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SUPERCENTRALIZING AUTOMORPHISMS ON PRIME SUPERALGEBRAS

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Abstract. Let $A = A_0 \oplus A_1$ be a noncommutative prime superalgebra over a commutative associative ring F with $\frac{1}{2} \in F$. Let $Z_s(A)$ be the supercenter of A. If an Z_2-preserving automorphism $\varphi : A \to A$ satisfies $[\varphi(x), x]_s \in Z_s(A)$ for all $x \in A$, then $\varphi = 1$, where 1 denotes the identity map of A. Moreover, if $A_1 \neq 0$, then A is a central order in a quaternion algebra. This gives a version of Mayne's theorem for superalgebras.

1. Introduction

Let R be a ring with center Z, and for $x, y \in R$, by $[x, y]$ we denote the usual commutator $xy - yx$. Let S be a subset of R. A map $f : S \to R$ is said to be *centralizing* if $[f(x), x] \in Z$ for all $x \in S$. In the special case where $[f(x), x] = 0$ for all $x \in S$, f is called *commuting*. The study of centralizing maps was initiated by a well-known theorem of Posner in [13] which states that the existence of a nonzero centralizing derivation in a prime ring R implies that R is commutative. An analogous result for centralizing automorphisms on prime rings was obtained by Mayne [10]. He proved that the existence of a nontrivial centralizing automorphism in a prime ring R implies that R is commutative. In [3] Brešar gave a description of all centralizing (commuting) additive maps of prime rings. Over the past few years a considerable part of the theory of associative rings has been extended to superalgebras by several authors (see, for examples, [1, 4, 5, 6, 7, 8, 9, 11, 12]).

Throughout the article, algebras are over a unital commutative associative ring F. We shall assume without further mentioning, that $\frac{1}{2} \in F$. Although this requirement is not always needed, it is assumed for the sake of simplicity.

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Let A be an associative algebra. We say that A is Z_2-*graded* if there are two F-submodules A_0 and A_1 of A such that $A = A_0 \oplus A_1$ and $A_i A_j \subseteq A_{i+j}$ (where indexes are computed modulo 2). We say that A_0 is the even, and A_1 is the odd part of A. In this case, A is called a *superalgebra* over F. If $A_1 = 0$ then A is said to be a trivial superalgebra.

Suppose that $A = A_0 \oplus A_1$ is a superalgebra. An element $a \in A_i$ $(i = 0, 1)$ is said to be *homogeneous of degree* i and this is indicated by $|a| = i$. For an F-submodule S of A, we put $S_i = S \bigcap A_i$, $i = 0, 1$, and say that S is *graded* if $S = S_0 + S_1$. A *graded ideal* of A is an ideal of A which is graded when considered as an F-module. Now, A is said to be *prime* if the product of any two nonzero graded ideals is nonzero. Further, A is called *semiprime* if it has no nonzero nilpotent graded ideals.

Form now on, let A be a superalgebra with the center $Z(A)$. Define for any $u, v \in A_0 \cup A_1$, the *super-commutator* $[u, v]_s = uv - (-1)^{|u||v|} vu$, and extend this product to A, additively. Thus,

$$[a, b]_s = [a_0, b_0]_s + [a_1, b_0]_s + [a_0, b_1]_s + [a_1, b_1]_s,$$

where $a = a_0 + a_1$, $b = b_0 + b_1$ and $a_i, b_i \in A_i$, for $i = 0, 1$. The *supercenter*, $Z_s(A)$, consists of the elements $a \in A$ such that $[a, b]_s = 0$ for all $b \in A$.

Let S be a subset of A and call a mapping $f : S \to A$ *supercentralizing* (*supercommuting*) on S if $[f(x), x]_s \in Z_s(A)$ ($[f(x), x]_s = 0$, respectively) for all $x \in S$. Let $i \in \{0, 1\}$. An F-linear mapping $d_i : A \to A$ is called a *superderivation of degree* $|d_i| = i$ if it satisfies $d_i(A_j) \subseteq A_{i+j}$, and

$$d_i(xy) = d_i(x)y + (-1)^{i|x|} x d_i(y) \quad \text{for all } x, y \in A_0 \cup A_1.$$

A *superderivation* is simply the sum of a superderivation of degree zero and a superderivation of degree 1. In [4] Chen gave a version of Posner's theorem for superderivation on graded-prime superalgebras. He proved that the existence of a nonzero supercentralizing superderivation in a graded-prime superalgebra A implies that A is commutative. An automorphism φ of A (as algebra) is called a Z_2-*preserving automorphism* of A if it preserves Z_2-gradation (i.e., $\varphi(A_i) \subseteq A_i$, for $i = 0, 1$).

The main purpose of this paper is to give a description of all supercentralizing (supercommuting) Z_2-preserving automorphisms in prime superalgebras.

2. The Main Results and Their Proofs

In this section, let $A = A_0 \oplus A_1$ be a semiprime superalgebra over F with its $Z(A)$ and its supercenter $Z_s(A)$. Clearly, $Z_s(A)$ is a subsuperalgebra of A. Note that $Z_s(A) = Z(A)_0$ (see [4, Lemma 2.4] or [12, Lemma 1.3(1)]). It is well known that A and A_0 are semiprime as algebras [11, Lemma 1.2]. By C we denote the extended centroid of A and Q the Martindale right ring of quotients of A. All these notions are explained in detail in the book [2, Chapter 2].

We define $\sigma : A \to A$ by $(a_0+a_1)^\sigma = a_0-a_1$. Note that σ is an automorphism of A such that $\sigma^2 = 1$. Conversely, given an algebra A and an automorphism σ of A with $\sigma^2 = 1$, A then becomes a superalgebra by defining $A_0 = \{a \in A \mid \sigma(a) = a\}$ and $A_1 = \{a \in A \mid \sigma(a) = -a\}$. Since σ can be extended to Q such that $\sigma^2 = 1$ on Q [2, Proposition 2.5.3]. Thus Q is also a semiprime superalgebra. It is well known that for any $a \in Q$ there exists an essential ideal I of A such that $aI \subseteq A$. We may assume that I is graded since otherwise we can replace it by $I \cap I^\sigma$. This fact will be used in the proof of main results.

We begin with some basic properties of prime superalgebras.

Lemma 2.1. [5, Lemma 2.1, (i) and (vii)]. *Let $A = A_0 \oplus A_1$ be a prime superalgebra. If $a \in A$ is such that $aA_1 = 0$ (or $A_1 a = 0$), then $a = 0$ or A is a trivial superalgebra. If $[A_0, A_1] = 0$ then either A is commutative (as an algebra) or it is a trivial superalgebra.*

The following result is a special case of a theorem of Brešar [3, Proposition 3.1].

Lemma 2.2. *Let R be a 2-torsion free semiprime ring R, and U a subring of R. If an additive mapping f of R into itself is centralizing on R, then f is commuting on U.*

The following result is a special case of [11, Lemma 1.8 (i)].

Lemma 2.3. *Let $A = A_0 \oplus A_1$ be a prime superalgebra. If $[a_1, A_0] = 0$ where $a_1 \in A_1$, then $a_1 \in Z(A)$.*

The following important result will be used in the next lemma.

Lemma 2.4. [5, Lemma 3.4]. *Let A be a prime superalgebra such that $C_1 = 0$. Let $k = 0$ or $k = 1$. Suppose that $a_{i_0}, b_{i_0} \in A_0$ and $a_{j_1}, b_{j_1} \in A_1$ are such that*

$$\sum_{i_0=1}^{n} a_{i_0} x_k b_{i_0} = \sum_{j_1=1}^{m} a_{j_1} x_k b_{j_1} \quad \text{for all } x_k \in A_k.$$

Then

$$\sum_{i_0=1}^{n} a_{i_0} x_k b_{i_0} = \sum_{j_1=1}^{m} a_{j_1} x_k b_{j_1} = 0 \quad \text{for all } x_k \in A_k.$$

We now give a crucial result for the proof of our main results, which is of independent interest.

Lemma 2.5. *Let $A = A_0 \oplus A_1$ be a nontrivial prime superalgebra. If $x_1^2 \in Z(A)_0$ for all $x_1 \in A_1$, then $[A_0, A_0] = 0$.*

Proof. We may assume without loss of generality that A is not commutative. Suppose first that $C_1 = 0$. By assumption we have

(1) $$x_1^2 \in Z(A)_0 \quad \text{for all } x_1 \in A_1.$$

A linearization of (1) gives

(2) $$x_1 y_1 + y_1 x_1 \in Z(A)_0 \quad \text{for all } x_1, y_1 \in A_1.$$

In particular, $[x_0, x_1 y_1 + y_1 x_1] = 0$ for all $x_0 \in A_0, x_1, y_1 \in A_1$. That is

$$x_0 x_1 y_1 + x_0 y_1 x_1 - x_1 y_1 x_0 - y_1 x_1 x_0 = 0.$$

Rewriting this equation yields

(3) $$x_0 y_1 x_1 - y_1 x_1 x_0 = x_1 y_1 x_0 - x_0 x_1 y_1.$$

Multiplying (3) by an arbitrary $t_1 \in A_1$ from the right we get

$$x_0 y_1 (x_1 t_1) - y_1 (x_1 x_0 t_1) = x_1 y_1 (x_0 t_1) - (x_0 x_1) y_1 t_1,$$

for all $y_1 \in A_1$. It follows from Lemma 2.4 that

$$x_0 y_1 x_1 t_1 - y_1 x_1 x_0 t_1 = 0.$$

By Lemma 2.1 we get that

$$x_0 y_1 x_1 - y_1 x_1 x_0 = 0 \quad \text{for all } x_0 \in A_0, x_1 \in A_1, y_1 \in A_1.$$

That is $[x_0, x_1 y_1] = 0$ for all $x_0 \in A_0, x_1, y_1 \in A_1$. Thus $[x_0, y_0 x_1 y_1] = 0$ and so $[x_0, y_0] x_1 y_1 = 0$ for all $x_0, y_0 \in A_0, x_1, y_1 \in A_1$. By Lemma 2.1 again we get that $[x_0, y_0] = 0$ for all $x_0, y_0 \in A_0$ as desired.

We next discuss the case when $C_1 \neq 0$. Pick a nonzero $\lambda_1 \in C_1$ and choose an essential graded ideal I of A such that $\lambda_1 I \subseteq A$. Since $\lambda_1 y_0 \in A_1$ for every $y_0 \in I_0$, it follows from (2) that

(4) $$x_1 \lambda_1 y_0 + \lambda_1 y_0 x_1 \in Z(A)_0 \quad \text{for all } x_1 \in A_1, y_0 \in I_0.$$

Note that every nonzero homogeneous element in C is invertible [5, Lemma 3.1]. We deduce from (4) that

(5) $$x_1y_0 + y_0x_1 \in Z(A) \quad \text{for all } x_1 \in A_1, y_0 \in I_0.$$

It follows from (2) and (5) that

$$x_1y + yx_1 \in Z(A) \quad \text{for all } x_1 \in A_1, y \in I.$$

In view of [2, Proposition 2.1.10 and Theorem 6.4.1] we get

(6) $$x_1y + yx_1 \in Z(A) \quad \text{for all } x_1 \in A_1, y \in Q.$$

Taking $y = 1$ in (6) we have that $2x_1 \in Z(A)$ for all $x_1 \in A_1$ and so $A_1 \subseteq Z(A)$. Since $A_1 \neq 0$ it follows from Lemma 2.1 that A is commutative, a contradiction. The proof of the lemma is now complete.

The *central closure* of A is the central associative superalgebra $Z(A)_0^{-1}A = \{z^{-1}a \mid z \in Z(A)_0 \setminus \{0\}, a \in A\}$ over the ring $Z(A)_0^{-1}Z(A)_0$. We say that A is a *central order* in $Z(A)_0^{-1}A$.

Lemma 2.6. [11, Lemma 1.9]. *Let $A = A_0 \oplus A_1$ be a prime superalgebra. If A_0 is commutative, then $Z(A)_0^{-1}A$ is one of the following superalgebras:*

1. *the field $\Omega(A) = Z(A)_0^{-1}Z(A)_0$, with trivial gradation;*
2. *a direct sum $\Omega(A) \oplus \Omega(A)$, with the gradation given by the exchange automorphism;*
3. *a field extension $\Delta = \Omega + \Omega u$, with $u^2 \in \Omega, \Delta_0 = \Omega$ and $\Delta_1 = \Omega u$;*
4. *a quaternion algebra $Q(\alpha, \beta)$ having an Ω-basis $1, u, v, uv$, with $u^2 = \alpha \in \Omega\setminus\{0\}$, $v^2 = \beta \in \Omega\setminus\{0\}$, $uv = -vu$, and the gradation given by $Q(\alpha,\beta)_0 = \Omega 1 + \Omega u$, $Q(\alpha,\beta)_1 = \Omega v + \Omega uv$.*

Now we are ready to prove our first main theorem.

Theorem 2.7. *Let $A = A_0 \oplus A_1$ be a noncommutative prime superalgebra over a commutative associative ring F with $\frac{1}{2} \in F$. Let $Z_s(A)$ be the supercenter of A. If an Z_2-preserving automorphism $\varphi : A \to A$ satisfies $[\varphi(x), x]_s \in Z_s(A)$ for all $x \in A$, then $\varphi = 1$, where 1 denotes the identity map of A. Moreover, if $A_1 \neq 0$, then A is a central order in a quaternion algebra.*

Proof. If $[A_0, A_1] = 0$, by Lemma 2.1 we get that $A_1 = 0$. Then the theorem follows from Mayne's theorem. Therefore, we may assume that $[A_0, A_1] \neq 0$. Our first goal is to show that $\varphi = 1$ on A, where 1 denotes the identity mapping of A.

We first show that $\varphi(x_0) = x_0$ for all $x_0 \in A_0$. By assumption we have $[\varphi(x_0), x_0] \in Z(A)_0 \subseteq Z(A_0)$, for all $x_0 \in A_0$. In view of Lemma 2.2 we get

$$[\varphi(x_0), x_0] = 0 \quad \text{for all } x_0 \in A_0. \tag{7}$$

A linearization of (7) gives

$$[\varphi(x_0), y_0] = [x_0, \varphi(y_0)] \quad \text{for all } x_0, y_0 \in A_0. \tag{8}$$

For any $x_0 \in A_0, x_1 \in A_1$, we deduce from our assumption that

$$\begin{aligned} &[\varphi(x_0) + x_1), x_0 + x_1]_s \\ &= [\varphi(x_0), x_0]_s + [\varphi(x_1), x_1]_s + [\varphi(x_0), x_1]_s + [\varphi(x_1), x_0]_s \in Z(A)_0. \end{aligned}$$

Since $[\varphi(x_i), x_i]_s \in Z(A)_0$ for all $x_i \in A_i, i = 0, 1$, it follows from this equation that

$$[\varphi(x_0), x_1]_s + [\varphi(x_1), x_0]_s \in Z(A)_0 \cap A_1 = 0.$$

That is, $[\varphi(x_0), x_1] + [\varphi(x_1), x_0] = 0$ for all $x_0 \in A_0, x_1 \in A_1$. Rewriting this equation we get

$$[\varphi(x_0), x_1] = [x_0, \varphi(x_1)] \quad \text{for all } x_0 \in A_0, x_1 \in A_1. \tag{9}$$

Combining (8) with (9) we deduce that

$$[\varphi(x_0), y] = [x_0, \varphi(y)] \quad \text{for all } x_0 \in A_0, y \in A. \tag{10}$$

Substituting x_0y for y in (10) we get

$$[\varphi(x_0), x_0y] = [x_0, \varphi(x_0)\varphi(y)],$$

for all $x_0 \in A_0, y \in A$. Expanding this equation we get

$$[\varphi(x_0), x_0]y + x_0[\varphi(x_0), y] = [x_0, \varphi(x_0)]\varphi(y) + \varphi(x_0)[x_0, \varphi(y)].$$

According to (7) and (10) we have

$$(\varphi(x_0) - x_0)[\varphi(x_0), y] = 0 \quad \text{for all } x_0 \in A_0, y \in A. \tag{11}$$

Substituting wy_1 for y in (11), where $w \in A, y_1 \in A_1$, we get

$$(\varphi(x_0) - x_0)([\varphi(x_0), w]y_1 + w[\varphi(x_0), y_1]) = 0.$$

In view of (11) we obtain

$$(\varphi(x_0) - x_0)A[\varphi(x_0), y_1] = 0 \quad \text{for all } x_0 \in A_0, y_1 \in A_1.$$

In view of [1, Lemma 2.1] it yields that $\varphi(x_0) = x_0$ or $[\varphi(x_0), A_1] = 0$ for all $x_0 \in A_0$. So A_0 is the union of two subgroups I_1 and I_2, where $I_1 = \{x_0 \in A_0 \mid \varphi(x_0) = x_0\}$ and $I_2 = \{x_0 \in A_0 \mid [\varphi(x_0), A_1] = 0\}$. It is impossible for a group to be the union of two proper subgroups; therefore, either $I_1 = A_0$ or $I_2 = A_0$. Thus $\varphi(x_0) = x_0$ for all $x_0 \in A_0$ or $[\varphi(A_0), A_1] = 0$. Since $\varphi(A_0) = A_0$ and $[A_0, A_1] \neq 0$, we obtain that $\varphi(x_0) = x_0$ for all $x_0 \in A_0$.

Next, we claim that $\varphi(x_1) = x_1$ for all $x_1 \in A_1$. Since $\varphi(x_0) = x_0$ for all $x_0 \in A_0$, it follows from (9) that

$$[x_0, x_1] = [x_0, \varphi(x_1)] \quad \text{for all } x_0 \in A_0, x_1 \in A_1.$$

That is, $[A_0, \varphi(x_1) - x_1] = 0$ for all $x_1 \in A_1$. By Lemma 2.3 we have

$$\varphi(x_1) - x_1 \in Z(A) \quad \text{for all } x_1 \in A_1. \tag{12}$$

Substituting x_0x_1 for x_1 ($x_0 \in A_0$) in (12) yields that $\varphi(x_0)\varphi(x_1) - x_0x_1 \in Z(A)$. Since $\varphi(x_0) = x_0$ for all $x_0 \in A_0$, we have

$$x_0(\varphi(x_1) - x_1) \in Z(A). \tag{13}$$

Suppose that $\varphi(x_1) - x_1 \neq 0$ for some $x_1 \in A_1$. Recall that nonzero homogeneous elements in C are invertible. According to (13), together with (12), we have $x_0 \in Z(A)$ for all $x_0 \in A_0$, a contradiction. So $\varphi(x_1) = x_1$ for all $x_1 \in A_1$ as claimed. Therefore, $\varphi = 1$ on A.

Since $\varphi = 1$ on A we have that $[x_1, x_1]_s \in Z(A)_0$ for all $x_1 \in A_1$. That is, $2x_1^2 \in Z(A)_0$ and so $x_1^2 \in Z(A)_0$ for all $x_1 \in A_1$. Thus, Lemma 2.5 tells us that $[A_0, A_0] = 0$. According to Lemma 2.6, the central closure $S = Z(A)_0^{-1}A$ has four possibilities, and in the first three cases S is commutative. Since A is a subsuperalgebra of S, A is commutative if S is one of the first three cases in Lemma 2.6. Since $[A_0, A_1] \neq 0$, we have that $S = Q(\alpha, \beta)$ and A is a central order in a quaternion algebra $Q(\alpha, \beta)$. The proof of the theorem is completed. ■

Having Theorem 2.7 in hand we can easily prove our second main theorem.

Theorem 2.8. *Let $A = A_0 \oplus A_1$ be a prime superalgebra over a commutative associative ring F with $\frac{1}{2} \in F$. If an Z_2-preserving automorphism $\varphi : A \to A$ satisfies $[\varphi(x), x]_s = 0$ for all $x \in A$, then A must be a trivial superalgebra. Moreover, if A is noncommutative then $\varphi = 1$, where 1 denotes the identity map of A.*

Proof. In view of Mayne's theorem [10] we only need to prove that A is a trivial superalgebra. By our assumption we have

$$\varphi(x_1)x_1 + x_1\varphi(x_1) = 0 \quad \text{for all } x \in A_1. \tag{14}$$

Suppose that A is commutative. It follows from (14) that $2\varphi(x_1)x_1 = 0$ and so $\varphi(x_1)x_1 = 0$. Since any nonzero homogeneous element in A has no nonzero divisor, it implies that $\varphi(x_1) = 0$ for all $x_1 \in A_1$, forcing $A_1 = 0$ as desired.

Suppose next that A is not commutative. Then $\varphi = 1$ on A in view of Theorem 2.7. It follows from (14) that $2x_1^2 = 0$ and so $x_1^2 = 0$ for all $x_1 \in A_1$. Hence

$$x_1y_1 + y_1x_1 = 0 \quad \text{for all } x_1, y_1 \in A_1. \tag{15}$$

For any $x_0 \in A_0, x_1, y_1 \in A_1$, we get from (15) that

$$-y_1x_1x_0 = x_1y_1x_0 = -y_1x_0x_1,$$

that is, $A_1[A_0, A_1] = 0$. It follows from Lemma 2.1 that $A_1 = 0$. Thus, in any case, A is a trivial superalgebra. This proves the theorem. ■

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