_1}^{(2)}\right| < 0; \mathcal{B} = \{i_1,\dots,i_k\}, \tau_1 = n; E\right)$$
$$+ P\left(\left|X_{\tau_1}^{(1)}\right| - \left|X_{\tau_1}^{(2)}\right| < 0; \mathcal{B} = \{i_1,\dots,i_k\}, \tau_1 = n; F\right). \tag{3.4}$$

Let us get an upper bound for the second term in (3.4).

$$P\left(\left|X_{\tau_1}^{(1)}\right| - \left|X_{\tau_1}^{(2)}\right| < 0; \mathcal{B} = \{i_1,\dots,i_k\}, \tau_1 = n; F\right)$$
$$= \sum_{\ell=1}^{n} P\left(\left|X_{\tau_1}^{(1)}\right| - \left|X_{\tau_1}^{(2)}\right| < 0; \mathcal{B} = \{i_1,\dots,i_k\}, \tau_1 = n, \sigma_2 = \ell; F\right)$$
$$\leq \sum_{\ell=1}^{n} P\left(\mathcal{B} = \{i_1,\dots,i_k\}, \tau_1 = n, \sigma_2 = \ell; F\right). \tag{3.5}$$

If $\ell \notin \{i_1,\dots,i_k\}$, then we have

$$P\left(\mathcal{B} = \{i_1,\dots,i_k\}, \tau_1 = n, \sigma_2 = \ell; F\right)$$
$$\leq P\left(\mathcal{B} = \{i_1,\dots,i_k\}, \tau_1 = n; U_\ell \in \bigcup_{m=1}^{n}\left(\frac{1}{\beta+1}, \frac{(Z_{1,m}'-Z_{2,m}')\beta}{(\beta+1)Z_{1,m}'} + \frac{1}{\beta+1}\right)\right)$$
$$= P\left(\mathcal{B}_{(\ell)} = \{i_1,\dots,i_k\}, \tau_{1(\ell)} = n, U_\ell \in \bigcup_{m=1}^{n}\left(\frac{1}{\beta+1}, \frac{(Z_{1,m}'-Z_{2,m}')\beta}{(\beta+1)Z_{1,m}'} + \frac{1}{\beta+1}\right)\right)$$
$$= P\left(\mathcal{B}_{(\ell)} = \{i_1,\dots,i_k\}, \tau_{1(\ell)} = n\right) P\left(U_\ell \in \bigcup_{m=1}^{n}\left(\frac{1}{\beta+1}, \frac{(Z_{1,m}'-Z_{2,m}')\beta}{(\beta+1)Z_{1,m}'} + \frac{1}{\beta+1}\right)\right)$$

$$\le n\frac{(\beta+1)}{\beta}P\left(\mathcal{B}_{(\ell)}=\{i_1,\dots,i_k\},\tau_{1(\ell)}=n,U_\ell\ge\frac{1}{\beta+1}\right)\cdot E\left[1-\frac{Z_2'}{Z_1'}\right]$$
$$\le 2nP\big(\mathcal{B}=\{i_1,\dots,i_k\},\tau_1=n\big)\cdot E\left[1-\frac{Z_2'}{Z_1'}\right]. \tag{3.6}$$

If $\ell\in\{i_1,\dots,i_k\}$, which happens only when $\eta_3\ge\eta_4$, we define

$$\begin{aligned}G_m&:=Z_{2,m}'[\eta_3-\eta_4]_+\\&=Z_{2,m}'\left[\left(\frac{1}{\beta+1}-\frac{1}{Z_{2,m}'\beta+1}\right)\frac{1}{Z_{2,m}'}-\left(\frac{1}{\beta+1}-\frac{1}{Z_{1,m}'\beta+1}\right)\frac{1}{Z_{1,m}'}\right]_+.\end{aligned}$$

Then, we get

$$\begin{aligned}&P\big(\mathcal{B}=\{i_1,\dots,i_k\},\tau_1=n,\sigma_2=\ell;F\big)\\&\quad\le P\left(\mathcal{B}=\{i_1,\dots,i_k\},\tau_1=n;U_\ell\in\bigcup_{m=1}^{n}\left(\frac{1}{\beta Z_{2,m}'+1},\frac{1}{\beta Z_{2,m}'+1}+G_m\right)\right)\\&\quad\le P\big(\mathcal{B}^{(\ell)}=\{i_1,\dots,i_k\},\tau_1^{(\ell)}=n\big)\cdot n\cdot E[G_m]\\&\quad=(\beta+1)P\left(\mathcal{B}^{(\ell)}=\{i_1,\dots,i_k\},\tau_1^{(\ell)}=n,U_\ell\le\frac{1}{\beta+1}\right)\cdot n\cdot E[G_m]\\&\quad=(\beta+1)P\big(\mathcal{B}=\{i_1,\dots,i_k\},\tau_1=n\big)\cdot n\cdot E[G_m].\end{aligned}$$

For a coupled (Z_1',Z_2'), we have (after a little computation)

$$\begin{aligned}&\left(\frac{1}{\beta+1}-\frac{1}{Z_2'\beta+1}\right)-\left(\frac{1}{\beta+1}-\frac{1}{Z_1'\beta+1}\right)\frac{Z_2'}{Z_1'}\\&\quad=\left(\frac{1}{\beta+1}\right)\left[1-\frac{(Z_1'-1)\beta}{Z_1'\beta+1}-\frac{\beta+1}{Z_2'\beta+1}+\left(\frac{(Z_1'-1)\beta}{Z_1'\beta+1}\right)\left(1-\frac{Z_2'}{Z_1'}\right)\right].\end{aligned}$$

It is easy to check that

$$1-\frac{(Z_1'-1)\beta}{Z_1'\beta+1}-\frac{\beta+1}{Z_2'\beta+1}=\frac{(Z_2'-Z_1')\beta+(Z_2'-Z_1')\beta^2}{(Z_1'\beta+1)(Z_2'\beta+1)}\le 0,$$

and

$$0\le\frac{(Z_1'-1)\beta}{Z_1'\beta+1}\left(1-\frac{Z_2'}{Z_1'}\right)\le 1-\frac{Z_2'}{Z_1'}.$$

Hence $E[G_m]\le(\frac{1}{\beta+1})E[(1-\frac{Z_2'}{Z_1'})]$ and therefore

$$\begin{aligned}&(\beta+1)P\big(\mathcal{B}=\{i_1,\dots,i_k\},\tau_1=n\big)\cdot n\cdot E[G_m]\\&\quad\le(\beta+1)P\big(\mathcal{B}=\{i_1,\dots,i_k\},\tau_1=n\big)\cdot n\cdot\frac{1}{\beta+1}E\left[1-\frac{Z_2'}{Z_1'}\right]\\&\quad=nP\big(\mathcal{B}=\{i_1,\dots,i_k\},\tau_1=n\big)E\left[1-\frac{Z_2'}{Z_1'}\right].\end{aligned}\tag{3.7}$$

So plugging (3.6) and (3.7) back into (3.5), we get

$$
\begin{aligned}
&P\big(\big|X_{\tau_1}^{(1)}\big|-\big|X_{\tau_1}^{(2)}\big|<0;\mathcal{B}=\{i_1,\dots,i_k\},\tau_1=n;F\big)\\
&\quad\le 2n^2P\big(\mathcal{B}=\{i_1,\dots,i_k\},\tau_1=n\big)E\bigg[1-\frac{Z_2'}{Z_1'}\bigg]\\
&\quad\le 2(3k+2)^2P\big(\mathcal{B}=\{i_1,\dots,i_k\},\tau_1=n\big)E\bigg[1-\frac{Z_2'}{Z_1'}\bigg].
\end{aligned}
$$

This takes care of the second term in (3.4). Finally, let us give an upper bound for the first term in (3.4). We omit some of the steps since they are similar. In the following computations, remember that $\sigma_1\in\mathcal{B}$.

$$
\begin{aligned}
&P\big(\big|X_{\tau_1}^{(1)}\big|-\big|X_{\tau_1}^{(2)}\big|<0;\mathcal{B}=\{i_1,\dots,i_k\},\tau_1=n;E\big)\\
&\quad\le\sum_{m=1}^{k}P\big(\mathcal{B}=\{i_1,\dots,i_k\},\tau_1=n;\sigma_1=i_m;E\big)\\
&\quad\le knP\big(\mathcal{B}=\{i_1,\dots,i_k\},\tau_1=n\big)\cdot(\beta+1)\cdot E\bigg[\frac{1}{Z_2'\beta+1}-\frac{1}{Z_1'\beta+1}\bigg]\\
&\quad\le k(3k+2)P\big(\mathcal{B}=\{i_1,\dots,i_k\},\tau_1=n\big)\cdot(\beta+1)\cdot E\bigg[\frac{\beta(Z_1'-Z_2')}{Z_1'Z_2'\beta^2}\bigg]\\
&\quad\le 2k(3k+2)P\big(\mathcal{B}=\{i_1,\dots,i_k\},\tau_1=n\big)\cdot E\bigg[\frac{1}{Z_2'}-\frac{1}{Z_1'}\bigg].
\end{aligned}
$$

Similar to our arguments in (3.2), we get

$$
\begin{aligned}
&\tilde{E}\big[\big|X_{\tau_1}^{(1)}\big|-\big|X_{\tau_1}^{(2)}\big|\big]\\
&\quad\ge 2\bigg(\frac{\beta}{\beta+1}\bigg)^4E\bigg[\frac{1}{Z_2'\beta+1}-\frac{1}{Z_1'\beta+1}\bigg]\\
&\qquad-\frac{1}{p_\infty}\sum_{k=2}^{\infty}2(3k+2)^2P\big(|\mathcal{B}|=k\big)E\bigg[1-\frac{Z_2'}{Z_1'}\bigg]\\
&\qquad-\frac{1}{p_\infty}\sum_{k=2}^{\infty}2k(3k+2)P\big(|\mathcal{B}|=k\big)\cdot E\bigg[\frac{1}{Z_2'}-\frac{1}{Z_1'}\bigg]\\
&\quad\ge\bigg(\frac{1}{2\beta}\bigg)\bigg(\frac{\beta}{\beta+1}\bigg)^4E\bigg[\frac{1}{Z_2'}-\frac{1}{Z_1'}\bigg]\\
&\qquad-\frac{c}{p_\infty}E\bigg[1-\frac{Z_2'}{Z_1'}\bigg]\sum_{k=2}^{\infty}2(3k+2)^2\bigg(\frac{27}{4(1+\beta)}\bigg)^k\\
&\qquad-\frac{c}{p_\infty}E\bigg[\frac{1}{Z_2'}-\frac{1}{Z_1'}\bigg]\sum_{k=2}^{\infty}2k(3k+2)\bigg(\frac{27}{4(1+\beta)}\bigg)^k. \qquad (3.8)
\end{aligned}
$$

As earlier, we conclude that for any $\delta>0$ there is a universal constant c_δ' such that

$$
\tilde{E}\big[\big|X_{\tau_1}^{(1)}\big|-\big|X_{\tau_1}^{(2)}\big|\big]>0,
$$

whenever

$$\beta > \max\left\{c'_\delta\left(\frac{E[Z'_2((1/Z'_2)-(1/Z'_1))]}{E[(1/Z_2)-(1/Z_1)]}+1\right), \frac{23}{4}+\delta\right\}.$$

$\square$

Proof of Corollary 5. We shall write Z_i for $Z_i^{(1)}$, for $i=1,2$ and p_j for $P_1\{j\}$. Let us first prove that $E[m_2^k/Z_1^{(k)}]\to 0$ as $k\to\infty$. Pick up some m_3 satisfying $m_2<m_3<m_1$. Then, we have

$$\begin{aligned} m_2^k E\left[\frac{1}{Z_1^{(k)}}\right] &= m_2^k \sum_{n\le m_3^k}\frac{1}{n}P\big(Z_1^{(k)}=n\big)+m_2^k\sum_{n>m_3^k}\frac{1}{n}P\big(Z_1^{(k)}=n\big) \\ &\le m_2^k P\big(Z_1^{(k)}\le m_3^k\big)+\frac{m_2^k}{m_3^k}. \end{aligned}$$

Therefore, it is sufficient to prove that $m_2^k P(Z_1^{(k)}\le m_3^k)\to 0$ as $k\to\infty$.

If W_i denotes the almost sure limit of the martingale $Z_i^{(k)}/m_i^k$, then under the assumption $E[Z_i\log^+ Z_i]<\infty$, W_i is a positive random variable for $i=1,2$ (see, e.g., [5] and [9]). Several other properties of W_i have been well studied in the literature. Recall that f is the generating function of Z_1, then $0<\alpha=-\log f'(0)/\log f'(1)$. Let us first consider the case $p_1>0$. Note that $\alpha<\infty$ when $p_1>0$. From [4] and the references therein, if $p_1>0$, then, there exists a positive constant D such that $P(W_1\le\varepsilon)\le D\varepsilon^\alpha$ as $\varepsilon\downarrow 0$.

Moreover, [2] proved that if there exists some $\theta>0$ such that $E[\mathrm{e}^{\theta Z_1}]<\infty$ and $p_j\ne 1$ for any $j\ge 1$, then there exist some constants C_1, C_2 such that

$$P\left(\left|\frac{Z_1^{(k)}}{m_1^k}-W_1\right|\ge\varepsilon\right)\le C_1\mathrm{e}^{-C_2\varepsilon^{2/3}m_1^{k/3}}.$$

Now, splitting $P(Z_1^{(k)}\le m_3^k)$ into two terms, we get

$$\begin{aligned} P\big(Z_1^{(k)}\le m_3^k\big) &= P\left(Z_1^{(k)}\le m_3^k, \left|\frac{Z_1^{(k)}}{m_1^k}-W_1\right|>\varepsilon^{(k)}\right) \\ &\quad + P\left(Z_1^{(k)}\le m_3^k, \left|\frac{Z_1^{(k)}}{m_1^k}-W_1\right|\le\varepsilon^{(k)}\right) \\ &\le P\left(\left|\frac{Z_1^{(k)}}{m_1^k}-W_1\right|>\varepsilon^{(k)}\right)+P\left(W_1\le\varepsilon^{(k)}+\frac{m_3^k}{m_1^k}\right). \end{aligned}$$

Let us choose $\varepsilon^{(k)}=m_2^{-(k/\alpha)-k\delta}$ for some $\delta>0$.

Using the results stated before from [4],

$$\begin{aligned} m_2^k P\left(W_1\le\varepsilon^{(k)}+\frac{m_3^k}{m_1^k}\right) &\le D m_2^k\left(\varepsilon^{(k)}+\frac{m_3^k}{m_1^k}\right)^\alpha \\ &= D\left(m_2^{-k\delta}+\left(\frac{m_2^{1/\alpha}m_3}{m_1}\right)^k\right)^\alpha\to 0, \end{aligned}$$

as $k\to\infty$ if we have $m_1>m_2^{1/\alpha}m_3$. Since it is valid for any $m_2<m_3<m_1$, the condition $m_1>m_2^{(1/\alpha)+1}$ is enough.

Using the results stated before from [2],

$$m_2^k P\left(\left|\frac{Z_1^{(k)}}{m_1^k}-W_1\right|>\varepsilon^{(k)}\right)\le m_2^k C_1\mathrm{e}^{-C_2(\varepsilon^{(k)})^{2/3}m_1^{k/3}}=m_2^k C_1\mathrm{e}^{-C_2 m_2^{-(2k/3\alpha)-(2/3)k\delta}m_1^{k/3}}\to 0,$$

as $k \to \infty$ if $m_1 > m_2^{(2/\alpha)+2\delta}$. Since we can pick up any $\delta > 0$, the condition $m_1 > m_2^{2/\alpha}$ is enough. This proves that $E[m_2^k/Z_1^{(k)}] \to 0$ as $k \to \infty$.

If $p_1 = 0$, then $\kappa := \min\{k > 0\colon\ p_k > 0\} \geq 2$ and from [4], we have $\log P(W_1 \leq \varepsilon) \leq -C\varepsilon^{-\beta/(1-\beta)}$, for some positive constant C and $\beta := \log\kappa/\log m_1$. In other words, $P(W_1 \leq \varepsilon)$ is exponentially small. Since $m_1 > m_2$, we can pick up some α' large enough such that $m_1 > m_2^{\max\{2/\alpha',(1/\alpha')+1\}}$ holds. Since $P(W_1 \leq \varepsilon)$ is exponentially small, we can find a positive constant D' such that $P(W_1 \leq \varepsilon) \leq D'\varepsilon^{\alpha'}$. Repeat the arguments as in the case $p_1 > 0$ replacing α by α' and D by D'. This proves that $E[m_2^k/Z_1^{(k)}] \to 0$ as $k \to \infty$.

Now, let us go back to the proof of the corollary. From (1.3), it suffices to show that

$$\frac{E[((1/Z_1^{(k)}) - (1/Z_2^{(k)}))1_{Z_1^{(k)}<Z_2^{(k)}}]}{E[(1/Z_2'^{(k)}) - (1/Z_1'^{(k)})]} \to 0, \quad \text{as } k \to \infty. \tag{3.9}$$

Since, $E\exp(\theta Z_1) < \infty$ (in particular $E[Z_i \log^+ Z_1] < \infty$), we have $\lim Z_i'^{(k)}/m_i^k > 0$ a.s. and hence

$$\frac{Z_2'^{(k)}}{Z_1'^{(k)}} = \frac{m_2^k}{m_1^k} \cdot \frac{Z_2'^{(k)}/m_2^k}{Z_1'^{(k)}/m_1^k} \to 0,$$

as $k \to \infty$, which implies that

$$\liminf_{k\to\infty} E\left[m_2^k\left(\frac{1}{Z_2'^{(k)}} - \frac{1}{Z_1'^{(k)}}\right)\right] = \liminf_{k\to\infty} E\left[\frac{m_2^k}{Z_2'^{(k)}}\left(1 - \frac{Z_2'^{(k)}}{Z_1'^{(k)}}\right)\right] \geq E\left[\frac{1}{W_2}\right] > 0.$$

Finally, notice that

$$m_2^k E\left[\left(\frac{1}{Z_1^{(k)}} - \frac{1}{Z_2^{(k)}}\right)1_{Z_1^{(k)}<Z_2^{(k)}}\right] = E\left[\frac{m_2^k}{Z_1^{(k)}}\left(1 - \frac{Z_1^{(k)}}{Z_2^{(k)}}\right)1_{Z_1^{(k)}<Z_2^{(k)}}\right] \leq E\left[\frac{m_2^k}{Z_1^{(k)}}\right].$$

Therefore, we proved (3.9). Given any $\beta > 23/4$ we can choose $\delta > 0$ such that $23/4 + \delta < \beta$ and then choose $k = k(P_1, P_2)$ large enough so that the maximum in (3.3) equals $23/4 + \delta$. $\square$

We now sketch a proof of Theorem 4.

Proof of Theorem 4. We begin with the independent sequences $\{U_i\}_{i\geq 1}$, $\{Z_{1,k}\}_{k\geq 1}$, $\{Z_{2,k}\}_{k\geq 1}$ and $\{(\widetilde{Z}'_{1,k}, \widetilde{Z}'_{2,k})\}_{k\geq 1}$ where the first three have the same meaning as in Section 2 and $\{(\widetilde{Z}'_{1,k}, \widetilde{Z}'_{2,k})\}_{k\geq 1}$ are i.i.d. copies of $((Z_1^{(1)}, \dots, Z_1^{(\ell)}), (Z_2^{(1)}, \dots, Z_2^{(\ell)}))$, the latter having the same meaning as in the statement of Theorem 4. We shall write $\widetilde{Z}'_{i,k} = (Z_{i,k}^{(1)}, \dots, Z_{i,k}^{(\ell)})$ for $i = 1, 2$.

We start both walks at the roots and when $X^{(i)}$ visits the jth distinct site at level k for the first time, we assign $Z_{i,k+1}^{(j)}$ many children to that site for $i = 1, 2$ and $j \leq l$. If one of the walks, say $X^{(1)}$ is visiting the jth distinct site at level k for the first time where $j > \ell$, then we assign $Z_{1,i}$ many children to that site for some i for which $Z_{1,i}$ has not been used before. At time n, we make the transition using the two rules explained in Section 2 according as the number of children of $X_n^{(1)}$ is larger or smaller than the number of children of $X_n^{(2)}$.

If $|\mathcal{B}| = k$, we have

(i) $0 \leq |X_{\tau_1}^{(1)}| - |X_{\tau_1}^{(2)}| \leq 2\ell$ when $k \leq \ell$.
(ii) $|X_{\tau_1}^{(1)}| - |X_{\tau_1}^{(2)}| \geq -2(k - \ell)$ when $k \geq \ell + 1$.

This can be argued as follows. Assume that $\mathcal{B} = \{i_1, \dots, i_k\}$ where $k \geq \ell$. If $|X_j^{(1)}| < |X_j^{(2)}|$ for some $j \leq i_\ell$, define $j_* := \min\{i\colon\ |X_i^{(1)}| < |X_i^{(2)}|\}$. Then $|X_{j_*-1}^{(1)}| = |X_{j_*-1}^{(2)}|$. Since $j_* - 1 < i_\ell$, none of the walks has visited any of the levels more than ℓ times up till time $j_* - 1$. We also have $\min\{Z_{1,k}^{(1)}, \dots, Z_{1,k}^{(\ell)}\} \geq \max\{Z_{2,k}^{(1)}, \dots, Z_{2,k}^{(\ell)}\}$ and hence the number of offsprings of $X^{(1)}_{j_*-1}$ is not smaller than the number of offsprings of $X^{(2)}_{j_*-1}$. But then $|X^{(1)}_{j_*}| \ge |X^{(2)}_{j_*}|$, a contradiction. Hence $|X^{(1)}_j| \ge |X^{(2)}_j|$ whenever $j \le i_\ell$, this implies the claims in (i) and (ii) stated above. A similar argument can be given for the case $|\mathcal{B}| < \ell$.

So if we carry out an analysis similar to the one given in the proof of Theorem 1, then instead of (3.2), we shall get

$$\begin{aligned}
&\tilde{E}\big[\big|X^{(1)}_{\tau_1}\big| - \big|X^{(2)}_{\tau_1}\big|\big] \\
&\quad \ge 2\left(\frac{\beta}{\beta+1}\right)^4 E\left[\frac{(Z_1' - Z_2')}{4Z_2'Z_1'\beta}\right] - \frac{c \cdot 27^{\ell+1}\ell^2\beta}{4^{\ell+1}(\beta-1)(\beta+1)^{\ell+1}} E\left[\left(\frac{1}{Z_1} - \frac{1}{Z_2}\right)1_{Z_1<Z_2}\right] \\
&\qquad \cdot \sum_{k=\ell+1}^{\infty} 16k(k-\ell)(3k+2)^2\left(\frac{27}{4(1+\beta)}\right)^{k-\ell-1},
\end{aligned}$$

and (3.8) can be modified similarly. □

Proof of Corollay 6. The proof is an extension and almost the same as the proof of Theorem 1. One needs to couple the two random walks on $GW(P_1)$ and $GW(P_2)$, with a $d\beta$-random walk on $\mathbb{Z}_{\ge 0}$. Formally we re-define the walk Y_n as $Y_0 := 0$ and for $n \ge 1$,

$$Y_n := \sum_{i=1}^{n}\{1_{U_i > 1/(d\beta+1)} - 1_{U_i \le 1/(d\beta+1)}\}, \quad n \in \mathbb{N}.$$

The walk on $GW(P_1)$ and $GW(P_2)$ should also be changed accordingly. Similar arguments, as in the proof of Theorem 1, give the counterparts to (3.2) and (3.8). Let us only present the latter, which is

$$\begin{aligned}
&\tilde{E}\big[\big|X^{(1)}_{\tau_1}\big| - \big|X^{(2)}_{\tau_1}\big|\big] \\
&\quad \ge \left(\frac{1}{2\beta}\right)\left(\frac{d\cdot\beta}{d\cdot\beta+1}\right)^4 E\left[\frac{1}{Z_2'} - \frac{1}{Z_1'}\right] \\
&\qquad - \frac{c}{p_\infty} E\left[1 - \frac{Z_2'}{Z_1'}\right]\sum_{k=2}^{\infty} 2(3k+2)^2\left(\frac{27}{4(1+d\cdot\beta)}\right)^k \\
&\qquad - \frac{c\cdot d}{p_\infty} E\left[\frac{1}{Z_2'} - \frac{1}{Z_1'}\right]\sum_{k=2}^{\infty} 2k(3k+2)\left(\frac{27}{4(1+d\cdot\beta)}\right)^k \\
&\quad \ge \left(\frac{1}{2\beta}\right)\left(\frac{d\cdot\beta}{d\cdot\beta+1}\right)^4 E\left[\frac{1}{Z_2'} - \frac{1}{Z_1'}\right] \\
&\qquad - \frac{c\cdot d}{p_\infty} E\left[1 - \frac{Z_2'}{Z_1'}\right]\sum_{k=2}^{\infty} 2(3k+2)^2\left(\frac{27}{4(1+d\cdot\beta)}\right)^k \\
&\qquad - \frac{c\cdot d}{p_\infty} E\left[\frac{1}{Z_2'} - \frac{1}{Z_1'}\right]\sum_{k=2}^{\infty} 2k(3k+2)\left(\frac{27}{4(1+d\cdot\beta)}\right)^k.
\end{aligned}$$

Now, note that this gives the same constant as in (3.8). Therefore, as long as $d \cdot \beta > \beta_0$ we have $\tilde{E}[|X^{(1)}_{\tau_1}| - |X^{(2)}_{\tau_1}|] > 0$ which completes the proof. □

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