

RISK MEASURING UNDER MODEL UNCERTAINTY

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The framework of this paper is that of risk measuring under uncertainty which is when no reference probability measure is given. To every regular convex risk measure on $\mathcal{C}_b(\Omega)$, we associate a unique equivalence class of probability measures on Borel sets, characterizing the riskless nonpositive elements of $\mathcal{C}_b(\Omega)$. We prove that the convex risk measure has a dual representation with a countable set of probability measures absolutely continuous with respect to a certain probability measure in this class. To get these results we study the topological properties of the dual of the Banach space $L^1(c)$ associated to a capacity c .

As application, we obtain that every G -expectation \mathbb{E} has a representation with a countable set of probability measures absolutely continuous with respect to a probability measure P such that $P(|f|) = 0$ if and only iff $\mathbb{E}(|f|) = 0$. We also apply our results to the case of uncertain volatility.

1. Introduction. The purpose of this paper is to introduce a very general framework enabling the study of risk measures and dynamic risk measures in a context of model uncertainty, which is when no reference probability measure is given.

In order to quantify the risk in finance, Artzner et al. [1] have introduced the notion of coherent (i.e., sublinear) risk measure in the context of finite probability spaces. This notion has been extended to general probability spaces [12] and then to the convex case [21] and [22]. The notion of conditional risk measure has been considered in [6] and [17]. Dynamic risk measures have then been studied in many papers, among them [7, 8, 11, 13, 25, 30]. For the particular case of dynamic risk measures on a Brownian filtration, one can cite [3, 14, 26]. Notice that in all these papers on dynamic risk measures, a reference probability space is fixed. This framework is rich enough to study models with stochastic volatility or models with jumps, but not to deal with model uncertainty.

What does uncertainty mean? Usually in mathematical finance, in order to compute the risk or the price associated to financial assets, one assumes that a reference family of liquid assets is given and that the dynamics of these reference assets is known. However, in a context of model uncertainty, the dynamics of the liquid reference assets is only assumed to belong to a certain class of models. A simple

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example is given, within the Brownian framework, by a class of models with uncertain volatility. That is, one considers a family of possible models of the form $dX_t^\sigma = b_t X_t^\sigma dt + \sigma_t X_t^\sigma dW_t$ where σ_t is allowed to vary inside an interval $[\underline{\sigma}, \bar{\sigma}]$. When σ describes the set of predictable processes varying inside this interval, the laws of the processes X_σ^t are not all absolutely continuous with respect to some probability measure. Avellaneda, Levy and Paras [2], Denis and Martini [16] and Denis, Hu and Peng [15] have considered the problem of pricing for this family of models. Only a few papers study convex risk measures in a context of uncertainty. Föllmer and Schied [21] have studied static risk measures defined on the vector space of all bounded measurable maps. This has been extended by Bion-Nadal to the conditional case in [6]. Kervarec [24] has studied static risk measures when model uncertainty is specified by a nondominated weakly compact set of probability measures.

In this paper, motivated by the general context of model uncertainty, we study regular convex risk measures defined on $\mathcal{C}_b(\Omega)$, the set of continuous bounded functions on a Polish space Ω . Regularity is here equivalent to continuity with respect to a certain capacity c . Considering the completion $L^1(c)$ of $\mathcal{C}_b(\Omega)$ with respect to the capacity c , this means that we study convex risk measures on the Banach space $L^1(c)$. Our main result is that for every regular convex risk measure on $\mathcal{C}_b(\Omega)$, there is a unique equivalence class of probability measures characterizing the riskless nonpositive elements of $\mathcal{C}_b(\Omega)$ and that the convex risk measure has a dual representation with a countable set of probability measures all absolutely continuous with respect to a certain probability measure belonging to this equivalence class. The tools of the proof are the capacities, topological properties of the dual of the Banach space $L^1(c)$ associated to a capacity c and convex duality for locally convex spaces.

The paper is organized as follows. First, in Section 2 we study the topological properties of the dual of $L^1(c)$. We prove that the nonnegative part of the dual ball of $L^1(c)$ is metric compact for the weak* topology $\sigma(L^1(c)^*, L^1(c))$.

Section 3 deals with convex risk measures on $L^1(c)$. We prove that they satisfy the representation formula

$$(1.1) \quad \rho(X) = \sup_{Q \in \mathcal{P}'} (E_Q[-X] - \alpha(Q)),$$

where \mathcal{P}' is a set of probability measures belonging to the dual of $L^1(c)$. There are two important results in this section. The first one is the characterization of convex risk measures on $L^1(c)$ admitting a representation of the form (1.1) having a compact set \mathcal{P}' of probability measures [for the weak* topology $\sigma(L^1(c)^*, L^1(c))$]. In this case, the supremum in (1.1) is a maximum. Moreover, making use of the topological results of Section 2, we prove that every convex risk measure on $L^1(c)$ has a dual representation of the form (1.1) with a countable set of probability measures.

In Section 4 we assume that the capacity is defined on $\mathcal{C}_b(\Omega)$ by $c_{\mathcal{P}, \mathcal{P}}(f) = \sup_{P \in \mathcal{P}} E_P(|f|^p)^{1/p}$ for some weakly relatively compact set \mathcal{P} of probability

measures. We prove that the capacity $c_{\mathcal{P}, \mathcal{P}}$ is equal to the capacity $c_{\mathcal{P}, \mathcal{Q}}$ defined using a certain countable subset \mathcal{Q} of \mathcal{P} . We introduce a new equivalence relation on the set of nonnegative measures belonging to the dual of $L^1(c_{\mathcal{P}, \mathcal{P}})$. When \mathcal{P} is a singleton, it coincides with the usual equivalence relation on nonnegative measures. The main result of Section 4 is the existence of an equivalence class of probability measures characterizing the null elements of $L^1(c_{\mathcal{P}, \mathcal{P}})_+$, that is, P belongs to this equivalence class if and only if for all f in $L^1(c_{\mathcal{P}, \mathcal{P}})$, $(E_P(|f|) = 0) \iff (c_{\mathcal{P}, \mathcal{P}}(|f|) = 0)$.

Section 5 deals with uniformly regular convex risk measures on $\mathcal{C}_b(\Omega)$. We prove that every such risk measure on $\mathcal{C}_b(\Omega)$ extends into a convex risk measure on $L^1(c)$ for a certain capacity c associated to a weakly compact set \mathcal{P} of probability measures: $c(f) = \sup_{P \in \mathcal{P}} E_P(f)$. Therefore, we can make use of the results obtained in Sections 4 and 3 in order to get the main result of the paper in Theorem 5.1: to every uniformly regular convex risk measure ρ on $\mathcal{C}_b(\Omega)$, one can associate a unique equivalence class of probability measures defined on the Borel sets, called c_ρ -class, characterizing the nonpositive elements of $\mathcal{C}_b(\Omega)$ with risk 0. The convex risk measure has then a dual representation with a countable set of probability measures all absolutely continuous with respect to a certain probability measure belonging to this c_ρ -class.

Section 6 deals with two examples. The first one is G -expectations introduced by Peng [27]. The capacity associated to a G -expectation \mathbb{E} is $c(f) = \mathbb{E}(|f|)$. As application of our results, we obtain that there is a unique equivalence class of probability measures characterizing the nonnegative elements f of $\mathcal{C}_b(\Omega)$ such that $\mathbb{E}(f) = 0$. The G -expectation \mathbb{E} has then a representation in terms of a countable set of probability measures all absolutely continuous with respect to a certain probability measure belonging to this class

$$(1.2) \quad \mathbb{E}(X) = \sup_{n \in \mathbb{N}} E_{Q_n}(X).$$

The second example, for which all our results apply, is the case where model uncertainty is characterized by a relatively weakly compact set of probability measures \mathcal{P} .

2. Topological properties of the dual space of $L^1(c)$.

2.1. *The ordered space $L^1(c)$.* Let Ω be a metrizable and separable space. One classical example, furthermore a Polish space, is $\Omega = \mathcal{C}_0([0, \infty[, \mathbb{R}^d)$ endowed with the topology of uniform convergence on compact subspaces. $\mathcal{B}(\Omega)$ denotes the Borel σ -algebra on Ω . Denote $\mathcal{M}(\Omega)$ the set of all bounded signed measures on $(\Omega, \mathcal{B}(\Omega))$ and $\mathcal{M}_+(\Omega)$ the subset of nonnegative finite measures.

In the following, \mathcal{L} denotes a linear vector subspace of $\mathcal{C}_b(\Omega)$ containing the constants, generating the topology of Ω and which is a vector lattice. Recall the following definition of a capacity.

DEFINITION 2.1. A capacity on \mathcal{L} is a semi-norm c defined on \mathcal{L} satisfying the following properties:

- (1) monotonicity: $\forall f, g \in \mathcal{L}$ such that $|f| \leq |g|$, $c(f) \leq c(g)$;
- (2) regularity along sequences: for every sequence $f_n \in \mathcal{L}$ decreasing to 0, $\inf c(f_n) = 0$.

The semi-norm c is extended as in [20], Section 2, to all real functions defined on Ω ,

$$(2.1) \quad \forall f \text{ l.s.c. } f \geq 0 \quad c(f) = \sup\{c(\phi) \mid 0 \leq \phi \leq f, \phi \in \mathcal{L}\},$$

$$(2.2) \quad \forall g \quad c(g) = \inf\{c(f) \mid f \geq |g|, f \text{ l.s.c.}\},$$

where l.s.c. means lower semi-continuous. $\mathcal{L}^1(c)$ denotes the closure of \mathcal{L} in the set $\{g \mid c(g) < \infty\}$. From Proposition 10 of [20], $\mathcal{L}^1(c)$ contains $\mathcal{C}_b(\Omega)$. Let $L^1(c)$ be the quotient of $\mathcal{L}^1(c)$ by the c null elements. It is a Banach space. The following result shows that $c(1_A)$ can be expressed as the limit of a monotone sequence $c(f_n)$ for continuous functions f_n with limit 1_A , as soon as A is either an open subset or a closed subset of Ω .

PROPOSITION 2.1. *Let V be an open subset of Ω . There is an increasing sequence of nonnegative continuous functions h_n on Ω such that $1_V = \lim_{n \rightarrow \infty} h_n$ and $c(1_V) = \lim_{n \rightarrow \infty} c(h_n)$.*

Let F be a closed subset of Ω . There is a decreasing sequence of continuous functions $g_n \leq 1$ on Ω such that $1_F = \lim_{n \rightarrow \infty} g_n$ and $c(1_F) = \lim_{n \rightarrow \infty} c(g_n)$.

PROOF. 1_V is a nonnegative bounded l.s.c. function. Thus, it is the limit of an increasing sequence of nonnegative continuous functions f_n . On the other hand, from definition of $c(1_V)$ [equation (2.1)], there is a sequence of continuous functions $g_n \leq 1_V$ such that $c(1_V) = \lim c(g_n)$. Let $h_1 = g_1$ and for every n , $h_{n+1} = \sup(h_n, f_n, g_n)$. h_n is an increasing sequence of continuous functions with limit 1_V and such that $c(1_V) = \lim c(h_n)$.

Let F be a closed subset of Ω . By definition of the capacity, $c(1_F) = \inf_{\{\psi \text{ l.s.c.}, 1_F \leq \psi\}} c(\psi)$. The infimum of two l.s.c. functions is also l.s.c., thus there is a decreasing sequence ψ_n greater or equal to 1_F such that $c(1_F) = \lim c(\psi_n)$. Thus, there is a strictly increasing sequence $k(n)$ such that for all n , $c(\psi_{k(n)}) \leq c(1_F) + \frac{1}{n^2}$. Let $\varepsilon_n > 0$ such that $(\frac{1}{1-\varepsilon_n})(c(1_F) + \frac{1}{n^2}) \leq c(1_F) + \frac{1}{n}$. Let $V_n = \{x \mid \psi_{k(n)}(x) > 1 - \varepsilon_n\} \cap \{x \in \Omega; \text{dist}(x, F) < \frac{1}{n}\}$. As $\psi_{k(n)}$ is l.s.c., V_n is an open set; furthermore, $F = \bigcap_{n \in \mathbb{N}^*} V_n$. For every n , there is a continuous function f_n such that $F \prec f_n \prec V_n$. One can thus construct a decreasing sequence of continuous functions g_n such that $1_F \leq g_n \leq 1_{V_n}$. Thus the sequence g_n is decreasing to 1_F . As $c(1_{V_n}) \leq \frac{1}{1-\varepsilon_n} c(\psi_{k(n)}) \leq c(1_F) + \frac{1}{n}$, it follows that $c(1_F) \leq c(g_n) \leq c(1_F) + \frac{1}{n}$. \square

Further definitions and results on capacities are recalled in the [Appendix](#). We refer also to [20].

Partial order on $L^1(c)$.

DEFINITION 2.2. Let $X \in L^1(c)$. We say that $X \geq 0$ if there is a sequence $(f_n)_{n \in \mathbb{N}}$, $f_n \in \mathcal{L}$, $f_n \geq 0$ such that for every $g \in \mathcal{L}^1(c)$ of class X , $\lim_{n \rightarrow \infty} c(g - f_n) = 0$.

LEMMA 2.1. Let $X, Y \in L^1(c)$. If $X \geq 0$ and $Y \geq 0$, then $X + Y \geq 0$.

If there is in the class of X a nonnegative function f , then $X \geq 0$.

Let $X \in L^1(c)$; $|X| \in L^1(c)$. Furthermore, $X \geq 0$ if and only if $X = |X|$ in $L^1(c)$.

PROOF. The first part of the lemma is trivial. The second point follows from the inequality

$$(2.3) \quad c(|f| - |f_n|) \leq c(f - f_n).$$

Thus, as $f = |f|$, $c(f - |f_n|) \leq c(f - f_n)$. One can deduce from (2.3) that for all $X \in L^1(c)$, $|X| \in L^1(c)$. From point 2, $|X| - X \geq 0$. Thanks to (2.3) and the inequality $c(|f| - f) \leq c(|f| - f_n) + c(f - f_n)$, it follows that $X \geq 0$ if and only if $X = |X|$ in $L^1(c)$. \square

PROPOSITION 2.2. The relation $X \leq Y$ defined by $Y - X \geq 0$ defines a partial order on $L^1(c)$.

PROOF. (1) Reflexivity is trivial; take $f_n = 0$ for all n .

(2) Antisymmetry. Let $X \geq Y$ and $Y \geq X$. Let h be in the class of $X - Y$. By definition, there are two sequences f_n and g_n of nonnegative functions in \mathcal{L} such that $\lim_{n \rightarrow \infty} c(f_n - h) = 0$ and $\lim_{n \rightarrow \infty} c(g_n + h) = 0$. It follows that $\lim_{n \rightarrow \infty} c(f_n + g_n) = 0$. As $0 \leq |f_n - g_n| \leq f_n + g_n$, it follows that $\lim_{n \rightarrow \infty} c(|f_n - g_n|) = 0$. However, $\lim_{n \rightarrow \infty} c(f_n - g_n - 2h) = 0$. Thus, $X - Y$, the class of h is equal to 0.

(3) Transitivity follows from the first part of Lemma 2.1. \square

2.2. *Topological properties of the nonnegative part of the unit ball of $L^1(c)^*$.* For the definition of a Prokhorov capacity, see the [Appendix](#).

PROPOSITION 2.3. Let c be a Prokhorov capacity on a metrizable and separable space Ω . Every continuous linear form L on $L^1(c)$ admits a representation,

$$(2.4) \quad L(f) = \int f d\mu \quad \forall f \in L^1(c),$$

where μ is a regular bounded signed measure defined on a σ -algebra containing the Borel σ -algebra of Ω .

If L is a nonnegative linear form, the regular measure μ is nonnegative finite.

Following [5], a bounded signed measure μ is called regular if for all Borel set A , for all $\varepsilon > 0$, there is a closed set F and an open set G such that $F \subset A \subset G$ and $|\mu|(G - F) < \varepsilon$.

Notice that in [20], the existence of a bounded measure μ satisfying (2.4) is proved. However, the statement of Proposition 11 of [20] does not give information on the σ algebra on which the measure μ is defined. Therefore, we have to go inside the proof.

PROOF OF PROPOSITION 2.3. A metrizable space is completely regular and c is a Prokhorov capacity so Proposition 11 of [20] gives the existence of a measure μ satisfying (2.4). We want to now prove that μ is defined on the Borel σ algebra. As in the proof of Proposition 11 of [20], let Z be a compactification of Ω and c' the capacity defined on Z by $c'(g) = c(g|_{\Omega})$. As c is a Prokhorov capacity, from Proposition 11 of [20], $c'(1_{Z-\Omega}) = 0$ and $L^1(c) = L^1(c')$.

As Z is a compact space, it follows from Theorem 3 of [19] that every nonnegative linear form on $L^1(c')$ can be represented by a nonnegative measure obtained from the Riesz representation theorem applied to $\mathcal{C}(Z)$. Therefore, this measure is defined on a σ -algebra containing the Borel sets of Z . From Theorem 6 of [19] every continuous linear form on $L^1(c)$ is the difference of two nonnegative linear forms, thus, the bounded measure μ satisfying (2.4) is defined on a σ -algebra \mathcal{B} containing the Borel σ -algebra of Z .

We want to prove that μ is defined on the Borel σ -algebra of Ω . μ is defined on the σ -algebra \mathcal{F} obtained by completion of \mathcal{B} with the μ -null sets. Notice that from Theorem 3 of [19], every c' -negligible set [i.e., $c'(1_A) = 0$] is also μ -negligible. This is, in particular, the case for $Z - \Omega$ which is therefore, μ -measurable. Every open set V of Ω can be written $V = U \cap \Omega$ for some open set U of Z . Therefore, V belongs to \mathcal{F} . It follows that the measure μ defined on \mathcal{F} is thus defined on the Borel σ -algebra of Ω . As Ω is a metric space and μ is defined on the Borel σ -algebra of Ω , μ is regular from Theorem 1.1 of [5]. \square

Recall that the weak topology on $\mathcal{M}_+(\Omega)$, the set of nonnegative finite measures on $(\Omega, \mathcal{B}(\Omega))$, is the coarsest topology for which the mappings

$$\mu \in \mathcal{M}_+(\Omega) \rightarrow \int f d\mu$$

are continuous for every given f in $\mathcal{C}_b(\Omega)$.

PROPOSITION 2.4. *Let c be a Prokhorov capacity on a metrizable separable space. The set of nonnegative linear forms on the Banach space $L^1(c)$ is a subset of $\mathcal{M}_+(\Omega)$. The weak* topology [i.e., the $\sigma(L^1(c)^*, L^1(c))$ topology] on the nonnegative part K_+ of the unit ball of $L^1(c)^*$ coincides with the restriction to K_+ of the weak topology on $\mathcal{M}_+(\Omega)$.*

PROOF. From Proposition 2.3, every nonnegative linear form on $L^1(c)$ belongs to $\mathcal{M}_+(\Omega)$. Let $\mu \in K_+$. As $\mathcal{C}_b(\Omega)$ is dense in the Banach space $L^1(c)$, the open sets

$$V_{f_1, f_2, \dots, f_n, \varepsilon}(\mu) = \{v \in K_+ \mid \forall i \in \{1, \dots, n\}, |\mu(f_i) - v(f_i)| < \varepsilon\}$$

with $f_i \in \mathcal{C}_b(\Omega)$ form a basis of neighborhoods of μ in K_+ for the weak* topology. Thus, the weak* topology on K_+ coincides with the weak topology. \square

PROPOSITION 2.5. *Let c be a Prokhorov capacity on a metrizable separable space Ω . The set K_+ is compact metrizable for the weak* topology [i.e., the $\sigma(L^1(c)^*, L^1(c))$ topology] as well as for the weak topology.*

PROOF. Prove first that K_+ is metrizable for the weak* topology. From Proposition 2.4, the weak* topology on K_+ coincides with the restriction to K_+ of the weak topology on $\mathcal{M}_+(\Omega)$. As Ω is metrizable and separable, $\mathcal{M}_+(\Omega)$ is also metrizable and separable for the weak topology from [10], Section 5. Thus, K_+ is metrizable for the weak* topology.

From the Banach–Alaoglu theorem ([18], Theorem V 4.2), the closed unit ball of the dual space of a Banach space is always compact for the weak* topology. As K_+ is a closed subset of this unit ball for the weak* topology, it is also compact. This proves the result for the weak* topology. From Proposition 2.4, K_+ is also metrizable compact for the weak topology. \square

COROLLARY 2.1. *Assume that Ω is a Polish space. For every capacity c on Ω , the set K_+ is compact metrizable for the weak* topology.*

PROOF. From [20] (see also the Appendix), every capacity on a Polish space is a Prokhorov capacity and thus the result follows from Proposition 2.5. \square

In the particular case of a compact metrizable space, we obtain the following stronger result.

PROPOSITION 2.6. *Let Ω be a metrizable compact space. Let c be a capacity on Ω . Then the Banach space $L^1(c)$ is separable and the unit ball of $L^1(c)^*$ is metrizable compact for the weak* topology.*

PROOF. As Ω is a metrizable compact space, $\mathcal{C}(\Omega)$ is separable from Theorem 1, Section 3, of [9]. Thus, for every capacity c on Ω , $L^1(c)$ is also separable. Then from [18], Theorem V 51, the unit ball of $L^1(c)^*$ (and not only its nonnegative part) is metrizable compact for the weak* topology. \square

3. Representation of a convex risk measure on $L^1(c)$. In this section, c denotes a Prokhorov capacity on a metrizable separable space Ω . Recall that a partial order has been defined on $L^1(c)$ in Section 2.1. We can define convex risk measures in the usual way as follows.

DEFINITION 3.1. Let $\rho : L^1(c) \rightarrow \mathbb{R}$.

- ρ is monotonic if $\rho(X) \geq \rho(Y)$ for every $X, Y \in L^1(c)$, such that $X \leq Y$.
- ρ is convex if for every $X, Y \in L^1(c)$, for every $0 \leq \lambda \leq 1$, $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$.
- ρ is translation invariant if $\rho(X + a) = \rho(X) - a$ for every $X \in L^1(c)$ and $a \in \mathbb{R}$.
- ρ is a convex risk measure if it satisfies all these conditions.

3.1. *Representation for convex risk measures.* Duality results for risk measures are well known in other settings. A duality result was first proved in the case of risk measures on L^∞ spaces assuming, furthermore, continuity from below. Duality results are based on the Fenchel–Legendre duality, generalized to the context of locally convex topological spaces by Rockafellar [29]. This is the generalized version that we need here. No additional hypothesis is needed in order to prove the dual representation result. The important and new discussion will be developed in Section 3.2 using the topological results proved in Section 2.2.

THEOREM 3.1. Let ρ be a convex risk measure on $L^1(c)$. Then, ρ is continuous and admits a representation of the form

$$(3.1) \quad \forall X \in L^1(c) \quad \rho(X) = \sup_{Q \in \mathcal{P}'} (E_Q[-X] - \alpha(Q)),$$

where

$$(3.2) \quad \alpha(Q) = \sup_{X \in L^1(c)} (E_Q[-X] - \rho(X)).$$

\mathcal{P}' is the set of probability measures on $(\Omega, \mathcal{B}(\Omega))$ belonging to $L^1(c)^*$.

PROOF. The continuity of ρ follows from Theorem 1 of [4].

We call α the function on $L^1(c)^*$ defined by

$$\forall \mu \in L^1(c)^* \quad \alpha(\mu) = \sup_{X \in L^1(c)} (\mu(X) - \rho(X)).$$

As the dual of $L^1(c)^*$ (with the weak* topology) is $L^1(c)$, the locally convex topological spaces $L^1(c)$ and $L^1(c)^*$ are paired in the sense of [29]. ρ is continuous; we can thus apply Theorem 5 in Rockafellar [29]. We get the following equality:

$$\forall X \in L^1(c) \quad \rho(X) = \sup_{\mu \in L^1(c)^*} (\mu(X) - \alpha(\mu)).$$

In the supremum above, we can obviously restrict to the elements μ of $L^1(c)^*$ such that $\alpha(\mu) < +\infty$.

Let $\mu_0 \in L^1(c)^*$ such that $\alpha(\mu_0) < +\infty$, we first prove that $-\mu_0$ is a positive linear form. Let $X \in L^1(c)$ such that $X \geq 0$. For all $\lambda > 0$, using the monotonicity of ρ , $\rho(\lambda X) \leq \rho(0)$, which implies that

$$\lambda\mu_0(X) - \alpha(\mu_0) \leq \rho(0).$$

$\rho(0)$ and $\alpha(\mu_0)$ are finite and the above inequality is satisfied for all $\lambda > 0$, thus, $\mu_0(X) \leq 0$.

From Proposition 2.3, $-\mu_0$ is represented by a finite nonnegative measure defined on $(\Omega, \mathcal{B}(\Omega))$. Thanks to the translation invariance of ρ , for all $\lambda \in \mathbb{R}$, $\rho(\lambda) = \rho(0) - \lambda$, which means that

$$\rho(0) = \lambda + \sup_{\mu \in L^1(c)^*} (\lambda\mu(1) - \alpha(\mu)) \geq \lambda(1 + \mu_0(1)) - \alpha(\mu_0).$$

We conclude as above that $1 + \mu_0(1) = 0$. Thus, $-\mu_0$ is a probability measure on $(\Omega, \mathcal{B}(\Omega))$ and $-\mu_0 \in L^1(c)^*$. \square

3.2. *Risk measures represented by a weakly relatively compact set of probability measures.* In this section we want to characterize risk measures ρ on $L^1(c)$ admitting a dual representation with a relatively compact set of probability measures for the weak* topology.

DEFINITION 3.2. A convex risk measure ρ on $L^1(c)$ is normalized if $\rho(0) = 0$.

PROPOSITION 3.1. Let $\rho : L^1(c) \rightarrow \mathbb{R}$ be a normalized convex risk measure. The following conditions are equivalent:

- (1) ρ is majorized by a sublinear risk measure;
- (2) $\forall X \in L^1(c), \sup_{\lambda > 0} \frac{\rho(\lambda X)}{\lambda} < \infty$;
- (3) there exists $K > 0$ such that $\forall X \in L^1(c), |\rho(X)| \leq Kc(X)$;
- (4) ρ is represented by a set \mathcal{Q} of probability measures in $L^1(c)^*$ relatively compact for the weak* topology, that is,

$$(3.3) \quad \forall X \in L^1(c) \quad \rho(X) = \sup_{Q \in \mathcal{Q}} (E_Q[-X] - \alpha(Q)).$$

Before giving the proof of the proposition, we prove the following lemma.

LEMMA 3.1. Let \mathcal{Q} be a set of probability measures on $(\Omega, \mathcal{B}(\Omega))$ such that $\mathcal{Q} \subset L^1(c)^*$. Assume that \mathcal{Q} is relatively compact for the weak* topology $\sigma(L^1(c)^*, L^1(c))$. Then \mathcal{Q} is contained in some closed ball of $L^1(c)^*$ and the weak* closure of \mathcal{Q} is also compact for the weak topology.

PROOF. Denote $\overline{\mathcal{Q}}$ the closure of \mathcal{Q} for the weak* topology. $\overline{\mathcal{Q}}$ is compact. Let $X \in L^1(c)$. The map $Q \rightarrow E_Q(X)$ is continuous for the weak* topology, thus, $\sup_{Q \in \overline{\mathcal{Q}}} |E_Q(X)| < \infty$. From Banach–Steinhaus theorem (cf. [31]) it follows that $\overline{\mathcal{Q}}$ is contained in some closed ball of $L^1(c)^*$ and thus in the nonnegative part of this closed ball. From Proposition 2.4, $\overline{\mathcal{Q}}$ is weakly compact. \square

We can now give the proof of Proposition 3.1.

PROOF OF PROPOSITION 3.1. Consider the dual representation of ρ given by (3.1). Denote $\mathcal{Q} = \{Q \in \mathcal{P}' \mid \alpha(Q) < \infty\}$. Then

$$(3.4) \quad \forall X \in L^1(c) \quad \rho(X) = \sup_{Q \in \mathcal{Q}} (E_Q(-X) - \alpha(Q)).$$

(1) implies (2). Let ρ_1 be a sublinear risk measure majorizing ρ . Then for every $\lambda \in \mathbb{R}_*^+$, $\rho(\lambda X) \leq \lambda \rho_1(X)$. Thus, $\sup_{\lambda > 0} \frac{\rho(\lambda X)}{\lambda} \leq \rho_1(X)$ and (2) is proved.

(2) implies (3). For every $X \in L^1(c)$, denote $\beta_X = \sup_{\lambda > 0} \frac{\rho(\lambda X)}{\lambda}$. From the dual representation (3.4), applied with λX for every $\lambda > 0$, it follows that $\forall Q \in \mathcal{Q}$, $E_Q(-X) \leq \beta_X$ and thus, $\sup_{Q \in \mathcal{Q}} E_Q(-X) \leq \beta_X < \infty$ for every $X \in L^1(c)$. With $X = -|Y|$, we get that

$$(3.5) \quad \forall Y \in L^1(c) \quad \sup_{Q \in \mathcal{Q}} |E_Q(Y)| < \infty.$$

$L^1(c)$ is a Banach space and from Theorem 3.1, every E_Q is a continuous linear form on $L^1(c)$. Denote $\|E_Q\|$ its norm. From Banach–Steinhaus theorem, equation (3.5) implies the existence of $K > 0$ such that $\sup_{Q \in \mathcal{Q}} \|E_Q\| \leq K$. Notice that from the normalization condition [$\rho(0) = 0$], it follows from (3.2) that for every Q , $\alpha(Q) \geq 0$. Thus, from the representation (3.4), for every $X \in L^1(c)$,

$$(3.6) \quad \rho(X) \leq Kc(X).$$

From the convexity, the monotonicity of ρ and $\rho(0) = 0$, it follows that

$$(3.7) \quad -\rho(X) \leq \rho(-X) \leq \rho(-|X|) \leq Kc(-|X|) = Kc(X).$$

Thus, from (3.6) and (3.7), for every $X \in L^1(c)$,

$$|\rho(X)| \leq Kc(X).$$

This proves (3).

(3) implies (4). From the representation of ρ , equation (3.4) applied with $-\lambda|X|$ for every $\lambda > 0$, it follows from hypothesis (3) that for every $Q \in \mathcal{Q}$ $\|E_Q\| \leq K$. This means that \mathcal{Q} is contained in a closed ball of the dual of $L^1(c)$. Every such closed ball is compact for the weak* topology (Banach–Alaoglu theorem). Thus, \mathcal{Q} is relatively compact for the weak* topology.

(4) implies (1). ρ is represented by a set of probability measures $\mathcal{Q} \subset L^1(c)^*$ relatively compact for the weak* topology. From Lemma 3.1, \mathcal{Q} is contained in

some closed ball of $L^1(c)^*$. Define ρ_1 by $\rho_1(X) = \sup_{Q \in \mathcal{Q}} E_Q(-X)$. As \mathcal{Q} is bounded, $\rho_1(X)$ is finite for every X in $L^1(c)$. It is easy to verify that ρ_1 is a sublinear risk measure and that ρ is majorized by ρ_1 . \square

THEOREM 3.2. *Let ρ be a convex risk measure on $L^1(c)$. Assume that ρ is represented by*

$$\rho(X) = \sup_{Q \in \mathcal{Q}} (E_Q(-X) - \alpha(Q)),$$

where \mathcal{Q} is a set of probability measures in $L^1(c)^*$ relatively compact for the weak* topology. Let $\bar{\mathcal{Q}}$ be the closure of \mathcal{Q} for the weak* topology. Then $\bar{\mathcal{Q}}$ is metrizable compact both for the weak* topology and the weak topology.

For every $X \in L^1(c)$, there is a probability measure $Q_X \in \bar{\mathcal{Q}}$ such that

$$(3.8) \quad \rho(X) = E_{Q_X}(-X) - \alpha(Q_X).$$

PROOF. From Lemma 3.1, $\bar{\mathcal{Q}}$ is contained in a closed ball of $L^1(c)^*$ and is compact both for the weak and the weak* topology. From Proposition 2.5 it is metrizable compact. Let $X \in L^1(c)$. Let Q_n be a sequence of elements in \mathcal{Q} such that for every n ,

$$(3.9) \quad \rho(X) - \frac{1}{n} < E_{Q_n}(-X) - \alpha(Q_n) \leq \rho(X).$$

As $\bar{\mathcal{Q}}$ is metrizable compact for the weak* topology, there is a subsequence $Q_{\phi(n)}$ converging to $\tilde{Q} \in \bar{\mathcal{Q}}$, satisfying the inequality

$$(3.10) \quad E_{\tilde{Q}}(-X) - \frac{1}{n} < E_{Q_{\phi(n)}}(-X) < E_{\tilde{Q}}(-X) + \frac{1}{n}.$$

From inequality (3.9), applied with $Q_{\phi(n)}$, inequality (3.10) and the inequality $\phi(n) \geq n$, it follows that

$$(3.11) \quad E_{\tilde{Q}}(-X) - \rho(X) - \frac{1}{n} < \alpha(Q_{\phi(n)}) < E_{\tilde{Q}}(-X) - \rho(X) + \frac{2}{n}.$$

Let $Y \in L^1(c)$. Let $\varepsilon > 0$. There is $N(Y)$ such that for every $n > N(Y)$, $E_{\tilde{Q}}(-Y) < E_{Q_{\phi(n)}}(-Y) + \varepsilon$. $N(Y)$ can be chosen such that $N(Y) \geq \frac{1}{\varepsilon}$. Then for $n \geq N(Y)$,

$$(3.12) \quad \begin{aligned} E_{\tilde{Q}}(-Y) - \rho(Y) &\leq \alpha(Q_{\phi(n)}) + \varepsilon \\ &\leq E_{\tilde{Q}}(-X) - \rho(X) + \frac{2}{n} + \varepsilon \\ &\leq E_{\tilde{Q}}(-X) - \rho(X) + 3\varepsilon. \end{aligned}$$

As the inequality is satisfied for every Y and every $\varepsilon > 0$, it follows that

$$\alpha(\tilde{Q}) = \sup_{Y \in L^1(c)} (E_{\tilde{Q}}(-Y) - \rho(Y)) \leq E_{\tilde{Q}}(-X) - \rho(X)$$

and thus,

$$\rho(X) = E_{\tilde{Q}}(-X) - \alpha(\tilde{Q}). \quad \square$$

PROPOSITION 3.2. *Let ρ be a normalized convex risk measure on $L^1(c)$ majorized by a sublinear risk measure. There is a countable set $\{R_n, n \in \mathbb{N}\}$ of probability measures belonging to $L^1(c)^*$ which is relatively compact for the weak* topology of $L^1(c)^*$ and also for the weak topology and such that*

$$(3.13) \quad \forall X \in L^1(c) \quad \rho(X) = \sup_{n \in \mathbb{N}} (E_{R_n}[-X] - \alpha(R_n)),$$

where

$$(3.14) \quad \alpha(R) = \sup_{X \in L^1(c)} (E_R[-X] - \rho(X)).$$

PROOF. From Proposition 3.1, there is a set Q of probability measures in $L^1(c)^*$, relatively compact for the weak* topology, such that equation (3.3) is satisfied. From Lemma 3.1, Q is contained in mK_+ , the nonnegative part of a certain closed ball of $L^1(c)^*$. From Proposition 2.6, mK_+ , is metrizable compact for the weak* topology. There is thus a countable dense set $(Q_n)_{n \in \mathbb{N}}$ in mK_+ . Denote d a distance on mK_+ defining the weak* topology. For every $Q \in mK_+$, let $B(Q, r) = \{R \in mK_+ | d(Q, R) \leq r\}$. The set $B(Q, r)$ is compact for the weak* topology. The penalty α defined on $L^1(c)^*$ by (3.14) is l.s.c. thus, for every $n \in \mathbb{N}$ and $k \in \mathbb{N}^*$, there is $R_{n,k}$ in $B(Q_n, \frac{1}{2^k})$ such that $\alpha(R_{n,k}) = \min\{\alpha(Q), Q \in B(Q_n, \frac{1}{2^k})\}$.

Let $X \in L^1(c)$. From Theorem 3.2, there is $Q_X \in Q$ such that $\rho(X) = E_{Q_X}(-X) - \alpha(Q_X)$. For all $\varepsilon > 0$, there is $\eta > 0$ such that $\forall Q \in B(Q_X, \eta)$, $|E_{Q_X}(-X) - E_Q(-X)| < \varepsilon$. Let k be such that $\frac{1}{2^{k-1}} < \eta$. Let n be such that $Q_X \in B(Q_n, \frac{1}{2^k})$ then $E_{R_{n,k}}(-X) - \alpha(R_{n,k}) > \rho(X) - \varepsilon$. It follows that $\{R_{n,k}, n \in \mathbb{N}, k \in \mathbb{N}^*\}$ is a countable set weakly relatively compact (as it is contained in mK_+) satisfying the required condition. \square

THEOREM 3.3. *Every convex risk measure on $L^1(c)$ can be represented by a countable set of probability measures $\{R_n, n \in \mathbb{N}\}$ belonging to $L^1(c)^*$.*

$$(3.15) \quad \forall X \in L^1(c) \quad \rho(X) = \sup_{n \in \mathbb{N}} (E_{R_n}(-X) - \alpha(R_n)),$$

where $\alpha(R)$ is given by (3.14).

PROOF. From Theorem 3.1, ρ has a dual representation given by (3.1). Denote then $\rho_m(X) = \sup_{Q \in mK_+} (E_Q(-X) - \alpha(Q))$. Even if ρ_m is not necessarily normalized, all the arguments of the proof of Proposition 3.2 apply as mK_+ is metrizable compact for the weak* topology and α is l.s.c. Thus, ρ_m has a representation with a countable set of probability measures. As $\rho = \sup_{m \in \mathbb{N}} \rho_m$, this gives the result. \square

4. Equivalence class of probability measures associated to a nondominated set of probability measures. Let Ω be a metrizable and separable space. In this section we study a capacity defined from a weakly relatively compact set of probability measures \mathcal{P} possibly nondominated.

DEFINITION 4.1. Let \mathcal{P} be a weakly relatively compact set of probability measures on $(\Omega, \mathcal{B}(\Omega))$. Let $1 \leq p < \infty$. The capacity $c_{p,\mathcal{P}}$ is defined on $\mathcal{C}_b(\Omega)$ by

$$(4.1) \quad c_{p,\mathcal{P}}(f) = \sup_{P \in \mathcal{P}} E_P(|f|^p)^{1/p}$$

and extended to every function on Ω as explained in Section 2.1, equations (2.1) and (2.2).

Notice that as \mathcal{P} is a weakly relatively compact set of probability measures, $c_{p,\mathcal{P}}$ is a capacity (see Proposition I.3 of [24] or the Appendix). The Banach space associated to the capacity $c_{p,\mathcal{P}}$ is denoted $L^1(c_{p,\mathcal{P}})$. When there is no ambiguity on the set \mathcal{P} we simply write c_p for $c_{p,\mathcal{P}}$.

When $\mathcal{P} = \{\mu_0\}$, $L^1(c_{p,\{\mu_0\}}) = L^1(\Omega, \mathcal{B}(\Omega), \mu_0)$. A nonnegative measure μ on $(\Omega, \mathcal{B}(\Omega))$ belongs to the (usual) equivalence class of the probability measure μ_0 if and only if $\forall A \in \mathcal{B}(\Omega), \mu(A) = 0 \iff \mu_0(A) = 0$.

Equivalently, for μ in the dual of $L^1(\Omega, \mathcal{B}(\Omega), \mu_0)$,

$$\mu \sim \mu_0 \iff \left[\forall X \in L^1(\Omega, \mathcal{B}(\Omega), \mu_0)_+, X = 0 \iff \int X d\mu = 0 \right].$$

We address the following question: when \mathcal{P} is weakly relatively compact, can one associate a probability measure P to $L^1(c_{p,\mathcal{P}})$ characterizing the null elements in the cone $L^1(c_{p,\mathcal{P}})_+$, that is, such that $\forall X \in L^1(c_{p,\mathcal{P}})_+, X = 0 \iff E_P(X) = \int X dP = 0$? If yes, can one define a natural equivalence relation so that one gets a unique equivalence class of such probability measures? Notice that when \mathcal{P} is not finite, characteristic functions of Borelian sets are not all in $L^1(c_{p,\mathcal{P}})$.

4.1. *Properties of the capacity.*

LEMMA 4.1. For all X in $L^1(c_{p,\mathcal{P}})$, $c_{p,\mathcal{P}}(X) = \sup_{Q \in \mathcal{P}} E_Q(|X|^p)^{1/p}$.

PROOF. Denote $c_p = c_{p,\mathcal{P}}$. For all f, g in $\mathcal{C}_b(\Omega)$, for all $Q \in \mathcal{P}$,

$$|E_Q(|f|^p)^{1/p} - E_Q(|g|^p)^{1/p}| \leq E_Q(|f - g|^p)^{1/p} \leq c_p(|f - g|).$$

As $\mathcal{C}_b(\Omega)$ is dense in $L^1(c_p)$ for the c_p norm, it follows that for every $X \in L^1(c_p)$, $g \in \mathcal{C}_b(\Omega)$ and $Q \in \mathcal{P}$,

$$(4.2) \quad |E_Q(|X|^p)^{1/p} - E_Q(|g|^p)^{1/p}| \leq c_p(|X - g|).$$

From (4.2) it follows that

$$(4.3) \quad E_Q(|X|^p)^{1/p} \leq c_p(X) \quad \forall Q \in \mathcal{P}.$$

For every $X \in L^1(c_p)$, for every $\varepsilon > 0$ there is $g \in \mathcal{C}_b(\Omega)$ such that

$$(4.4) \quad c_p(X - g) \leq \varepsilon.$$

From Definition 4.1, there is $Q_0 \in \mathcal{P}$ such that

$$(4.5) \quad c_p(g) \leq E_{Q_0}(|g|^p)^{1/p} + \varepsilon.$$

As $c_p(X) \leq c_p(g) + \varepsilon$, it follows from (4.2), (4.4) and (4.5) that $c_p(X) \leq \sup_{Q \in \mathcal{P}} E_Q(|X|^p)^{1/p}$. The result follows from (4.3). \square

THEOREM 4.1. *Assume that Ω is a Polish space. There is a countable subset \mathcal{Q} of \mathcal{P} , $\mathcal{Q} = \{P_n, n \in \mathbb{N}\}$, such that for every $X \in L^1(c_{p,\mathcal{P}})$, for every $p \in [1, \infty[$,*

$$(4.6) \quad c_{p,\mathcal{P}}(X) = \sup_{n \in \mathbb{N}} (E_{P_n}(|X|^p))^{1/p}.$$

The capacities $c_{p,\mathcal{P}}$ and $c_{p,\mathcal{Q}}$ defined on $\mathcal{C}_b(\Omega)$ by (4.1) and extended to real functions using formulas (2.1) and (2.2) are equal. The associated Banach spaces are equal: $L^1(c_{p,\mathcal{P}}) = L^1(c_{p,\mathcal{Q}})$.

PROOF. From the previous lemma, applied with $p = 1$, it follows that the set \mathcal{P} is contained in K_+ , the nonnegative part of the unit ball of the dual of $L^1(c_{1,\mathcal{P}})$. Ω is a Polish space, so from Corollary 2.1, K_+ is metrizable compact for the weak* topology. Thus $\overline{\mathcal{P}}$, the closure of \mathcal{P} for the weak* topology, is metrizable compact. There is then in \mathcal{P} a countable set $(P_n)_{n \in \mathbb{N}}$ dense in $\overline{\mathcal{P}}$ for the weak* topology. It follows that for every $X \in L^1(c_{1,\mathcal{P}})$, $\sup_{Q \in \mathcal{P}} E_Q(|X|) = \sup_{n \in \mathbb{N}} E_{P_n}(|X|)$. The equation (4.6) follows for every $p \geq 1$ for every $X \in \mathcal{C}_b(\Omega)$.

The two capacities

$$c_{p,\mathcal{P}}(f) = \sup_{P \in \mathcal{P}} E_P(|f|^p)^{1/p} \quad \text{and} \quad c_{p,\mathcal{Q}} = \sup_{Q \in \mathcal{Q}} E_Q(|f|^p)^{1/p}$$

coincide on $\mathcal{C}_b(\Omega)$. By definition of the extension of a capacity to the set of all functions on Ω , these extensions are the same. Therefore, $L^1(c_{p,\mathcal{P}}) = L^1(c_{p,\mathcal{Q}})$. \square

In the following proposition we study possible extensions of (4.1).

PROPOSITION 4.1. *Let $c_p = c_{p,\mathcal{P}}$. For every nonnegative bounded lower semi-continuous map g ,*

$$(4.7) \quad c_p(g) = \sup_{Q \in \mathcal{P}} E_Q(g^p)^{1/p}.$$

For every Borelian map f ,

$$(4.8) \quad \sup_{Q \in \mathcal{P}} E_Q(|f|^p)^{1/p} \leq c_p(f).$$

PROOF. The proof of the first part of Proposition 2.1, which was given for the characteristic function of an open set, applies without any change to every nonnegative bounded l.s.c. function g . Thus, there is an increasing sequence of continuous functions h_n with limit g and such that $c_p(g) = \lim c_p(h_n)$. As g is bounded, $c_p(g)$ is finite. Let $\varepsilon > 0$. There is n such that $c_p(g) - \varepsilon \leq c_p(h_n) \leq c_p(g)$. By definition of c_p on $\mathcal{C}_b(\Omega)$, there is Q_n in \mathcal{P} such that $c_p(h_n) - \varepsilon \leq E_{Q_n}(h_n^p)^{1/p} \leq c_p(h_n)$. Thus,

$$(4.9) \quad E_{Q_n}(g^p)^{1/p} \geq c_p(g) - 2\varepsilon.$$

On the other hand, for all Q in \mathcal{P} , $E_Q(h_n^p)^{1/p} \leq c_p(h_n) \leq c_p(g)$. From the monotone convergence theorem it follows that

$$(4.10) \quad \forall Q \in \mathcal{P} \quad E_Q(g^p)^{1/p} \leq c_p(g).$$

Thus, from (4.9) and (4.10) we get that

$$(4.11) \quad c_p(g) = \sup_{Q \in \mathcal{P}} E_Q(g^p)^{1/p}.$$

Let f be a Borelian map. If $c_p(f) = +\infty$, the result is trivial. Assume that $c_p(f) < \infty$. Let $\varepsilon > 0$. By definition of $c_p(f)$, equation (2.2), there is g l.s.c., $g \geq |f|$ such that $c_p(g) < c_p(f) + \varepsilon$. As g is l.s.c., we already know that $\sup_{Q \in \mathcal{P}} E_Q(|g|^p)^{1/p} = c_p(g)$. As f is Borel measurable, for all $Q \in \mathcal{P}$, $E_Q(|f|^p)^{1/p}$ is defined. As $g \geq |f|$ it follows that $E_Q(|f|^p)^{1/p} \leq c_p(f) + \varepsilon$. This inequality is true for every ε and every $Q \in \mathcal{P}$. This proves the announced result for every f Borel measurable. \square

REMARK 1. For every open subset V of Ω , 1_V is lower semi-continuous, so from Proposition 4.1, $c_p(1_V) = \sup_{Q \in \mathcal{P}} Q(V)^{1/p}$. However, there are Borelian subsets of Ω for which the equality $c_p(1_A) = \sup_{Q \in \mathcal{P}} Q(A)^{1/p}$ is not satisfied.

For example, let $\Omega = [0, 1]$. Let $x_n \in]0, 1[$ be a sequence converging to 0. Let $A = [0, 1] - \{x_n, n \in \mathbb{N}\}$. Let $Q_n = \delta_{x_n}$. Let $\mathcal{P} = \{Q_n, n \in \mathbb{N}\}$. \mathcal{P} is weakly relatively compact. Let f l.s.c. such that $1_A \leq f \leq 1$. For every $\eta > 0$, $V = \{x \mid f(x) > 1 - \eta\}$ is an open set containing A . As $0 \in A$, there is $\varepsilon > 0$ such that $[0, \varepsilon[\subset V$. So there is $N \in \mathbb{N}$ such that $x_n \in V \forall n \geq N$. So $E_{Q_n}(f^p) = (f(x_n))^p > (1 - \eta)^p$. From (4.7), $1 \geq c_p(f) = \sup_{n \in \mathbb{N}} (E_{Q_n}(f^p))^{1/p} > 1 - \eta$ for every $\eta > 0$. Thus, $c_p(f) = 1$. It follows that $c_p(1_A) = 1$. On the other hand, $Q_n(1_A) = 0$ for all $n \in \mathbb{N}$. Therefore, $\sup_{Q \in \mathcal{P}} Q(A)^{1/p} = 0$. This gives a counterexample.

4.2. *Canonical equivalence class of nonnegative measures associated to c_p .* In all this section, we assume that Ω is a Polish space. We denote c_p the capacity defined on $\mathcal{C}_b(\Omega)$ by $c_p(f) = \sup_{Q \in \mathcal{P}} E_Q(|f|^p)^{1/p}$.

DEFINITION 4.2. $\mathcal{M}^+(c_p)$ is the set of nonnegative finite measures on $(\Omega, \mathcal{B}(\Omega))$ defining an element of $L^1(c_p)^*$.

In the following, we identify an element μ of $\mathcal{M}^+(c_p)$ with its associated linear form on $L^1(c_p)$.

REMARK 2. A nonnegative finite measure μ on $(\Omega, \mathcal{B}(\Omega))$ belongs to $\mathcal{M}^+(c_p)$ if and only if there is a constant $K > 0$ such that $\forall f \in \mathcal{C}_b(\Omega), |\mu(f)| \leq K c_p(f)$. It follows easily that every element in the weak closure of the convex hull of \mathcal{P} defines an element of $\mathcal{M}^+(c_p)$.

DEFINITION 4.3. Define on $\mathcal{M}^+(c_p)$ the relation \mathcal{R}_{c_p} by

$$(4.12) \quad \begin{aligned} \mu \mathcal{R}_{c_p} \nu &\iff \{X \in L^1(c_p), X \geq 0 \mid \mu(X) = 0\} \\ &= \{X \in L^1(c_p), X \geq 0 \mid \nu(X) = 0\}. \end{aligned}$$

The following lemma is trivial.

LEMMA 4.2. \mathcal{R}_{c_p} defines an equivalence relation on $\mathcal{M}^+(c_p)$.

DEFINITION 4.4. Let $\mu \in \mathcal{M}^+(c_p)$. The c_p -class of μ is the equivalence class of μ for the equivalence relation \mathcal{R}_{c_p} .

THEOREM 4.2. To every weakly relatively compact set \mathcal{P} of probability measures on $(\Omega, \mathcal{B}(\Omega))$, possibly nondominated, can be associated canonically a c_p -class of nonnegative measures on $(\Omega, \mathcal{B}(\Omega))$ such that an element μ of $\mathcal{M}^+(c_p)$ belongs to this class if and only if

$$\forall X \in L^1(c_p), X \geq 0 \quad \{\mu(X) = 0\} \iff \{X = 0 \text{ in } L^1(c_p)\}.$$

This class is referred to as the canonical c_p -class.

For every set $\{Q_n, n \in \mathbb{N}\}$ of probability measures on $(\Omega, \mathcal{B}(\Omega))$ such that equality (4.6) is satisfied for all $X \in L^1(c_p)$, for $\alpha_n > 0$ such that $\sum_{n \in \mathbb{N}} \alpha_n = 1$, the probability measure $\sum_{n \in \mathbb{N}} \alpha_n Q_n$ belongs to the canonical c_p -class.

PROOF. Let $p \in [1, \infty[$. Let $\{Q_n\}$ be a countable set of probability measures such that equality (4.6) is satisfied. Let $\mathcal{Q} = \{Q_n, n \in \mathbb{N}\}$. Let $P = \sum_{n \in \mathbb{N}} \alpha_n Q_n$. Let $X \in L^1(c_p), X \geq 0$, that is, from Lemma 2.1, $X = |X|$. $E_P(X) = 0$ if and only if $E_{Q_n}(|X|) = 0$ for all $n \in \mathbb{N}$.

From (4.6), it follows that for $X \geq 0$, $E_P(X) = 0$ if and only if $c_p(X) = 0$, and only if $X = 0$ in $L^1(c_p)$.

This proves that the canonical c_p -class is well defined (as it is not empty) and that $\sum_{n \in \mathbb{N}} \alpha_n Q_n$ belongs to the canonical c_p -class. \square

LEMMA 4.3. Let P be a probability measure belonging to the canonical c_p -class. Let X be an element of $L^1(c_p)$. Then $X \geq 0$ [for the order in $L^1(c_p)$] if and only $X \geq 0$ P a.s.

PROOF. For every $X \in L^1(c_p)$, $|X| - X \geq 0$. From Lemma 2.1 $X \geq 0$ if and only if $|X| - X = 0$ in $L^1(c_p)$. By definition of the canonical c_p -class, this is equivalent to $|X| - X = 0$ P a.s., that is, $X \geq 0$ P a.s. \square

REMARK 3. When $\mathcal{P} = \{P\}$, the canonical c_p -class is the restriction to $\mathcal{M}^+(c_p)$ of the usual equivalence class of the probability measure P .

When \mathcal{P} is a finite set, $\mathcal{P} = \{P_1, \dots, P_n\}$ the canonical c_p -class is the restriction to $\mathcal{M}^+(c_p)$ of the equivalence class (in the usual sense) of the probability measure $P = \frac{\sum_{1 \leq i \leq n} P_i}{n}$.

Our next goal is to give a description of $L^1(c_p)^*$.

THEOREM 4.3. *There is a regular probability measure P belonging to the canonical c_p -class and a countable subset $\mathcal{D} = \{L_n, n \in \mathbb{N}\}$ of the set $L^1(c_p)^*_+$ of nonnegative continuous linear forms on $L^1(c_p)$ such that:*

- $\{L_n, n \in \mathbb{N}\}$ is dense in $L^1(c_p)^*_+ = \mathcal{M}^+(c_p)$ for the weak* topology.
- Every L_n is represented by a nonnegative measure on $(\Omega, \mathcal{B}(\Omega))$ absolutely continuous with respect to P .

Every continuous linear form Φ on $L^1(c_p)$ is the weak* limit of a sequence Φ_n where every Φ_n is the difference of two elements of \mathcal{D} .

Furthermore, for every $X \geq 0$ in $L^1(c_p)$, $X = 0$ iff $P(X) = 0$, iff $L_n(X) = 0$ for all $n \in \mathbb{N}$.

PROOF. Denote $nK_+ = \{L \in L^1(c_p)^*, L \geq 0 \text{ and } \|L\| \leq n\}$. From Corollary 2.1, every nK_+ is metrizable compact for the weak* topology. There is then in nK_+ a dense countable set \mathcal{D}_n . Thus, $\mathcal{D} = \bigcup_{n \in \mathbb{N}} \mathcal{D}_n$ is countable and dense in $L^1(c_p)^*_+$ for the weak* topology. Enumerate the elements of \mathcal{D} , $\mathcal{D} = \{L_n, n \in \mathbb{N}\}$. From Proposition 2.3, every L_n is represented by a nonnegative finite measure μ_n on $(\Omega, \mathcal{B}(\Omega))$. Let $\alpha_n > 0$ such that $\sum_{n \in \mathbb{N}} \alpha_n \|L_n\| < \infty$. Then $\tilde{L} = \sum_{n \in \mathbb{N}} \alpha_n L_n \in L^1(c_p)^*_+$. From Proposition 2.3, \tilde{L} is represented by a nonnegative finite measure μ . Denote P the probability measure $P = \frac{\mu}{\mu(\Omega)}$. P is a probability measure on $(\Omega, \mathcal{B}(\Omega))$, $P \in \mathcal{M}^+(c_p)$. Furthermore, every μ_n is absolutely continuous with respect to P and P is regular from Theorem 1.1 of [5].

We prove now that P belongs to the canonical c_p -class. Every L_n belongs to $L^1(c_p)^*$. Thus, for every X in $L^1(c_p)$ such that $X = 0$ in $L^1(c_p)$, $L_n(X) = 0$ and thus $\tilde{L}(X) = 0$. It follows that $P(X) = 0$. Conversely let $X \geq 0$ in $L^1(c_p)$ such that $P(X) = 0$. It follows that $\tilde{L}(X) = 0$. Every L_n belongs to $L^1(c_p)^*_+$ and $X \geq 0$, thus, $L_n(X) \geq 0$ for all n . From the equality $\tilde{L}(X) = 0$, it follows that $L_n(X) = 0 \forall n \in \mathbb{N}$. $\{L_n, N \in \mathbb{N}\}$ is dense in $L^1(c_p)^*_+$ for the weak* topology, therefore, $L(X) = 0$ for all $L \in L^1(c_p)^*_+$. From the representation result of continuous linear forms on $L^1(c_p)$ (Proposition 2.3) and the Jordan decomposition

of bounded signed measures on $(\Omega, \mathcal{B}(\Omega))$, it follows that every $\Phi \in L^1(c_p)^*$ is represented by a bounded measure $\mu = \mu^+ - \mu^-$. There is a Borelian set A such that $\int f d\mu^+ = \int f 1_A d\mu$ for every $f \in \mathcal{C}_b(\Omega)$. $|\mu| = \mu^+ + \mu^-$ is defined on $(\Omega, \mathcal{B}(\Omega))$ and is thus regular from Theorem 1.1 of [5].

$$(4.13) \quad \forall \varepsilon > 0, \exists V \text{ open}, A \subset V \quad \text{such that } |\mu|(1_V - 1_A) \leq \frac{\varepsilon}{2}.$$

1_V is lower semi-continuous so it is the increasing limit of a sequence of continuous functions h_n . From the monotone convergence theorem and equation (4.13), it follows that

$$(4.14) \quad \forall \varepsilon > 0, \exists h \in \mathcal{C}_b(\Omega), 0 \leq h \leq 1_V \quad \text{such that } \int |1_A - h| d|\mu| < \varepsilon.$$

Thus,

$$(4.15) \quad \left| \int f 1_A d\mu - \int f h d\mu \right| < \|f\|_\infty \varepsilon.$$

By definition of μ ,

$$(4.16) \quad \forall f \in \mathcal{C}_b(\Omega) \quad \left| \int f h d\mu \right| < \|\Phi\|_{c_p}(fh) \leq \|\Phi\|_{c_p}(f).$$

From (4.15) and (4.16), we get $|\int f d\mu^+| = |\int f 1_A d\mu| \leq \|\Phi\|_{c_p}(f)$. It follows that μ^+ defines an element of $L^1(c_p)_+^*$. It is the same for μ^- . Thus, for every $\Phi \in L^1(c_p)^*$, $\Phi(X) = 0$. From Hahn–Banach theorem, it follows that $X = 0$ in $L^1(c_p)$. This proves that P belongs to the canonical c_p -class.

We have proved that every $\Phi \in L^1(c_p)^*$ can be written $\Phi = \Phi^+ - \Phi^-$, $\Phi^+, \Phi^- \in L^1(c_p)_+^*$. The result follows then from the density of \mathcal{D} in $L^1(c_p)_+^*$. \square

The results of the previous section on convex risk measures on $L^1(c)$ can be specified when the capacity is $c_p = c_{p,\mathcal{P}}$.

PROPOSITION 4.2. *Let ρ be a convex risk measure on $L^1(c_p)$. There is a probability measure Q in the canonical c_p -class and a countable set $\{Q_n, n \in \mathbb{N}\}$ of probability measures all absolutely continuous with respect to Q such that*

$$(4.17) \quad \rho(X) = \sup_{n \in \mathbb{N}} [E_{Q_n}(-X) - \alpha(Q_n)] \quad \forall X \in L^1(c_p).$$

PROOF. From Theorem 3.3, there is a countable set $\{Q_n, n \in \mathbb{N}\}$ of probability measures such that equation (4.17) is satisfied. From Theorem 4.2 there is a probability measure P in the canonical c_p -class. Let $Q = \frac{P}{2} + \sum_{n \in \mathbb{N}} \frac{Q_n}{2^{n+2}}$. It is easy to verify that Q satisfies the required conditions. \square

REMARK 4. Even if the capacity c_p is defined from a weakly relatively compact set of probability measures, the set of probability measures $\{Q_n, n \in \mathbb{N}\}$ in the above dual representation (4.17) of a convex risk measure ρ on $L^1(c_p)$ is not always relatively compact for the weak* topology. From Proposition 3.3, $\{Q_n, n \in \mathbb{N}\}$ is relatively compact iff ρ is majorized by a sublinear risk measure.

5. Regular risk measures on $\mathcal{C}_b(\Omega)$.

5.1. *Regularity.* Notice that in a context of uncertainty, which is when no reference probability measure is given, it is natural to consider risk measures defined on the space $\mathcal{C}_b(\Omega)$ or more generally on a lattice vector subspace of $\mathcal{C}_b(\Omega)$. As in Section 2.1, \mathcal{L} denotes a linear vector subspace of $\mathcal{C}_b(\Omega)$ containing the constants, generating the topology of Ω and which is a vector lattice.

DEFINITION 5.1. $\rho: \mathcal{L} \rightarrow \mathbb{R}$ is a convex risk measure on \mathcal{L} if it satisfies the axioms of Definition 3.1, replacing everywhere $L^1(c)$ by \mathcal{L} . It is normalized if $\rho(0) = 0$.

A sublinear risk measure ρ on \mathcal{L} is regular if for every decreasing sequence X_n of elements of \mathcal{L} with limit 0, $\rho(-X_n)$ tends to 0. A normalized convex risk measure is uniformly regular if for all $X \sup_{\lambda>0} \frac{\rho(\lambda X)}{\lambda} < \infty$ and for every decreasing sequence X_n of elements of \mathcal{L} with limit 0, $\frac{\rho(-\lambda X_n)}{\lambda}$ converges to 0 uniformly in λ .

REMARK 5. For sublinear risk measures, the two notions of regularity and uniform regularity are equivalent.

From now on in this section ρ is a normalized convex risk measure on \mathcal{L} .

LEMMA 5.1. Assume that ρ is uniformly regular. $\rho_{\min}(X) = \sup_{\lambda>0} \frac{\rho(\lambda X)}{\lambda}$ defines a regular sublinear risk measure on \mathcal{L} . It is the minimal sublinear risk measure on \mathcal{L} majorizing ρ .

PROOF. The convexity, monotonicity and translation invariance of ρ_{\min} follow easily from the same properties of ρ . The homogeneity of ρ_{\min} follows from its definition. Thus, ρ_{\min} is a sublinear risk measure on \mathcal{L} majorizing ρ . The regularity of ρ_{\min} follows from the uniform regularity of ρ . For every sublinear risk measure ρ_1 majorizing ρ , for every $X \in \mathcal{L}$, $\rho_{\min}(X) \leq \rho_1(X)$. Thus, ρ_{\min} is minimal. \square

LEMMA 5.2. For every Y in \mathcal{L} , for every sequence λ_n of real numbers decreasing to 1, the sequence $\rho(\lambda_n Y)$ converges to the limit $\rho(Y)$.

PROOF. As λ_n is a decreasing sequence with limit 1, one can assume that $2 > \lambda_n \geq 1$. Write $\lambda_n = 1 + \varepsilon_n$, $0 \leq \varepsilon_n < 1$. From the convexity of ρ and $\rho(0) = 0$, it follows that

$$(5.1) \quad \rho((1 + \varepsilon_n)Y) \geq (1 + \varepsilon_n)\rho(Y),$$

$(1 + \varepsilon_n)Y = (1 - \varepsilon_n)Y + \varepsilon_n(2Y)$. Using the convexity of ρ , it follows that

$$(5.2) \quad \rho((1 + \varepsilon_n)Y) \leq (1 - \varepsilon_n)\rho(Y) + \varepsilon_n\rho(2Y).$$

From (5.1) and (5.2),

$$(5.3) \quad (1 + \varepsilon_n)\rho(Y) \leq \rho((1 + \varepsilon_n)Y) \leq (1 - \varepsilon_n)\rho(Y) + \varepsilon_n\rho(2Y).$$

Passing now to the limit in inequality (5.3), it follows that the sequence $\rho((1 + \varepsilon_n)Y)$ has a limit equal to $\rho(Y)$. \square

Using the preceding lemma, we prove now that every normalized uniformly regular convex risk measure can be extended into a convex risk measure on $L^1(c)$ for some capacity c . Therefore, we will be able to apply the representation results of Section 3.

LEMMA 5.3. *Assume that ρ is uniformly regular. Denote ρ_1 a regular sublinear risk measure on \mathcal{L} such that $\rho \leq \rho_1$.*

- $c(X) = \rho_1(-|X|)$ defines a capacity on \mathcal{L} .
- ρ_1 has a unique continuous extension into a sublinear risk measure $\bar{\rho}_1$ on $L^1(c)$.
- ρ has a unique continuous extension into a normalized convex risk measure $\bar{\rho}$ on $L^1(c)$ majorized by $\bar{\rho}_1$.

PROOF. The sublinearity, monotonicity and regularity of ρ_1 imply that c is a capacity on \mathcal{L} . As usual, this leads to the Banach space $L^1(c)$. As ρ_1 is sublinear, for every $X, Y \in \mathcal{L}$, $\rho_1(X) \leq \rho_1(Y) + \rho_1(X - Y)$.

Exchanging X and Y and using the monotonicity of ρ_1 and the definition of c , it follows that $|\rho_1(X) - \rho_1(Y)| \leq c(X - Y)$. Thus, ρ_1 is uniformly continuous on \mathcal{L} for the c semi-norm. It extends uniquely into a continuous function $\bar{\rho}_1$ on $L^1(c)$. $\bar{\rho}_1$ is a sublinear risk measure. Let $\varepsilon_n > 0$ decreasing to 0:

$$X = \frac{1}{1 + \varepsilon_n}[(1 + \varepsilon_n)Y] + \frac{\varepsilon_n}{1 + \varepsilon_n} \left[\frac{1 + \varepsilon_n}{\varepsilon_n}(X - Y) \right].$$

From the convexity of ρ , the majoration of ρ by ρ_1 and the homogeneity of ρ_1 (cf. ρ_1 is sublinear), it follows that

$$(5.4) \quad \rho(X) \leq \frac{1}{1 + \varepsilon_n} \rho((1 + \varepsilon_n)Y) + \rho_1(X - Y).$$

From inequality (5.4) and Lemma 5.2 applied with $(1 + \varepsilon_n)Y$, passing to the limit, it follows then that $\rho(X) - \rho(Y) \leq \rho_1(X - Y) \leq c(X - Y)$. Exchanging X and Y , this proves the uniform continuity of ρ for the c semi-norm. ρ extends then uniquely into a continuous function $\bar{\rho}$ on $L^1(c)$. $\bar{\rho}$ is a convex risk measure on $L^1(c)$ majorized by $\bar{\rho}_1$. \square

DEFINITION 5.2. Let ρ be a normalized uniformly regular convex risk measure on \mathcal{L} . The capacity c_ρ defined as $c_\rho(X) = \rho_{\min}(-|X|)$ is called the capacity canonically associated with ρ .

5.2. *Representation of uniformly regular convex risk measures.* In this section, we assume that Ω is a Polish space. Taking into account the liquidity risk in a financial market, we introduce the following definition for a riskless asset, which means that all investment in this asset is risk-free.

DEFINITION 5.3. A nonpositive element X of $\mathcal{C}_b(\Omega)$ is riskless if for all $\lambda > 0$, $\rho(\lambda X) = 0$ [or equivalently for all $\lambda > 0$, $\rho(\lambda X) \leq 0$].

THEOREM 5.1. Let ρ be a normalized uniformly regular convex risk measure on \mathcal{L} .

Then ρ extends uniquely to $\mathcal{C}_b(\Omega)$ and admits the following representation,

$$(5.5) \quad \forall X \in \mathcal{C}_b(\Omega) \quad \rho(X) = \sup_{n \in \mathbb{N}} (E_{Q_n}(-X) - \alpha(Q_n))$$

for a certain weakly relatively compact set $\{Q_n, n \in \mathbb{N}\}$ of probability measures. Furthermore, for $\alpha_n > 0$ such that $\sum_{n \in \mathbb{N}} \alpha_n = 1$, the probability measure $P = \sum_{n \in \mathbb{N}} \alpha_n Q_n$ characterizes the riskless nonnegative elements of $\mathcal{C}_b(\Omega)$, that is, $X \leq 0$ is riskless iff $X = 0$ P a.s.

For every $X \in \mathcal{C}_b(\Omega)$ there is a probability measure Q_X in the weak closure of $\{Q_n, n \in \mathbb{N}\}$, such that

$$(5.6) \quad \rho(X) = E_{Q_X}(-X) - \alpha(Q_X).$$

PROOF. Let $c_\rho(X) = \rho_{\min}(-|X|)$ be the capacity canonically associated with ρ (Definition 5.2). As Ω is a Polish space, every capacity is a Prokhorov capacity. Denote ρ (resp., ρ_{\min}) the extensions of ρ (resp., ρ_{\min}) to $L^1(c_\rho)$ given by Lemma 5.3.

As ρ is majorized by ρ_{\min} , the representation result with a countable weakly relatively compact set $\mathcal{Q} = \{Q_n\}$ follows from Proposition 3.2. We can, of course, restrict to Q_n such that $\alpha(Q_n) < \infty$. Then $c_\rho(X) = \sup_{n \in \mathbb{N}} E_{Q_n}(|X|)$, that is, $c_\rho = c_{1, \mathcal{Q}}$. From Theorem 4.2 the probability measure $P = \sum_{n \in \mathbb{N}} \alpha_n Q_n$ belongs to the canonical c_ρ -class. Let $X \leq 0$ in $\mathcal{C}_b(\Omega)$, X is riskless iff $\rho(\lambda X) = 0 \forall \lambda > 0$, iff $c_\rho(-X) = 0$, iff $X = 0$ P a.s. The existence of Q_X follows from Theorem 3.2. \square

6. Examples.

6.1. *G-expectations.* In this section, $\Omega = \mathcal{C}_0([0, \infty[, \mathbb{R}^d)$, the set of continuous functions f defined on $[0, \infty[$ with values in \mathbb{R}^d such that $f(0) = 0$. $\mathcal{C}_0([0, \infty[, \mathbb{R}^d)$ endowed with the topology of uniform convergence on compact spaces is a Polish space.

Peng introduced the notion of sublinear expectation and of G -expectations [27, 28] defined on a vector lattice \mathcal{H} of real functions containing 1 and included

in $C_b(\Omega)$. For the definition of a sublinear expectation \mathbb{E} on \mathcal{H} we refer to [15], Section 3. G -expectations are defined from solutions of P.D.E. in [27] and [28]. A G -expectation is up to a minus sign a sublinear risk measure.

It is proved in [15] and [23] that every G -expectation \mathbb{E} has a representation with respect to a weakly relatively compact set of probability measures \mathcal{P} : $\mathbb{E}(f) = \sup_{P \in \mathcal{P}} E_P(f)$ for all f in \mathcal{H} . \mathbb{E} extends naturally to $C_b(\Omega)$,

$$(6.1) \quad \mathbb{E}(f) = \sup_{P \in \mathcal{P}} E_P(f) \quad \forall f \in C_b(\Omega).$$

As \mathcal{P} is weakly relatively compact, $\rho(f) = \mathbb{E}(-f)$ is a sublinear regular risk measure on $C_b(\Omega)$. Denote $c_{\mathbb{E}} = c_{\rho}$ the corresponding capacity $c_{\mathbb{E}}(X) = \mathbb{E}(|X|) \forall X \in C_b(\Omega)$.

Notice that, alternatively, regularity could be proved directly for G -expectations. Theorem 5.1 would thus give the representation result [equation (6.1)].

PROPOSITION 6.1. *There is a countable weakly relatively compact set $\{Q_n, n \in \mathbb{N}\}$ of probability measures, $Q_n \in \mathcal{P}$ such that*

$$(6.2) \quad \forall X \in C_b(\Omega) \quad \mathbb{E}(X) = \sup_{n \in \mathbb{N}} E_{Q_n}(X).$$

Let $P = \sum_{n \in \mathbb{N}^*} \frac{Q_n}{2^{n+1}}$. For all $f \geq 0$ in $C_b(\Omega)$, $\mathbb{E}(f) = 0$ iff $f = 0$ P a.s.

For every $X \in C_b(\Omega)$, there is a probability measure Q_X in the weak closure of $\{Q_n, n \in \mathbb{N}^*\}$, such that $\mathbb{E}(X) = E_{Q_X}(X)$.

PROOF. The result follows from Theorem 5.1. \square

6.2. Risk measure in context of uncertain volatility. We consider a framework introduced in [16]. Let $\Omega = C_0([0, T], \mathbb{R}^d)$ the space of continuous functions on $[0, T]$ null in zero. For every $t \leq T$, let $\Omega_t = C_0([0, t], \mathbb{R}^d)$. Ω_t is identified with the subset of Ω of elements which are constant on $[t, T]$. Let \mathcal{B}_t be the σ -algebra on Ω generated by the open sets of Ω_t . Denote B_t the coordinate process. A probability measure Q on $(\Omega, \mathcal{B}(\Omega))$ is called an orthogonal martingale measure if the coordinate process (B_t) is a martingale with respect to \mathcal{B}_t under Q and if the martingales $((B_i)_t)_{1 \leq i \leq d}$ are orthogonal in the sense that for all $i \neq j$, $\langle B_i, B_j \rangle_t^Q = 0$ Q a.s. $\langle B_i, B_j \rangle^Q$ denotes the quadratic covariational process corresponding to B^i and B^j , under Q and $\langle B \rangle^Q$ the quadratic variation of B under Q . Fix for all $i \in \{1, \dots, d\}$ two finite deterministic Hölder-continuous measures $\underline{\mu}_i$ and μ_i on $[0, T]$ and consider the set \mathcal{P} of orthogonal martingale measures such that

$$\forall i \in \{1, \dots, d\} \quad d\underline{\mu}_{i,t} \leq d\langle B_i \rangle_t^Q \leq d\mu_{i,t}.$$

Kervarec has proved in [24], Lemma 1.3, that the set \mathcal{P} is weakly relatively compact. Thus, $c_1(f) = \sup_{Q \in \mathcal{P}} E_Q(|f|)$ defines a capacity on $C_b(\Omega)$ (see the Appendix). As in Section 4, $L^1(c_1)$ denotes the corresponding Banach space containing $C_b(\Omega)$ as a dense subset. From Theorems 4.1 and 4.2, there is a countable set

$(P_n)_{n \in \mathbb{N}}$, $P_n \in \mathcal{P}$ such that $\forall X \in L^1(c_1)$, $c_1(X) = \sup_{n \in \mathbb{N}} E_{P_n}(|X|)$ and such that $P = \sum_{n \in \mathbb{N}} \frac{P_n}{2^n}$ belongs to the canonical c_1 -class.

LEMMA 6.1. *For every probability measure R defining an element of $L^1(c_1)^*$,*

$$\forall i \in \{1, \dots, d\} \quad d\underline{\mu}_{i,t} \leq d\langle B_i \rangle_t^R \leq d\mu_{i,t}.$$

Notice that a probability measure R in $L^1(c)^*$ does not necessarily belong to \mathcal{P} and therefore the result is not trivial.

PROOF OF LEMMA 6.1. From [16], $(B_i)_s^2 \in L^1(c_1)$ for every s , thus, $\int_0^t (B_i)_s d(B_i)_s$ can be defined as an element of $L^1(c_1)$. We thus define the quadratic variation of B in $L^1(c_1)$ by

$$(6.3) \quad \langle B_i \rangle_t^{c_1} = (B_i)_t^2 - 2 \int_0^t (B_i)_s d(B_i)_s.$$

This equation is satisfied in $L^1(c_1)$ thus it is satisfied R a.s. for every probability measure R defining an element of $L^1(c_1)^*$. Let $s \leq t$. Let $A = \{\omega | \langle B_i \rangle_t^{c_1} - \langle B_i \rangle_s^{c_1} > \mu_i[s, t]\} \cup \{\omega | \langle B_i \rangle_t^{c_1} - \langle B_i \rangle_s^{c_1} < \underline{\mu}_i[s, t]\}$. By hypothesis $P_n(A) = 0$. Thus, $P(A) = 0$. The inequality

$$(6.4) \quad \mu_i[s, t] \geq \langle B_i \rangle_t^{c_1} - \langle B_i \rangle_s^{c_1} \geq \underline{\mu}_i[s, t]$$

is thus satisfied P a.s. From Lemma 4.3, inequality (6.4) is then satisfied in $L^1(c_1)$ and then also R a.s. for every probability measure defining an element of $L^1(c_1)^*$. □

PROPOSITION 6.2. *The set \mathcal{P} is convex metrizable compact for the weak* topology $\sigma(L^1(c_1)^*, L^1(c_1))$ and also for the weak topology.*

PROOF. The convexity of \mathcal{P} is obvious. Denote as in Section 2, K_+ the non-negative part of the unit ball of $L^1(c)^*$. From the definition of c_1 it follows that $\mathcal{P} \subset K_+$. Thus, the weak* closure $\overline{\mathcal{P}}$ of \mathcal{P} is a subset of K_+ . From Lemma 6.1 it follows that every element $Q \in \overline{\mathcal{P}}$ satisfies

$$\forall i \in \{1, \dots, d\} \quad d\underline{\mu}_{i,t} \leq d\langle B_i \rangle_t^Q \leq d\mu_{i,t}.$$

From Corollary 2.1, K_+ is metrizable compact for the weak* topology thus, for every $Q \in \overline{\mathcal{P}}$, there is a sequence Q_n , $Q_n \in \mathcal{P}$ converging to Q for the weak* topology.

From [16], $|(B_i)_t|^k \in L^1(c_1)$ for $k = 1$ or 2 , so $(E_{Q_n} - E_Q)(|(B_i)_t|^k) \rightarrow 0$. Passing to the limit, $E_Q(|(B_i)_t|) \leq c_1(|(B_i)_t|)$ and

$$(6.5) \quad E_Q(|(B_i)_t|^2) \leq c_1(|(B_i)_t|^2).$$

Let g in $\mathcal{C}_b(\Omega_s)$. g can be identified with the element \tilde{g} of $\mathcal{C}_b(\Omega)$ defined by $\tilde{g}(x) = g(x|_{[0,s]})$. It follows from the inequality $c_1(Xg) \leq \|g\|_\infty c_1(|X|)$ that $\forall u \geq s, (B_i)_u g \in L^1(c_1)$, so $\forall g \in \mathcal{C}_b(\Omega_s) \forall \lambda \in \mathbb{R}$,

$$(6.6) \quad (E_{Q_n} - E_Q)((B_i)_u(g + \lambda)) \rightarrow 0.$$

$(B_i)_t$ is a martingale for Q_n , thus passing to the limit in (6.6), with $u = t$ and $u = s$, we obtain $\forall g \in \mathcal{C}_b(\Omega_s) \forall \lambda \in \mathbb{R}$,

$$(6.7) \quad E_Q((B_i)_t(g + \lambda)) = E_Q((B_i)_s(g + \lambda)).$$

From (6.5), $(B_i)_u \in L^2(\Omega, \mathcal{B}_u, Q)$ for $u = t, s$, and $\{g + \lambda, g \in \mathcal{C}_b(\Omega_s), \lambda \in \mathbb{R}\}$ is dense in $L^2(\Omega, \mathcal{B}_s, Q)$ thus, equality (6.7) is satisfied for every $g \in L^2(\Omega, \mathcal{B}_s, Q)$. This proves that $(B_i)_t$ is a martingale for Q . A very similar proof leads to the fact that the martingales $(B_i)_t$ and $(B_j)_t$ are mutually orthogonal for $i \neq j$. Thus, \mathcal{P} is closed for the weak* topology. As $\mathcal{P} \subset K_+$, \mathcal{P} is metrizable compact for the weak* topology. The result follows from Proposition 2.4 for the weak topology. \square

For every $P \in \mathcal{P}$ let $\beta(P) \geq 0$. Let ρ be defined by

$$(6.8) \quad \forall X \in \mathcal{C}_b(\Omega) \quad \rho(X) = \sup_{P \in \mathcal{P}} (E_P(-X) - \beta(P)).$$

As \mathcal{P} is metrizable compact for the weak topology, $\rho - \rho(0)$ is a uniformly regular convex risk measure. Thus, Theorem 5.1 applies.

The link between the two previous examples is studied in [15]. The convex weakly compact set characterizing the G -expectation \mathbb{E} is in fact contained in the set \mathcal{P} of orthogonal martingale measures introduced in [16] and considered in Section 6.2.

APPENDIX

Let Ω be a metrizable separable space and \mathcal{L} as in Section 2 a lattice of continuous bounded functions, containing constants and generating the topology of Ω . We now recall some definitions and propositions proved in Section 2 of [20]. A capacity is defined as in Definition 2.1, Section 2.

DEFINITION A.1. A capacity c defined on \mathcal{L} is regular if it satisfies the following:

For all decreasing net $f_\alpha \in \mathcal{L}$ converging to 0, $\inf c(f_\alpha) = 0$.

DEFINITION A.2. A capacity c defined on \mathcal{L} is a Prokhorov capacity if for all $\varepsilon > 0$, there exists a compact set K such that $c(f) \leq \varepsilon$ for all $f \in \mathcal{L}$ such that $|f| \leq \mathbf{1}_{\Omega \setminus K}$.

PROPOSITION A.3. *If Ω is a Lindelöf space, then every capacity is a regular capacity.*

PROPOSITION A.4. *If Ω is locally compact or a Polish space, then every regular capacity is a Prokhorov capacity.*

REMARK 6. If Ω is a Polish space, then it is a Lindelöf space and thus every capacity is a Prokhorov capacity.

PROPOSITION A.5. *If \mathcal{P} is weakly relatively compact, c defined on $\mathcal{C}_b(\Omega)$ by $c(f) = \sup_{P \in \mathcal{P}} (E_P[|f|^p])^{1/p}$ is a capacity.*

The proof follows from Dini's theorem (see [24], Proposition I.3, for more details).

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REFERENCES

- [1] ARTZNER, P., DELBAEN, F., EBER, J.-M. and HEATH, D. (1999). Coherent measures of risk. *Math. Finance* **9** 203–228. [MR1850791](#)
- [2] AVELLANEDA, M., LEVY, A. and PARAS, A. (1995). Pricing and hedging derivative securities in markets with uncertain volatilities. *Appl. Math. Finance* **2** 73–88.
- [3] BARRIEU, P. and EL KAROUI, N. (2009). Pricing, hedging and optimally designing derivatives via minimization of risk measures. In *Volume on Indifference Pricing* (R. Carmona, ed.). Princeton Univ. Press, Princeton, NJ.
- [4] BIAGINI, S. and FRITTELLI, M. (2009). On the extension of the Namioka–Klee theorem and on the Fatou property for risk measures. In *Optimality and Risk—Modern Trends in Mathematical Finance: The Kabanov Festschrift* (F. Delbaen, M. Rásonyi and C. Stricker, eds.) 1–28. Springer, Berlin. [MR2648595](#)
- [5] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. Wiley, New York. [MR1700749](#)
- [6] BION-NADAL, J. (2004). Conditional risk measure and robust representation of convex conditional risk measures. Preprint CMAP 557.
- [7] BION-NADAL, J. (2008). Dynamic risk measures: Time consistency and risk measures from BMO martingales. *Finance Stoch.* **12** 219–244. [MR2390189](#)
- [8] BION-NADAL, J. (2009). Time consistent dynamic risk processes. *Stochastic Process. Appl.* **119** 633–654. [MR2494007](#)
- [9] BOURBAKI, N. (1958). *Eléments de Mathématiques, Topologie Générale*, Chapter X, 2nd ed. Hermann, Paris.
- [10] BOURBAKI, N. (1969). *Eléments de Mathématiques, Integration*, Chapter 9. Hermann.
- [11] CHERIDITO, P., DELBAEN, F. and KUPPER, M. (2006). Dynamic monetary risk measures for bounded discrete-time processes. *Electron. J. Probab.* **11** 57–106. [MR2199055](#)
- [12] DELBAEN, F. (2002). Coherent risk measures on general probability spaces. In *Advances in Finance and Stochastics. Essays in Honour of Dieter Sondermann* (K. Sandmann and P. J. Schönbucher, eds.) 1–37. Springer, Berlin. [MR1929369](#)

- [13] DELBAEN, F. (2006). The structure of m -stable sets and in particular of the set of risk neutral measures. In *In Memoriam Paul-André Meyer: Séminaire de Probabilités XXXIX. Lecture Notes in Math.* **1874** 215–258. Springer, Berlin. [MR2276899](#)
- [14] DELBAEN, F., PENG, S. and ROSAZZA GIANIN, E. (2010). Representation of the penalty term of dynamic concave utilities. *Finance Stoch.* **14** 449–472. [MR2670421](#)
- [15] DENIS, L., HU, M. and PENG, S. (2011). Function spaces and capacity related to a sublinear expectation: Application to G -Brownian motion paths. *Potential Anal.* **34** 139–161.
- [16] DENIS, L. and MARTINI, C. (2006). A theoretical framework for the pricing of contingent claims in the presence of model uncertainty. *Ann. Appl. Probab.* **16** 827–852. [MR2244434](#)
- [17] DETLEFSEN, K. and SCANDOLO, G. (2005). Conditional and dynamic convex risk measures. *Finance Stoch.* **9** 539–561. [MR2212894](#)
- [18] DUNFORD, N. and SCHWARTZ, J. T. (1958). *Linear Operators. Part I. General Theory*. Wiley, New York. [MR1009162](#)
- [19] FEYEL, D. and DE LA PRADELLE, A. (1977). Topologies fines et compactifications associées à certains espaces de Dirichlet. *Ann. Inst. Fourier (Grenoble)* **27** 121–146. [MR0508018](#)
- [20] FEYEL, D. and DE LA PRADELLE, A. (1989). Espaces de Sobolev Gaussiens. *Ann. Inst. Fourier (Grenoble)* **39** 875–908. [MR1036336](#)
- [21] FÖLLMER, H. and SCHIED, A. (2004). *Stochastic Finance: An Introduction in Discrete Time*, 2nd ed. *de Gruyter Studies in Mathematics* **27**. de Gruyter, Berlin. [MR2169807](#)
- [22] FRITTELLI, M. and ROSAZZA GIANIN, E. (2002). Putting order in risk measures. *Journal of Banking Finance* **26** 1473–1486.
- [23] HU, M.-S. and PENG, S.-G. (2009). On representation theorem of G -expectations and paths of G -Brownian motion. *Acta Math. Appl. Sin. Engl. Ser.* **25** 539–546. [MR2506990](#)
- [24] KERVAREC, M. (2008). Modèles non dominés en mathématiques financières. Thèse de Doctorat en Mathématiques, Univ. d'Evry, Evry, France.
- [25] KLÖPPEL, S. and SCHWEIZER, M. (2007). Dynamic indifference valuation via convex risk measures. *Math. Finance* **17** 599–627. [MR2352907](#)
- [26] PENG, S. (2004). Nonlinear expectations, nonlinear evaluations and risk measures. In *Stochastic Methods in Finance. Lecture Notes in Math.* **1856** 165–253. Springer, Berlin. [MR2113723](#)
- [27] PENG, S. (2007). G -expectation, G -Brownian motion and related stochastic calculus of Itô type. In *Stochastic Analysis and Applications. Abel Symp.* **2** 541–567. Springer, Berlin. [MR2397805](#)
- [28] PENG, S. (2008). Multi-dimensional G -Brownian motion and related stochastic calculus under G -expectation. *Stochastic Process. Appl.* **118** 2223–2253. [MR2474349](#)
- [29] ROCKAFELLAR, R. T. (1974). *Conjugate Duality and Optimization*. SIAM, Philadelphia, PA. [MR0373611](#)
- [30] ROORDA, B. and SCHUMACHER, J. M. (2007). Time consistency conditions for acceptability measures, with an application to Tail Value at Risk. *Insurance Math. Econom.* **40** 209–230. [MR2286942](#)
- [31] RUDIN, W. (1966). *Real and Complex Analysis*. McGraw-Hill, New York. [MR0210528](#)

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