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ON THE BOUNDARY OF FOURIER AND COMPLEX ANALYSIS: THE POMPEIU PROBLEM

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Dedicated to Professor Hwai-Chiuan Wang

Abstract. In this paper, we first survery some Pompeiu-type theorems in $\mathbb{R}^n$. As consequences of some of these results, we obtain some Morera-type theorems in one and several complex variables. We also use the heat kernel of the sub-Laplacian to obtain a Pompeiu-type theorem for $L^{p,q}$ functions in the Heisenberg group.

1. Pompeiu Pproblem in $\mathbb{R}^n$

We first recall the following general version of the Pompeiu problem. Let X be a locally compact topological vector space, μ a nonnegavtive Radon measure on X, $\{\gamma_j\}_{j=1}^N$ a finite collection of compact subsets of X, and G a topological group acting on X and leaving the measure μ invariant. Consider the *Pompeiu transform*:

$$P : C(X) \to (C(G))^N$$

defined by

$$(P_j f)(\mathbf{g}) := \int_{\mathbf{g}\gamma_j} f d\mu,$$

where P_j is the jth component of P and we denote $\mathbf{g} \cdot \mathbf{x}$ the action of the element $\mathbf{g} \in G$ on the point $\mathbf{x} \in X$. We say that the family $\{\gamma_j\}_{j=1}^N$ has the *Pompeiu property* if P is injective. Note that for this condition to be nontrivial one essentially

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needs the action of G to be transitive. The **Pompeiu problem** is to decide *as explicitly as possible* whether the family has the Pompeiu property. One can also consult the monograph written by Agranovsky [1] for a more functional analytical view of this subject. Let us begin with a simple observation.

Let μ_r be the normalized surface measure on the sphere $C_r(0) = \{x \in \mathbb{R}^n : |x| = r\}$. Then we want to know whether

$$f * \mu_r = 0 \Rightarrow f \equiv 0.$$

$f * \mu_r$ are called the spherical means of f.

If $f \in L^p(\mathbb{R}^n)$, $1 \le p \le 2$, then by taking Fourier transform

$$\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx,$$

we have

$$\widehat{f * \mu_r} = \hat{f} \cdot \hat{\mu}_r = 0 \Rightarrow f \equiv 0,$$

since $\hat{\mu}$ is an entire function of exponential type by Paley-Wiener theorem. Obviously, the sphere $C_r(0)$ satisfies the Pompeiu property whenever $f \in L^2(\mathbb{R}^n)$. This is usually called a one radius theorem.

If we take

$$f(x) = \phi_\lambda(x) = c_n \frac{J_{\frac{n}{2}-1}(\lambda|x|)}{(\lambda|x|)^{\frac{n}{2}-1}},$$

where J_α is the Bessel function of order α, then it is well known that

$$f * \mu_r(x) = \phi_\lambda(r)\phi_\lambda(x),$$

where $\phi_\lambda(r)$ stands for $\phi_\lambda(y)$ with $|y| = r$. Therefore, if

$$r \in \mathbb{R} \quad \text{is such that} \quad J_{\frac{n}{2}-1}(\lambda r) = 0, \text{ then } f * \mu_r = 0.$$

Note that $\phi_\lambda \in L^p(\mathbb{R}^n)$ for $p > 2n/(n-1)$. Therefore, we have:

1. If $p > 2n/(n-1)$, one radius theorem is not true for $L^p(\mathbb{R}^n)$.
2. If $1 \le p \le 2n/(n-1)$, $C_r(0)$ indeed satisfies the Pompeiu property for **any** $r > 0$ by a theorem of Thangavelu [40].

When $n = 2$, then by Green's theorem,

$$f * \mu_r(\zeta) = \int_{C_r(0)} f(z-\zeta)dz = 2i \iint_{D_r(0)} \frac{\partial f}{\partial \bar{z}}(z-\zeta)dV(z)$$

for $f \in C^1(\mathbb{R}^2)$. Here $D_r(0) = \{z \in \mathbb{C} : |z| \le r\}$. Therefore, in $\mathbb{C}$, Pompeiu property implies the analyticity of the function f.

Let us recall the famous Morera theorem:

Theorem 1.1. *Let $f \in C(\mathbb{C})$. Then f is holomorphic in $\mathbb{C}$ if*

$$\int_\gamma f(z)dz = 0$$

for all piecewise C^1 Jordan curves γ in $\mathbb{C}$.

In fact, the condition for the Morera theorem is too strong. Of course, there is no L^p entire function in $\mathbb{C}$. Obviously, one radius is definitely not enough. However, Zalcman [43, 44] had proved the following beautiful result.

Theorem 1.2. *Let $f \in C(\mathbb{R}^2)$ and suppose that there exist positive numbers r_1, r_2 such that*

$$\int_{C_r(\zeta)} f(z)dz = 0$$

for every $\zeta \in \mathbb{C}$ and $r = r_1, r_2$. Then f is an entire function so long as

$$\frac{r_1}{r_2} \notin \mathcal{Q}(J_1) = \left\{ \frac{\eta}{\xi} : J_1(\eta) = J_1(\xi) = 0 \right\}.$$

Here J_1 is the Bessel function of order 1. When $r_1/r_2 \in \mathcal{Q}(J_1)$ then there exists a function f such that f need not be holomorphic **anywhere** *in $\mathbb{C}$.*

Similarly, let $f \in C(\mathbb{R}^n)$ and suppose there exist positive numbers r_1, r_2 such that

$$f * \mu_r = 0$$

for $r = r_1, r_2$. Then $f \equiv 0$ as long as $r_1/r_2 \notin \mathcal{Q}(J_{n/2})$. From this, Theorem 1.2 can be generalized to $\mathbb{C}^n$ as follows (see Berenstein [6]):

Theorem 1.3. *Let $f \in C(\mathbb{C}^n)$ and suppose that there exist positive numbers r_1, r_2 such that*

$$\int_{C_r(0)} f(z - \zeta)dz = 0$$

for every $\zeta \in \mathbb{C}^n$ and $r = r_1, r_2$. Then f is an entire function so long as $r_1/r_2 \notin \mathcal{Q}(J_n)$.

A different sort of variation is obtained by placing appropriate restrictions on a Jordan curve γ_0. In [17], Brown, Schreiber and Taylor obtained the following theorem:

Theorem 1.4. *Let $f \in C(\mathbb{C})$ and let γ_0 be a "generic" Jordan curve. If*

$$\int_{\sigma(\gamma_0)} f(z)dz = 0 \qquad for\ all \quad \sigma \in M(2),$$

then f is an entire function. Here $M(2)$ is the Euclidean motion group in $\mathbb{R}^2$.

Remark 1. The notion of genericity used in Theorem 1.4 consists in the nonexistence of positive eigenvalues α for the following overdetermined Neumann problem:

$$\begin{aligned} \Delta u + \alpha u = 0 \qquad & \text{in} \quad \Omega, \\ u = 1, \quad \frac{\partial u}{\partial \vec{n}} = 0 \qquad & \text{on} \quad \partial\Omega = \gamma_0. \end{aligned}$$

Here Ω is the interior of the region bounded by γ_0 such that $\Omega^c = \mathbb{R}^n \setminus \Omega$ is connected and $\vec{n}$ is the unit outward normal.

From the above discussion, we know that if $\Omega =$ ball, then γ_0 fails to have the Pompeiu property. However, from the work of Cafarelli, Williams [42] has proved the following result:

Corollary 1.5. *If γ_0 is Lipschitz but not real analytic everywhere, then Ω has the Pompeiu property in $\mathbb{R}^n$ (for the Euclidean motion group $M(n)$).*

Question 1. *The hypothesis in Corollary 1.5 is only a sufficient condition. It is known that*

$$\gamma_0 = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a \neq b \right\}$$

satisfies the Pompeiu property. However, γ_0 is real analytic everywhere. It is interesting to find a necessary and sufficient condition for γ_0 so that γ_0 satisfies the Pompeiu property.

It turns out that the key element of the proof of the above theorems is the reduction of the Pompeiu problem to the spectral synthesis problem in $C_*(\mathbb{R}^n)$, the space of radially symmetric continuous functions in $\mathbb{R}^n$.

If only translations are allowed, a single Jordan curve does not imply analyticity. In 1977, Berenstein and Taylor [18] proved a result, the so-called *three squares theorem*, which asserts that any continuous function defined in the plane is completely determined by its averages over a family of three squares γ_1, γ_2, γ_3 of sides parallel to the coordinate axes and size ℓ_1, ℓ_2, ℓ_3 if and only if ℓ_1/ℓ_2, ℓ_2/ℓ_3, and ℓ_3/ℓ_1 are not rational numbers.

In fact, the above condition for ℓ_j is equivalent to

$$\{\xi \in \mathbb{C}^n : \hat{\chi}_{\gamma_1}(\xi) = \hat{\chi}_{\gamma_2}(\xi) = \hat{\chi}_{\gamma_3}(\xi) = 0\} = \emptyset, \tag{1}$$

where χ_{γ_j} is the characteristic function of the square γ_j, $\hat{\chi}_{\gamma_j}$ its Fourier transform. The Pompeiu property is just the statement that for any $f \in C(\mathbb{R}^n)$,

$$\chi_{\gamma_1} * f = \chi_{\gamma_2} * f = \chi_{\gamma_3} * f = 0 \Rightarrow f = 0. \tag{2}$$

However, the failure of the spectral synthesis does not guarantee that (1) implies (2) for characteristic functions of three **arbitrary** compact sets γ_1, γ_2, γ_3. On the other hand, it was first proved by Berenstein and Taylor [18] that *if* γ_1 *is a square* (*or a rectangle*) *then* (1) *does imply* (2) *for any other collection* γ_2, γ_3 (*or more generally*, $\gamma_1, \ldots, \gamma_N$).

In fact, the three squares theorem is closely related to the *multisensor deconvolution problem*: Given a collection of compactly supported distributions $\{\mu_j\}_{j=1}^N$ on $\mathbb{R}^n$, find a collection of compactly supported distributions $\{\nu_j\}_{j=1}^N$ such that

$$\sum_{j=1}^{N} \mu_j * \nu_j = \delta. \tag{3}$$

A theorem of Hörmander [28] asserts that (3) has a solution if and only if $\{\mu_j\}_{j=1}^N$ satisfies the *strongly coprime* condition

$$\sum_{j=1}^{N} |\hat{\mu}_j(\xi)| \geq A(1+|\xi|)^{-M} e^{-B|\mathrm{Im}\,\xi|} \quad \text{for all} \quad \xi \in \mathbb{C}^n,$$

for some constants A, B, $M > 0$.

Question 2. *Can we generalize Hörmander's theorem to nilpotent Lie groups or symmetric spaces*?

For $n = 2$, let $\mu_j = \chi_{\gamma_j}$, $j = 1, 2, 3$. Then (3) implies the "global" three squares theorem since

$$\sum_{j=1}^{3} (f * \mu_j) * \nu_j = \sum_{j=1}^{3} f * (\mu_j * \nu_j) = f * \left(\sum_{j=1}^{3} \mu_j * \nu_j \right) = f * \delta = f.$$

On the other hand, by Fourier transform, (3) is equivalent to

$$\hat{\mu}_1 \hat{\nu}_1 + \cdots + \hat{\mu}_N \hat{\nu}_N = 1, \tag{4}$$

so that

$$\{\xi \in \mathbb{C}^n : \hat{\mu}_1(\xi) = \cdots = \hat{\mu}_N(\xi) = 0\} = \emptyset \tag{5}$$

is clearly a necessary condition for (4). Once we pose the problem this way, we can see that deconvolution is relevant to the Pompeiu problem for the translation group. In fact, we have the following theorem:

Theorem 1.6. *Let γ be a hyper-pyramid in $\mathbb{R}^n$ centered at the origin with side length $\ell > 0$. Then there exist rotations $\sigma_1, \sigma_2, \ldots, \sigma_{n+1} \in U(n)$ with $\sigma_1 =$ Identity map such that the family $\{\sigma_j\gamma\}_{j=1}^{n+1}$ has the Pompeiu property, i.e., if $f \in C(\mathbb{R}^n)$ satisfies for every j, $1 \leq j \leq n+1$,*

$$\int_{\mathbf{y}+\sigma_j\gamma} f(\mathbf{x})d\mathbf{x} = 0, \quad \textit{for all} \quad \mathbf{y} \in \mathbb{R}^n,$$

then $f \equiv 0$.

For example, when $n = 2$, let γ_1 be an equilaterial triangle, and γ_2 and γ_3 the triangles obtained by rotating γ_1 through $\pi/6$ and $\pi/4$, respectively. If $f \in C(\mathbb{C})$ satisfies

$$\int_{z+\gamma_j} f(\zeta)d\zeta = 0, \qquad j = 1, 2, 3,$$

for all $z \in \mathbb{C}$, then f is entire.

There are local versions of the above theorems too. For example, the following results were proved by Berenstein-Zalcman [19] and Berenstien-Gay [9], respectively:

Theorem 1.7. *Let $f \in C(\mathbb{D})$ and let γ_0 be a generic Jordan curve contained in $\mathbb{D}$. If*

$$\int_{\sigma(\gamma_0)} f(z)dz = 0 \qquad \forall\, \sigma \in \mathcal{B},$$

then f is holomorphic in $\mathbb{D}$. Here $\mathcal{B}$ is the Möbius group of conformal automorphisms of $\mathbb{D}$.

Remark 2. The notation of genericity used in Theorem 1.7 is formally the same as in Remark 1 except that the Euclidean Laplacian is replaced by the Laplace-Baltrami operator on $\mathbb{D}$ (viewed as the hyperbolic plane). In fact, the conditions in REMARK 1 may be restated as saying that if $\alpha > 0$, then the equation

$$\Delta u + \alpha u = -\chi_\Omega$$

has no solution. Here χ_Ω is the characteristic function of the compact set Ω. It remains the case that a sufficient condition for Theorem 1.7 to hold is that γ_0 be Lipschitz but not real analytic everywhere.

Theorem 1.8. *Let $f \in C(\mathbb{D})$ and let γ_0 be a piecewise smooth Jordan curve in $\{z \in \mathbb{C} : |z| < 1/2\}$ which is not real analytic everywhere (for instance, a rectangle). Suppose that*

$$\int_\gamma f(z)dz = 0$$

for all $\gamma \subset \mathbb{D}$ congruent to γ_0 (in the Euclidean sense). Then f is holomorphic in $\mathbb{D}$.

The constant 1/2 in Theorem 1.8 is sharp. Indeed, suppose that the Jordan region bounded by γ_0 contains a circle of radius $r > 1/2$. Then the disc

$$\Delta = \{z \in \mathbb{C} : |z| < 2r - 1\}$$

contains no point of any curve congruent to γ_0 which lies entirely in $\mathbb{D}$. Thus f may be prescribed arbitrarily inside Δ which satisfies

$$\int_\gamma f(z)dz = 0.$$

Similarly, we can generalize the above result to the following: Let $r_1, r_2 > 0$, $r_1 + r_2 < 1/2$, $r_1/r_2 \notin \mathcal{Q}(J_1)$. Then if $f \in C^1(\mathbb{C})$ satisfies

$$\int_{|w-z|=r_1} f(z)dz = 0 \quad \text{for all} \quad w \in B(0; 1 - r_1)$$

and

$$\int_{|w-z|=r_2} f(z)dz = 0 \quad \text{for all} \quad w \in B(0; 1 - r_2),$$

then f is holomorphic in $\mathbb{D}$. We also have a "local" three squares theorem as follows (see Berenstein, Gay and Yger [14, 15]):

Theorem 1.9. *Let γ_j, $1 \le j \le n+1$, be $n+1$ cubes in $\mathbb{R}^n$ centered at the origin with side length ℓ_j respectively and let $R > \ell_1 + \ell_2 + \cdots + \ell_{n+1}$. Then the following are equivalent:*

1. *The collection $\{\ell_j\}_{j=1}^{n+1}$ satisfies $\ell_j/\ell_k \notin \mathbb{Q}$, $j \neq k$.*
2. *If $f \in L^2(Q_R(0))$ satisfies for every j, $1 \le j \le n+1$,*

$$\int_{\mathbf{y}+\gamma_j} f(\mathbf{x})d\mathbf{x} = 0 \quad for \quad \max_{1\le k\le n} |y_k| < R - \ell_j,$$

then $f \equiv 0$. Here $Q_R(0)$ is the cube $[-R, R]^n$.

Applying the above theorem, we immediately have the following result:

Corollary 1.10. *Let* $0 < \ell_1 < \cdots < \ell_{n+1}$ *be* $n+1$ *numbers such that* $\ell_j/\ell_k \notin \mathbb{Q}$ *for* $j \neq k$. *Assume* Ω *is an open set in* $\mathbb{R}^n$ *with the property that for every* $\mathbf{x} \in \Omega$ *there is* $\mathbf{y} \in \Omega$ *and* $R_{\mathbf{y}} > 0$ *such that* $\mathbf{x} \in Q_{R_{\mathbf{y}}}(\mathbf{y}) \subset \Omega$ *and* $R_{\mathbf{y}} > \ell_1 + \cdots + \ell_{n+1}$. *Then the only continuous function* f *in* Ω *that satisfies*

$$\int_{\mathbf{y}+\gamma_j} f(\mathbf{x})d\mathbf{x} = 0 \quad \textit{for} \quad 1 \leq j \leq n+1$$

and every $\mathbf{y} \in \Omega$ *such that* $\mathbf{y} + \gamma_j \subset \Omega$, *is* $f \equiv 0$. *Here* $Q_{R_{\mathbf{y}}}(\mathbf{y}) = \prod_{k=1}^n [-R_{\mathbf{y}} + y_k, R_{\mathbf{y}} - y_k]$.

Question 3. *The reason for us to use cubes in Theorem* 1.9 *and Corollary* 1.10 *is a technical one since we know exactly what is the set which is generated by* $n+1$ *cubes. In fact, we may replace cubes by other sets, e.g., hyper-pyramids. How does* Ω *look like now?*

2. Pompeiu Problem on the Heisenberg Group

The Heisenberg group $\mathbf{H}^n$ is the simplest noncommutative nilpotent Lie group with underlying manifold $\mathbb{C}^n \times \mathbb{R}$ and the group law

$$(\mathbf{z}, t) \cdot (\mathbf{w}, s) = \left(\mathbf{z} + \mathbf{w}, t + s + 2\mathrm{Im} \sum_{j=1}^n z_j \bar{w}_j \right).$$

In fact, $\mathbf{H}^n \equiv \partial \mathcal{D}_{n+1}$ (topologically), where

$$\mathcal{D}_{n+1} = \left\{ (\mathbf{z}, z_{n+1}) \in \mathbb{C}^{n+1} : \mathrm{Im}\, z_{n+1} > \sum_{j=1}^n |z_j|^2 \right\}$$

by means of the homeomorphism

$$\Phi : \mathbf{H}^n \to \partial \mathcal{D}_{n+1}, \quad (\mathbf{z}, t) \mapsto (\mathbf{z}, t + i|\mathbf{z}|^2).$$

We endow $\mathbf{H}^n$ with the CR structure obtained by transporting the natural CR structure of $\partial \mathcal{D}_{n+1}$ to $\mathbf{H}^n$ via Φ. Then the left-invariant vector fields $\mathbf{Z}_j$ and $\bar{\mathbf{Z}}_j$ on $\mathbf{H}^n$, $j = 1, \ldots, n$, defined by

$$\mathbf{Z}_j = \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t}, \quad \bar{\mathbf{Z}}_j = \frac{\partial}{\partial \bar{z}_j} - iz_j \frac{\partial}{\partial t}$$

form a basis of the subbundle $T^{(1,0)} \oplus T^{(0,1)}$ of the complex tangent bundle $\mathbb{C}T\mathbf{H}^n$. It is easy to see that

$$[\mathbf{Z}_j, \mathbf{Z}_k] = [\bar{\mathbf{Z}}_j, \bar{\mathbf{Z}}_k] = 0,$$

and

$$[\mathbf{Z}_j, \bar{\mathbf{Z}}_k] = -2i\delta_{jk}\frac{\partial}{\partial t}.$$

Denote $\mathbf{T} = \partial/\partial t$ is the "missing" direction. A continuous function $f \in C(\mathbf{H}^n)$ is a CR function if and only if $\bar{\mathbf{Z}}_j f = 0$, $j = 1, \dots, n$, in the sense of distributions.

Before we go further, let us give some background of Laguerre calculus on $\mathbf{H}^n$. The readers can consult Berenstein-Chang-Tie's book [8] for detailed discussions on this subject. For $f \in L^1(\mathbf{H}^n)$, denote by

$$\tilde{f}_\lambda(\mathbf{z}) = \hat{f}(\mathbf{z}, \cdot)(\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}} f(z,t)e^{-i\lambda t}dt.$$

For $f, g \in L^1(\mathbf{H}^n)$, define the convolution $f * g$ of f and g by

$$(f * g)(\mathbf{x}) = \int_{\mathbf{H}^n} f(\mathbf{y})g(\mathbf{y}^{-1}\cdot\mathbf{x})d\mathbf{y}.$$

For $\lambda \in \mathbb{R}^* = \mathbb{R}\setminus\{0\}$, we define the twisted convolution of f and g by

$$(f *_\lambda g)(\mathbf{z}) = \int_{\mathbb{C}^n} f(\mathbf{z}-\mathbf{w})g(\mathbf{w})e^{-2i\lambda \mathrm{Im}\mathbf{z}\cdot\bar{\mathbf{w}}}dm(\mathbf{w}).$$

Here dm is the Lebesgue measure on $\mathbb{C}^n$. Then we have $\widetilde{(f * g)}_\lambda = \tilde{f}_\lambda *_\lambda \tilde{g}_\lambda$. The generalized Laguerre polynomials $L_k^{(p)}(x)$ are defined by their usual generating function formula:

$$\sum_{k=1}^{\infty} L_k^{(p)}(x)w^k = \frac{1}{(1-w)^{p+1}}\exp\left\{-\frac{xw}{1-w}\right\}$$

for $p \in \mathbb{Z}_+$, $x \geq 0$, and $|w| < 1$. From the Laguerre polynomials, we can define the Laguerre functions:

$$\ell_k^{(p)}(x) = \left[\frac{\Gamma(k+1)}{\Gamma(k+p+1)}\right]^{\frac{1}{2}} x^{\frac{p}{2}} L_k^{(p)}(x)e^{-\frac{x}{2}},$$

where $x \geq 0$ and $p, k \in \mathbb{Z}_+$. It is well-known that $\{\ell_k^{(p)}(x), k \in \mathbb{Z}_+\}$ forms a complete orthonormal basis of the space $L^2([0,\infty))$ for $p = 0, 1, 2, \dots$.

Let $z = |z|e^{i\theta}$ and $k, p \in \mathbb{Z}_+$. Then we define the exponential Laguerre functions as follows:

$$\widetilde{\mathcal{W}}_k^{(\pm p)}(z,\lambda) = (\pm 1)^p \frac{2|\lambda|}{\pi}\ell_k^{(p)}(2|\lambda||z|^2)e^{\pm ip\theta}.$$

Let us first recall a result due to Ogden and V'agi [34]:

Theorem 2.1. *Let* $p, k, q, m = 1, 2, \ldots$. *Then*

$$\widetilde{\mathcal{W}}^{(p-k)}_{p\wedge k-1} *_{|\lambda|} \widetilde{\mathcal{W}}^{(q-m)}_{q\wedge m-1} = \delta^{(q)}_{k} \cdot \widetilde{\mathcal{W}}^{(p-m)}_{p\wedge m-1},$$

and

$$\widetilde{\mathcal{W}}^{(p-k)}_{p\wedge k-1} *_{-|\lambda|} \widetilde{\mathcal{W}}^{(q-m)}_{q\wedge m-1} = \delta^{(p)}_{m} \cdot \widetilde{\mathcal{W}}^{(q-k)}_{q\wedge k-1},$$

where $a \wedge b = \min(a, b)$.

Let $\mathcal{W}^{(p)}_{k}(z,t)$, $\pm p, k = 0, 1, 2, \ldots$, be the inverse Fourier transform of $\widetilde{\mathcal{W}}^{(p)}_{k}(z,\lambda)$ with respect to λ, *i.e.*,

$$\mathcal{W}^{(p)}_{k}(z,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda t}\widetilde{\mathcal{W}}^{(p)}_{k}(z,\lambda)d\lambda.$$

These are the kernels of the generalized Cauchy-Szegö projection operators on $\mathbf{H}^1$. In particular,

$$\mathcal{W}^{(0)}_{0}(z,t) = S_+ + S_-$$

where

$$S_{\pm} = \frac{1}{\pi^2}\frac{1}{(|z|^2 \mp it)^2}.$$

Let K induce a left-invariant convolution operator $\mathbf{K}$ on $\mathbf{H}^n$,

$$\mathbf{K}(\phi)(\mathbf{x}) = \int_{\mathbf{H}^n} K(\mathbf{y})\phi(\mathbf{y}^{-1}\cdot\mathbf{x})d\mathbf{y}.$$

Now, $\tilde{K}(z,\lambda)$ has a Laguerre series expansion:

$$\tilde{K}(z,\lambda) = \sum_{|\mathbf{p}|,|\mathbf{k}|=1}^{\infty} K^{(\mathbf{p})}_{\mathbf{k}}(\lambda)\prod_{j=1}^{n}\widetilde{\mathcal{W}}^{(p_j)}_{k_j}(z_j,\lambda).$$

Define the Laguerre tensor $\mathcal{M}(K)$ of K:

$$\mathcal{M}(K) = \mathcal{M}_+(K) \oplus \mathcal{M}_-(K),$$

where

$$\mathcal{M}_+(K) = \left(K^{(\mathbf{p})}_{\mathbf{k}}(\lambda)\right), \quad \lambda > 0,$$

and

$$\mathcal{M}_-(K) = \left(K^{(\mathbf{p})}_{\mathbf{k}}(\lambda)\right)^T, \quad \lambda < 0.$$

The following theorem is the cornerstone for Laguerre calculus on $\mathbf{H}^n$, which was first proved by Greiner (see [8]):

Theorem 2.2. *Let F and G induce two convolution operators on $\mathbf{H}^n$. $\mathcal{M}(F)$ and $\mathcal{M}(G)$ denote the Laguerre tensors of F and G respectively. Then*

$$\mathcal{M}(F * G) = \mathcal{M}_+(F) \cdot \mathcal{M}_+(G) \oplus \mathcal{M}_-(F) \cdot \mathcal{M}_-(G).$$

Corollary 2.3. *The identity operator $\mathbf{I}$ on $C_0^\infty(\mathbf{H}^n)$ is induced by the identity Laguerre tensor*

$$\mathcal{M}_\pm(\mathbf{I}) = \left(\delta_{k_1}^{(p_1)} \cdots \delta_{k_n}^{(p_n)}\right).$$

Let $f \in L^p(\mathbf{H}^n)$, $1 < p < \infty$. Then

$$\lim_{r \to 1^-} \sum_{|\mathbf{k}|=0}^{\infty} r^{|\mathbf{k}|} f * \mathcal{W}_{\mathbf{k}}^{(\mathbf{0})} = f$$

in L^p-norm (see Chang-Tie [20] and Strichartz [38]).

A left-invariant differential operator $\mathcal{P}$ on $\mathbf{H}^n$ is a polynomial $\mathcal{P}(\mathbf{Z}, \bar{\mathbf{Z}}, \mathbf{T})$ with constant coefficents. Then

$$\mathcal{P} = \mathcal{P}\mathbf{I} = \sum_{|\mathbf{k}|=0}^{\infty} \mathcal{P}\mathcal{W}_{k_1,\dots,k_n}^{(0,\dots,0)} *$$

where $\mathbf{I} = \sum_{|\mathbf{k}|=0}^{\infty} \mathcal{W}_{\mathbf{k}}^{(\mathbf{0})} *$ is the identity operator on $C_0^\infty(\mathbf{H}^n)$.

In particular, we have the following:

1. $\mathcal{M}(\mathbf{T}) = i\tau(\delta_{k_1}^{(p_1)} \cdots \delta_{k_n}^{(p_n)})$.
2. $\mathcal{M}(\mathbf{Z}_j) = \mathcal{M}_+(\mathbf{Z}_j) \oplus \mathcal{M}_-(\mathbf{Z}_j)$, where

$$\mathcal{M}_-(\mathbf{Z}_j)_{k_1,\dots,k_n}^{(p_1,\dots,p_n)} = \sqrt{2|\lambda| p_j}\,\delta_{k_1}^{(p_1)} \cdots \delta_{k_j}^{(p_j+1)} \cdots \delta_{k_n}^{(p_n)}$$

   and $\mathcal{M}_+(\mathbf{Z}_j) = \mathcal{M}_-(\mathbf{Z}_j)^T$.
3. $\mathcal{M}(\bar{\mathbf{Z}}_j) = -\mathcal{M}(\mathbf{Z}_j)^T$.

Example 1. When $n = 1$, we have

$$\mathcal{M}_+(\mathbf{Z}_1) = \sqrt{2|\tau|} \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{6}$$

and

$$\mathcal{M}_-(\mathbf{Z}_1) = [\mathcal{M}_+(\mathbf{Z}_1)]^t.$$

Now we may set

$$\mathcal{M}_+(K) = \frac{1}{\sqrt{2|\tau|}} \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{\sqrt{1}} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{7}$$

and

$$\mathcal{M}_-(K) = [\mathcal{M}_+(K)]^t.$$

Thus

$$\tilde{K}_\pm(z,\tau) = \frac{1}{\sqrt{2|\tau|}} \sum_{k=0}^{\infty} \frac{1}{\sqrt{k+1}} \widetilde{\mathcal{W}}^{(1)}_{\pm,k}(z,\tau).$$

Using the defintion of $\widetilde{\mathcal{W}}^{(1)}_{\pm,k}(z,\tau)$, we sum the series

$$\tilde{K}(z,\tau) = \frac{2|\tau| z e^{-|\tau||z|^2}}{\pi} \int_0^1 \sum_{k=0}^{\infty} r^k L_k^{(1)}(2|\tau||z|^2) dr.$$

But we know that

$$\sum_{k=0}^{\infty} r^k L_k^{(1)}(x) = \frac{e^x}{(1-r)^2} e^{-x/(1-r)}.$$

Therefore,

$$\tilde{K}(z,\tau) = \frac{1}{\pi} \frac{e^{-|\tau||z|^2}}{\bar{z}}$$

and

$$K(z,t) = \frac{1}{2\pi^2 \bar{z}} \int_{\mathbb{R}} e^{it\tau - |\tau||z|^2} d\tau = \frac{z}{\pi^2(|z|^4 + t^2)}.$$

This recovers the Greiner, Kohn and Stein Theorem [26] on the Heisenberg group:

$$\mathbf{Z}_1 K = \mathbf{I} - \mathcal{W}^{(0)}_{-,0} = \mathbf{I} - S_-,$$
$$K\mathbf{Z}_1 = \mathbf{I} - \mathcal{W}^{(0)}_{+,0} = \mathbf{I} - S_+.$$

We define the sub-Laplacian on $\mathbf{H}^n$ as follows:

$$\mathcal{L} = \left(-\frac{1}{2}\right) \sum_{j=1}^{n} (\mathbf{Z}_j \bar{\mathbf{Z}}_j + \bar{\mathbf{Z}}_j \mathbf{Z}_j).$$

This operator is a sum of squares of $2n$ “horizontal” vector fields, and it is therefore not elliptic, but since the rank of the Lie algebra generated by $\{\mathbf{Z}_1, \ldots, \mathbf{Z}_n, \bar{\mathbf{Z}}_1, \ldots, \bar{\mathbf{Z}}_n\}$ equals $2n+1$ by a well-known result of Hörmander [29], we know that it is hypoelliptic, *i.e.*, the solution u of $\mathcal{L}u = f$ is smooth whenever $f \in C^\infty(\mathbf{H}^n)$.

The Heisenberg group and its sub-Laplacian are at the cross-roads of many domains of analysis and geometry: nilpotent Lie group theory, hypoelliptic second-order partial differential equations, strongly pseudoconvex domains in complex analysis, probability theory of degenerate diffusion process, sub-Riemannian geometry, control theory and semiclassical analysis of quantum mechanics; see, *e.g.*, [8, 10, 11, 12, 23, 24, 39, 40].

We define the joint L^p spectrum of the pair $(\mathcal{L}, i\mathbf{T})$ as the complement of the set of $(\alpha, \beta) \in \mathbb{C}^2$ for which there exist L^p bounded operators A and B with

$$A(\alpha\mathbf{I} - \mathcal{L}) + B(\beta\mathbf{I} - i\mathbf{T}) = \mathbf{I}.$$

This implies that the spectrum should be the set of $(\alpha, \beta) \in \mathbb{C}^2$ for which neither $(\alpha\mathbf{I} - \mathcal{L})$ nor $(\beta\mathbf{I} - i\mathbf{T})$ is invertible. By using Laguerre calculus, we obtained the Laguerre tensor for the operators $\alpha\mathbf{I} - \mathcal{L}$ and $\beta\mathbf{I} - i\mathbf{T}$ have the following form (see [8, 20, 38]):

$$\begin{aligned}
\mathcal{M}(\alpha\mathbf{I} - \mathcal{L}) &= \left([\alpha - |\lambda|(n + 2|\mathbf{k}|)]\delta_{k_1}^{(p_1)} \cdots \delta_{k_n}^{(p_n)}\right), \\
\mathcal{M}(\beta\mathbf{I} - i\mathbf{T}) &= \left([\beta + \lambda]\delta_{k_1}^{(p_1)} \cdots \delta_{k_n}^{(p_n)}\right).
\end{aligned}$$

Hence $\alpha\mathbf{I} - \mathcal{L}$ is invertible if and only if $\alpha - |\lambda|(n + 2|\mathbf{k}|) \neq 0$ for all $\mathbf{k} \in (\mathbb{Z}_+)^n$, $\lambda \in \mathbb{R}$, and $\beta\mathbf{I} - i\mathbf{T}$ is invertible if and only if $\beta + \lambda \neq 0$ for all $\lambda \in \mathbb{R}$. Hence the joint spectrum of $(\mathcal{L}, i\mathbf{T})$ is the union of

$$\left\{(\alpha, \beta) \in \mathbb{C}^2 : \alpha = |\beta|(n + 2|\mathbf{k}|) \text{ and } \beta \in \mathbb{R}\right\}$$

over the set $\mathbf{k} \in (\mathbb{Z}_+)^n$, *i.e.*,

$$\sigma(\mathcal{L}, i\mathbf{T}) = \cup_{k \in \mathbb{Z}_+} \left\{(\alpha, \beta) \in \mathbb{C}^2 : \, \alpha \geq 0, \, \varepsilon = \pm 1, \beta = \frac{\varepsilon\alpha}{n + 2k}\right\}.$$

This set is called the Heisenberg fan.

Next we will find the eigenfunction corresponding to $(\alpha, \beta) \in \sigma(\mathcal{L}, i\mathbf{T})$, *i.e.*, we want to find the function $\phi_{k,\varepsilon}^{(\alpha)}(\mathbf{z}, t)$ such that

$$(\alpha - \mathcal{L})\phi_{\mathbf{k},\varepsilon}^{(\alpha)}(\mathbf{z}, t) = 0, \quad (\beta - i\mathbf{T})\phi_{\mathbf{k},\varepsilon}^{(\alpha)}(\mathbf{z}, t) = 0,$$

where

$$\beta = \frac{\varepsilon\alpha}{\sum_{j=1}^n (2k_j + 1)} = \varepsilon\alpha n + 2|\mathbf{k}|$$

with $\varepsilon = \pm 1$ and $\alpha \geq 0$. From Theorem 2.2, we have

$$(\alpha - \widetilde{\mathcal{L}}) \prod_{j=1}^{n} \widetilde{\mathcal{W}}_{k_j}^{(0)}(z_j, \lambda) = \left[\alpha - |\lambda| \sum_{j=1}^{n} (2k_j + 1) \right] \prod_{j=1}^{n} \widetilde{\mathcal{W}}_{k_j}^{(0)}(z_j, \lambda)$$

and

$$(\beta - i\tilde{\mathbf{T}}) \prod_{j=1}^{n} \widetilde{\mathcal{W}}_{k_j}^{(0)}(z_j, \lambda) = (\beta + \lambda) \prod_{j=1}^{n} \widetilde{\mathcal{W}}_{k_j}^{(0)}(z_j, \lambda).$$

Hence, if we set $\beta = -\lambda$ and $\alpha = |\lambda| \sum_{j=1}^{n} (2k_j + 1) = |\lambda|(2|\mathbf{k}| + n) = |\lambda|(2k + n)$, then

$$(\alpha - \widetilde{\mathcal{L}}) \prod_{j=1}^{n} \widetilde{\mathcal{W}}_{k_j}^{(0)}(z_j, \lambda) = 0 \quad \text{and} \quad (\beta - i\tilde{\mathbf{T}}) \prod_{j=1}^{n} \widetilde{\mathcal{W}}_{k_j}^{(0)}(z_j, \lambda) = 0.$$

This yields that

$$\tilde{\phi}_{\mathbf{k},\varepsilon}^{(\alpha)}(\mathbf{z}, \lambda) = \delta \left(\lambda + \frac{\varepsilon \alpha}{\sum_{j=1}^{n} (2k_j + 1)} \right) \cdot \prod_{j=1}^{n} \widetilde{\mathcal{W}}_{k_j}^{(0)}(z_j, \lambda).$$

This satisfies the condition $(\alpha - \tilde{\mathcal{L}}) \tilde{\phi}_{\mathbf{k},\varepsilon}^{(\alpha)}(\mathbf{z}, \lambda) = 0$ and $(\beta + \lambda) \tilde{\phi}_{\mathbf{k},\varepsilon}^{(\alpha)}(\mathbf{z}, \lambda) = 0$.

Therefore the eigenfunction corresponding to $(\alpha, \beta) \in \sigma(\mathcal{L}, i\mathbf{T})$ is just the inverse Fourier transform of $\tilde{\phi}_{\mathbf{k},\varepsilon}^{(\alpha)}$ in λ:

$$\begin{aligned}
\phi_{\mathbf{k},\varepsilon}^{(\alpha)}(\mathbf{z}, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda} \tilde{\phi}_{\mathbf{k},\varepsilon}^{(\alpha)}(\mathbf{z}, \lambda) d\lambda \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda} \delta \left(\lambda + \frac{\varepsilon \alpha}{\sum_{j=1}^{n} (2k_j + 1)} \right) \times \prod_{j=1}^{n} \widetilde{\mathcal{W}}_{k_j}^{(0)}(z_j, \lambda) d\lambda \\
&= \frac{1}{2\pi} \exp \left\{ - \frac{i \varepsilon \alpha t}{\sum_{j=1}^{n} (2k_j + 1)} \right\} \prod_{j=1}^{n} \mathcal{W}_{k_j}^{(0)} \left(z_j, - \frac{\varepsilon \alpha}{\sum_{j=1}^{n} (2k_j + 1)} \right).
\end{aligned}$$

Applying the definition of $\widetilde{\mathcal{W}}_{k_j}^{(0)}(z_j, \lambda)$,

$$\widetilde{\mathcal{W}}_{k_j}^{(0)}(z_j, \lambda) = \frac{2|\lambda|}{\pi} e^{-|\lambda||z_j|^2} L_{k_j}^{(0)}(2|\lambda||z_j|^2),$$

to the last formula, one first obtains

$$\begin{aligned}
&\prod_{j=1}^{n} \mathcal{W}_{k_j}^{(0)} \left(z_j, \frac{-\varepsilon \alpha}{\sum_{j=1}^{n} (2k_j + 1)} \right) \\
&= \left(\frac{2\alpha}{\pi \sum_{j=1}^{n} (2k_j + 1)} \right)^n \exp \left\{ \frac{-\alpha \cdot |\mathbf{z}|^2}{\sum_{j=1}^{n} (2k_j + 1)} \right\} \prod_{j=1}^{n} L_{k_j}^{(0)} \left(\frac{2\alpha |z_j|^2}{\sum_{j=1}^{n} (2k_j + 1)} \right).
\end{aligned} \tag{8}$$

Substitutes (8) into the last formula for $\phi^{(\alpha)}_{\mathbf{k},\varepsilon}(\mathbf{z},t)$. Then this yields

$$\begin{aligned}\phi^{(\alpha)}_{\mathbf{k},\varepsilon}(\mathbf{z},t) =: \phi^{(\alpha)}_{k,\varepsilon}(\mathbf{z},t) &= \frac{(2\pi)^{-n-1}\alpha^n}{[n+2k]^n} e^{-\frac{\alpha(|\mathbf{z}|^2+i\varepsilon t)}{n+2k}} \prod_{j=1}^{n} L^{(0)}_{k_j}\left(\frac{2\alpha|z_j|^2}{n+2|\mathbf{k}|}\right) \\ &= \frac{\alpha^n}{(2\pi)^{n+1}[n+2k]^n} e^{-\frac{\alpha(|\mathbf{z}|^2+i\varepsilon t)}{n+2k}} L^{(n-1)}_{k}\left(\frac{2\alpha|\mathbf{z}|^2}{n+2k}\right).\end{aligned}$$

More precisely, we have

$$\begin{aligned}\mathcal{L}f * \phi^{(\alpha)}_{k,\varepsilon} &= \alpha f * \phi^{(\alpha)}_{k,\varepsilon}, \\ -i\mathbf{T}f * \phi^{(\alpha)}_{k,\varepsilon} &= \frac{\varepsilon\alpha}{n+2k} f * \phi^{(\alpha)}_{k,\varepsilon}.\end{aligned}$$

We now consider the pointwise spectral projection induced by the eigenfunction $\phi^{(\lambda)}_{\mathbf{k},\varepsilon}$:

$$\begin{aligned}\Phi^{(\alpha)}_{k,\varepsilon}(f)(\mathbf{z},t) &= f * \phi^{(\alpha)}_{k,\varepsilon} \\ &= \int_{\mathbf{H}^n} \phi^{(\alpha)}_{k,\varepsilon}((\mathbf{w},s)^{-1}\cdot(\mathbf{z},t)) f(\mathbf{w},s)\,dm(\mathbf{w})ds, \quad \text{for } f \in L^p(\mathbf{H}^n).\end{aligned}$$

The variables t and $\mathbf{z}$ in $\phi^{(\alpha)}_{k,\varepsilon}$ are separated, and we can write $\phi^{(\alpha)}_{k,\varepsilon} = \psi_1(t)\psi_2(\mathbf{z})$ with

$$\psi_1(t) = e^{-i\mu t} \quad \text{and} \quad \psi_2(\mathbf{z}) = \frac{1}{(2\pi)^{n+1}}|\mu|^n e^{-|\mu|\cdot|\mathbf{z}|^2} \prod_{j=1}^{n} L^{(0)}_{k_j}(2|\mu||z_j|^2),$$

where

$$\mu = \frac{\alpha\varepsilon}{n+2k}, \quad \text{with} \quad k = \sum_{j=1}^{n} k_j.$$

To simplify the problem, we consider $f(\mathbf{z},t) \in L^p(\mathbf{H}^n)$ such that $f = f_1(t)f_2(\mathbf{z})$ with $f_1(t) \in L^p(\mathbb{R})$ and $f_2(\mathbf{z}) \in L^p(\mathbb{C}^n)$. Then we obtain:

$$\begin{aligned}\Phi^{(\alpha)}_{k,\varepsilon}(f)(\mathbf{z},t) &= \int_{\mathbb{R}} f_1(s)e^{-i\mu(t-s)}ds \int_{\mathbb{C}^n} f_2(\mathbf{w})\psi_2(\mathbf{z}-\mathbf{w})e^{-i\mu\langle\mathbf{z},\mathbf{w}\rangle}dm(\mathbf{w}) \\ &= \tilde{f}_1(-\mu)e^{-i\mu t}[(f_2 *_\mu \psi_2)(\mathbf{z})],\end{aligned}$$

where $\tilde{f}_1$ is the Fourier transform of f_1 and $[(f_2 *_\mu \psi_2)(\mathbf{z})]$ is exactly the twisted convolution of f_2 and ψ_2. The identity $|\Phi^{(\alpha)}_{k,\varepsilon}(f)(\mathbf{z},t)| = |\tilde{f}_1(-\mu)||[(f_2 *_\mu \psi_2)(\mathbf{z})]|$, which is independent of the variable t, yields that $\Phi^{(\alpha)}_{k,\varepsilon}(f)(\mathbf{z},t)$ cannot be in $L^p(\mathbf{H}^n)$

for any finite p. Therefore, the pointwise spectrum projection is not bounded on $L^p(\mathbf{H}^n)$ for any $1 \le p < \infty$.

Instead, we may consider the projection operator on the ray $\alpha > 0$:

$$\mathbf{P}_{k,\varepsilon}(f)(\mathbf{z},t) = \int_0^\infty f * \phi^{(\alpha)}_{k,\varepsilon}(\mathbf{z},t)d\alpha.$$

It can be calculated that the kernel $K_{k,\varepsilon}$ of the projection operator $\mathbf{P}_{k,\varepsilon}$ is

$$\begin{aligned} K_{k,\varepsilon}(\mathbf{z},t) &= \int_0^\infty \phi^{(\alpha)}_{k,\varepsilon}(\mathbf{z},t)d\alpha \\ &= \frac{n+2k}{[2\pi(|\mathbf{z}|^2+i\varepsilon t)]^{n+1}} \sum_{|\mathbf{l}|=0}^{k} (n+|\mathbf{l}|)! \prod_{j=1}^{n} \left(-\frac{2|z_j|^2}{|\mathbf{z}|^2+i\varepsilon t}\right)^{l_j}. \end{aligned}$$

The kernel $K_{k,\varepsilon} \in C^\infty(\mathbf{H}^n \setminus \{0\})$ is homogeneous of degree $-2n-2$ with respect to the Heisenberg dilations:

$$\delta \cdot (\mathbf{z},t) \mapsto (\delta\mathbf{z}, \delta^2 t).$$

Moreover,

$$\int_{(|\mathbf{z}|^4+t^2)^{1/4}=1} K_{k,\varepsilon}(\mathbf{z},t)d\sigma(\mathbf{z},t) = 0.$$

Therefore,

$$\mathbf{P}_{k,\varepsilon} : H^p(\mathbf{H}^n) \to H^p(\mathbf{H}^n)$$

for $0 < p < \infty$, $k \in \mathbb{Z}_+$, $\varepsilon = \pm 1$.

For $f \in L^2(\mathbf{H}^n)$, we have the following Plancherel formula:

$$\begin{aligned} \|f\|^2_{L^2} &= 2\pi \sum_{k=0}^\infty (n+2k) \int_0^\infty \left| f * [\phi^{(\alpha)}_{k,+1} + \phi^{(\alpha)}_{k,-1}](\mathbf{z},0)\right|^2 dm(\mathbf{z})d\alpha \\ &= 2\pi \sum_{k=0}^\infty (n+2k) \int_{-\infty}^\infty \left| f * \phi^{(\alpha)}_{k}(\mathbf{z},0)\right|^2 dm(\mathbf{z})d\alpha. \end{aligned}$$

For $f \in L^p(\mathbf{H}^n)$, $1 < p < \infty$, we have

$$\begin{aligned} f(\mathbf{z},t) &= \lim_{r\to 1^-} \sum_{k=0}^\infty r^k \int_{-\infty}^\infty f * \phi^{(\alpha)}_{k}(\mathbf{z},t)d\alpha \\ &= \lim_{r\to 1^-} \sum_{k=0}^\infty r^k f * \mathbf{P}_k(\mathbf{z},t) \\ &= \lim_{r\to 1^-} \sum_{k=0}^\infty r^k f * [K_{k,+1} + K_{k,-1}](\mathbf{z},t), \end{aligned}$$

where the limit is in the L^p-norm.

Let μ_r be the surface measure on the sphere $\{(\mathbf{z}, 0) \in \mathbf{H}^n : |\mathbf{z}| = r\}$ and define the spherical mean on $\mathbf{H}^n$ by

$$f * \mu_r(\mathbf{z}, t) = \int_{|\mathbf{w}|=r} f(\mathbf{z} - \mathbf{w}, t - s - 2\mathrm{Im}\, \mathbf{z} \cdot \bar{\mathbf{w}}) d\mu_r(\mathbf{w}).$$

Let

$$\psi_k^{(\alpha)}(\mathbf{z}, t) = e^{i\alpha t} e^{-|\alpha| \cdot |\mathbf{z}|^2} L_k^{(n-1)}(2|\alpha||\mathbf{z}|^2)$$

be the generalized Laguerre function of type $n-1$. Then we have the following:

$$\psi_k^{(\alpha)} * \mu_r(\mathbf{z}, t) = \frac{k!(n-1)!}{(k+n-1)!} \psi_k^{(-\alpha)}(r, 0) \psi_k^{(\alpha)}(\mathbf{z}, t),$$

where

$$\psi_k^{(-\alpha)}(r, 0) = e^{-|\alpha| r^2} L_k^{(n-1)}(2|\alpha| r^2).$$

This follows from the fact that $\psi_k^{(\alpha)}(\mathbf{z}, t)$ are spherical functions on $\mathbf{H}^n$ (see Agranovsky-Berenstein-Chang-Pascuas [5, 8]). Now the condition $f * \mu_r = 0$ implies that

$$\lim_{r \to 1^-} \sum_{k=0}^{\infty} C_{n,k} r^k \int_{-\infty}^{\infty} \psi_k^{(-\alpha)}(r, 0) [f * \psi_k^{(\alpha)}](\mathbf{z}, t) |\alpha|^n d\alpha = 0.$$

Since the projection operator $\mathbf{P}_k$ is bounded on $L^p(\mathbf{H}^n)$ for $1 < p < \infty$, it follows that

$$\int_{-\infty}^{\infty} \psi_k^{(-\alpha)}(r, 0) [f * \psi_k^{(\alpha)}](\mathbf{z}, t) |\alpha|^n d\alpha = 0 \quad \text{for} \quad f \in L^p(\mathbf{H}^n).$$

We may choose a sequence $\{f_j\} \subset \mathcal{S}(\mathbf{H}^n)$ such that $f_j \to f$ in L^p norm. Hence we have

$$\lim_{j \to \infty} \int_{-\infty}^{\infty} \psi_k^{(-\alpha)}(r, 0) [f_j * \psi_k^{(\alpha)}](\mathbf{z}, t) |\alpha|^n d\alpha = 0.$$

Therefore,

$$\lim_{j \to \infty} \int_{-\infty}^{\infty} \psi_k^{(-\alpha)}(r, 0) \widetilde{(\mathbf{P}_k f_j)}_\alpha(\mathbf{z}) e^{i\alpha t} d\alpha = 0.$$

Here $\widetilde{(\mathbf{P}_k f_j)}$ denotes the partial Fourier transform of $\mathbf{P}_k f_j$ in the t-variable. That is, since the above sequence converges to 0 in the L^p-norm, the sequence of the partial Fourier transform converges to 0 in the sense of distributions:

$$\lim_{j \to \infty} \psi_k^{(-\alpha)}(r, 0) \widetilde{(\mathbf{P}_k f_j)}_\alpha(\mathbf{z}) = 0.$$

We also know that

$$\lim_{j\to\infty} \widetilde{(\mathbf{P}_k f_j)}_\alpha(\mathbf{z}) = \widetilde{(\mathbf{P}_k f)}_\alpha(\mathbf{z})$$

as distributions. It follows that

$$\psi_k^{(-\alpha)}(r,0)\widetilde{(\mathbf{P}_k f)}_\alpha(\mathbf{z}) = 0.$$

This means that the support of $\widetilde{(\mathbf{P}_k f)}_\alpha(\mathbf{z})$ in the α-variable is contained in the zero set of $\psi_k^{(-\alpha)}(r,0)$, which is a discrete set. As $\mathbf{P}_k f \in L^p$, this is not possible unless $\mathbf{P}_k f = 0$. Since this holds for all k, we conclude that $f = 0$. Therefore, we have the following one radius theorem (see [4, 40]). For $k = 1, \ldots, n$, consider the following differential forms on $\mathbb{C}^n$:

$$\omega_k(\mathbf{z}) = dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \widehat{d\bar{z}_k} \cdots \wedge d\bar{z}_n.$$

Theorem 2.4. *Let $r_1, \ldots, r_n > 0$ and let $f \in L^p(\mathbf{H}^n)$ where $1 < p < \infty$. Then f is a CR function on $\mathbf{H}^n$ if and only if*

$$\int_{|\mathbf{z}|=r_k} L_{\mathbf{y}} f(\mathbf{z},0)\omega_k(\mathbf{z}) = \int_{|\mathbf{z}|=r_k} f(\mathbf{y}\cdot(\mathbf{z},0))\omega_k(\mathbf{z}) = 0$$

for every $\mathbf{y} \in \mathbf{H}^n$ and $k = 1, \ldots, n$.

When $p = 1$, choose an approximation to the identity $\phi_j \in C_0^\infty(\mathbf{H}^n)$. Then $\phi_j * f \in L^p$ for all $p \geq 1$ and also $\phi_j * f$ satisfies $\phi_j * f * \mu_r = 0$ for every j. By Theorem 2.4, we know that $\phi_j * f = 0$ for all j. On the other hand, since $\phi_j * f \to f$ in $L^1(\mathbf{H}^n)$, it follows that $f \equiv 0$. This concludes that Theorem 2.4 is also true for $p = 1$.

Remark 3. Note that the above theorem requires only one radius per variable. We can even assume that $r_1 = r_2 = \cdots = r_n = r$, *i.e.*, $C_r(0)$ has the Pompeiu property for $L^p(\mathbf{H}^n)$, $1 \leq p < \infty$. Here $C_r(0) = \{(z,0) \in \mathbf{H}^n : |z| = r\}$.

Question 4. *Can we prove a theorem similar to Theorem* 2.4 *by using other sets*, *e.g.*, *Korányi ball*, $\{(\mathbf{z},t) \in \mathbf{H}^n : |\mathbf{z}|^4 + t^2 = r^4\}$ *for* $r > 0$?

In fact, we may consider the Pompeiu problem in a more general setting (see Berenstein-Zalcman [19] and Sajith-Rathakumar [36]). Let K be a compact subgroup of Aut($\mathbf{H}^n$), the automorphism group of $\mathbf{H}^n$. There is a natural action of K on $L^1(\mathbf{H}^n)$ defined by

$$L_{\mathbf{y}}(f)(\mathbf{x}) = (\mathbf{y}\cdot f)(\mathbf{x}) := f(\mathbf{y}\cdot\mathbf{x}) \qquad \text{for} \quad \mathbf{y} \in K.$$

We say that the pair $(\mathbf{H}^n, K)$ is a Gelfand pair if the subalgebra $L^1_K(\mathbf{H}^n)$ of K-invariant functions in $L^1(\mathbf{H}^n)$ under this action is commutative with respect to the usual convolution. There are many proper subgroups of $U(n)$ for which $(\mathbf{H}^n, K)$ forms a Gelfand pair. For example, $(\mathbf{H}^n, SU(n))$, $(\mathbf{H}^n, sp(k))$. In [15], Benson, Jenkin and Ratcliff characterized all the compact subgroup K of Aut$(\mathbf{H}^n)$ such that $(\mathbf{H}^n, K)$ is a Gelfand pair. Our approach is to exploit the general theory of commutative Banach algebra for $L^1_K(\mathbf{H}^n)$.

A K-invariant complex-valued function ψ on $\mathbf{H}^n$ is called K-spherical if $\psi(\mathbf{0}) = 1$ and ψ is a joint eigenfunction for all left $\mathbf{H}^n$-invariant and right K-invariant differential operators on $\mathbf{H}^n$.

K-spherical functions can also be characterized as the nontrivial continuous functions on $\mathbf{H}^n$ satisfying the functional equation:

$$\int_K \psi(\mathbf{xg} \cdot \mathbf{y}) d\mathbf{g} = \psi(\mathbf{x})\psi(\mathbf{y}),$$

where $d\mathbf{g}$ denotes the normalized Haar measure on K.

The complex homomorphisms of the commutative Banach algebra $L^1_K(\mathbf{H}^n)$ are all given by the Gelfand transform of f:

$$f \mapsto \mathcal{G}(f) = \int_{\mathbf{H}^n} f(\mathbf{x})\psi(\mathbf{x}) d\mathbf{x},$$

where ψ is a bounded K-spherical function on $\mathbf{H}^n$. This way the maximal ideal space of $L^1_K(\mathbf{H}^n)$ is identified with the set of bounded K-spherical functions on $\mathbf{H}^n$. For example, the bounded $U(n)$-spherical functions on $\mathbf{H}^n$ are the eigenfunctions $\phi^{(\alpha)}_{k,\varepsilon}(\mathbf{z}, t)$ and $c_n\ J_{n-1}(\rho|\mathbf{z}|)/(\rho|\mathbf{z}|)^{n-1}$ (see [5] and Korányi [32]).

Observe that the sphere $\{(\mathbf{z}, t) \in \mathbf{H}^n : |\mathbf{z}| = r\}$ is the $U(n)$-orbit and $\{(\mathbf{z}, t) \in \mathbf{H}^n : |z_j| = r,\ j = 1, \ldots, n\}$ is the $T(n)$-orbit of a point $(\mathbf{z}, t)$, respectively. In general, let $(\mathbf{z}_0, t_0) \in \mathbf{H}^n$ and let

$$K_{(\mathbf{z}_0, t_0)} = \{(\mathbf{g} \cdot \mathbf{z}_0, t_0),\ \mathbf{g} \in K\}$$

denote the K-orbit of the point $(\mathbf{z}_0, t_0)$. Since K is a compact subgroup of $U(n)$, it is easy to see that $K_{(\mathbf{z}_0, t_0)}$ is a smooth compact manifold in $\mathbb{C}^n \times \{t_0\} \subset \mathbf{H}^n$ homeomorphic to $K/\mathcal{I}(\mathbf{z}_0)$, where $\mathcal{I}(\mathbf{z}_0)$ is the isotropic subgroup for $(\mathbf{z}_0, t_0)$, *i.e.*, $\mathcal{I}(\mathbf{z}_0) = \{\mathbf{g} \in K : \mathbf{g} \cdot \mathbf{z}_0 = \mathbf{z}_0\}$. Let $\mu_{(\mathbf{z}_0, t_0)}$ denote the normalized surface measure on the K-orbit of the point $(\mathbf{z}_0, t_0)$. Then we have the "one radius theorem":

Theorem 2.5. *Let $\mu_{(\mathbf{z}_0, t_0)}$ be the normalized surface measure on the K-orbit of the point $(\mathbf{z}_0, t_0)$. If $f \in L^p(\mathbf{H}^n)$, $1 \le p < \infty$, satisfies $f * \mu_{(\mathbf{z}_0, t_0)} = 0$, then $f \equiv 0$.*

The above theorem can be generalized to a wider class of K-invariant probability measures on $\mathbf{H}^n$. Let us borrow an idea from Choquet [20] to construct an example as follows. Let $t_0 \in \mathbb{R}$ be fixed. Let us define an equivalent relation $\sim$ on $\mathbb{C}^n \times \{t_0\} \subset \mathbf{H}^n$ by setting $(\mathbf{z}_1, t_0) \sim (\mathbf{z}_2, t_0)$ if there exists a $\mathbf{g} \in K$ such that $\mathbf{z}_1 = \mathbf{g} \cdot \mathbf{z}_2$. It follows that the equivalent classes are precisely the K-orbits and the set of equivalent classes $\mathbb{C}^n \times \{t_0\}/ \sim = \mathcal{X}$ can be identified with a subset of $\mathbb{C}^n$. Obviously, this is an unbounded set, which contains $\mathbb{R}_+ = [0, \infty)$. Let ν be a probability measure on $\mathcal{X}$ such that $\int_{\mathcal{X}} \mu_{(\mathbf{z}_0,t_0)} d\nu(\mathbf{z}_0)$ converges in $(C_0(\mathbb{C}^n))^*$ in the weak $*$ topology. Then

$$\mu = \int_{\mathcal{X}} \mu_{(\mathbf{z}_0,t_0)} d\nu(\mathbf{z}_0)$$

defines a K-invariant probability measure on $\mathbf{H}^n$. It can be shown that

$$\phi^{(\alpha)}_{k,j} * \mu = C_{k,j} \int_{\mathcal{X}} \phi^{(-\alpha)}_{k,j}(\mathbf{z}_0, t_0) d\nu(\mathbf{z}_0) \cdot \phi^{(\alpha)}_{k,j},$$

where $\phi^{(\alpha)}_{k,j}$ are K-spherical functions. Therefore, the above theorem holds for all such measures μ provided zeros of $\mu(\phi^{(-\alpha)}_{k,j})$ as a function of α form a discrete set since $f * \mu \in L^p(\mathbf{H}^n)$ whenever $f \in L^p(\mathbf{H}^n)$. In particular, when $K = U(n)$, we have $\mathcal{X} = [0, \infty)$. Let $\mu_{r,t}$ denote the normalized surface measure on the sphere $\{(\mathbf{z}, t) \in \mathbf{H}^n : |\mathbf{z}| = r\}$ in $\mathbf{H}^n$. Let ν be any probability measure on $[0, \infty)$ such that $\int_0^\infty r^2 d\nu(r) < \infty$. Then

$$\mu_t = \int_{\mathcal{X}} \mu_{r,t} d\nu(r)$$

is a $U(n)$-invariant probability measure. In view of the integrability condition on the measure ν, it is not difficult to see that $\mu_t(\phi^{(-\alpha)}_{k,j})$ extends to the half plane $\{d : \mathrm{Re}\,(\alpha) > 0\}$ as a holomorphic function of α. Consequently, the zeros of $\mu_t(\phi^{(-\alpha)}_{k,j})$ form a discrete set and hence Theorem 2.5 holds. Readers can consult [3] and [36] for detailed discussions.

As we can see, the method employed in the case $p < \infty$ does not work for $p = \infty$ since there is no spectral decomposition for bounded functions on $\mathbf{H}^n$. Inspired by a computation in Stein and Weiss [31], we have the following example (see also Berenstein-Chang-Pascuas-Zalcman [7]):

Example 2. Let $\alpha > 0$, and consider the function

$$f(\mathbf{z}, t) = f(\mathbf{z}) = e^{-i\pi\alpha \mathrm{Re}\,(z_n)}, \qquad (\mathbf{z}, t) \in \mathbf{H}^n.$$

Clearly, $f \in L^\infty(\mathbf{H}^n)$ but $f \notin L^p(\mathbf{H}^n)$ for $1 \le p < \infty$. Since f does not depend on t and

$$\frac{\partial f}{\partial \bar{z}_n}(\mathbf{z}, t) = -i\pi\alpha f(\mathbf{z}, t) \neq 0, \qquad (\mathbf{z}, t) \in \mathbf{H}^n.$$

It follows that f is not a CR function on $\mathbf{H}^n$. On the other hand, if $\mathbf{y} = (\mathbf{w}, s) \in \mathbf{H}^n$, then

$$L_{\mathbf{y}} f(\mathbf{z}, 0) = f(\mathbf{z} + \mathbf{w}) = f(\mathbf{z}) f(\mathbf{w}),$$

so that

$$\int_{|\mathbf{z}|=r} L_{\mathbf{y}} f(\mathbf{z}, 0) \omega_k(\mathbf{z}) = f(\mathbf{w}) \int_{|\mathbf{z}|=r} f(\mathbf{z}) \omega_k(\mathbf{z}).$$

By Stokes's theorem we have

$$\begin{aligned}\int_{|\mathbf{z}|=r} f(\mathbf{z}) \omega_k(\mathbf{z}) &= c_n \int_{|\mathbf{z}|<r} \partial f \partial \bar{z}_k dm(\mathbf{z}) \\ &= -\frac{i\pi}{2} c_n \delta_{n,k} \alpha \int_{|\mathbf{z}|<r} f(\mathbf{z}) dm(\mathbf{z}) \\ &= C_{n,k} \alpha \left(\frac{r}{\alpha}\right)^n J_n(\alpha r).\end{aligned}$$

Now we may choose $r \in \mathbb{R}$ such that $J_n(\alpha r) = 0$. Then it is easy to see that one radius theorem is not true for $L^\infty(\mathbf{H}^n)$.

In [2, 3, 5], we studied the injectivity properties for $U(n)$ and $T(n)$ spherical averages for the bounded continuous functions on $\mathbf{H}^n$. One of the main ingredients in the proof for L^∞ function is the following Wiener-Tauberian theorem. We may imitate the idea in Hulanicki and Ricci [31] to prove this theorem.

Theorem 2.6. *Let $\mathcal{R}$ be a family of K-invariant compactly supported Radon measures on $\mathbf{H}^n$. Assume that the family $\mathcal{R}$ is large enough so that for any bounded K-spherical function ψ there exists a $\mu \in \mathcal{R}$ such that*

$$\int \psi d\mu \neq 0.$$

Then if $f \in L^\infty(\mathbf{H}^n) \cap C(\mathbf{H}^n)$ is such that

$$f * \mu = 0 \quad \textit{for all} \quad \mu \in \mathcal{R}, \quad \textit{then} \quad f \equiv 0.$$

Moreover, if the above condition fails to hold, then there exists $f \in L^\infty(\mathbf{H}^n) \cap C(\mathbf{H}^n)$, $f \neq 0$, such that $f * \mu = 0$.

Now we may apply the above theorem to prove the following theorem:

Theorem 2.7. *Let f be a bounded continuous function on $\mathbf{H}^n$ satisfying the condition $f * \mu_{(\mathbf{z}_j, t_j)} = 0$ for m points $(\mathbf{z}_1, t_1)$,...,$(\mathbf{z}_m, t_m)$ in $\mathbf{H}^n$. Let Ψ and Φ be the functions defined by*

$$\Phi(\alpha, k) = \sum_{j=1}^m \left| L_k^{(n-1)} \left(\frac{2\alpha |\mathbf{z}_j|^2}{n + 2k} \right) \right| \quad \textit{and} \quad \Psi(\mathbf{w}) = \sum_{j=1}^m |\widehat{\mu_{K_\mathbf{w}}}(\mathbf{z}_j)|.$$

Then $f \equiv 0$ *if and only if both* Φ *and* Ψ *never vanish.*

Remark 4. It is interesting to observe that this condition is the same as saying that $\mathbf{z}_j/\mathbf{z}_k$ are not quotients (with suitable interpretation) of zeros of K-spherical functions. This is analogous to the condition for the two radii theorem below.

As we mentioned in Section 1, in the classical situation of $\mathbb{C}^n$ the Morera problem can be related via Stokes's formula to the Pompeiu problem. However, there does not seem to be such clear-cut relation between Pompeiu and Morera problems in $\mathbf{H}^n$. Thus, the proof of Morera's theorem becomes rather tricky. We have to understand how the left-inavraint vector fields and right-invariant vector fields act on K-spherical functions. However, when $K = U(n)$ or $T(n)$, we do conquer these difficulties. Here we just state two results and the proofs of these two theorems can be found in [2] and [8]. We will not go through the details here.

Theorem 2.8. *Let* $r_1, r_2 > 0$ *be such that the following conditions hold* :

1. $(r_1/r_2)^2 \notin \cup_{k\in\mathbb{Z}_+} \mathcal{Q}(L_k^{(n)})$;
2. $(r_1/r_2) \notin \mathcal{Q}(J_n)$.

Let f *be a bounded* C^1*-function on* $\mathbf{H}^n$ *with the property that*

$$\int_{|\mathbf{z}|=r_j} L_{\mathbf{y}} f(\mathbf{z},0)\omega_k(\mathbf{z}) = 0, \qquad j = 1, 2, \tag{9}$$

for every $\mathbf{y} \in \mathbf{H}^n$ *and* $k = 1, \dots, n$. *Then* f *is a CR function. Conversely, if one of conditions* 1 *and* 2 *fails to hold, then there is a bounded* C^1 *function* f *which satisfies* (9), *but* f *is not a CR function.*

Theorem 2.9. *Let us consider* m *square-type tori* $T(r_j)$, $j = 1, \dots, n+1$, *such that the following conditions hold* :

1. *For all* $\mathbf{k} \in (\mathbb{Z}_+)^n$, $|\mathbf{k}| = 1$, *the functions*

$$\mathcal{P}_j^{(\mathbf{k})}(\lambda;\nu) = \prod_{i=1}^n L_{\nu_i}^{(k_i)}(\lambda r_j^2), \qquad j = 1, \dots, n+1,$$

   have no common zero $(\lambda;\nu) \in (0,\infty) \times (\mathbb{Z}_+)^n$.

2. *For all* $\mathbf{k} \in (\mathbb{Z}_+)^n$, $|\mathbf{k}| = 1$, *the functions*

$$\mathcal{J}_j^{(\mathbf{k})}(\rho) = \prod_{i=1}^n J_{k_i}(\rho_i r_j), \qquad j = 1, \dots, n+1,$$

   have no common zero $\rho \in (\mathbb{R}_+)^n$.

Let f be a bounded C^1-function on $\mathbf{H}^n$ with the property that

$$\int_{T(r_j)} L_{\mathbf{y}} f(\mathbf{z}, 0)\mathbf{z}^{\mathbf{k}} d\sigma_{r_j}(\mathbf{z}) = 0, \qquad j = 1, \ldots, n+1, \tag{10}$$

for every $\mathbf{y} \in \mathbf{H}^n$ and all $\mathbf{k} \in (\mathbb{Z}_+)^n$ with $|\mathbf{k}| = 1$. Then f is a CR function.

Conditions 1 and 2 are necessary in the following sense: if one of them fails, then there exists a bounded C^1 function f which satisfies (10), but f is not a CR function.

Question 5. *Can we prove Theorems 2.8 and 2.9 by using m K-orbits $K_{(\mathbf{z}_j,t_j)} = \{(\mathbf{g}\cdot\mathbf{z}_j, t_j) : \mathbf{g} \in K\}$, $1 \le j \le m$?*

3. A Morera Type Theorem in $\mathbf{B}_n$

As we have seen that the hypotheses of Theorems 2.4, 2.7 and 2.8 are in some sense as weak as possible since we integrate over spheres, *i.e.*, $2n+1$ dimensional sets. Integral conditions on lower dimensional sets, *e.g.*, circles embedded on $\mathbf{H}^n$, are stronger. In particular, when Ω is the unit ball $\mathbf{B}_n$ of $\mathbb{C}^n$, we only need to check the hypotheses for a restricted family of k-planes in order to insure holomorphy. The following result was first proved in [7]. Since the situation in $\mathbf{B}_n$ is quite different from $\mathbb{R}^n$ and $\mathbf{H}^n$, here we give detailed discussions again.

Theorem 3.1. *Let $n \ge 2$, $1 \le k \le n-1$, and $0 < r < 1$. Assume that $f \in C(\partial\mathbf{B}_n)$ satisfies*

$$\int_{\Lambda\cap\partial\mathbf{B}_n} f\beta = 0 \tag{11}$$

for every complex k-plane Λ at distance r from the origin and for every $(k, k-1)$-form β with constant coefficients on $\mathbb{C}^n$. Let E be the set of all r's, $0 < r < 1$, such that $r^2/(1-r^2)$ is a root of one of the following polynomials :

$$P_{p,q}(x) = \sum_{\ell=\max\{p+1-q,0\}}^{p} \frac{(-1)^\ell x^\ell}{\ell!(p-\ell)!(\ell+q-p-1)!(n+p-\ell-1)!}$$

for $p \ge 0$, and $q \ge 1$. Suppose that one of the following conditions holds :

1. $k < n-1$.
2. $k = n-1$ *and* $r \notin E$.

Then f is a CR function.

Moreover, if $r \in E$ then there exists $f \in C(\partial\mathbf{B}_n)$ which is not a CR function but satisfies (11) *for every $(n-1)$-plane Λ and for every $(n-1,n-2)$-form β with constant coefficients on $\mathbb{C}^n$.*

Proof. Let Y be the subspace of all functions $f \in C(\partial\mathbf{B}_n)$ satisfying (11). Let $\mathcal{H}(p,q)$ be the space of all harmonic homogeneous polynomials of total degree p in the variables $z_1,\dots,z_n$ and of total degree q in the variables $\bar{z}_1,\dots,\bar{z}_n$ for every $p,q \geq 0$. Let U be the unitary group on $\mathbb{C}^n$. It is clear that Y is a U-invariant subspace of $C(\partial\mathbf{B}_n)$. Then by a result of Nagel and Rudin [p. 63, 45], every function in Y is a CR function if and only if $\mathcal{H}(p,q) \not\subset Y$ for each $p \geq 0$ and $q \geq 1$. (Note that if $\mathcal{H}(p,q) \not\subset Y$, then $\mathcal{H}(p,q) \cap Y = \{0\}$.)

Case (1): $k < n-1$. Let us check that the function $f(z) = z_n^p \bar{z}_k^q$ does not belong to Y for $p \geq 0$, $q \geq 1$. Let

$$\beta = dz_1 \wedge \cdots \wedge dz_k \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{k-1}.$$

Consider the k-plane $\Lambda = \mathbf{u}(\Lambda_0)$, where

$$\Lambda_0 = \{\zeta \in \mathbb{C}^n : \zeta_{k+1} = r, \zeta_j = 0 \ (k+1 < j \leq n)\},$$

and $\mathbf{u} \in U$, which will be selected later in order to assure that the integral in (11) does not vanish. Then

$$\int_{\Lambda\cap\partial\mathbf{B}_n} f\beta = \int_{\Lambda_0\cap\partial\mathbf{B}_n} \mathbf{u}^*(f\beta) = \int_{\Lambda_0\cap\partial\mathbf{B}_n} (f\circ\mathbf{u})\mathbf{u}^*\beta.$$

Then by Stokes's theorem, we have

$$\begin{aligned}\int_{\Lambda\cap\partial\mathbf{B}_n} f\beta &= (-q)\Delta\left[\sum_{\ell=1}^{k}(-1)^{k+\ell}\bar{u}_{k,\ell}\bar{\Delta}_\ell\right] \textstyle\int_{\Lambda_0\cap\mathbf{B}_n}(\mathbf{u}_n(\zeta))^p(\overline{\mathbf{u}_k(\zeta)})^{q-1}\omega \\ &= (-q)|\Delta|^2 \int_{\sqrt{1-r^2}\mathbf{B}_k} (V_n(\zeta))^p(\overline{V_n(\zeta)})^{q-1}\omega,\end{aligned}$$

where $\omega = d\zeta_1 \wedge \cdots \wedge d\zeta_k \wedge d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_k$, Δ is the determinant of the matrix $(u_{j,\ell})_{j,\ell=1,\dots,k}$, and Δ_ℓ is the minor obtained from the above matrixby by deleting its last row and ℓ^{th} column.

The binomial expansion yields that the last integral is $p!(q-1)!$ times

$$\begin{aligned}&\sum \frac{r^{p_{k+1}+q_{k+1}}}{p_1!q_1!\cdots p_{k+1}!q_{k+1}!} u_{n,1}^{p_1}\bar{u}_{k,1}^{q_1}\cdots u_{n,k+1}^{p_{k+1}}\bar{u}_{k,k+1}^{q_{k+1}} \\ &\int_{\sqrt{1-r^2}\mathbf{B}_k} \zeta_1^{p_1}\bar{\zeta}_1^{q_1}\cdots\zeta_k^{p_k}\bar{\zeta}_k^{q_k}\omega,\end{aligned} \tag{12}$$

where the sum is over $p_1 + \cdots + p_{k+1} = p$ and $q_1 + \cdots + q_{k+1} = q - 1$. Since the last integral is zero when $(p_1, \ldots, p_k) \neq (q_1, \ldots, q_k)$, it turns out that (12) equals $p!(q-1)!$ times

$$\sum \frac{r^{p_{k+1}+q_{k+1}}(u_{n,1}\bar{u}_{k,1})^{p_1} \cdots (u_{n,k}\bar{u}_{k,k})^{p_k} u_{n,k+1}^{p_{k+1}} \bar{u}_{k,k+1}^{q_{k+1}}}{(p_1!)^2 \cdots (p_k!)^2 p_{k+1}! q_{k+1}!}$$
$$\int_{\sqrt{1-r^2}\mathbf{B}_k} |\zeta_1|^{2p_1} \cdots |\zeta_k|^{2p_k} \omega,$$

the sum being over $p_1 + \cdots + p_k = p - p_{k+1} = q - 1 - q_{k+1}$. But we know that

$$\begin{aligned}\int_{\sqrt{1-r^2}\mathbf{B}_k} |\zeta_1|^{2p_1} \cdots |\zeta_k|^{2p_k} \omega &= c_k \left(\sqrt{1-r^2}\right)^{2k+p_1+\cdots+p_k} \int_{\mathbf{B}_k} |\zeta_1|^{2p_1} \cdots |\zeta_k|^{2p_k} dm \\ &= c_k \left(\sqrt{1-r^2}\right)^{2k+p_1+\cdots+p_k} \frac{p_1! \cdots p_k!}{(k+p-p_{k+1})!} \\ &= c_k \left(\sqrt{1-r^2}\right)^{2k+p-p_{k+1}+q-1-q_{k+1}} \frac{p_1! \cdots p_k!}{(k+p-p_{k+1})!}.\end{aligned}$$

Here dm is the Lebesgue measure on $\mathbb{C}^k$.

Hence we know that $\int_{\Lambda \cap \partial \mathbf{B}_n} f\beta$ equals

$$-c_k |\Delta|^2 p! q! \left(\sqrt{1-r^2}\right)^{2k+p+q-1}$$
$$\sum \frac{(u_{n,1}\bar{u}_{k,1})^{p_1} \cdots (u_{n,k}\bar{u}_{k,k})^{p_k} u_{n,k+1}^{p_{k+1}} \bar{u}_{k,k+1}^{q_{k+1}}}{p_1! \cdots p_k! p_{k+1}! q_{k+1}! (k+p-p_{k+1})!} \left(\frac{r}{\sqrt{1-r^2}}\right)^{p_{k+1}+q_{k+1}}.$$

We may pick $\mathbf{u} \in U$ such that

$$\Delta \neq 0 \quad \text{and} \quad u_{n,1} = \cdots = u_{n,k+1} = u_{k,1} = \cdots = u_{k,k+1} = \lambda \in \mathbb{R}^* (= \mathbb{R} \setminus \{0\}).$$

Then $\int_{\Lambda \cap \partial \mathbf{B}_n} f\beta$ equals $-c_k |\Delta|^2 p! q! \left(\sqrt{1-r^2}\right)^{2k+p+q-1} \lambda^{p+q-1}$ times

$$\sum \frac{\left(\frac{r}{\sqrt{1-r^2}}\right)^{p_{k+1}+q_{k+1}}}{p_2! \cdots p_k! p_{k+1}! q_{k+1}! (k+p-p_{k+1})! j_2! \cdots j_{k+1}!},$$

which obviously is nonzero.

Case (2): $k = n-1$. Let $f(z) = z_{n-1}^p \bar{z}_n^q$ for $p \geq 0$ and $q \geq 1$. We will check that $f \in Y$ if and only if $P_{p,q}(r^2/(1-r^2)) = 0$. Consider a generic generating $(n-1, n-2)$-form with constant coefficients on $\mathbb{C}^n$:

$$\beta = dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_{j_1}} \wedge \cdots \wedge \widehat{d\bar{z}_{j_2}} \wedge \cdots \wedge d\bar{z}_n$$

with $1 \leq i \leq n$ and $1 \leq j_1 < j_2 \leq n$. Let $\Lambda_0 = \{\zeta \in \mathbb{C}^n : \zeta_n = r\}$. Then a generic $(n-1)$-plane is $\Lambda = \mathbf{u}(\Lambda_0)$, where $\mathbf{u} \in U$. Similar computations as in Case (1) show that $\int_{\Lambda\cap\partial\mathbf{B}_n} f\beta$ equals $-c_n p! q! (\sqrt{1-r^2})^{2n+p+q-3}$ times

$$\Delta_i \bar{\Delta}_J \sum \frac{(u_{n-1,1}\bar{u}_{n,1})^{p_1}\cdots(u_{n-1,n-1}\bar{u}_{n,n-1})^{p_{n-1}} u_{n-1,n}^{p_n}\bar{u}_{n,n}^{q_n}}{p_1!\cdots p_n! q_n!(n+p-p_n-1)!} \times \left(\frac{r}{\sqrt{1-r^2}}\right)^{p_n+q_n},$$

the sum being over $p_1 + \cdots + p_{n-1} = p - p_n = q - 1 - q_n$, where

$$\Delta_i = \begin{vmatrix} u_{1,1} & \cdots & u_{1,n-1} \\ \vdots & \ddots & \vdots \\ \widehat{u_{i,1}} & \cdots & \widehat{u_{i,n-1}} \\ \vdots & \ddots & \vdots \\ u_{n,1} & \cdots & u_{n,n-1} \end{vmatrix} \tag{13}$$

and

$$\Delta_J = \begin{vmatrix} u_{1,1} & \cdots & u_{1,n-1} \\ \vdots & \ddots & \vdots \\ \widehat{u_{j_1,1}} & \cdots & \widehat{u_{j_1,n-1}} \\ \vdots & \ddots & \vdots \\ \widehat{u_{j_2,1}} & \cdots & \widehat{u_{j_2,n-1}} \\ \vdots & \ddots & \vdots \\ u_{n,1} & \cdots & u_{n,n-1} \\ u_{n,1} & \cdots & u_{n,n-1} \end{vmatrix}, \quad J = (j_1, j_2). \tag{14}$$

It is clear that $\Delta_J = 0$ when $j_2 \neq n$. Therefore we have only to consider the case $J = (j_1, n)$, for $1 \leq j < n$, and then $\Delta_J = \Delta_j$.

Let

$$\mathcal{V} = \cup_{1\leq j<n, 1\leq i\leq n} \mathcal{V}_{i,j},$$

where

$$\mathcal{V}_{i,j} = \{\mathbf{u} \in U : \Delta_i = \Delta_i(\mathbf{u}) \neq 0, \quad \Delta_j = \Delta_j(\mathbf{u}) \neq 0\}.$$

It is clear that $\mathcal{V}$ is a nonempty open set of U. Consider the following real-analytic function on U:

$$F_r(\mathbf{u}) = \sum \frac{(u_{n-1,1}\bar{u}_{n,1})^{p_1}\cdots(u_{n-1,n-1}\bar{u}_{n,n-1})^{p_{n-1}} u_{n-1,n}^{p_n}\bar{u}_{n,n}^{q_n}}{p_1!\cdots p_n! q_n!(n+p-p_n-1)!} \left(\frac{r}{\sqrt{1-r^2}}\right)^{p_n+q_n},$$

where the sum is over $p_1 + \cdots + p_{n-1} = p - p_n = q - 1 - q_n$. Then from the above discussion, $f \in Y$ is equivalent to $F_r \equiv 0$ on $\mathcal{V}$. Since U is connected (see e.g., [1, 27]), this last condition means that $F_r \equiv 0$ on U, by analytic continuation.

Now for $\mathbf{u} \in U$, we have

$$u_{n-1,1}\bar{u}_{n,1} + \cdots + u_{n-1,n-1}\bar{u}_{n,n-1} = -u_{n-1,n}\bar{u}_{n,n}.$$

Thus, using the binomial expansion, we obtain

$$\begin{aligned} F_r(U) &= \sum_{p-p_n=q-1-q_n} \frac{(-1)^{p-p_n}}{p_n!q_n!(n+p-p_n-1)!(p-p_n)!} \left(\frac{r}{\sqrt{1-r^2}}\right)^{p_n+q_n} u^p_{n-1,n}\bar{u}^{q-1}_{n,n} \\ &= (-1)^p \left(\frac{r}{\sqrt{1-r^2}}\right)^{q-1-p} P_{p,q}\left(\frac{r^2}{1-r^2}\right) u^p_{n-1,n}\bar{u}^{q-1}_{n,n}. \end{aligned}$$

Finally, since there exists $\mathbf{u} \in U$ such that $u_{n-1,n} = u_{n,n} \neq 0$, the above identity shows that $F_r \equiv 0$ on U if and only if $P_{p,q}(r^2/(1-r^2)) = 0$. ■

When $n = 2$, Theorem 3.1 is a result of Globevnik and Stout [33]. Note that in this case only hyperplanes arise ($n = 2$, $k = 1$). Observe also that the cases $k < n-1$ and $k = n-1$ exhibit completely different behaviors, in the sense that in the first case there are no exceptional r's, while in the second case such exceptional r's do appear. In fact, we have $E \neq \emptyset$, since it is easy to check that $P_{p,q}$ has at least one positive root for p odd and $p+1 \leq q$. The simplest example is $p = 1$ and $q = 2$; then the corresponding value of r is $1/\sqrt{n}$.

Let us show that when $r = 0$ Theorem 3.1 **never** holds.

Example 3. Let $f(z) = z_j^m \bar{z}_\ell$, $1 \leq j, \ell \leq n$, $m \geq 1$. It is clear that f is not a CR function on $\partial\mathbf{B}_n$. But it satisfies condition (11) for every k-plane Λ passing through the origin and for every $(k, k-1)$-form β with constant coefficients on $\mathbb{C}^n$. In fact, carrying out computations similar to those in the proof of Theorem 3.1, we see that the integral in (11) is a constant times

$$\int_{\mathbf{B}_k} (u_{j,1}\zeta_1 + \cdots + u_{j,k}\zeta_k)^p \beta,$$

which obviously vanishes.

Question 6. *Can we prove a local theorem similar to Theorems* 1.8 *and* 1.9 *in* $\mathbf{H}^n$ *or* $\partial\mathbf{B}_n$?

4. Pompeiu Problem Related to the Heat Kernel

In this section, we will discuss some Pompeiu problem related to the heat kernel of the sub-Laplacian:

$$\frac{\partial u}{\partial s} + \mathcal{L}u = 0, \quad \text{for} \quad (\mathbf{z}, t; s) \in \mathbf{H}^n \times \mathbb{R}^+, \quad \text{and} \quad u(\mathbf{z}, t; 0) = f(\mathbf{z}, t).$$

The heat kernel was independently studied by Beals-Greiner [13], Gaveau [25] and Hulanicki [30]. Here, we will see that $h_s(\mathbf{z}, t)$ can be obtained easily via the Laguerre calculus.

We first take the Fourier transform with respect to the t-variable and write the heat kernel $\tilde{h}_s(\mathbf{z}, \tau)$ as a twisted convolution operator. The detailed calculation can be found in Chapter 2 of [3].

$$\begin{aligned}\tilde{h}_s(\mathbf{z}, \tau) &= \exp\{-s\widetilde{\mathcal{L}}\}\tilde{\mathbf{I}} = \sum_{|\mathbf{k}|=0}^{\infty} \exp\{-s\widetilde{\mathcal{L}}\} \left[\prod_{j=1}^{n} \widetilde{\mathcal{W}}_{k_j}^{(0)}(z_j, \tau)\right] \\ &= \sum_{|\mathbf{k}|=0}^{\infty} e^{-s|\tau|(2|\mathbf{k}|+n)} \prod_{j=1}^{n} \widetilde{\mathcal{W}}_{k_j}^{(0)}(z_j, \tau) \\ &= \frac{1}{\pi^n}\left(\frac{|\tau|}{\sinh(|\tau|s)}\right)^n \exp\left\{-|\tau|\,|\mathbf{z}|^2\coth(|\tau|s)\right\}.\end{aligned}$$

Since

$$\left(\frac{|\tau|}{\sinh(|\tau|s)}\right)^n = \left(\frac{\tau}{\sinh(\tau s)}\right)^n \quad \text{and} \quad |\tau|\coth(|\tau|s) = \tau\coth(\tau s),$$

we can simplify the above identity by removing the absolute sign for τ and have

$$\tilde{h}_s(\mathbf{z}, \tau) = \frac{1}{\pi^n}\left[\frac{\tau}{\sinh(\tau s)}\right]^n e^{-\tau\gamma(\tau s;\mathbf{z})},$$

where $\gamma(\tau s; \mathbf{z}) = |\mathbf{z}|^2\coth(\tau s)$.

Now, we take the inverse Fourier transform with respect to the τ-variable and obtain the heat kernel in the integral form:

$$\begin{aligned}h_s(\mathbf{z}, t) &= \frac{1}{2\pi^{n+1}}\int_{-\infty}^{+\infty}\left(\frac{\tau}{\sinh(\tau s)}\right)^n e^{\tau(it-\gamma(\tau s;\mathbf{z}))}d\tau \\ &= \frac{1}{2(\pi s)^{n+1}}\int_{-\infty}^{+\infty}\left(\frac{\tau}{\sinh(\tau s)}\right)^n e^{-\frac{tau}{s}(\gamma(\tau;\mathbf{z})-it)}d\tau.\end{aligned}$$

We substitute τ by 2τ and rewrite the heat kernel in terms of the complex distance g and volume element ν of the Heisenberg group:

$$h_s(\mathbf{z}, t) = \frac{1}{(\pi s)^{n+1}}\int_{-\infty}^{+\infty}\nu(\tau)\tau^n e^{-\frac{2\tau}{s}g(\tau;\mathbf{z},t)}d\tau. \tag{15}$$

Here

$$\nu(\tau) = \left(\frac{1}{\sinh(\tau s)}\right)^n \quad \text{and} \quad g(\tau; \mathbf{z}, t) = |\mathbf{z}|^2\coth(\tau s) - it.$$

Now we have the following theorems:

Theorem 4.1. *Let* $1 \le p < \infty$, $1 \le q \le 2$. *Let* $\Gamma_r = S_r \times \mathbb{R} = \{(\mathbf{z}, t) \in \mathbf{H}^n : |\mathbf{z}| = r,\, t \in \mathbb{R}\}$ *and let* $\tilde{\Gamma}_r = T_r \times \mathbb{R} = \{(\mathbf{z}, t) \in \mathbf{H}^n : |z_j| = r, 1 \le j \le n,\, t \in \mathbb{R}\}$. *Then* Γ_r *and* $\tilde{\Gamma}_r$ *satisfy the Pompeiu property for* $L^{p,q}(\mathbf{H}^n)$ *in* $\mathbf{H}^n$, *i.e., if*

$$\int_{\mathbb{R}} \int_{S_r} f(\mathbf{g} \cdot (\mathbf{z}, t)) d\sigma(\mathbf{z}) dt = 0 \quad \textit{for all} \quad \mathbf{g} \in \mathbf{H}^n$$

or

$$\int_{\mathbb{R}} \int_{T_r} f(\mathbf{g} \cdot (z_1, \ldots, z_n, t)) \prod_{j=1}^{n} d\sigma_j(z_j) dt = 0 \quad \textit{for all} \quad \mathbf{g} \in \mathbf{H}^n,$$

then $f \equiv 0$. *Here*

$$L^{p,q}(\mathbf{H}^n) = \left\{ f \in L^1_{\text{loc}}(\mathbf{H}^n) : \; \|f\|_{L^{p,q}} = \int_{\mathbb{C}^n} \left(\int_{\mathbb{R}} |f(\mathbf{z}, t)|^q dt \right)^{\frac{p}{q}} dV(\mathbf{z}) < \infty \right\}.$$

Theorem 4.2. *Let* $f \in L^{p,q}(\mathbf{H}^n)$ *with* $1 \le p < \infty$ *and* $1 \le q \le 2$. *If* $u(\mathbf{z}, t; s) = f * h_s(\mathbf{z}, t) = 0$ *for all* $s > 0$ *on a cylinder* Γ_r *or* $\tilde{\Gamma}_r$, *then* $f \equiv 0$.

Before we go further, let us recall some basic notations. For each pair $(p, q) \in (\mathbb{Z}_+)^2$, let $\mathcal{P}(p, q)$ be the collection of all polynomials P in $\mathbf{z}$ and $\bar{\mathbf{z}}$ of the form

$$P(\mathbf{z}) = \sum_{|\alpha|=p} \sum_{|\beta|=q} c_{\alpha\beta} z^\alpha \bar{z}^\beta.$$

We denote $\mathcal{H}(p, q) = \{P \in \mathcal{P}(p, q),\ \Delta P = 0\}$, where Δ is the Euclidean Laplacian on $\mathbb{C}^n$. Let

$$\mathcal{S}(p, q) = \{P \in \mathcal{H}(p, q),\ \text{with } |\mathbf{z}| = 1\}$$

be the space of all spherical harmonic polynomials on the unit sphere S^{2n-1}. It can be shown that the space $L^2(S^{2n-1})$ is the orthogonal direct sum of $\mathcal{S}(p, q)$ as (p, q) ranges over $(\mathbb{Z}_+)^2$. Given a continuous function f on $\mathbb{C}^n$, we can expand the function $f(\rho\omega)$, where $\rho > 0$, $\omega \in S^{2n-1}$, in terms of elements in $\mathcal{S}(p, q)$, *i.e.*,

$$f(\rho\omega) = \sum_{k=0}^{\infty} \sum_{p+q=k} f_{pq}(\rho\omega) \quad \text{with} \quad f_{pq} \in \mathcal{S}(p, q).$$

The natural action of $U(n)$ on the unit sphere S^{2n-1} defines a unitary representation of $U(n)$ on the Hilbert space $L^2(S^{2n-1})$. When we restrict this action to $\mathcal{S}(p, q)$, it defines an irreducible representation, which we denote by σ_{pq}. Let χ_{pq} be the character associated to σ_{pq}. We may imitate the argument of Proposition 2.7 in

Helgason's book [27] where the ordinary spherical harmonic expansion has been considered, and obtain the following result.

Lemma 4.3. *Given a continuous function f on $\mathbb{C}^n$, the projections f_{pq} appearing in the spherical harmonic expansion*

$$f(\rho\omega) = \sum_{k=0}^{\infty} \sum_{p+q=k} f_{pq}(\rho\omega) \quad \textit{with} \quad f_{pq} \in \mathcal{S}(p,q)$$

are given by

$$f_{pq}(\mathbf{z}) = d_{pq} \int_{U(n)} \chi_{pq}(\mathbf{u}) f(\mathbf{u} \circ \mathbf{z}) d\mathbf{u},$$

where $\mathbf{u}$ is the normalized Haar measure of the group $U(n)$ and $d_{pq} = \dim \mathcal{S}(p,q)$.

We also have the following result via direct computations.

Lemma 4.4. *Let $g(\mathbf{z}) = g(|\mathbf{z}|)$ be a radial function on $\mathbb{C}^n$ and $P \in \mathcal{H}(p,q)$. We assume further that $gP \in L^1(\mathbb{C}^n) \cup L^2(\mathbb{C}^n)$. Then the following identity holds*:

$$(gP) *_\lambda \Phi_k^\lambda(\mathbf{z}) = \frac{(k-p)!(n-1)!}{(n-1+p+q)!} P(\mathbf{z}) L_{k-p}^{(n-1+p+q)}(2|\lambda||\mathbf{z}|^2) e^{-|\lambda||\mathbf{z}|^2}.$$

Here $\Phi_k^\lambda(\mathbf{z}) = L_k^{(n-1)}(2|\lambda||\mathbf{z}|^2) e^{-|\lambda||\mathbf{z}|^2}$.

In order to prove Theorems 4.1 and 4.2, we need the following lemma.

Lemma 4.5. *Let $f \in L^p(\mathbb{C}^n)$, $1 \le p \le \infty$. If*

$$f *_1 H_s(\mathbf{z}) = (2\pi)^{-n} (\sinh s)^{-n} \int_{\mathbb{C}^n} f(\mathbf{z} - \mathbf{w}) e^{-(\coth s)|\mathbf{w}|^2} e^{-2i \mathrm{Im}\,(\mathbf{z} \cdot \bar{\mathbf{w}})} dm(\mathbf{w})$$

vanishes on a sphere for all $s > 0$, then so does

$$f *_1 \Phi_k(\mathbf{z}) = f *_1 L_k^{(n-1)}(2|\mathbf{z}|^2) e^{-|\mathbf{z}|^2}$$

for all $k \ge 0$.

Proof. Note that

$$H_s(\mathbf{z}) = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-(2k+n)s} L_k^{(n-1)}(2|\mathbf{z}|^2) e^{-|\mathbf{z}|^2} = (2\pi)^{-n} (\sinh s)^{-n} e^{-(\coth s)|\mathbf{z}|^2}$$

is the heat kernel for the operator $L = \Delta_{\mathbf{z}} - |\mathbf{z}|^2 - i \sum_{j=1}^n (x_j(\partial/\partial y_j) - y_j(\partial/\partial x_j))$, which can be extended as a holomorphic function to the half plane $\mathrm{Re}\,(s) > 0$. It is a striaghtforward computation to show that $|H_s(\mathbf{z})|$ has exponential decay in $\mathbf{z}$ in the same region. It follows that $f *_1 H_s(\mathbf{z})$ is well-defined and holomorphic in $\mathrm{Re}\,(s) > 0$. Since $f *_1 H_s(\mathbf{z}) = 0$ on a sphere for all $s > 0$, we know that $f *_1 H_{s+i\eta}(\mathbf{z}) = 0$ on the same sphere for all η if $s > 0$, *i.e.*,

$$0 = f *_1 H_{s+i\eta}(\mathbf{z}) = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-(2k+n)s} e^{-(2k+n)i\eta} f *_1 \Phi_k(\mathbf{z})$$

which vanishes for all η on a sphere S_r for $s > 0$ fixed. By calculating the Fourier coefficients of $f *_1 H_{s+i\eta}(\mathbf{z})$ as a function of η, we obatin $f *_1 \Phi_k(\mathbf{z}) = 0$ on the same sphere. This completes the proof of the lemma. ∎

Now we are in a position to prove the main results of this section.

Proof of Theorem 4.1. If f is a radial function, then

$$f *_1 \Phi_k(\mathbf{z}) = c_k \left(\int_{\mathbb{C}^n} f(\mathbf{w}) L_k^{(n-1)}(2|\mathbf{w}|^2) e^{-|\mathbf{w}|^2} dm(\mathbf{w}) \right) \cdot L_k^{(n-1)}(2|\mathbf{z}|^2) e^{-|\mathbf{z}|^2}.$$

Suppose that $f *_1 \Phi_k(\mathbf{z})$ vanishes on the sphere S_r. Since the Laguerre polynomial $L_k^{(n-1)}$ has distinct zeros, $L_k^{(n-1)}(2r^2)$ can vanish for at most one value, say, $k = m$, and hence the above integral vanishes for all $k \neq m$. This means that

$$f(\mathbf{z}) = C_m \Phi_m(\mathbf{z}) = C_m L_m^{(n-1)}(2|\mathbf{z}|^2) e^{-|\mathbf{z}|^2}.$$

But then $f(\mathbf{z}) e^{|\mathbf{z}|^2}$ cannot be in any $L^p(\mathbb{C}^n)$.

In general, without loss of generality, we may assume that f is continuous. Now we may expand f in terms of spherical functions, *i.e.*, $f(\mathbf{z}) = \sum_{p,q} f_{pq}(\mathbf{z})$, where

$$f_{pq}(\mathbf{z}) = \sum_{\ell=1}^{d_{pq}} f_{pq}^{\ell}(|\mathbf{z}|) P_{pq}^{\ell}(\mathbf{z}) \quad \text{with} \quad P_{pq}^{\ell} \in \mathcal{H}(p,q). \tag{16}$$

Consider

$$f_{pq} *_1 \Phi_k(\mathbf{z}) = \int_{\mathbb{C}^n} L_k^{(n-1)}(2|\mathbf{w}|^2) e^{-|\mathbf{w}|^2} e^{-2i\mathrm{Im}\,(\mathbf{z}\cdot\bar{\mathbf{w}})} f_{pq}(\mathbf{z} - \mathbf{w}) dm(\mathbf{w}).$$

Then Lemma 4.3 implies that the above is equal to

$$\int_{\mathbb{C}^n} \int_{U(n)} L_k^{(n-1)}(2|\mathbf{z} - \mathbf{w}|^2) e^{-|\mathbf{z}-\mathbf{w}|^2} e^{-2i\mathrm{Im}\,(\mathbf{z}\cdot\bar{\mathbf{w}})} \chi_{pq}(\mathbf{u}) f_{pq}(\mathbf{u} \circ \mathbf{w}) d\mathbf{u} dm(\mathbf{w}).$$

Since $\Phi_k(\mathbf{z})$ is radial and $\mathrm{Im}\,(\mathbf{z} \cdot \bar{\mathbf{w}}) = \mathrm{Im}\,[(\mathbf{u} \circ \mathbf{z}) \cdot (\mathbf{u} \circ \bar{\mathbf{w}})]$ for all $\mathbf{u} \in U(n)$, we have

$$f_{pq} *_1 \Phi_k(\mathbf{z}) = \int_{U(n)} f *_1 \Phi_k(\mathbf{u} \circ \mathbf{z}) \chi_{pq}(\mathbf{u}) d\mathbf{u}.$$

Therefore, if $f *_1 \Phi_k$ vanishes on a sphere S_r then so does $f_{pq} *_1 \Phi_k$ for any ordered pair $(p, q) \in (\mathbb{Z}_+)^2$. Since f_{pq} is given by the expression (16), by Lemma 4.4, we have

$$f_{pq} *_1 \Phi_k(\mathbf{z}) = \left(\sum_{\ell=1}^{d_{pq}} C^{\ell}_{pq}(k) P^{\ell}_{pq}(\mathbf{z}) \right) L^{(n-1+p+q)}_{k-p}(2|\mathbf{z}|^2) e^{-|\mathbf{z}|^2},$$

where C^{ℓ}_{pq} is given by the integral

$$\frac{(k-p)!(n-1)!}{(n-1+k+p)!} \int_{\mathbb{C}^n} f^{\ell}_{pq}(|\mathbf{z}|) L^{(n-1+p+q)}_{k-p}(2|\mathbf{z}|^2) e^{-|\mathbf{z}|^2} |\mathbf{z}|^{2(p+q)} dm(\mathbf{z}).$$

Now if $f_{pq} *_1 \Phi_k(\mathbf{z})$ vanishes on S_r, then as before since the zeros of $L^{(n-1+p+q)}_{k-p}(2|\mathbf{z}|^2)$ are distinct, we conclude that

$$\sum_{\ell=1}^{d_{pq}} C^{\ell}_{pq}(k) P^{\ell}_{pq}(\mathbf{z}) = 0$$

for all values of k except possibly for one value $k = m$. The restrictions of P^{ℓ}_{pq} to the unit sphere are othonormal and so the above implies $C^{\ell}_{pq}(m) = 0$ for all ℓ when $k \neq m$. This means that $f_{pq} *_1 \Phi_k(\mathbf{z}) = 0$ for all $k \neq m$ and hence

$$f_{pq}(\mathbf{z}) = \left(\sum_{\ell=1}^{d_{pq}} C^{\ell}_{pq}(m) P^{\ell}_{pq}(\mathbf{z}) \right) L^{(n-1+p+q)}_{m-p}(2|\mathbf{z}|^2) e^{-|\mathbf{z}|^2}.$$

Note that the condition $f(\mathbf{z}) e^{|\mathbf{z}|^2} \in L^p(\mathbb{C}^n)$ holds also for f_{pq} and hence the above is possible only when $f_{pq} = 0$. Since p, q are arbitrary, we conclude that $f \equiv 0$. The proof of Theorem 4.1 is therefore complete. ■

Proof of Theorem 4.2. The proof of this theorem is similar. Without loss of generality, we may assume that $f \in C(\mathbf{H}^n)$. Since $f * h_s(\mathbf{z}, t) = 0$ on $\Gamma_r = S_r \times \mathbb{R}$ for all $s > 0$, by taking Fourier transform in the t-variable, we have $\tilde{f}_\lambda *_\lambda \tilde{h}_s(\mathbf{z}, \lambda) = 0$ for all $\lambda \in \mathbb{R}^*$, $s > 0$ and $\mathbf{z} \in S_r$. Since $\tilde{h}_s(\mathbf{z}, \lambda) = C \cdot H_{\lambda s}(\sqrt{\lambda}\mathbf{z})$, by Lemma 4.5, we obtain the equation

$$\tilde{f}_\lambda *_\lambda \Phi^\lambda_k(\mathbf{z}) = 0 \quad \text{on} \quad S_r, \quad \text{for} \quad \lambda \in \mathbb{R}^* \quad \text{and} \quad k \in \mathbb{Z}_+.$$

Now we expand $\tilde{f}_\lambda(\mathbf{z})$ in terms of spherical harmonic functions getting

$$\tilde{f}_\lambda(\mathbf{z}) = \sum_{p,q} f^\lambda_{pq}(\mathbf{z})$$

and as before this leads to the equations $f^\lambda_{pq} *_\lambda \Phi^\lambda_k(\mathbf{z}) = 0$ on S_r. It follows that

$$f^\lambda_{pq}(\mathbf{z}) = \sum_{\ell=1}^{d_{pq}} f^{\lambda,\ell}_{pq}(|\mathbf{z}|)P^\ell_{pq}(\mathbf{z}) \quad \text{with} \quad P^\lambda_{pq} \in \mathcal{H}(p,q).$$

Hence,

$$f^\lambda_{pq} *_\lambda \Phi^\lambda_k(\mathbf{z}) = \left(\sum_{\ell=1}^{d_{pq}} C^{\lambda,\ell}_{pq}(k) P^\lambda_{pq}(\mathbf{z}) \right) L^{(n-1+p+q)}_{k-p}(2|\lambda||\mathbf{z}|^2)e^{-|\lambda||\mathbf{z}|^2}.$$

Therefore, the condition $f^\lambda_{pq} *_\lambda \Phi^\lambda_k(\mathbf{z}) = 0$ on S_r implies that for each k, $L^{(n-1+p+q)}_{k-p}(2|\lambda|r^2) = 0$ only for finitely many values of λ. Thus, there is a countable set N such that when λ is not in N,

$$\sum_{\ell=1}^{d_{pq}} C^{\lambda,\ell}_{pq}(k) P^\lambda_{pq}(\mathbf{z}) = 0 \quad \text{for all} \quad k \in \mathbb{Z}_+.$$

This in turn leads to the vanishing of $C^{\lambda,\ell}_{pq}(k)$ for all ℓ and k. Consequently, $f^\lambda_{pq} = 0$ for all $(p,q) \in (\mathbb{Z}_+)^2$, which means that $\tilde{f}_\lambda(\mathbf{z}) = 0$ for all $\lambda \notin N$ and $\mathbf{z} \in \mathbb{C}^n$. But then the Fourier transform of $f(\mathbf{z},t)$ in the t-variable is supported on the set N, which contradicts the assumption that $f \in L^{p,q}(\mathbf{H}^n)$ unless $f \equiv 0$. The proof of Theorem 4.2 is therefore complete. ∎

Remark 5. Here we just proved that the set Γ_r satisfies the Pompeiu property for $L^{p,q}(\mathbf{H}^n)$. In fact, using the above results, we may obtain several corollaries related to twisted spherical means. The readers can consult the paper by Narayanan and Thangavelu [35] for a detailed discussion.

For the case of $\tilde{\Gamma}_r$, the proofs of the above theorems are slightly complicated. Assume that $f * \mu_r(\mathbf{z},t) = 0$ for all $\mathbf{z} \in T_r$. (Here $\mu_r(\mathbf{z},t) = \prod_{j=1}^n d\sigma_j(z_j) \times dt$.) Then we know that $f *_1 \Psi_{\mathbf{k}}(\mathbf{z}) = 0$ on T_r as well. Here $\Psi_{\mathbf{k}}(\mathbf{z}) = \prod_{j=1}^n C_j L^{(0)}_{k_j}(2|z_j|^2)e^{-|z_j|^2}$. Let us consider the $\mathbf{m}$-homogenization of f and call it $f_{\mathbf{m}}$. We then have $f_{\mathbf{m}} *_1 \Psi_{\mathbf{k}}(\mathbf{z}) = 0$ on T_r. As f is $\mathbf{m}$-homogeneous and the special Hermite function $\Phi_{\mathbf{p},\mathbf{q}}$ is $\mathbf{p} - \mathbf{q}$ homogeneous. It follows that

$$f_{\mathbf{m}} *_1 \Psi_{\mathbf{k}} = \sum_{|\mathbf{p}|=|\mathbf{k}|} \langle f, \Phi_{\mathbf{p}-\mathbf{m},\mathbf{p}} \rangle \Phi_{\mathbf{p}-\mathbf{m},\mathbf{p}}(\mathbf{z}), \tag{17}$$

which vanishes on T_r. Now $\Phi_{\mathbf{p}-\mathbf{m},\mathbf{p}}(\mathbf{z})$ can be written in terms of certain Laguerre polynomials. We can see the implication of the vanishing of all the coefficients in the expression (17) and conclude that $f \equiv 0$. We can also generalize these reaults to non-isotropic Heisenberg group, *i.e.*, the group law on $\mathbf{H}^n$ is given as

$$(\mathbf{z}, t) \cdot (\mathbf{w}, s) = \left(\mathbf{z} + \mathbf{w}, t + s + 2\text{Im} \sum_{j=1}^{n} a_j z_j \bar{w}_j \right)$$

with $a_j > 0$ for $j = 1, \ldots, n$. For details, see Chang and Thangavelu [21].

Question 7. *The restriction on* $1 \leq q \leq 2$ *is imposed for technical reasons since we have to take Fourier transform in the t-variable. We believe that Theorems* 4.1 *and* 4.2 *are ture for all* q *with* $1 \leq q < \infty$. *How do we remove this restriction*?

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