TAIWANESE JOURNAL OF MATHEMATICS
Vol. 14, No. 4, pp. 1377-1384, August 2010
This paper is available online at http://www.tjm.nsysu.edu.tw/

NOTES ON DYNAMICS OF THE ADJOINT OF A WEIGHTED COMPOSITION OPERATOR

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Abstract. In the present paper, we study the hypercyclicity of the adjoint of a weighted composition operator acting on some holomorphic function spaces.

1. Introduction

A weighted composition operator $C_{\varphi,\psi}$ is an operator that maps $f \in H(\mathbb{U})$, the F-space of holomorphic functions on the unit disk $\mathbb{U}$, into $C_{\varphi,\psi}(f)(z) = \varphi(z)f(\psi(z))$, where φ and ψ are analytic functions defined in $\mathbb{U}$ such that $\psi(\mathbb{U}) \subseteq \mathbb{U}$. When $\varphi \equiv 1$, we just have the composition operator C_ψ defined by $C_\psi(f) = f \circ \psi$. A bounded linear operator T on an F-space X is said to be hypercyclic if there is a vector $x \in X$ such that the orbit $\{T^n x : n \geq 0\}$ is dense in X and in this case we refer to x as a hypercyclic vector for T. Each of the following classes of linear maps contains hypercyclic operators: Backward unilateral and bilateral weighted shifts [11, 12], translation and differentiation operators [1, 4], adjoint of multiplication operators [7], composition operators [2, 3, 13] and weighted composition operators [15]. For a complete survey of hypercyclicity, the reader must refer to [8, 9].

Studying the dynamics of weighted composition operators entails a study of the iterating behavior of holomorphic self-maps. But the iterate of holomorphic self-map can be identify by the Denjoy-Wolff Theorem.

For simplicity, throughout this paper we use the notation ψ_n for indicating the nth iterate of ψ and by $\psi'(\alpha)$ we denote the angular derivative of ψ at α. Note that if $\alpha \in \mathbb{U}$, then $\psi'(\alpha)$ has the natural meaning of derivative.

Denjoy-Wolff Theorem

Suppose ψ is a holomorphic self-map of $\mathbb{U}$ that is not an automorphism.

Received January 28, 2008, accepted September 11, 2008.
Communicated by Bor-Luh Lin.
2000 *Mathematics Subject Classification*: Primary 47B37; Secondary 47A25.
Key words and phrases: Weighted composition operator, Hypercyclic operator, Linear-Fractional Model Theorem.

(i) If ψ has a fixed point $\alpha \in \mathbb{U}$, then $\psi_n(z) \to \alpha$ and $|\psi'(\alpha)| < 1$.

(ii) If ψ has no fixed point in $\mathbb{U}$, then there is a point $\alpha \in \mathbb{U}$ such that $\psi_n(z) \to \alpha$ and the angular derivative of ψ exists at α, with $0 < \psi'(\alpha) \leq 1$.

We call the unique attracting point α, the Denjoy-Wolff point of ψ. By the Denjoy- Wolff Iteration Theorem, a general classification for holomorphic self-maps of $\mathbb{U}$ can be given: The *elliptic* type has a fixed point in $\mathbb{U}$, *hyperbolic* type, if it has no fixed point in $\mathbb{U}$ and has derivative < 1 at its Denjoy-Wolff point; *parabolic* type if it has no fixed point in $\mathbb{U}$ and has derivative $= 1$ at its Denjoy-Wolff point.

The first section of the present paper by some useful criteria, provides hypercyclicity for the adjoint of a weighted composition operator on general Banach space of analytic function.

In the next section, we severely limit the kinds of maps that can produce hypercyclicity for the adjoint of a weighted composition operator and give some conditions which leads to the non-hypercyclicity.

2. Hypercyclicity on Banach Space

Throughout this section let X be a separable reflexive Banach space of holomorphic functions on the open unit disk $\mathbb{U}$ which contains the bounded functions on X and for each $z \in \mathbb{U}$, the evaluation functional $k_z : X \to \mathbb{C}$ defined by $k_z(f) = f(z)$ is bounded on X. Note that X may be, for example, the classical Bergman and Hardy Hilbert space of $\mathbb{U}$. Moreover, we suppose that φ, ψ are holomorphic maps on $\mathbb{U}$, such that $\psi(\mathbb{U}) \subseteq \mathbb{U}$ and $C_{\varphi,\psi}$ acts boundedly on X.

A useful tool for showing the hypercyclicity of many operators on each Frechlet space is the Hypercyclicity Criterion. This Criterion is developed independently by Kitai [10] and Gethner and Shapiro [7]. We use this criterion for indicating the hypercyclicity of the adjoint of $C_{\varphi,\psi}$.

Hypercyclicity Criterion

Suppose that X is a separable Banach space and that T is a continuous linear mapping on X. If there exist two dense subsets Y and Z in X and a sequence (n_k) such that:

(1) $T^{n_k} y \to 0$ for every $y \in Y$,

(2) there exist functions $S_k : Z \to X$ such that for every $z \in Z, S_k z \to 0,$ and $T^{n_k} S_k z \to z$

then T is hypercyclic

When ψ is an elliptic automorphism, authors in [15], by this criterion, proved the hypercyclicity of $C^*_{\varphi,\psi}$ on general Hilbert space of analytic function:

Theorem 2.1. *Let ψ be of elliptic automorphism type with interior fixed point w. If $|\varphi(w)| < 1 < \lim_{|z|\to 1^-} \inf |\varphi(z)|$ then $C^*_{\varphi,\psi}$ is hypercyclic on X^*.*

The next theorem extends this result for when ψ is either hyperbolic or parabolic automorphism.

Theorem 2.2. *Assume that ψ is either hyperbolic or parabolic automorphism and α, β are Denjoy-Wolff points of ψ and ψ^{-1}, respectively. Then $C^*_{\varphi,\psi}$ is hypercyclic on X^* whenever*

$$\limsup_{z\to\alpha} |\varphi(z)| < \liminf_{z\to\alpha} \frac{\|k_z\|}{\|k_{\psi(z)}\|} \tag{1}$$

and

$$\liminf_{z\to\beta} |\varphi(z)| > \liminf_{z\to\beta} \frac{\|k_z\|}{\|k_{\psi(z)}\|} > 0. \tag{2}$$

Proof. Let $Y = \vee\{k_z : z \in \mathbb{U}\}$, then Y is a dense subset of X. Put $T = C^*_{\varphi,\psi}$. Note that for each $z \in \mathbb{U}$, the linear functional $k_z : X \to \mathbb{C}$ by $k_z(f) = f(z)$ is bounded on X and for every $z \in \mathbb{U}$ we get

$$C^*_{\varphi,\psi}(k_z)(f) = k_z \circ C_{\varphi,\psi}(f) = k_z(\varphi.f \circ \psi) = \varphi(z)(f \circ \psi)(z) = \varphi(z)k_{\psi(z)}(f)$$

for all $f \in X$. Thus $T(k_z) = \varphi(z)k_{\psi(z)}$. In general, for every positive integer n we have

$$T^n(k_z) = \prod_{i=0}^{n-1} \varphi(\psi_i(z))k_{\psi_n(z)} \tag{3}$$

By (1), we can choose two real numbers a and $r < 1$, satisfying:

$$\limsup_{z\to\alpha} |\varphi(z)| < a < r \liminf_{z\to\alpha} \frac{\|k_z\|}{\|k_{\psi(z)}\|}$$

Hence $|\varphi(z)| < r\frac{\|k_z\|}{\|k_{\psi(z)}\|}$ when z sufficiently near to α. Now for each $z \in \mathbb{U}$, since $\psi_i(z) \to \alpha$, for some positive integer N, we get

$$|\varphi(\psi_i(z)| < r\frac{\|k_{\psi_i(z)}\|}{\|k_{\psi_{i+1}(z)}\|} \qquad (i \geq N) \tag{4}$$

Thus from (3) and (4) we obtain

$$\begin{aligned}\|T^n(k_z)\| &= \prod_{i=0}^{n-1} |\varphi(\psi_i(z))| \cdot \|k_{\psi_n(z)}\| \\ &= \prod_{i=0}^{N-1} |\varphi(\psi_i(z))| \prod_{i=N}^{n-1} |\varphi(\psi_i(z))| \cdot \|k_{\psi_n(z)}\| \\ &\leq \prod_{i=0}^{N-1} |\varphi(\psi_i(z))| \cdot \|k_{\psi_N(z)}\| r^{n-N-1}\end{aligned}$$

Therefore the sequence $\{T^n\}$ converges pointwise to zero on the dense subset Y. Now define a sequence of linear maps $S_n : Y \to X$ by

$$S_n k_z = \prod_{i=0}^{n-1} (\varphi(\psi_i^{-1}(z)))^{-1} k_{\psi_n^{-1}(z)} \qquad (z \in \mathbb{U}, n \geq 1) \tag{5}$$

Similarly, by (2), we can choose two real scalers b and $s > 1$ such that

$$\liminf_{z\to\beta} |\varphi(z)| > b > s \liminf_{z\to\beta} \frac{\|k_z\|}{\|k_{\psi(z)}\|}$$

Hence $|\varphi(z)| > s\frac{\|k_z\|}{\|k_{\psi(z)}\|}$ when z sufficiently near to β. Now for each $z \in \mathbb{U}$, since $\psi_i^{-1}(z) \to \beta$, for some positive integer N, we get

$$|\varphi(\psi_i^{-1}(z))| > s\frac{\|k_{\psi_i^{-1}(z)}\|}{\|k_{\psi_{i-1}^{-1}(z)}\|} \qquad (i \geq N) \tag{6}$$

Consequently from (5) and (6) we get

$$\begin{aligned}\|S_n(k_z)\| &= \prod_{i=0}^{n-1} \frac{1}{|\varphi(\psi_i^{-1}(z))|} \cdot \|k_{\psi_n^{-1}(z)}\| \\ &= \prod_{i=0}^{N-1} \frac{1}{|\varphi(\psi_i^{-1}(z))|} \prod_{i=N}^{n-1} \frac{1}{|\varphi(\psi_i^{-1}(z))|} \cdot \|k_{\psi_n^{-1}(z)}\| \\ &\leq \prod_{i=0}^{N-1} \frac{1}{|\varphi(\psi_i^{-1}(z))|} \cdot \|k_{\psi_N^{-1}(z)}\| (\frac{1}{s})^{(n-N-1)}\end{aligned}$$

Thus the sequence $\{S_n\}$ converges pointwise to zero on the dense subset Y. Also note that $T^n S_n k_z = k_z$ on Y which implies that $T^n S_n$ is identity on the dense subset Y. Therefore T satisfies the hypothesis of Hypercyclicity Criterion and so is hypercyclic. ∎

Recall that the classical Hardy space $H^p(\mathbb{U}), 1 \leq p < \infty$, consisting of all holomorphic functions $f : \mathbb{U} \to \mathbb{C}$ such that

$$\|f\|_p^p = \sup_{0<r<1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < \infty$$

It is well known such spaces are reflexive for $1 < p < \infty$ and

$$\|k_z\|_p = (1 - |z|^2)^{\frac{-1}{p}} \qquad (z \in \mathbb{U}, 1 \leq p < \infty)$$

Also note that

$$\liminf_{z \to \alpha} \frac{\|k_z\|}{\|k_{\psi(z)}\|} = \liminf_{z \to \alpha} \frac{(1 - |\psi(z)|^2)^{\frac{1}{p}}}{(1 - |z|^2)^{\frac{1}{p}}} = (\psi'(\alpha))^{\frac{1}{p}}$$

and

$$\liminf_{z \to \beta} \frac{\|k_z\|}{\|k_{\psi(z)}\|} = \liminf_{z \to \beta} \frac{(1 - |\psi(z)|^2)^{\frac{1}{p}}}{(1 - |z|^2)^{\frac{1}{p}}} = (\psi'(\beta))^{\frac{1}{p}}.$$

The above facts about Hardy space H^p with the Theorem 2.2 give the following Corollary:

Corollary 2.3. *Let ψ be of hyperbolic automorphism type and α and β are the Denjoy-Wolff point of ψ and ψ^{-1} respectively. If φ is continuous at both α and β, and*

$$|\varphi(\alpha)| < (\psi'(\alpha))^{\frac{1}{p}}, \quad (\psi'(\beta))^{\frac{1}{p}} < |\varphi(\beta)|$$

then $C^*_{\varphi,\psi}$ is hypercyclic on $H^p(\mathbb{U})$ for $1 < p < +\infty$.

Examples 2.4. Let $1 < p < +\infty,\ 0 < a < 1,\ \psi(z) = \frac{(1+a)z+1-a}{(1-a)z+1+a}$ and $\varphi(z) = \exp(\lambda z)$ where $\lambda < \ln(a^{\frac{1}{p}})$. Then ψ is a hyperbolic automorphism which fixes ± 1 and $\psi'(1) = a$ and $\psi'(-1) = \frac{1}{a}$. On the other hand $\varphi(1) < a^{\frac{1}{p}} = (\psi'(1))^{\frac{1}{p}}$ and $\varphi(-1) > (\frac{1}{a^{\frac{1}{p}}}) = (\psi'(-1))^{\frac{1}{p}}$. Therefore by Corollary 2.3, $C^*_{\varphi,\psi}$ is hypercyclic on $H^p(\mathbb{U})$.

3. Non-hypercyclicity on Banach Space

In this section we severely limits the kinds of maps that can produce hypercyclic weighted composition operator adjoint. In fact we show in linear-fractional setting, just automorphism can induce hypercyclic operator. Before stating the main Theorem of this section we need some information about the fixed point theory of linear fractional self-maps of the unit disk.

Each linear-fractional self-maps of $\mathbb{U}$ has one or two fixed points. They fall into distinct classes determined by their fixed-point properties [13]. *Parabolic maps* have just one fixed point on the boundary of $\mathbb{U}$ that are conjugate to translations of the right half-plane into itself, with the automorphisms corresponding to the pure imaginary translations. The other possibility is that they have two distinct fixed points, ones in $\bar{\mathbb{U}}$ and others outside of $\mathbb{U}$. Maps with an interior fixed point is called *elliptic*, with a boundary fixed point is called *hyperbolic*.

Our argument hinges on the following simple observation:

Lemma 3.1. *Suppose X is a Banach space and T a bounded linear operator on X. If there exists $\Lambda \neq 0$ in X^* such that the orbit $\{T^{*n}\Lambda\}$ is bounded in X^*, then T is not hypercyclic.*

Proposition 3.2. *Suppose $C^*_{\varphi,\psi}$ is hypercyclic on X^*, then:*

(i) *ψ is automorphism*

(ii) $\varphi(\mathbb{U}) \cap \partial\mathbb{U} \neq \emptyset$

Proof. Assume that ψ is not automorphism and w is the Denjoy-Wolff point of ψ. If ψ is not parabolic, and w_0 is the other fixed point of ψ, then w is necessarily in the closure of $\mathbb{U}$ and w_0 is necessarily outside the closure of $\mathbb{U}$ (see [13], sec 0.4 or [3], pages 6 and 7). Consider the mapping $\sigma(z) = \frac{z-w}{N(z-w_0)}$, for some integer N with $N(|w_0|-1) > 2$. Note that $\sigma \circ \psi \circ \sigma^{-1}$ is a linear fractional map which fixes both 0 and ∞ so it must have the form $\sigma \circ \psi \circ \sigma^{-1}(z) = \lambda z$ for some nonzero complex number λ and for all z. Note that $\lambda^n \sigma(z) = \sigma(\psi_n(z))$ and by Denjoy-Wolff point theorem $\psi_n \to w$. Hence $|\lambda|^n \to 0$. Let $M = \|\varphi\|_\infty + 1$ and m be chosen in such a way that $|\lambda|^m < \frac{1}{M}$. Put $f = \sigma$, then f is bounded and so $f \in X$. On the other hand

$$|f \circ \psi_m(z)| = |\sigma(\psi_m(z))| = |\lambda|^m |\sigma(z)| < \frac{1}{M}|f(z)|. \qquad (z \in \mathbb{U})$$

More generally

$$|f \circ \psi_n(z)| \leq (\frac{1}{M})^{n-m+1}|f(z)|. \qquad (z \in \mathbb{U}, n \geq m) \tag{7}$$

From (1) we obtain:

$$\|(C_{\varphi,\psi})^n f\| = \|\prod_{i=0}^{n-1} \varphi \circ \psi_i . f \circ \psi_n\| \leq \|\varphi\|^n_\infty \|(\frac{1}{M})^{n-m+1} < M^{m-1}. \quad (n \geq m) \tag{8}$$

Hence the orbit of $C_{\varphi,\psi}$ under the vector $f \in X$ is bounded and its adjoint can not be hypercyclic on X^* by Lemma 3.1.

Now suppose ψ is parabolic. The Linear-Fractional Model Theorem [13] then provides a function σ holomorphic on $\mathbb{U}$ with values in the right half-plane $\mathbb{P}$ such that $\sigma \circ \psi = \sigma + b$ for some real b with $Reb > 0$. Hence more generally

$$\sigma \circ \psi_n = \sigma + nb$$

Let λ be a real number satisfying $\lambda Reb < \log(\frac{1}{M}) < 0$ and m be a positive integer m such that $\exp(m\lambda b) < \frac{1}{M}$. Let $g = \exp(\lambda\sigma)$ then g is bounded and so $g \in X$. On the other hand for all $z \in \mathbb{U}$ we have

$$\begin{aligned} |g \circ \psi_m(z)| &= |\exp(\lambda\sigma \circ \psi_m(z))| = |\exp(\lambda\sigma(z) + m\lambda b)| \\ &= \exp(\lambda Re\sigma(z) + m\lambda Reb) < \frac{1}{M}|g(z)| \end{aligned}$$

As before both relations (1) and (2) are hold for g instead of f. Again the orbit of $C_{\varphi,\psi}$ under a vector in X is bounded and so its adjoint is not hypercyclic on X^* by Lemma 3.1.

For the proof of (ii), suppose by contradiction that $\varphi(\mathbb{U}) \cap \partial\mathbb{U} = \emptyset$. Then $\varphi(\mathbb{U})$ lies either inside $\mathbb{U}$ or outside $\overline{\mathbb{U}}$. In the first case $\|\varphi\|_\infty < 1$ and orbit $\{(C_{\varphi,\psi})^n f\}$ is bounded for every bounded function $f \in X$. In the further case, $\|\varphi^{-1}\|_\infty < 1$ and M_φ is invertible. On the other hand by the first part of theorem ψ is also automorphism. Thus $C_{\varphi,\psi}$ and consequently its adjoint are invertible and the orbit $\{(C_{\varphi,\psi})^n f\}$ is bounded for every bounded function $f \in X$. This implies that $(C^*_{\varphi,\psi})^{-1}$ is not hypercyclic by the preceding Lemma. Since an invertible operator is hypercyclic if and only if its inverse is hypercyclic, so $C^*_{\varphi,\psi}$ is also not hypercyclic on X^*. Therefore in each case we get a contradiction and so the proof is complete. ■

It is well known the condition $\varphi(\mathbb{U}) \cap \partial\mathbb{U} \neq \emptyset$ is necessary and sufficient for the hypercyclicity of adjoint M_φ [7]. By this fact the following corollary is immediate.

Corollary 3.3. If $C^*_{\varphi,\psi}$ is hypercyclic on X^*, then M^*_φ is also hypercyclic on X^*.

Our discussion will be finished with the following question:

Question 3.4. Can parabolic automorphism map, induce hypercyclic weighted composition operator adjoint?

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