Brazilian Journal of Probability and Statistics
2011, Vol. 25, No. 1, 90–98
DOI: 10.1214/09-BJPS111

Covariance analysis of the squares of the purely diagonal bilinear time series models

Iheanyi S. Iwueze[a] and Ohakwe Johnson[b]

[a]*Renaissance University*
[b]*Abia State University*

Abstract. The covariance structure among other properties of the square of the purely diagonal bilinear time series model is obtained. The time series properties of these squares are compared with those of the linear moving average time series model. It was discovered that the square of a linear moving average process is also identified as a moving average process whereas, while the nonlinear purely diagonal bilinear process is identified as a linear moving average process, its square is identified as an autoregressive moving average process.

1 Introduction

The majority of work on time series has been centered on linear models. The traditional search has been for a class of models that can be analyzed with reasonable ease, yet are sufficiently general to be able to well approximate most series that arise in the real world. The class of models considered by Box and Jenkins (1976), known as autoregressive moving average (ARMA) models have this property and have been used with considerable success in many scientific fields, including economic forecasting [see Granger and Newbold (1977)]. Nevertheless ARMA models are linear models and are evaluated using only criteria appropriate to such models, whereas in most disciplines, the theory may be on nonlinear relationships between variables.

A special class of nonlinear models which have been found useful and gained wide acceptability is the bilinear time series model proposed by Granger and Andersen (1978) and studied further by Subba Rao (1981). Let e_t, $t \in Z$, $Z\{\ldots, -1, 0, 1, \ldots\}$ be a sequence of independent identically distributed random variables with $E(e_t) = 0$ and $E(e_t^2) = \sigma^2 < \infty$. Let $a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_h$, θ_{ij}, $1 \le i \le m$, $1 \le j \le l$, be real constants. The general bilinear autoregressive moving average process of order (r, h, m, l), denoted by BARMA(r, h, m, l) as defined by Granger and Andersen (1978) is given by

$$X_t = \sum_{j=1}^{r} a_j X_{t-j} + \sum_{j=1}^{h} b_j e_{t-j} + \sum_{i=1}^{m} \sum_{j=1}^{\ell} \theta_{ij} X_{t-i} e_{t-j} + e_t \tag{1.1}$$

Key words and phrases. Purely diagonal bilinear model, linear moving average model, first-order moments, covariance function.

Received August 2008; accepted November 2009.

for every $t \in Z$. Special cases of (1.1) are discussed by the following authors, amongst others; Granger and Andersen (1978), Subba Rao (1981), Subba Rao and Gabr (1981), Tong (1981), Bhashara Rao, Subba Rao and Walker (1983), Akamanam, Bhaskara Rao and Subramanyam (1986).

A bilinear model is one which is linear in both X_t, $t \in Z$, and e_t, $t \in Z$, but not in those variables jointly. The first part of (1.1) is identifiable as the autoregressive (AR) part, the second as the moving average (MA) part and the third part is the pure bilinear part of the process X_t, $t \in Z$. When $\theta_{ij} = 0$ for all i and j in (1.1), we obtain the linear ARMA process. Thus a study of bilinear models subsumes the study of AR models as well as MA and ARMA models.

The model (1.1) is said to be a purely bilinear process when $a_j = b_j = 0$ for all j. A purely bilinear process is said to be a purely diagonal bilinear process if in addition $\theta_{ij} = 0$ for all $i \neq j$. Let X_t, $t \in Z$, and e_t, $t \in Z$, be two stochastic processes defined on some probability space (Ω, A, P), $X_t, t \in Z$, is said to be a purely diagonal bilinear process of order q (PDB(q)) with respect to the process $e_t, t \in Z$, if

$$X_t = \sum_{j=1}^{q} \theta_{jj} X_{t-j} e_{t-j} + e_t \tag{1.2}$$

for every $t \in Z$. Akamanam (1983) established that the covariance function of (1.2) is the same as that of the moving average process of order q (MA(q)) given by

$$X_t = \beta_0 + \sum_{j=1}^{q} \beta_j \mu_{t-j} + \mu_t, \tag{1.3}$$

where $\mu_t, t \in Z$, may not be a purely random process. The necessary and sufficient condition of stationarity of (1.2) has been discussed in Terdik (1999). For a time series model to be useful for forecasting purposes, it is necessary that it should be invertible. A sufficient condition for the invertibility of model (1.2) have been derived by Guegan and Pham (1987).

We have seen that linear and bilinear models have the same covariance structure under second-order analysis. The problem of differentiating a linear ARMA process from a nonlinear BARMA process has therefore engaged the attention of many authors. Third-order moments and cumulants are the widely accepted method [see Gabr (1988), Sessay and Subba Rao (1991), Oyet and Iwueze (1993), Iwueze and Chikezie (2006)] of differentiating between the two competing models. Little attention has been paid to the fourth moments or the time series properties of the squares of the series.

The purpose of this paper is to carry out a second-order analysis on the squares ($Y_t = X_t^2$) of the purely diagonal bilinear process (1.2) and that of the linear moving average process (1.3), with a view to providing an alternate differentiation technique. Section 2 will consider the first-order and second-order moments of $Y_t = X_t^2$ for (1.2), while Section 3 will consider the same moments for (1.3). Section 4 contains the concluding remarks.

2 Covariance analysis for the square of the diagonal completely bilinear process

Given that $X_t, t \in Z$, satisfies (1.2), we obtain

$$
\begin{aligned}
Y_t = X_t^2 = \sum_{j=1}^{q} \theta_{jj}^2 X_{t-j}^2 e_{t-j}^2 + 2 \sum_{i<j}^{q} \sum^{q} \theta_{ii}\theta_{jj} X_{t-i} e_{t-i} X_{t-i} e_{t-j} \\
+ 2 \sum_{j=1}^{q} \theta_{jj} X_{t-j} e_{t-j} e_t + e_t^2.
\end{aligned} \tag{2.1}
$$

In obtaining the expressions in this section, we assume that the random variables $e_t, t \in Z$, are Gaussian with $E(e_t) = 0$ and $E(e_t^2) = \sigma^2 < \infty$. We shall also use the assumption that by expression (1.2), $e_t, t \in Z$, is independent of $X_h, h < t$. Based on these assumptions, it can easily be shown that

$$E(X_t e_t) = \sigma^2, \tag{2.2}$$

$$E(X_t^2 e_t^2) = \sigma^2 E(X_t^2) + 2\sigma^4, \tag{2.3}$$

$$E(X_{t-i} e_{t-i} X_{t-j} e_{t-j}) = \sigma^4. \tag{2.4}$$

Thus

$$E(Y_t) = E(X_t^2) = \sigma^2 + 2\sigma^4 \sum_{i<j}^{q} \sum^{q} \theta_{ii}\theta_{jj} + \sum_{j=1}^{q} \theta_{jj}^2 [\sigma^2 E(X_t^2) + 2\sigma^4].$$

Therefore

$$\left(1 - \sigma^2 \sum_{j=1}^{q} \theta_{jj}^2\right) E(X_t) = \sigma^2 + 2\sigma^4 \sum_{i<j}^{q} \sum^{q} \theta_{ii}\theta_{jj} + 2\sigma^4 \sum_{j=1}^{q} \theta_{jj}^2.$$

This implies that

$$E(Y_t) = E(X_t^2) = \frac{\sigma^2 (1 + 2\sigma^2 \sum^q \sum^q{}_{i<j} \theta_{ii}\theta_{jj} + 2\sigma^2 \sum_{j=1}^q \theta_{jj}^2)}{(1 - \sigma^2 \sum_{j=1}^q \theta_{jj}^2)} = \mu_y \tag{2.5}$$

provided, $\sigma^2 \sum_{j=1}^q \theta_{jj}^2 < 1$. It is clear from (2.5) that

$$\left(1 - \sigma^2 \sum_{j=1}^{q} \theta_{jj}^2\right) = \frac{\sigma^2 (1 + 2\sigma^2 \sum^q \sum^q{}_{i<j} \theta_{ii}\theta_{jj} + 2\sigma^2 \sum_{j=1}^q \theta_{jj}^2)}{\mu_y}. \tag{2.6}$$

We now proceed to find the autocorrelation function, $R_y(k), k = 0, 1, 2, \dots$ for the variable $Y_t = X_t^2$, where

$$R_y(k) = E(Y_t Y_{t+k}) - \mu_y^2 = E(X_t^2 X_{t+k}^2) - (E(X_t^2))^2. \tag{2.7}$$

Before we proceed, it is very important to highlight the fact that after a series of algebraic manipulations using (2.6) the following results were established:

(i) For $k < q$, we have that

$$
\begin{aligned}
E(X_t^2 X_{t+k}^2) = {} & \theta_{kk}^2 E(X_t^4 e_t^2) + \sum_{j=1, j\neq k} \theta_{jj}^2 E(X_{t+k-j}^2 e_{t+k-j}^2 X_t^2) \\
& + 2 \sum_{(i=k)<j} \theta_{kk}\theta_{jj} E(X_t^3 X_{t+k-j} e_{t+k-j} e_t) \\
& + 2 \sum_{i<(k=j)} \theta_{ii}\theta_{kk}\sigma^2 E(X_t^3 e_t) \\
& + 2 \sum_{i<j} \sum_{i\neq k, j, j\neq k} \theta_{ii}\theta_{jj} E(X_{t+k-i} e_{t+k-i} X_{t+k-j} e_{t+k-j} X_t^2) \\
& + \sigma^2 E(X_t^2)
\end{aligned}
$$

and consequently it was found that $R_y(k)$ does not satisfy the Yule–Walker type difference equation.

(ii) For $k = q$, we have that

$$
R_y(k) = R_y(q) = \sum_{j=1}^{q} \theta_{jj}^2 \sigma^2 R(q-j) + 2\sigma^4 \left[2 \sum_{j=1}^{q-1} \theta_{ii}\theta_{qq} + 5\theta_{qq}^2 \right] \mu_y. \tag{2.8}
$$

Having noted these findings, we shall now proceed to considering when $k > q$. In this case

$$
\begin{aligned}
E(X_t^2 X_{t-k}^2) &= E(X_t^2 X_{t+k}^2) \\
&= \sum_{j=1}^{q} \theta_{jj}^2 E(X_t^2 e_t^2 X_{t+k-j}^2) \\
&\quad + 2 \sum_{i<j}^{q} \sum^{q} \theta_{ii}\theta_{jj}\sigma^4 E(X_t^2) + \sigma^2 E(X_t^2).
\end{aligned} \tag{2.9}
$$

Using equations (2.2) through (2.3) via a series of algebraic manipulations it was established that

$$
E(X_t^2 e_t^2 X_{t+k-j}^2) = \sigma^2 E(X_t^2 X_{t+k-j}^2) + 2\sigma^2 E(X_t^2). \tag{2.10}
$$

Thus,

$$
\begin{aligned}
R_y(k) = {} & \sum_{j=1}^{q} \theta_{jj}^2 \sigma^2 E(X_t^2 X_{t+k-j}^2) - \sum_{j=1}^{q} \theta_{jj}^2 \sigma^2 \mu_y^2 + \sum_{j=1}^{q} \theta_{jj}^2 \sigma^2 \mu_y^2 \\
& + 2\sigma^4 \sum_{j=1}^{q} \theta_{jj}^2 \mu_y + 2 \sum_{i<j}^{q} \sum^{q} \theta_{ii}\theta_{jj}\sigma^4 \mu_y + \sigma^2 \mu_y - \mu_y^2.
\end{aligned} \tag{2.11}
$$

Note that in (2.11), we added and subtracted the term $(\sum_{j=1}^{q}\theta_{jj}^2\sigma^2\mu_y^2)$ where $\mu_y = E(Y_t) = E(X_t^2)$ thus

$$R_y(k) = \sum_{j=1}^{q}\theta_{jj}^2\sigma^2[E(X_t^2X_{t+k-j}^2) - \mu_y^2] - \mu_y^2\Big[1 - \sum\theta_{jj}^2\sigma^2\Big]$$

$$-\sigma^2\left[1 + 2\sigma^2\sum_{j=1}^{q}\theta_{jj}^2 + 2\sum_{i<j}^{q}\sum^{q}\theta_{ii}\theta_{jj}\sigma^2\right]\mu_y.$$

Using equations (2.6) and (2.7), we have that

$$R_y(k) = \sum_{j=1}^{q}\theta_{jj}^2\sigma^2R(k-j) - \sigma^2\left[1 + 2\sigma^2\sum_{j=1}^{q}\theta_{jj}^2 + 2\sum_{i<j}^{q}\sum^{q}\theta_{ii}\theta_{jj}\sigma^2\right]\mu_y$$

$$+\sigma^2\left[1 + 2\sigma^2\sum_{j=1}^{q}\theta_{jj}^2 + 2\sum_{i<j}^{q}\sum^{q}\theta_{ii}\theta_{jj}\sigma^2\right]\mu_y.$$

Since from (2.6),

$$\left(1 - \sum_{j=1}^{q}\theta_{jj}^2\sigma^2\right)\mu_y^2 = \sigma^2\left[1 + 2\sigma^2\sum_{j=1}^{q}\theta_{jj}^2 + 2\sum_{i<j}^{q}\sum^{q}\theta_{ii}\theta_{jj}\sigma^2\right]\mu_y.$$

Thus

$$R_y(k) = \sum_{j=1}^{q}\theta_{jj}^2\sigma^2R(k-j), \qquad k = q+1, q+2, \ldots. \tag{2.12}$$

Equation (2.12) is a Yule–Walker equation for ARMA(q,q). Thus our study of the square of model (2.1) has lead to the following important theory needed for model identification.

Theorem 1. *Let $e_t, t \in Z$, be a sequence of independent and identically distributed random variables with $E(e_t) = 0$ and $E(e_t^2) = \sigma^2 < \infty$. Suppose there exists a stationary and invertible process $X_t, t \in Z$, satisfying $X_t = \sum_{j=1}^{q}\theta_{jj}X_{t-j}e_{t-j} + e_t$ for every $t \in Z$ for some constants $\theta_1, \theta_2, \ldots, \theta_q$, for $q > 0$. Then $X_t^2, t \in Z$, will be an* ARMA(q,q).

3 Covariance analysis for the square of the linear moving average process

Here we shall show that if $X_t, t \in Z$, is an MA(q), then $X_t^2, t \in Z$, is also an MA(q). Before doing that, we shall first obtain the mean of $X_t^2, t \in Z$, as follows:

Given model (1.3), we have that

$$Y_t = X_t^2 = \sum_{j=1}^{q} \beta_j^2 e_{t-j}^2 + 2\sum_{i<j}\sum \beta_i \beta_j e_{t-i} e_{t-j} + 2\sum_{j=1}^{q} \beta_j e_{t-j} e_t + e_t^2. \tag{3.1}$$

From (3.1) the following results were gotten:

$$E(X_t^2 e_t^2) = \sigma^4 \left(3 + \sum_{j=1}^{q} \beta_j^2\right), \tag{3.2}$$

$$E(X_t^2) = \sigma^2 \left(1 + \sum_{j=1}^{q} \beta_j^2\right) \tag{3.3}$$

and

$$\begin{aligned} X_t^2 X_{t+k}^2 = {} & \sum_{j=1}^{q} \beta_j^2 X_t^2 e_{t+k-j}^2 + 2\sum_{i<j}\sum \beta_i \beta_j X_t^2 e_{t+k-j} e_{t+k-i} \\ & + 2\sum_{j=1}^{q} \beta_j X_t^2 e_{t+k-j} e_{t+j} + X_t^2 e_{t+k}^2. \end{aligned} \tag{3.4}$$

In order to obtain the autocovariance function, we shall consider two cases, namely, when $k \le q$ and $k > q$.

Case I: When $k \le q$.
Here, we have that

$$\begin{aligned} X_t^2 X_{t+k}^2 = {} & \beta_k^2 X_t^2 + \sum_{j=1, j\neq k}^{q} \beta_j^2 X_t^2 e_{t+k-j}^2 + 2\sum_{k<j}\sum \beta_k \beta_j X_t^2 e_t e_{t+k-j} \\ & + 2\sum_{k<i}\sum \beta_k \beta_i X_t^2 e_t e_{t+k-i} + 2\sum_{i<j} \sum_{i\neq k, j\neq k} \beta_i \beta_j X_t^2 e_{t+k-i} e_{t+k-j} \\ & + 2\beta_k X_t^2 e_t e_{t+k} + X_t^2 e_{t+k}^2. \end{aligned}$$

Therefore

$$\begin{aligned} E(X_t^2 X_{t+k}^2) & = \beta_k^2 E(X_t^2 e_t^2) + \sigma^2 \sum_{j=1, j\neq k} \beta_j^2 E(X_t^2) + \sigma^2 E(X_t^2) \\ & = \beta_k^2 E(X_t^2 e_t^2) + \sigma^2 E(X_t^2)\left[1 + \sum_{j=1, j\neq k} \beta_j^2\right]. \end{aligned}$$

Thus when $k \le q$, $R_y(k)$ is given by

$$\theta_k^2 E(X_t^2 e_t^2) + \sigma^2 E(X_t^2)\left[1 + \sum_{j=1, j\neq k} \beta_j^2\right] - \left[\sigma^2\left(1 + \sum_{j=1}^{q} \beta_j^2\right)\right]. \tag{3.5}$$

Case II: When $k > q$.
In this case we have that,

$$X_t^2 X_{t+k}^2 = \sum_{j=1}^{q} \beta_j^2 X_t^2 e_{t+k-j}^2 + 2\sum\sum_{i<j} \beta_i \beta_j X_t^2 e_{t+k-i} e_{t+k-j} + 2\sum_{j=1}^{q} \beta_j X_t^2 e_{t+k-i} e_{t+k-j} + X_t^2 e_{t+k}.$$

This implies that

$$E(X_t^2 X_{t+k}^2) = \sum_{j=1}^{q} \beta_j^2 E(X_t^2 e_{t+k-j}^2) + 2\sum\sum_{i<j} \beta_i \beta_j E(X_t^2 e_{t+k-i} e_{t+k-j}) + 2\sum_{j=1}^{q} \beta_j E(X_t^2 e_{t+k-i} e_{t+k-j}) + \sigma^2 E(X_t^2) \tag{3.6}$$

therefore

$$E(X_t^2 X_{t+k}^2) = \sigma^2 \sum_{j=1}^{q} \beta_j^2 E(X_t^2) + \sigma^2 E(X_t^2) = \sigma^2 E(X_t^2)\left[1 + \sum_{j=1}^{q} \beta_j^2\right]. \tag{3.7}$$

Thus

$$R_y(k) = \sigma^2 E(X_t^2)\left[1 + \sum_{j=1}^{q} \beta_j^2\right] - \left[\sigma^2\left(1 + \sum_{j=1}^{q} \beta_j^2\right)\right]^2. \tag{3.8}$$

If we substitute (3.3), that is, $E(X_t^2) = \sigma^2(1 + \sum_{j=1}^{q} \beta_j^2)$ into (3.8), we have that

$$R_y(k) = 0. \tag{3.9}$$

Theorem 2. *If $X_t, t \in Z$, is an* MA(q) *process*, $X_t^2, t \in Z$, *is also an* MA(q) *process*.

The result of Theorem 2 follows easily from the fact that if X_t is a MA(q) and e_t is iid then X_t is q-dependent, therefore X_t^2 (and some more functions of X_t as well) is q-dependent, hence MA(q), provided the second-order moments exist; see Section 3.2 (Brockwell and Davies, 1987).

4 Conclusion

The fundamental findings of this study are that the squares of the linear moving average process of order q (MA(q)) is also identified as a moving average process of order q, whereas while the nonlinear purely diagonal bilinear process of order

q (PDB(q)) is identified as a linear moving average process of order q, its square is identified as some linear autoregressive moving average process ARMA(q, q) process.

Considering the sameness in covariance structures of the linear moving average processes and that of the nonlinear purely diagonal bilinear processes, these findings will provide a powerful differentiating technique.

Acknowledgments

The authors wish to thank the referees for the comment and suggestions that helped to improve the quality of the paper.

References

Akamanam, S. I. (1983). Some contributions to the study of bilinear time series model. Ph.D. Thesis, University of Sheffield.

Akamanam, S. I., Bhaskara Rao, M. and Subramanyam, K. (1986). On the ergodicity of bilinear time series models. *Journal of Time Series Analysis* **7**, 157–163. MR0857246

Box, G. E. P. and Jenkins, G. M. (1976). *Time Series Analysis, Forecasting and Control*, revised ed. *Holden-Day Series in Time Series Analysis*. San Fransisco, CA: Holden-Day. MR0436499

Brockwell, P. J. and Davis, R. A. (1987). *Time Series: Theory and Methods*. *Springer Series in Statistics.* New York: Springer-Verlag. MR0868859

Gabr, M. M. (1988). On the third order moments structure and bispectral analysis of some bilinear time series. *Journal of Time Series Analysis* **9**, 11–20. MR0931914

Granger, C. W. J. and Andersen, A. P. (1978). *An Introduction to Bilinear Time Series Models*. Gottingen: Vanderhoeck and Reprecht. MR0483231

Granger, C. W. J. and Newbold, P. (1977). *Forecast Economic Time Series*. New York: Academic Press.

Guegan, K. and Pham, T. D. (1987). Minimalite et inversibilite des models bilinearies a temps discet. *C. R. Acad. Sci. Series 1* **448**, 159–162. MR0880122

Iwueze, I. S. and Chikezie, D. C. (2006). Cumulant structure for the bilinear multiplicative seasonal ARIMA$(0, d, 0) \times (1, D, 1)_S$ time series model. *Global Journal of Mathematical Sciences* **5**, 25–34.

Oyet, A. J. and Iwueze, I. S. (1993). Difference equations for third order moments and cumulants of seasonal bilinear time series. *Journal of Nigerian Mathematical Society* **1**, 71–83. MR1328781

Rao, M. B., Subba Rao, T. and Walker, A. M. (1983). On the existence of some bilinear time series models. *Journal of Time Series Analysis* **4**, 95–110. MR0724541

Sessay, S. A. O. and Subba Rao, T. (1991). Difference equations for higher order moments and cumulants for the bilinear time series model BL$(p, 0, p, 1)$. *Journal of Time Series* **12**, 156–176. MR1108798

Subba Rao, T. (1981). On the theory of bilinear time series models. *Journal of the Royal Statistical Society, Series B* **43**, 244–255. MR0626772

Subba Rao, T. and Gabr, M. M. (1981). The properties of bilinear time series models and their usefulness in forecasting—with examples. Technical Report No. 151, Department of Mathematics, Institute of Sciences and Technology, University of Manchester.

Terdik, G. (1999). *Bilinear Stochastic Models and Related Problems of Nonlinear Time Series Analysis; A Frequency Domain Approach*. *Lecture Notes in Statistics* **142**. New York: Springer-Verlag. MR1702281

Tong, H. (1981). A note on Markovian bilinear stochastic process in discrete time. *Journal of Time Series Analysis* **4**, 279–284. MR0648732

Department of Physical Sciences
College of Natural and Applied Sciences
Renaissance University
P.M.B 01183, Ugbawka, Enugu State
Nigeria

Department of Statistics
Faculty of Biological and Physical Sciences
Abia State University
P.M.B. 2000, Uturu, Abia State
Nigeria