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A STRONG AND WEAK CONVERGENCE THEOREM FOR RESOLVENTS OF ACCRETIVE OPERATORS IN BANACH SPACES

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Abstract. In this paper, we first introduce an iterative sequence of Mann's type and Halpern's type for finding a zero point of an m-accretive operator in a real Banach space. Then we obtain the strong and weak convergence by changing control conditions of the sequence. The result improves and extends a strong convergence theorem and a weak convergence theorem obtained by Kamimura and Takahashi [9], simultaneously.

1. INTRODUCTION

Let E be a real Banach space and let $A \subset E \times E$ be an m-accretive operator. Then the problem of finding a solution $v \in H$ with $0 \in Av$ has been investigated by many researchers.

One well-known method for solving the equation $0 \in Av$ in E is the following: $x_0 = x \in E$ and

$$x_{n+1} = J_{\lambda_n} x_n, \quad n = 0, 1, 2, \cdots, \tag{1}$$

where $\{\lambda_n\} \subset (0, \infty)$ and $J_{\lambda_n} = (I + \lambda_n A)^{-1}$. This method is called the *proximal point algorithm*. Rockafellar [21] proved that if E is a Hilbert space, $\liminf_{n\to\infty} \lambda_n > 0$ and $A^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1) converges weakly to an element of $A^{-1}0$. Later, many researchers have studied the convergence of (1); Brézis and Lions [1], Güler [5], Reich [14, 18], Pazy [13], Nevanlinna and Reich [11], Jung and Takahashi [7] and these references mentioned therein. Some of them dealt with the weak convergence of (1) and others proved

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strong convergence theorems by imposing strong assumptions on A. See also Bruck [3], Reich [15-17, 19], Passty [12] and Bruck and Passty [4]. On the other hand, motivated by Halpern [6] and Mann [10], Kamimura and Takahashi [9] introduced the following two iterative schemes,

$$x_{n+1} = \alpha_n x + (1-\alpha_n) J_{\lambda_n} x_n, \quad n = 0, 1, 2, \cdots \tag{2}$$

and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) J_{\lambda_n} x_n, \quad n = 0, 1, 2, \cdots, \tag{3}$$

where $x_0 = x \in E$, $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{\lambda_n\}$ is a sequence in $(0,\infty)$. Then, under additional conditions, they proved that the sequence $\{x_n\}$ generated by (2) converges strongly to some $v \in A^{-1}0$ and the sequence $\{x_n\}$ generated by (3) converges weakly to some $v \in A^{-1}0$.

In this paper, motivated by Kamimura and Takahashi [9], we introduce the following iterative sequence: $x_0 = x \in E$ and

$$x_{n+1} = \alpha_n x + \beta_n x_n + \gamma_n J_{\lambda_n} x_n, \quad n = 0, 1, 2, \cdots, \tag{4}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\} \subset [0,1]$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\lambda_n\} \subset (0,\infty)$. And, by changing control conditions of the sequence, we prove a convergence theorem which improves and extends a strong convergence theorem and a weak convergence theorem obtained by Kamimura and Takahashi [9], simultaneously.

Finally, using this result, we consider the problem of finding a minimizer of a convex function in a real Hilbert space H.

2. Preliminaries

Throughout this paper, we denote the set of all nonnegative integers by $\mathcal{N}$. Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We also know that if C is a closed convex subset of a uniformly convex Banach space E, then for each $x \in E$, there exists a unique element $u = Px \in C$ with $\|x-u\| = \inf\{\|x-y\| : y \in C\}$. Such a P is called the metric projection of E onto C. The duality mapping J from E into 2^{E^*} is defined by

$$J(x) = \{y^* \in E^* : \langle x, y^* \rangle = \|x\|^2 = \|y^*\|^2\}, \quad x \in E.$$

Let $S(E) = \{x \in E : \|x\| = 1\}$. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in S(E)$, the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{5}$$

is attained uniformly for $x \in S(E)$. The norm of E is said to be Fréchet differentiable if for each $x \in S(E)$, (5) is attained uniformly for $y \in S(E)$. It is also said to be uniformly Fréchet differentiable if the limit (5) is attained uniformly for $x, y \in S(E)$. In such a case, E is called uniformly smooth. It is known that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping J is single-valued and uniformly norm to weak* continuous on each bounded subset of E. If E is uniformly smooth, then the duality mapping J is uniformly norm to norm continuous on each bounded subset of E.

Let C be a closed convex subset of E. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote the set of all fixed points of T by $F(T)$. A closed convex subset C of E is said to have the fixed point property for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset D of C into itself has a fixed point in D. A nonempty closed convex subset of a uniformly convex Banach space E has the fixed point property for nonexpansive mappings. Let D be a subset of C. A mapping P of C into D is said to be sunny if $P(Px + t(x - Px)) = Px$ whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. A mapping P of C into itself is called a retraction if $P^2 = P$. We denote the closure of the convex hull of D by $\overline{\text{co}}D$.

Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be accretive if for each $x_1, x_2 \in D(A)$ and $y_1 \in Ax_1, y_2 \in Ax_2$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq 0$. If A is accretive, then we have $\|x_1 - x_2\| \leq \|x_1 - x_2 + r(y_1 - y_2)\|$ for all $x_1, x_2 \in D(A)$, $y_1 \in Ax_1, y_2 \in Ax_2$ and $r > 0$. An accretive operator A is said to be m-accretive if $R(I + rA) = E$ for all $r > 0$. If A is accretive, then we can define, for each $r > 0$, a nonexpansive single valued mapping $J_r : R(I+rA) \to D(A)$ by $J_r = (I+rA)^{-1}$. It is called the resolvent of A. We also define the Yosida approximation A_r by $A_r = (I - J_r)/r$. We know that $A_r x \in AJ_r x$ for all $x \in R(I+rA)$ and $\|A_r x\| \leq \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$. We also know that for an m-accretive operator A, we have $A^{-1}0 = F(J_r)$ for all $r > 0$. An operator $A \subset E \times E^*$ is called monotone if for any $(x_1, y_1), (x_2, y_2) \in A, \langle x_1 - x_2, y_1 - y_2\rangle \geq 0$. A monotone operator $A \subset E \times E^*$ is called maximal if its graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. In a real Hilbert space, an operator A is m-accretive if and only if A is maximal monotone; see [24, 25] for more details.

3. Main Theorems

Let $A \subset E \times E$ be an m-accretive operator and let $J_r : E \to E$ be the resolvent of A for each $r > 0$. Then we consider the following algorithm. The sequence

$\{x_n\}$ is generated by

$$\begin{cases} x_0 = x \in E, \\ x_{n+1} = \alpha_n x + \beta_n x_n + \gamma_n J_{\lambda_n} x_n + e_n, \quad n \in \mathcal{N}, \end{cases} \tag{6}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0,1], \{\lambda_n\} \subset (0,\infty)$ and $\{e_n\} \subset E$.

The following lemmas are useful in the proof of our main theorem.

Lemma 3.1. (Reich [19] and Takahashi-Ueda [27]). *Let E be a reflexive Banach space whose norm is uniformly Gâteaux differentiable. Suppose that E has the fixed point property for nonexpansive mappings. If $A^{-1}0 \neq \emptyset$, then the strong $\lim_{t\to\infty} J_t x$ exists and belongs to $A^{-1}0$ for all $x \in E$. Further, if $Px = \lim_{t\to\infty} J_t x$ for each $x \in E$, then P is a sunny nonexpansive retraction of E onto $A^{-1}0$.*

Lemma 3.2. (Browder [2]). *Let C be a bounded closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself. If $\{x_n\}$ converges weakly to $z \in C$ and $\{x_n - Tx_n\}$ converges strongly to 0, then $Tz = z$.*

Lemma 3.3. (Reich [18] and Takahashi-Kim [26]). *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable, let C be a nonempty closed convex subset of E and let $\{T_0, T_1, T_2, \ldots\}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=0}^{\infty} F(T_n)$ is nonempty. Let $x \in C$ and $S_n = T_n T_{n-1} \cdots T_0$ for all $n \in \mathcal{N}$. Then the set $\bigcap_{n=0}^{\infty} \overline{co}\{S_m x : m \geq n\} \cap U$ consists of at most one point, where $U = \bigcap_{n=0}^{\infty} F(T_n)$.*

Lemma 3.4. (Xu [29]). *Let E be a uniformly convex Banach space. Then for each $r > 0$, there exists a strictly increasing, continuous and convex function $g : [0,\infty) \to [0,\infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)$$

for all $x, y \in \{z \in E : \|z\| \leq r\}$ and $\lambda \in [0,1]$.

Using these results, we first prove the following theorem. The proof is mainly due to Kamimura and Takahashi [9].

Theorem 3.1. *Let E be a uniformly convex Banach space whose norm is uniformly smooth and let $A \subset E \times E$ be an m-accretive operator. Let $x_0 = x \in E$ and let $\{x_n\}$ be a sequence generated by (6). Assume that $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathcal{N}$, $\sum_{n=0}^{\infty} \|e_n\| < \infty$ and $A^{-1}0 \neq \emptyset$. Then we have the following (i) and (ii):*

(i) *If*

$$\sum_{n=0}^{\infty}\alpha_n=\infty,\ \lim_{n\to\infty}\alpha_n=0,\ \lim_{n\to\infty}\beta_n=0,\ \textit{and}\ \lim_{n\to\infty}\lambda_n=\infty,$$

then $\{x_n\}$ *converges strongly to an element of* $A^{-1}0$. *Further, if* $Px=\lim_{n\to\infty}x_n$ *for each* $x\in E$, *then* P *is a sunny nonexpansive retraction of* E *onto* $A^{-1}0$.

(ii) *If*

$$\sum_{n=0}^{\infty}\alpha_n<\infty,\ \limsup_{n\to\infty}\beta_n<1,\ \textit{and}\ \liminf_{n\to\infty}\lambda_n>0,$$

then $\{x_n\}$ *converges weakly to* $v\in A^{-1}0$.

Proof. We first show that $\{x_n\}$ generated by (6) is bounded. In fact, from $A^{-1}0\neq\emptyset$, there exists $u\in A^{-1}0$ such that $J_su=u$ for all $s>0$. Then we have

$$\begin{aligned}\|x_1-u\|&=\|\alpha_0x+\beta_0x_0+\gamma_0J_{\lambda_0}x_0+e_0-u\|\\&\le\alpha_0\|x-u\|+\beta_0\|x_0-u\|+\gamma_0\|J_{\lambda_0}x_0-u\|+\|e_0\|\\&\le(\alpha_0+\beta_0)\|x-u\|+\gamma_0\|x_0-u\|+\|e_0\|\\&\le\|x-u\|+\|e_0\|.\end{aligned}$$

If $\|x_k-u\|\le\|x-u\|+\sum_{i=0}^{k-1}\|e_i\|$ holds for some $k\in\mathcal{N}$, we can similarly show $\|x_{k+1}-u\|\le\|x-u\|+\sum_{i=0}^{k}\|e_i\|$. Therefore, from $\sum_{n=0}^{\infty}\|e_n\|<\infty$, $\{x_n\}$ is bounded. Hence $\{J_{\lambda_n}x_n\}$ is also bounded.

(i) Let $z_t=J_tx, y_n=J_{\lambda_n}x_n$ and $u\in A^{-1}0$, where $t>0$. By Lemma 3.1, the strong $\lim_{t\to\infty}z_t$ exists and belongs to $A^{-1}0$. Putting $z=\lim_{t\to\infty}z_t$, we shall prove

$$\limsup_{n\to\infty}\langle x-z,J(x_n-z)\rangle\le 0. \tag{7}$$

To prove this, it is sufficient to show

$$\limsup_{n\to\infty}\langle x-z,J(y_n-z)\rangle\le 0. \tag{8}$$

In fact, since $x_{n+1}-y_n=\alpha_n(x-y_n)+\beta_n(x_n-y_n)+e_n$, we have $x_{n+1}-y_n\to 0$. This yields

$$\lim_{n\to\infty}\|J(x_{n+1}-z)-J(y_n-z)\|=0$$

because J is uniformly continuous. Then (8) implies (7). Now, we know that $(x-z_t)/t \in Az_t$ and $A_{\lambda_n}x_n \in Ay_n$. Since A is accretive, we obtain

$$\left\langle A_{\lambda_n}x_n - \frac{x-z_t}{t}, J(y_n-z_t)\right\rangle \geq 0$$

and hence

$$\langle x-z_t, J(y_n-z_t)\rangle \leq t\,\langle A_{\lambda_n}x_n, J(y_n-z_t)\rangle\,.$$

From $\lambda_n \to \infty$, we also have

$$\lim_{n\to\infty}\|A_{\lambda_n}x_n\| = \lim_{n\to\infty}\left\|\frac{x_n-y_n}{\lambda_n}\right\| = 0.$$

Then we have

$$\limsup_{n\to\infty}\langle x-z_t, J(y_n-z_t)\rangle \leq 0. \tag{9}$$

for all $t>0$. Since $z_t \to z$ as $t\to\infty$ and J is uniformly continuous, for any $\varepsilon>0$, there exists $t_0>0$ such that for all $t \geq t_0$ and $n \in \mathcal{N}$,

$$|\langle z-z_t, J(y_n-z_t)\rangle| \leq \frac{\varepsilon}{2} \quad \text{and} \quad |\langle x-z, J(y_n-z_t)-J(y_n-z)\rangle| \leq \frac{\varepsilon}{2}.$$

This implies that for $t \geq t_0$ and $n \in \mathcal{N}$,

$$\begin{aligned} &|\langle x-z_t, J(y_n-z_t)\rangle - \langle x-z, J(y_n-z)\rangle| \\ &\qquad \leq |\langle x-z_t, J(y_n-z_t)\rangle - \langle x-z, J(y_n-z_t)\rangle| \\ &\qquad\qquad + |\langle x-z, J(y_n-z_t)\rangle - \langle x-z, J(y_n-z)\rangle| \\ &\qquad = |\langle z-z_t, J(y_n-z_t)\rangle| + |\langle x-z, J(y_n-z_t)-J(y_n-z)\rangle| \\ &\qquad \leq \varepsilon. \end{aligned} \tag{10}$$

Hence, from (9) and (10), we have

$$\limsup_{n\to\infty}\langle x-z, J(y_n-z)\rangle \leq \limsup_{n\to\infty}\langle x-z_t, J(y_n-z_t)\rangle + \varepsilon \leq \varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we obtain (8).

Let $\varepsilon>0$. From $\sum_{n=0}^{\infty}\|e_n\| < \infty$ and (7), there exists $m \in \mathcal{N}$ such that for all $n \geq m$,

$$M\sum_{i=m}^{\infty}\|e_i\| \leq \frac{\varepsilon}{2} \qquad \text{and} \qquad \langle x-z, J(x_n-z)\rangle \leq \frac{\varepsilon}{4},$$

where $M = 2\sup_{n\in\mathcal{N}} \|x_n - z\|$. Since $\beta_n(x_n - z) + \gamma_n(y_n - z) = (x_{n+1} - z) - \alpha_n(x - z) - e_n$, we have

$$\|\beta_n(x_n - z) + \gamma_n(y_n - z)\|^2 \geq \|x_{n+1} - z\|^2 - 2\left\langle \alpha_n(x - z) + e_n, J(x_{n+1} - z)\right\rangle,$$

which yields

$$\begin{aligned}\|x_{n+1}-z\|^2 &\leq \|\beta_n(x_n-z) + \gamma_n(y_n-z)\|^2 + 2\left\langle \alpha_n(x-z) + e_n, J(x_{n+1}-z)\right\rangle \\ &\leq (\beta_n\|x_n-z\| + \gamma_n\|y_n-z\|)^2 + 2\alpha_n\left\langle x-z, J(x_{n+1}-z)\right\rangle + M\|e_n\| \\ &\leq (\beta_n\|x_n-z\| + \gamma_n\|x_n-z\|)^2 + 2\alpha_n\left\langle x-z, J(x_{n+1}-z)\right\rangle + M\|e_n\| \\ &\leq (1-\alpha_n)\|x_n-z\|^2 + 2\alpha_n\left\langle x-z, J(x_{n+1}-z)\right\rangle + M\|e_n\|.\end{aligned}$$

Hence for all $n \in \mathcal{N}$, we have

$$\|x_{n+m+1} - z\|^2 \leq (1 - \alpha_{n+m})\|x_{n+m} - z\|^2 + \alpha_{n+m}\frac{\varepsilon}{2} + M\|e_{n+m}\|.$$

By induction, we obtain

$$\|x_{n+m+1} - z\|^2 \leq \prod_{i=m}^{n+m}(1-\alpha_i)\|x_m - z\|^2 + \left\{1 - \prod_{i=m}^{n+m}(1-\alpha_i)\right\}\frac{\varepsilon}{2} + M\sum_{i=m}^{n+m}\|e_i\|$$

for all $n \in \mathcal{N}$. So, we obtain

$$\limsup_{n\to\infty}\|x_n - z\|^2 = \limsup_{n\to\infty}\|x_{n+m+1} - z\|^2 \leq \frac{\varepsilon}{2} + M\sum_{i=m}^{\infty}\|e_i\| \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we can conclude that $\{x_n\}$ converges strongly to z.

(ii) First we prove the result in the case of $\alpha_n \equiv 0$ and $e_n \equiv 0$, that is,

$$\text{(11)} \qquad \begin{cases} u_0 = x \in E; \\ u_{n+1} = \beta_n u_n + (1-\beta_n)J_{\lambda_n}u_n, \quad n \in \mathcal{N}. \end{cases}$$

Let $y_n = J_{\lambda_n}u_n$ and $v \in A^{-1}0$. For $l = \|x - v\|$, the set $D = \{z \in E : \|z - v\| \leq l\}$ is a nonempty bounded closed convex subset of E which is invariant under J_s for all $s > 0$. So $\{u_n\} \subset D$ is bounded and we can show that $\{J_{\lambda_n}u_n\}$ is also bounded. From

$$\begin{aligned}\|u_{n+1} - v\| &= \|\beta_n u_n + (1-\beta_n)y_n - v\| \\ &\leq \beta_n\|u_n - v\| + (1-\beta_n)\|y_n - v\| \\ &\leq \|u_n - v\|,\end{aligned}$$

$\lim_{n\to\infty} \|u_n - v\|$ exists. Since A is accretive and $A_{\lambda_n} u_n = (u_n - J_{\lambda_n} u_n)/\lambda_n = (u_n - y_n)/\lambda_n$, we have

$$\begin{aligned}
\|y_n - v\|^2 &\leq \left\| y_n - v + \frac{\lambda_n}{2}(A_{\lambda_n} u_n - 0)\right\|^2 \\
&= \left\| y_n - v + \frac{1}{2}(u_n - y_n)\right\|^2 \\
&= \left\| \frac{1}{2}(u_n - v) + \frac{1}{2}(y_n - v)\right\|^2 \\
&\leq \frac{1}{2}\|u_n - v\|^2 + \frac{1}{2}\|y_n - v\|^2 - \frac{1}{4} g(\|u_n - y_n\|) \\
&\leq \|u_n - v\|^2 - \frac{1}{4} g(\|u_n - y_n\|)
\end{aligned}$$

and hence

$$\begin{aligned}
&(1-\beta_n)\frac{1}{4} g(\|u_n - y_n\|) \\
&\qquad \leq (1-\beta_n)(\|u_n - v\| - \|y_n - v\|)(\|u_n - v\| + \|y_n - v\|) \\
&\qquad = (\|u_n - v\| - \beta_n \|u_n - v\| - (1-\beta_n)\|y_n - v\|)(\|u_n - v\| + \|y_n - v\|) \\
&\qquad \leq (\|u_n - v\| - \|u_{n+1} - v\|)(\|u_n - v\| + \|y_n - v\|).
\end{aligned}$$

Since $\limsup_{n\to\infty} \beta_n < 1$ and $\lim_{n\to\infty} \|u_n - v\|$ exists, from Lemma 3.4 we obtain $u_n - y_n \to 0$. So, from

$$\begin{aligned}
\|y_n - J_1 y_n\| &= \|(I - J_1) y_n\| \\
&= \|A_1 y_n\| \\
&\leq \inf\{\|z\| : z \in A y_n\} \\
&\leq \|A_{\lambda_n} u_n\| \\
&= \left\| \frac{u_n - y_n}{\lambda_n} \right\|
\end{aligned}$$

and $\liminf_{n\to\infty} \lambda_n > 0$, we have $y_n - J_1 y_n \to 0$. Further, letting $w \in E$ be a weak subsequential limit of $\{u_n\}$ such that $u_{n_i} \rightharpoonup w$, we get $y_{n_i} \rightharpoonup w$. Then it follows from Lemma 3.2 that $w \in F(J_1) = A^{-1}0$. Since E has a uniformly smooth norm, putting $T_n = \beta_n I + (1-\beta_n) J_{\lambda_n}$ and $S_n = T_n T_{n-1} \cdots T_0$, we have $\bigcap_{n=0}^{\infty} F(T_n) = A^{-1}0$ and $\{w\} = \bigcap_{n=0}^{\infty} \overline{\text{co}}\{u_m : m \geq n\} \cap A^{-1}0$ by Lemma 3.3. Therefore $\{u_n\}$ converges weakly to an element of $A^{-1}0$.

Finally, we show the theorem in the case of (ii). Our discussion follows an idea of Brézis and Lion [1]. Note that $\{J_{\lambda_n} x_n\}$ are bounded. Define $U_n z =$

$T_nz+\alpha_n(x-J_{\lambda_n}z)+e_n$ for all $z \in E$ and $n \in \mathcal{N}$, where $T_n = \beta_n I+(1-\beta_n)J_{\lambda_n}$. We know that $\{u_n\}$ defined by (11) converges weakly to some $u \in A^{-1}0$ and the sequence $\{x_n\}$ generated by (6) satisfies $x_{n+1} = U_n x_n$. We define, for every $m \in \mathcal{N}$, the sequence $\{z_n^m\}$ by $z_0^m = x_m$ and $z_{n+1}^m = T_{n+m}z_n^m, n \in \mathcal{N}$. Then, putting $u_0 = x_m$ and $u_n = z_n^m$, we have that $\{z_n^m\}$ converges weakly to some $z^m \in A^{-1}0$ as $n \to \infty$. From the definition of $\{z_n^m\}$, we also have

$$\begin{aligned}\left\|z_n^{m+1} - z_{n+1}^m\right\| &= \|T_{n+m}T_{n+m-1}\cdots T_{m+1}x_{m+1} - T_{n+m}T_{n+m-1}\cdots T_m x_m\|\\ &\le \|x_{m+1} - T_m x_m\| = \|\alpha_m(x - J_{\lambda_m}x_m) + e_m\|\\ &\le \alpha_m\|x - J_{\lambda_m}x_m\| + \|e_m\|\end{aligned}$$

for all $m, n \in \mathcal{N}$. Since $z_n^{m+1} \rightharpoonup z^{m+1}$ and $z_n^m \rightharpoonup z^m$ as $n \to \infty$, we have that $\left\|z^{m+1} - z^m\right\| \le \alpha_m\|x - J_{\lambda_m}x_m\| + \|e_m\|$ for all $m \in \mathcal{N}$. From $\sum_{n=0}^\infty \alpha_n < \infty$ and $\sum_{n=0}^\infty \|e_n\| < \infty$, $\{z^m\}$ is a Cauchy sequence and hence $\{z^m\}$ converges strongly to some $z \in A^{-1}0$. Since

$$\begin{aligned}\left\|x_{n+m+1}-z_{n+1}^m\right\| &= \|U_{n+m}U_{n+m-1}\cdots U_m x_m - T_{n+m}T_{n+m-1}\cdots T_m x_m\|\\ &= \|T_{n+m}U_{n+m-1}U_{n+m-2}\cdots U_m x_m\\ &\qquad + \alpha_{n+m}(x-J_{\lambda_{n+m}}U_{n+m-1}U_{n+m-2}\cdots U_m x_m) + e_{n+m}\\ &\quad -T_{n+m}T_{n+m-1}\cdots T_m x_m\|\\ &= \|T_{n+m}U_{n+m-1}U_{n+m-2}\cdots U_m x_m\\ &\quad + \alpha_{n+m}(x-J_{\lambda_{n+m}}x_{n+m}) + e_{n+m} - T_{n+m}T_{n+m-1}\cdots T_m x_m\|\\ &\le \|U_{n+m-1}U_{n+m-2}\cdots U_m x_m - T_{n+m-1}T_{n+m-2}\cdots T_m x_m\|\\ &\quad + \alpha_{n+m}\left\|x-J_{\lambda_{n+m}}x_{n+m}\right\| + \|e_{n+m}\|\\ &\le \cdots \le \sum_{i=m}^{n+m}\{\alpha_i\|x-J_{\lambda_i}x_i\| + \|e_i\|\},\end{aligned}$$

we have

$$\begin{aligned}|\langle x_{n+m+1} - z, h\rangle| &= |\langle x_{n+m+1} - z_{n+1}^m, h\rangle + \langle z_{n+1}^m - z^m, h\rangle + \langle z^m - z, h\rangle|\\ &\le \left(\sum_{i=m}^{n+m}\{\alpha_i\|x - J_{\lambda_i}x_i\| + \|e_i\|\}\right)\|h\| + \left|\left\langle z_{n+1}^m - z^m, h\right\rangle\right|\\ &\qquad + |\langle z^m - z, h\rangle|\end{aligned}$$

for all $h \in E^*$ and $m, n \in \mathcal{N}$. Since $z_{n+1}^m - z^m \to 0$ as $n \to \infty$, this implies

$$\limsup_{n\to\infty} |\langle x_n - z, h\rangle| = \limsup_{n\to\infty} |\langle x_{n+m+1} - z, h\rangle|$$
$$\leq \left(\sum_{i=m}^{\infty}\{\alpha_i \|x - J_{\lambda_i}x_i\| + \|e_i\|\}\right)\|h\| + |\langle z^m - z, h\rangle|$$

for all $h \in E^*$ and $m \in \mathcal{N}$. Since $z^m \to z$ as $m \to \infty$, $\sum_{n=0}^{\infty}\alpha_n < \infty$ and $\sum_{n=0}^{\infty}\|e_n\| < \infty$, $\{x_n\}$ converges weakly to $z \in A^{-1}0$. ∎

Using Theorem 3.1, we obtain the following two theorems proved by Kamimura and Takahashi [8].

Theorem 3.2. ([8]) *Let H be a real Hilbert space and let $A \subset H \times H$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$. Let $x_0 = x \in H$ and let $\{x_n\}$ be a sequence generated by*

$$y_n \approx J_{\lambda_n}x_n, \quad x_{n+1} = \alpha_n x + (1-\alpha_n)y_n, \quad n \in \mathcal{N}, \tag{12}$$

where $\|y_n - J_{\lambda_n}x_n\| \leq \delta_n$, $\sum_{n=0}^{\infty}\delta_n < \infty$, and $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy

$$\lim_{n\to\infty}\alpha_n = 0, \ \sum_{n=0}^{\infty}\alpha_n = \infty \ \textit{and} \ \lim_{n\to\infty}\lambda_n = \infty.$$

Then $\{x_n\}$ converges strongly to Px, where P is the metric projection of H onto $A^{-1}0$.

Proof. Letting $e_n = (1-\alpha_n)(y_n - J_{\lambda_n}x_n)$ in (12), we have

$$x_{n+1} = \alpha_n x + (1-\alpha_n)J_{\lambda_n}x_n + e_n$$

for all $n \in \mathcal{N}$. And we also have

$$\sum_{n=0}^{\infty}\|e_n\| = \sum_{n=0}^{\infty}(1-\alpha_n)\|y_n - J_{\lambda_n}x_n\| \leq \sum_{n=0}^{\infty}\|y_n - J_{\lambda_n}x_n\| \leq \sum_{n=0}^{\infty}\delta_n < \infty.$$

So, if we put $\beta_n = 0$ for every $n \in \mathcal{N}$ in Theorem 3.1, we get the conclusion. ∎

Theorem 3.3. ([8]) *Let H be a real Hilbert space and let $A \subset H \times H$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$ and let P be the metric projection of H onto $A^{-1}0$. Let $x \in H$ and let $\{x_n\}$ be a sequence generated by*

$$y_n \approx J_{\lambda_n}x_n, \quad x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \quad n \in \mathcal{N}, \tag{13}$$

where $\|y_n - J_{\lambda_n}x_n\| \leq \delta_n$, $\sum_{n=0}^{\infty}\delta_n < \infty$, and $\{\beta_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy

$$\limsup_{n\to\infty}\beta_n < 1 \ \textit{and} \ \liminf_{n\to\infty}\lambda_n > 0.$$

Then $\{x_n\}$ *converges weakly to* $v \in A^{-1}0$.

Proof. As in the proof of Theorem 3.2, put $e_n = (1-\beta_n)(y_n - J_{\lambda_n}x_n)$ in (13). So, if we put $\alpha_n = 0$ for every $n \in \mathcal{N}$ in Theorem 3.1, we get the conclusion. ■

Remark 1. As in the proofs of Theorems 3.2 and 3.3, we can also show the strong and weak convergence theorems of Xu [30, Theorems 5.1 and 5.2].

4. Applications

Let H be a real Hilbert space and let $f : H \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. Then, we can define the subdifferential of f as follows:

$$\partial f(x) = \{z \in H : f(y) \geq \langle z, y - x\rangle + f(x),\ y \in H\}$$

for all $x \in H$. In this section, we apply our algorithm to the case of $A = \partial f$. In such a case, we know that $A = \partial f$ is a maximal monotone operator; see [24, 25]. Our discussion follows Rockafellar [21]. If $A = \partial f$, the algorithm (6) is reduced to the following:

$$\text{(14)} \qquad \begin{cases} x_0 = x \in H, \\ y_n \approx \operatorname*{argmin}_{z \in H} \left\{ f(z) + \dfrac{1}{2\lambda_n} \|z - x_n\|^2 \right\} = J_{\lambda_n} x_n, \\ x_{n+1} = \alpha_n x + \beta_n x_n + \gamma_n y_n, \quad n \in \mathcal{N}, \end{cases}$$

where $\|y_n - J_{\lambda_n}x_n\| \leq \delta_n$, $J_{\lambda_n} = (I + \lambda_n \partial f)^{-1}$, $\sum_{n=0}^{\infty} \delta_n < \infty$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0,1]$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\lambda_n\} \subset (0, \infty)$. Using Theorem 3.1, we can prove the following theorem.

Theorem 4.1. *Let* $f : H \to (-\infty, \infty]$ *be a proper lower semicontinuous convex function with* $(\partial f)^{-1}0 \neq \emptyset$. *Let* $x_0 = x \in H$ *and let* $\{x_n\}$ *be a sequence generated by* (14). *Then we have the following* (*i*) *and* (*ii*):

(i) *Suppose that*

$$\sum_{n=0}^{\infty} \alpha_n = \infty,\ \lim_{n\to\infty} \alpha_n = 0,\ \lim_{n\to\infty} \beta_n = 0 \textit{ and } \lim_{n\to\infty} \lambda_n = \infty.$$

Then $\{x_n\}$ *converges strongly to* $v \in (\partial f)^{-1}0$, *where* $v = P_{(\partial f)^{-1}0}x$.

(ii) *Suppose that*

$$\sum_{n=0}^{\infty}\alpha_n < \infty,\ \limsup_{n\to\infty}\beta_n < 1,\ \textit{and}\ \liminf_{n\to\infty}\lambda_n > 0.$$

Then $\{x_n\}$ *converges weakly to* $v \in (\partial f)^{-1}0$.

Proof. (i) Putting $g_n(z) = f(z) + \|z - x_n\|^2 / 2\lambda_n$, we obtain

$$\partial g_n(z) = \partial f(z) + \frac{1}{\lambda_n}(z - x_n)$$

for all $z \in H$ and

$$J_{\lambda_n}x_n = (I + \lambda_n \partial f)^{-1}x_n = \operatorname{argmin}_{z\in H} g_n(z).$$

It follows from Theorem 3.1 that $\{x_n\}$ converges strongly to $v \in (\partial f)^{-1}0$, where $v = P_{(\partial f)^{-1}0}x$.

(ii) As in the proof of (i), we can prove (ii). ∎

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