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ON RANK 2 GEOMETRIES OF THE MATHIEU GROUP M_{23}

Nayil Kilic

Abstract. In this paper we determine all rank 2 geometries for the Mathieu group M_{23} for which object stabilizers are maximal subgroups.

1. Introduction

In this paper, we give the list of all rank 2 geometries for M_{23} that are firm, residually connected and flag transitive. We get 170 geometries of rank 2. These results were checked using a series of MAGMA [1] programs. The paper is organized as follows. In section 2, we recall the basic definitions needed in order to understand this paper. In section 3, we give the list of geometries we obtained. Finally, in section 4, we give the diagram of some rank 2 geometries for M_{23}.

2. Definitions and Notation

We begin by reviewing geometries and some standard notation. A geometry is a triple $(\Gamma, I, \star)$ where Γ is a set, I an index set and $\star$ a symmetric incidence relation on Γ which satisfy

(i) $\Gamma = \dot{\bigcup}_{i\in I}\Gamma_i$; and

(ii) if $x \in \Gamma_i$, $y \in \Gamma_j (i, j \in I)$ and $x \star y$, then $i \neq j$.

The elements of Γ_i are called objects of type i, and $|I|$ is the rank of the geometry Γ (as is usual we use Γ is place of the triple $(\Gamma, I, \star)$). A flag F of Γ is a subset of Γ in which every two elementof F are incident. The rank of F is $|F|$, the corank of F is $|I\backslash F|$ and the type of F is $\{i \in I | F \cap \Gamma_i \neq \emptyset\}$. A chamber of Γ is a flag of type I. All geometries we consider are assumed to contain at least one flag of

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rank $|I|$. The automorphism group of Γ, $Aut\Gamma$, consists of all permutations of Γ which preserve the sets Γ_i and the incidence relation $\star$. Let G be a subgroup of $Aut\Gamma$. We call Γ a flag transitive geometry for G if for any two flags F_1 and F_2 of Γ having the same type, there exists $g \in G$ such that $F_1{}^g = F_2$. For $\Delta \subseteq \Gamma$, the residue of Δ, denoted Γ_Δ, is defined to be $\{x \in \Gamma | x \star y$ for all $y \in \Delta\}$. A geometry Γ is called residually connected if for all flags F of Γ of corank 2 the incidence graph of Γ_F is connected. We call Γ firm provided that every flag of rank $|I| - 1$ is contained in at least two chambers. The diagram of a firm, residually connected, flag-transitive geometry Γ is a complete graph K, whose vertices are the elements of the set of type I of Γ, provided with some additional structure which is further described as follows. To each vertex $i \in I$, we attach the order s_i which is $|\Gamma_F| - 1$ where F is any flag of type $I \backslash \{i\}$, and the *number* n_i of *varieties* of *type* i, which is the index of G_i in G, and the subgroup G_i. To every edge $\{i, j\}$ of K, we associate three positive integers d_{ij}, g_{ij} and d_{ji} where g_{ij} (the gonality) is equal to half the girth of the incidence graph of a residue Γ_F of type $\{i, j\}$, and d_{ij} (resp. d_{ji}), the $i - diameter$ (resp. $j - diameter$) is the greatest distance from some fixed $i-$ element (resp. $j-$ element) to any other element in Γ_F.

On a picture of the diagram, this structure will often be depicted as follows.

$$\begin{array}{ccc} & d_{ij} \quad g_{ij} \quad d_{ji} & \\ \circ & \text{———————} & \circ \\ s_i & & s_j \\ n_i & & n_j \\ G_i & & G_j \\ & B = G_{ij} & \end{array}$$

Now suppose that Γ is a flag transitive geometry for the group G. As is well-known we may view Γ in terms of certain cosets of G. This is the approach we shall follow here. For each $i \in I$ choose an $x_i \in \Gamma_i$ and set $G_i = Stab_G(x_i)$. Let $\mathcal{F}$ = $\{G_i : i \in I\}$. We now define a geometry $\Gamma(G, \mathcal{F})$ where the objects of type i in $\Gamma(G, \mathcal{F})$ are the right cosets of G_i in G and for $G_i x$ and $G_j y$ $(x, y \in G, i, j \in I)$ $G_i x \star G_j y$ whenever $G_i x \cap G_j y \neq \emptyset$. Also by letting G act upon $\Gamma(G, \mathcal{F})$ by right multiplication we see that $\Gamma(G, \mathcal{F})$ is a flag transitive geometry for G. Moreover Γ and $\Gamma(G, \mathcal{F})$ are isomorphic geometries for G. So we shall be studying geometries of the form $\Gamma(G, \mathcal{F})$, where $G \cong M_{23}$ and G_i is a maximal subgroup of G for all $i \in I$

Buekenhout, in [2], sought to give a wider view of geometries so as to encompass configurations observed in the finite sporadic simple groups. An outgrowth of this has been attemps to catalogue various subcollections of geometries for the finite sporadic simple groups (and other related groups). So - called minimal parabolic geometries and maximal 2-local geometries were investigated in [17] and [18] while geometries satisfying certain additional conditions for a number of (relatively) small order simple groups have been exhaustively examined. See, for example, [3, 6, 11, 12, 13, 16, 14, 15] and [19]. the results in most of these papers were obtained using various computer algebra systems. All rank 2 and rank 3 residually connected geometries of M_{22} (the Mathieu Group of degree 22) were investigated in [9]. Again Kilic, in [10], calculated all rank 2 geometries for the Mathieu group M_{24} for which object stabilizers are maximal subgroups. Now we determine all rank 2 geometries of M_{23} (the Mathieu Group of degree 23) whose object stabilizer are maximal subgroups.

For the remainder of this paper G will denote M_{23}, the Mathieu Group of degree 23. Also Ω will denote a 24 element set possessing the Steiner system $S(24, 8, 5)$ as described by Curtis's MOG [5]. We will follow the notation of [5].

So $\Omega = \begin{array}{|c|c|c|} \hline O_1 & O_2 & O_3 \\ \hline \end{array} = \begin{array}{|cc|cc|cc|} \hline \infty & 14 & 17 & 11 & 22 & 19 \\ 0 & 8 & 4 & 13 & 1 & 9 \\ 3 & 20 & 16 & 7 & 12 & 5 \\ 15 & 18 & 10 & 2 & 21 & 6 \\ \hline \end{array}$, where O_1, O_2 and O_3 are the heavy bricks of the MOG. Here M_{24} is the Mathieu group of degree 24 which leaves invariant the Steiner system $S(24, 8, 5)$ on Ω. Set $\Lambda = \Omega \setminus \{24\}$

An octad of Ω is just an 8-element block of the Steiner system and a subset of Ω is called a dodecad if it is the symmetric difference of two octads of Ω which intersect in a set of size two. Corresponding to each 4 points of Ω there is a partition of the 24 points into 6 tetrads with the property that the union of any two tetrads is an octad, this configuration will be called a sextet. The following sets will appear when we describe geometries for G.

(i) $\mathcal{D}$=$\{X \subseteq \Lambda \| X| = 2\}$ (duads of Λ).

(ii) $\mathcal{H}$=$\{X \subseteq \Lambda | X \cup \{\infty\}$ is an octad of Ω $\}$(heptads of Λ).

(iii) $\mathcal{O}$=$\{X \subseteq \Omega | X$ is an octad of $\Omega\}$ (octads of Λ).

(iv) $\mathcal{D}_{\mathrm{o}}$=$\{X \subseteq \Lambda | X$ is a dodecad of $\Omega\}$ (dodecads of Λ).

(v) $\mathcal{S}$=$\{X_i \subseteq \Omega \| X_i| = 4$ (for each $i \in I$), $X_i \cup X_j$ is an octad $(i \neq j)$ and $\Omega = \dot{\bigcup}_{i \in I} X_i$, $i \in I = \{1 ... 6\}\}$ (sextets of Ω).

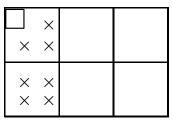
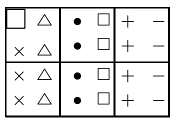
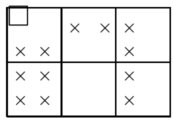
From the [4], the conjugacy classes of the maximal subgroups of G are as follows:

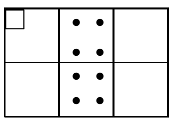
$Order$	$Index$	M_i	$Description$
443520	23	$M_1 \cong M_{22}$	$M_1 = Stab_G\{a\}$, $a \in \Lambda$
40320	253	$M_2 \cong L_3(4) : 2b$	$M_2 = Stab_G\{X\}$, $X \in \mathcal{D}$
40320	253	$M_3 \cong 2^4 : A_7$	$M_3 = Stab_G\{X\}$, $X \in \mathcal{H}$
20160	506	$M_4 \cong A_8$	$M_4 = Stab_G\{X\}$, $X \in \mathcal{O}$
7920	1288	$M_5 \cong M_{11}$	$M_5 = Stab_G\{X\}$, $X \in \mathcal{D}_{\mathrm{o}}$
5760	1771	$M_6 \cong 2^4 : (3 \times A_5) : 2$	$M_6 = Stab_G\{X\}$, $X \in \mathcal{S}$
253	40320	$M_7 \cong 23 : 11$	

For $i \in \{1, ..., 7\}$, we let $\mathfrak{M}_{\mathrm{i}}$ denote the conjugacy class of M_i, M_i as given in the previous table. We also set $\mathfrak{M}$=$\bigcup_{i=1}^{7} \mathfrak{M}_{\mathrm{i}}$; so $\mathfrak{M}$ consist of all maximal subgroups of G. In [5] and [10], we can find further information about 23:11. Also put $\mathfrak{X} = \Lambda \cup \mathcal{D} \cup \mathcal{H} \cup \mathcal{O} \cup \mathcal{D}_{\mathrm{o}} \cup \mathcal{S}$.

Suppose G_1 and G_2 are maximal subgroups of G with $G_1 \neq G_2$. Set $G_{12} = G_1 \cap G_2$. We use $\mathfrak{M}_{\mathrm{ij}}(t)$ to describe $\{G_1, G_2, G_1 \cap G_2\}$ according to the following scheme: $G_1 \in \mathfrak{M}_{\mathrm{i}}$, $G_2 \in \mathfrak{M}_{\mathrm{j}}$ (and so $G_1 = Stab_G(X_1)$ and $G_2 = Stab_G(X_2)$ for some appropriate subsets X_1 and X_2 of Ω in $\mathfrak{X}$) with $|X_1 \cap X_2| = t$. When listing up the rank 2 geometries of G in Theorem 1 the notation $\mathfrak{M}_{\mathrm{ij}}(t)$ is not sufficient enough to describe the geometries up to conjugacy in $AutG$. All calculations in $2^4 : (3 \times A_5) : 2$ and $23 : 11$ we can not use this notation, we shall use the following notation; $\mathfrak{M}_{46}(1)$ means the first case of the intersection of octad and sextet, $\mathfrak{M}_{46}(2)$ means the second case of the intersection of octad and sextet, using the same kind of idea we can define the remaining geometries. In addition, up to conjugacy, there are two geometries, we denote them the same notation as in $\mathfrak{M}_{47}(1)$(where $G_{12} = 1$ for both geometries). Similarly, up to conjugacy, we find 5 geometries for $\mathfrak{M}_{57}(1)$, 2 geometries for $\mathfrak{M}_{66}(1)$, 87 geometries for $\mathfrak{M}_{77}(1)$, 7 geometries for $\mathfrak{M}_{67}(1)$, 2 geometries for $\mathfrak{M}_{77}(2)$.

Two geometries $\Gamma_1(G, \{G_1, G_2\})$ and $\Gamma_2(G, \{H_1, H_2\})$ are conjugate iff there exists an element $g \in G$ such that $\{G_1, G_2\}^g = \{H_1, H_2\}$. In this paper, $N.G.$ denotes the number of geometries up to conjugacy.

Below we give certain subsets of Λ which will be encountered frequently in our list. $H_1 =$ , $S_1 =$ , $D_1 =$ 

$O_2 =$ , where $H_1 \in \mathcal{H}$, $S_1 \in \mathcal{S}$, $D_1 \in \mathcal{D}_{\mathrm{o}}$ and $O_2 \in \mathcal{O}$.

Our notation is as in the [4] with the following addition: F_n a Frobenious group of order n and $(S_n \times S_m)^+$ is the group of even permutation in the permutation group $S_n \times S_m$.

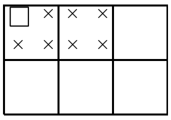
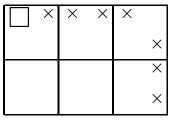
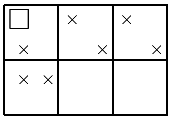
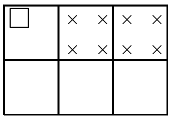
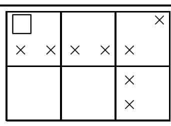
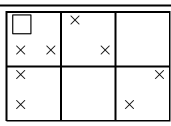
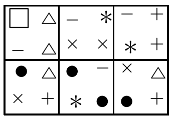
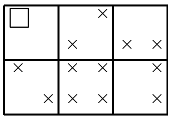
3. Rank 2 Geometries of M_{23}

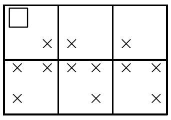
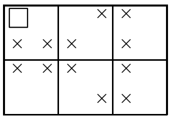
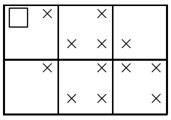
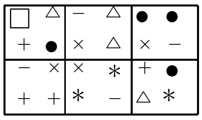
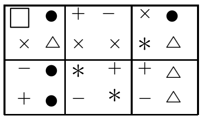
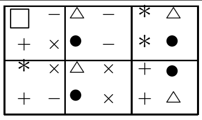
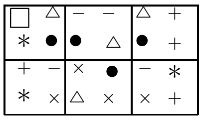
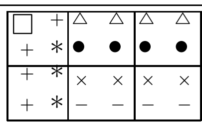
In this section, we give, up to conjugacy, all rank 2 geometries for M_{23} that are firm, residually connected and flag transitive.

Theorem 1. *Up to conjugacy in $AutG$ there are 170 firm, residually connected and flag transitive rank 2 geometries of $\Gamma = \Gamma(G, \{G_1, G_2\})$ with $G_1, G_2 \in \mathfrak{M}$. These together with the shape and order of G_{12} are listed in the following Table.*

Γ	G_{12}	$\mid G_{12} \mid$	*N.G.*	Γ	G_{12}	$\mid G_{12} \mid$	*N.G*
$\mathfrak{M}_{11}(0)$	M_{21}	20160	1	$\mathfrak{M}_{12}(1)$	$L_3(4)$	20160	1
$\mathfrak{M}_{12}(0)$	$2^4.S_5$	1920	1	$\mathfrak{M}_{13}(1)$	$2^4.A_6$	5760	1
$\mathfrak{M}_{13}(0)$	A_7	2520	1	$\mathfrak{M}_{14}(1)$	A_7	2520	1
$\mathfrak{M}_{14}(0)$	$2^3.L_3(2)$	1344	1	$\mathfrak{M}_{15}(0)$	$L_2(11)$	660	1
$\mathfrak{M}_{15}(1)$	M_{10}	720	1	$\mathfrak{M}_{16}(1)$	$2^4.3^2.2$	288	1
$\mathfrak{M}_{16}(2)$	$2^4.S_5$	1920	1	$\mathfrak{M}_{17}(1)$	C_{11}	11	1
$\mathfrak{M}_{22}(0)$	$2^4.D_{12}$	192	1	$\mathfrak{M}_{22}(1)$	$2^4.SL(2,4)$	960	1
$\mathfrak{M}_{23}(0)$	$2 \times L_3(2)$	336	1	$\mathfrak{M}_{23}(1)$	A_6	360	1
$\mathfrak{M}_{23}(2)$	$2^4.S_5$	1920	1	$\mathfrak{M}_{24}(0)$	$2^4.D_{12}$	192	1
$\mathfrak{M}_{24}(2)$	S_6	720	1	$\mathfrak{M}_{24}(1)$	$L_3(2)$	168	1
$\mathfrak{M}_{25}(0)$	$M_9.2$	144	1	$\mathfrak{M}_{25}(2)$	S_5	120	1
$\mathfrak{M}_{25}(1)$	A_5	60	1	$\mathfrak{M}_{26}(1)$	$2^4.D_{12}$	192	1
$\mathfrak{M}_{26}(2)$	$3^2.2^2$	36	1	$\mathfrak{M}_{26}(3)$	$2^4.S_5$	1920	1
$\mathfrak{M}_{26}(4)$	$2^4.S_3$	96	1	$\mathfrak{M}_{27}(1)$	1	1	1
$\mathfrak{M}_{33}(3)$	$2^2.(S_3 \times S_4)^+$	288	1	$\mathfrak{M}_{33}(1)$	A_6	360	1
$\mathfrak{M}_{34}(4)$	$2^2.(S_3 \times S_4)^+$	288	1	$\mathfrak{M}_{34}(2)$	S_5	120	1
$\mathfrak{M}_{34}(0)$	$2^3.L_3(2)$	1344	1	$\mathfrak{M}_{35}(6)$	A_6	360	1
$\mathfrak{M}_{35}(4)$	$M_8.S_3$	48	1	$\mathfrak{M}_{35}(2)$	S_5	120	1
$\mathfrak{M}_{36}(1)$	$2^4.(S_4 \times S_3)^+$	1152	1	$\mathfrak{M}_{36}(2)$	$2 \times S_4$	48	1
$\mathfrak{M}_{36}(3)$	$(S_4 \times S_3)^+$	72	1	$\mathfrak{M}_{36}(4)$	S_5	120	1
$\mathfrak{M}_{37}(1)$	1	1	1				
$\mathfrak{M}_{44}(0)$	$2^3.L_3(2)$	1344	1	$\mathfrak{M}_{44}(4)$	$2^4.S_3$	96	1
$\mathfrak{M}_{44}(2)$	$(S_4 \times S_3)^+$	72	1	$\mathfrak{M}_{45}(2)$	S_5	120	1
$\mathfrak{M}_{45}(4)$	S_4	24	1	$\mathfrak{M}_{45}(6)$	$3^2.D_8$	72	1
$\mathfrak{M}_{46}(1)$	$2^4.3^2.2^2$	576	1	$\mathfrak{M}_{46}(2)$	$2^2.D_{12}$	48	1
$\mathfrak{M}_{46}(3)$	S_4	24	1	$\mathfrak{M}_{46}(4)$	$(S_3 \times S_5)^+$	360	1
$\mathfrak{M}_{46}(5)$	$2 \times S_4$	48	1	$\mathfrak{M}_{47}(1)$	1	1	2
$\mathfrak{M}_{55}(8)$	S_4	24	1	$\mathfrak{M}_{55}(6)$	D_{10}	10	1
$\mathfrak{M}_{55}(4)$	$M_8.S_3$	48	1	$\mathfrak{M}_{56}(1)$	$3^2.2^2$	36	1
$\mathfrak{M}_{56}(2)$	D_{12}	12	1	$\mathfrak{M}_{56}(3)$	F_{20}	20	1
$\mathfrak{M}_{56}(4)$	S_4	24	1	$\mathfrak{M}_{56}(5)$	$M_8.S_3$	48	1
$\mathfrak{M}_{57}(1)$	C_{11}	11	5	$\mathfrak{M}_{57}(2)$	1	1	1
$\mathfrak{M}_{66}(1)$	D_{12}	12	2	$\mathfrak{M}_{66}(2)$	$3^2.2^2$	36	1
$\mathfrak{M}_{66}(3)$	$2^4.2^2$	64	1	$\mathfrak{M}_{66}(4)$	$2^4.S_3$	96	1
$\mathfrak{M}_{66}(5)$	$2^4.3^2.2$	288	1	$\mathfrak{M}_{67}(1)$	1	1	7
$\mathfrak{M}_{77}(1)$	1	1	87	$\mathfrak{M}_{77}(2)$	C_{11}	11	2

We now define some subgroups for $\mathfrak{M}_{\mathrm{ij}}(\mathrm{x})$ $(x \in \{1, 2..., 5\}, i \in \{1, 2..., 7\}, j = \{6\})$ by giving the $X_l \in \mathfrak{X}$ such that $G_l = Stab_G X_l$ $(l = 1, 2)$.

$\mathfrak{M}_{\mathrm{ij}}(\mathrm{x})$	X_1	X_2
$\mathfrak{M}_{16}(1)$	$\{14\}$	S_1
$\mathfrak{M}_{16}(2)$	$\{0\}$	S_1
$\mathfrak{M}_{26}(1)$	$\{14, 8\}$	S_1
$\mathfrak{M}_{26}(2)$	$\{14, 17\}$	S_1
$\mathfrak{M}_{26}(3)$	$\{0, 3\}$	S_1
$\mathfrak{M}_{26}(4)$	$\{0, 14\}$	S_1
$\mathfrak{M}_{36}(1)$	H_1	S_1
$\mathfrak{M}_{36}(2)$		S_1
$\mathfrak{M}_{36}(3)$		S_1
$\mathfrak{M}_{36}(4)$		S_1
$\mathfrak{M}_{46}(1)$	O_2	S_1
$\mathfrak{M}_{46}(2)$		S_1
$\mathfrak{M}_{46}(3)$		S_1
$\mathfrak{M}_{46}(4)$		S_1
$\mathfrak{M}_{46}(5)$	O_2	
$\mathfrak{M}_{56}(1)$	D_1	S_1
$\mathfrak{M}_{56}(2)$		S_1

$\mathfrak{M}_{56}(3)$		S_1
$\mathfrak{M}_{56}(4)$		S_1
$\mathfrak{M}_{56}(5)$		S_1
$\mathfrak{M}_{66}(1)$	S_1	
$\mathfrak{M}_{66}(2)$	S_1	
$\mathfrak{M}_{66}(3)$	S_1	
$\mathfrak{M}_{66}(4)$	S_1	
$\mathfrak{M}_{66}(5)$	S_1	

4. Diagrams for Rank 2 Geometries

In this section, we give the diagrams of the following geometries.

Γ	*Diagrams*	Γ	*Diagrams*	Γ	*Diagrams*
$\mathfrak{M}_{11}(0)$	4.1	$\mathfrak{M}_{12}(1)$	4.2	$\mathfrak{M}_{12}(0)$	4.3
$\mathfrak{M}_{13}(0)$	4.4	$\mathfrak{M}_{13}(1)$	4.5	$\mathfrak{M}_{14}(0)$	4.6
$\mathfrak{M}_{14}(1)$	4.7	$\mathfrak{M}_{15}(1)$	4.8	$\mathfrak{M}_{15}(0)$	4.9
$\mathfrak{M}_{16}(2)$	4.10	$\mathfrak{M}_{16}(1)$	4.11	$\mathfrak{M}_{22}(0)$	4.12
$\mathfrak{M}_{22}(1)$	4.13	$\mathfrak{M}_{23}(0)$	4.14	$\mathfrak{M}_{23}(1)$	4.15
$\mathfrak{M}_{23}(2)$	4.16	$\mathfrak{M}_{24}(0)$	4.17	$\mathfrak{M}_{24}(2)$	4.18

(Continued)

Γ	*Diagrams*	Γ	*Diagrams*	Γ	*Diagrams*
$\mathfrak{M}_{24}(1)$	4.19	$\mathfrak{M}_{25}(0)$	4.20	$\mathfrak{M}_{25}(2)$	4.21
$\mathfrak{M}_{25}(1)$	4.22	$\mathfrak{M}_{26}(1)$	4.23	$\mathfrak{M}_{26}(2)$	4.24
$\mathfrak{M}_{26}(3)$	4.25	$\mathfrak{M}_{26}(4)$	4.26	$\mathfrak{M}_{33}(3)$	4.27
$\mathfrak{M}_{33}(1)$	4.28	$\mathfrak{M}_{34}(4)$	4.29	$\mathfrak{M}_{34}(2)$	4.30
$\mathfrak{M}_{34}(0)$	4.31	$\mathfrak{M}_{35}(6)$	4.32	$\mathfrak{M}_{35}(4)$	4.33
$\mathfrak{M}_{35}(2)$	4.34	$\mathfrak{M}_{36}(1)$	4.35	$\mathfrak{M}_{36}(2)$	4.36
$\mathfrak{M}_{36}(3)$	4.37	$\mathfrak{M}_{36}(4)$	4.38	$\mathfrak{M}_{44}(0)$	4.39
$\mathfrak{M}_{44}(4)$	4.40	$\mathfrak{M}_{44}(2)$	4.41	$\mathfrak{M}_{45}(2)$	4.42
$\mathfrak{M}_{45}(4)$	4.43	$\mathfrak{M}_{45}(6)$	4.44	$\mathfrak{M}_{46}(1)$	4.45
$\mathfrak{M}_{46}(2)$	4.46	$\mathfrak{M}_{46}(3)$	4.47	$\mathfrak{M}_{46}(4)$	4.48
$\mathfrak{M}_{46}(5)$	4.49	$\mathfrak{M}_{55}(8)$	4.50	$\mathfrak{M}_{55}(6)$	4.51
$\mathfrak{M}_{55}(4)$	4.52	$\mathfrak{M}_{56}(1)$	4.53	$\mathfrak{M}_{56}(2)$	4.54
$\mathfrak{M}_{56}(3)$	4.55	$\mathfrak{M}_{56}(4)$	4.56	$\mathfrak{M}_{56}(5)$	4.57
$\mathfrak{M}_{66}(3)$	4.58	$\mathfrak{M}_{66}(1)$	4.59	$\mathfrak{M}_{66}(2)$	4.60
$\mathfrak{M}_{66}(4)$	4.61	$\mathfrak{M}_{66}(5)$	4.62	$\mathfrak{M}_{77}(1)$	4.63
$\mathfrak{M}_{77}(2)$	4.64	$\mathfrak{M}_{77}(1)^*$	4.65	$\mathfrak{M}_{57}(2)$	4.66
$\mathfrak{M}_{67}(1)$	4.67	$\mathfrak{M}_{57}(1)$	4.68	$\mathfrak{M}_{47}(1)$	4.69

In above table, $\mathfrak{M}_{77}(1)$ and $\mathfrak{M}_{77}(1)^*$ denote different geometries.

4.1 O —3—2—3— O
21 21
23 23
M_{22} M_{22}
$B = L_3(4)$

4.2 O —4—3—3— O
21 1
253 23
$L_3(4):2b$ M_{22}
$B = L_3(4)$

Due to Leemans (see [16], geometry 2.1).

4.3 O —3—2—3— O
230 20
253 23
$L_3(4):2b$ M_{22}
$B = 2^4.S_5$

4.4 O —3—2—3— O
175 15
253 23
$2^4:A_7$ M_{22}
$B = A_7$

4.5 O —3—2—3— O
76 6
253 23
$2^4:A_7$ M_{22}
$B = 2^4.A_6$

4.6 O —3—2—3— O
329 14
506 23
A_8 M_{22}
$B = 2^3.L_3(2)$

4.7 O—4 2 3—O

175	7
506	23
A_8	M_{22}

$B = A_7$

4.8 O—3 2 3—O

615	10
1288	23
M_{11}	M_{22}

$B = M_{10}$

4.9 O—3 2 3—O

671	11
1288	23
M_{11}	M_{22}

$B = L_2(11)$

4.10 O—4 2 3—O

230	2
1771	23
$2^4 : (3 \times A_5) : 2$	M_{22}

$B = 2^4.S_5$

4.11 O—3 2 3—O

1539	19
1771	23
$2^4 : (3 \times A_5) : 2$	M_{22}

$B = 2^4.3^2.2$

4.12 O—3 2 3—O

209	209
253	253
$L_3(4) : 2b$	$L_3(4) : 2b$

$B = 2^4.D_{12}$

4.13 O—3 2 3—O

41	41
253	253
$L_3(4) : 2b$	$L_3(4) : 2b$

$B = 2^4.SL(2, 4)$

4.14 O—3 2 3—O

119	119
253	253
$2^4 : A_7$	$L_3(4) : 2b$

$B = 2 \times L_3(2)$

4.15 O—3 2 3—O

111	111
253	253
$L_3(4) : 2b$	$2^4 : A_7$

$B = A_6$

4.16 O—4 2 3—O

20	20
253	253
$2^4 : A_7$	$L_3(4) : 2b$

$B = 2^4.S_5$

4.17 O—3 2 3—O

209	104
506	253
A_8	$L_3(4) : 2b$

$B = 2^4.D_{12}$

4.18 O—4 2 3—O

55	27
506	253
A_8	$L_3(4) : 2b$

$B = S_6$

4.19 $\circ\overset{3\quad 2\quad 3}{\rule{4em}{0.4pt}}\circ$

239	119
506	253
A_8	$L_3(4):2b$

$B = L_3(2)$

4.20 $\circ\overset{3\quad 2\quad 3}{\rule{4em}{0.4pt}}\circ$

54	54
1288	253
M_{11}	$L_3(4):2b$

$B = M_9.2$

4.21 $\circ\overset{3\quad 2\quad 3}{\rule{4em}{0.4pt}}\circ$

335	65
1288	253
M_{11}	$L_3(4):2b$

$B = S_5$

4.22 $\circ\overset{3\quad 2\quad 3}{\rule{4em}{0.4pt}}\circ$

671	131
1288	253
M_{11}	$L_3(4):2b$

$B = A_5$

4.23 $\circ\overset{4\quad 2\quad 3}{\rule{4em}{0.4pt}}\circ$

209	29
1771	253
$2^4:(3\times A_5):2$	$L_3(4):2b$

$B = 2^4.D_{12}$

4.24 $\circ\overset{3\quad 2\quad 3}{\rule{4em}{0.4pt}}\circ$

1119	159
1771	253
$2^4:(3\times A_5):2$	$L_3(4):2b$

$B = 3^2.2^2$

4.25 $\circ\overset{6\quad 3\quad 5}{\rule{4em}{0.4pt}}\circ$

20	2
1771	253
$2^4:(3\times A_5):2$	$L_3(4):2b$

$B = 2^4.S_5$

4.26 $\circ\overset{3\quad 2\quad 3}{\rule{4em}{0.4pt}}\circ$

419	59
1771	253
$2^4:(3\times A_5):2$	$L_3(4):2b$

$B = 2^4.S_3$

Due to Leemans (see [16], geometry 2.2).

4.27 $\circ\overset{3\quad 2\quad 3}{\rule{4em}{0.4pt}}\circ$

139	139
253	253
$2^4:A_7$	$2^4:A_7$

$B = 2^2.(S_3\times S_4)^+$

4.28 $\circ\overset{3\quad 2\quad 3}{\rule{4em}{0.4pt}}\circ$

111	111
253	253
$2^4:A_7$	$2^4:A_7$

$B = A_6$

4.29 $\circ\overset{4\quad 2\quad 3}{\rule[0.5ex]{6em}{0.4pt}}\circ$

139	69
506	253
A_8	$2^4 : A_7$

$B = 2^2.(S_3 \times S_4)^+$

4.30 $\circ\overset{3\quad 2\quad 3}{\rule[0.5ex]{6em}{0.4pt}}\circ$

335	167
506	253
A_8	$2^4 : A_7$

$B = S_5$

4.31 $\circ\overset{4\quad 2\quad 4}{\rule[0.5ex]{6em}{0.4pt}}\circ$

29	14
506	253
A_8	$2^4 : A_7$

$B = 2^3.L_3(2)$

4.32 $\circ\overset{4\quad 2\quad 3}{\rule[0.5ex]{6em}{0.4pt}}\circ$

111	21
1288	253
$2^4 : A_7$	M_{11}

$B = A_6$

4.33 $\circ\overset{3\quad 2\quad 3}{\rule[0.5ex]{6em}{0.4pt}}\circ$

839	164
1288	253
M_{11}	$2^4 : A_7$

$B = M_8.S_3$

4.34 $\circ\overset{3\quad 2\quad 3}{\rule[0.5ex]{6em}{0.4pt}}\circ$

335	65
1288	253
$2^4 : A_7$	M_{11}

$B = S_5$

4.35 $\circ\overset{4\quad 3\quad 4}{\rule[0.5ex]{6em}{0.4pt}}\circ$

34	4
1771	253
$2^4 : (3 \times A_5) : 2$	$2^4 : A_7$

$B = 2^4.(S_3 \times S_4)^+$

4.36 $\circ\overset{3\quad 2\quad 3}{\rule[0.5ex]{6em}{0.4pt}}\circ$

839	119
1771	253
$2^4 : A_7$	$2^4 : (3 \times A_5) : 2$

$B = 2 \times S_4$

4.37 $\circ\overset{3\quad 2\quad 3}{\rule[0.5ex]{6em}{0.4pt}}\circ$

559	79
1771	253
$2^4 : (3 \times A_5) : 2$	$2^4 : A_7$

$B = (S_3 \times S_4)^+$

4.38 $\circ\overset{4\quad 2\quad 3}{\rule[0.5ex]{6em}{0.4pt}}\circ$

335	47
1771	253
$2^4 : A_7$	$2^4 : (3 \times A_5) : 2$

$B = S_5$

4.39 $\circ\overset{5\quad 3\quad 5}{\rule[0.5ex]{6em}{0.4pt}}\circ$

14	14
506	506
A_8	A_8

$B = 2^3.L_3(2)$

4.40 $\circ\overset{3\quad 2\quad 3}{\rule[0.5ex]{6em}{0.4pt}}\circ$

209	209
506	506
A_8	A_8

$B = 2^4.S_3$

Due to Leemans (see [16], geometry 2.4).

4.41 $\circ \overset{3 \quad 2 \quad 3}{\overline{\qquad\qquad\qquad}} \circ$

279	279
506	506
A_8	A_8

$B = (S_3 \times S_4)^+$

4.42 $\circ \overset{4 \quad 2 \quad 4}{\overline{\qquad\qquad\qquad}} \circ$

167	65
1288	506
M_{11}	A_8

$B = S_5$

4.43 $\circ \overset{3 \quad 2 \quad 3}{\overline{\qquad\qquad\qquad}} \circ$

839	329
1288	506
M_{11}	A_8

$B = S_4$

4.44 $\circ \overset{3 \quad 2 \quad 4}{\overline{\qquad\qquad\qquad}} \circ$

279	109
1288	506
M_{11}	A_8

$B = 3^2.D_8$

4.45 $\circ \overset{5 \quad 2 \quad 5}{\overline{\qquad\qquad\qquad}} \circ$

9	34
506	1771
A_8	$2^4 : (3 \times A_5) : 2$

$B = 2^4.3^2.2^2$

4.46 $\circ \overset{3 \quad 2 \quad 3}{\overline{\qquad\qquad\qquad}} \circ$

119	419
506	1771
A_8	$2^4 : (3 \times A_5) : 2$

$B = 2^2.D_{12}$

4.47 $\circ \overset{3 \quad 2 \quad 3}{\overline{\qquad\qquad\qquad}} \circ$

239	839
506	1771
A_8	$2^4 : (3 \times A_5) : 2$

$B = S_4$

4.48 $\circ \overset{5 \quad 2 \quad 5}{\overline{\qquad\qquad\qquad}} \circ$

15	55
506	1771
A_8	$2^4 : (3 \times A_5) : 2$

$B = (S_3 \times S_5)^+$

4.49 $\circ \overset{4 \quad 2 \quad 3}{\overline{\qquad\qquad\qquad}} \circ$

119	419
506	1771
A_8	$2^4 : (3 \times A_5) : 2$

$B = 2 \times S_4$

4.50 $\circ \overset{3 \quad 2 \quad 3}{\overline{\qquad\qquad\qquad}} \circ$

329	329
1288	1288
M_{11}	M_{11}

$B = S_4$

4.51 $\circ \overset{3 \quad 2 \quad 3}{\text{———————}} \circ$

791	791
1288	1288
M_{11}	M_{11}

$B = D_{10}$

4.52 $\circ \overset{5 \quad 2 \quad 5}{\text{———————}} \circ$

164	164
1288	1288
M_{11}	M_{11}

$B = M_8.S_3$

4.53 $\circ \overset{3 \quad 2 \quad 3}{\text{———————}} \circ$

159	219
1288	1771
M_{11}	$2^4 : (3 \times A_5) : 2$

$B = 3^2.2^2$

4.54 $\circ \overset{3 \quad 2 \quad 3}{\text{———————}} \circ$

479	659
1288	1771
M_{11}	$2^4 : (3 \times A_5) : 2$

$B = D_{12}$

4.55 $\circ \overset{3 \quad 2 \quad 3}{\text{———————}} \circ$

287	395
1288	1771
M_{11}	$2^4 : (3 \times A_5) : 2$

$B = F_{20}$

4.56 $\circ \overset{3 \quad 2 \quad 4}{\text{———————}} \circ$

239	329
1288	1771
M_{11}	$2^4 : (3 \times A_5) : 2$

$B = S_4$

4.57 $\circ \overset{3 \quad 2 \quad 4}{\text{———————}} \circ$

119	164
1288	1771
M_{11}	$2^4 : (3 \times A_5) : 2$

$B = M_8.S_3$

4.58 $\circ \overset{3 \quad 2 \quad 3}{\text{———————}} \circ$

89	89
1771	1771
$2^4 : (3 \times A_5) : 2$	$2^4 : (3 \times A_5) : 2$

$B = 2^4.2^2$

4.59 $\circ \overset{3 \quad 2 \quad 3}{\text{———————}} \circ$

479	479
1771	1771
$2^4 : (3 \times A_5) : 2$	$2^4 : (3 \times A_5) : 2$

$B = D_{12}$

4.60 $\circ \overset{4 \quad 2 \quad 4}{\text{———————}} \circ$

159	159
1771	1771
$2^4 : (3 \times A_5) : 2$	$2^4 : (3 \times A_5) : 2$

$B = 3^2.2^2$

4.61 $\circ \overset{4 \quad 2 \quad 4}{\text{———————}} \circ$

59	59
1771	1771
$2^4 : (3 \times A_5) : 2$	$2^4 : (3 \times A_5) : 2$

$B = 2^4.S_3$

4.62 $\circ \overset{7 \quad 2 \quad 7}{\text{———————}} \circ$

19	19
1771	1771
$2^4 : (3 \times A_5) : 2$	$2^4 : (3 \times A_5) : 2$

$B = 2^4.3^2.2$

4.63 O—5 2 5—O

252	252
40320	40320
$23:11$	$23:11$

$B = 1$

4.64 O—7 3 7—O

22	22
40320	40320
$23:11$	$23:11$

$B = C_{11}$

4.65 O—4 2 4—O

252	252
40320	40320
$23:11$	$23:11$

$B = 1$

4.66 O—3 2 3—O

7919	252
40320	1288
$23:11$	M_{11}

$B = 1$

4.67 O—3 2 3—O

5759	252
40320	1771
$23:11$	$2^4:(3 \times A_5):2$

$B = 1$

4.68 O—4 2 4—O

719	22
40320	1288
$23:11$	M_{11}

$B = C_{11}$

4.69 O—3 2 3—O

20159	252
40320	506
$23:11$	A_8

$B = 1$

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Nayil Kilic
Department of Mathematics,
Arts and Science Faculty,
Harran University,
Sanliurfa 63000,
Turkey
E-mail: nayilkilic@harran.edu.tr