TAIWANESE JOURNAL OF MATHEMATICS
Vol. 24, No. 3, pp. 709–722, June 2020
DOI: 10.11650/tjm/190604

Optimality Conditions in Set-valued Optimization Problem with Respect to a Partial Order Relation via Directional Derivative

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Abstract. In this study, a new directional derivative is defined by using Minkowski difference. Some properties and existence theorems of this directional derivative are given. Moreover, necessary and sufficient optimality conditions are presented for set-valued optimization problems with respect to m_1 order relation via directional derivative.

1. Introduction

Optimization is always used in our life. When optimization problems are represented in a mathematical problem, an objective function arises. If the objective function is vector valued then the problem is called vector optimization problem. If the objective function is a set-valued map, then the problem is called set-valued optimization problem. Game theory, finance, control theory, engineering are some examples of the research areas that consist of some applications of set-valued optimization [4, 7, 15, 16, 24, 27]. So, set-valued optimization has become a popular subject in optimization theory, recently. There are different types of solution approaches in set-valued optimization problems such as set approach [8, 10, 12, 14, 15, 18–20], vector approach [5, 8, 9] and lattice structure [22]. One can find more information about set-valued optimization problem with respect to these approaches in [1, 5, 8–10, 12–15, 17–20, 22, 23, 28]. In this work we focus on set approach.

Kuroiwa introduced the set approach which depends on comparisons among values of the set-valued objective map [18,19]. So, this approach requires order relations to compare sets. In the literature, there are some order relations for this purpose [12,21]. None of these order relations are partial order. Karaman et al. [14] introduced the first partial order relation named m_1-order relation on the family of nonempty bounded sets via Minkowski difference. This partial order enables us to revise the definition of efficient set. By this revision the process of finding efficient set in two steps is reduced to one step. In addition, a scalarization was presented to obtain optimality conditions for $(m_1 - \mathrm{SOP})$. Ansari et al. [2] studied minimal element theorem, Ekeland's variational principle, Caristi's fixed

Received October 30, 2018; Accepted June 17, 2019.
Communicated by Jein-Shan Chen.
2010 *Mathematics Subject Classification.* 80M50, 90C26.
Key words and phrases. set-valued optimization, partial order, directional derivative, optimality conditions.
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point theorem, Takahashi's minimization theorem by using these partial order relations. Also, they gave some characterizations of these partial order relations in terms of oriented distance function.

Until now, by using directional derivative optimality conditions haven't been obtained for set-valued optimization problems with respect to m_1 order relation. For this reason, we define M-directional derivative. Optimality conditions of set-valued optimization problems with respect to m_1 partial order relation are obtained. Recently, some optimality conditions obtained by using other directional derivatives for set-valued optimization problems with respect to set approach in [6, 11, 26].

In this study, a directional derivative is defined by using Minkowski difference. Some properties of this directional derivative are examined and some existence theorems are obtained. Moreover, this directional derivative and the directional derivative, which is defined by Dempe and Pilecka [6], are compared. Some optimality conditions are presented via directional derivative for set-valued optimization with respect to m_1 order relation, which is a partial order. An application of one of the optimality conditions is explained on an example.

Section 2 is reserved for preliminaries. Section 3 is dedicated to the definitions and the properties of the directional derivative. Optimality conditions and the application are given in Section 4.

2. Preliminaries

Throughout this paper, $C \subset \mathbb{R}^p$ is a convex, closed, pointed cone, containing 0 with nonempty interior and $\mathbb{R}^p$ is ordered by C. The family of nonempty bounded subsets of $\mathbb{R}^p$ and the family of nonempty subsets of $\mathbb{R}^p$ are denoted by $\mathcal{B}^*(\mathbb{R}^p)$ and $\mathcal{P}_0(\mathbb{R}^p)$, respectively. For a set $A \subset \mathbb{R}^p$, we denote closure of A by $\mathrm{cl}(A)$, interior of A by $\mathrm{int}(A)$. $B(x, \varepsilon)$ is the open ball centered at x with radius ε for an $x \in \mathbb{R}^p$. Open unit ball in $\mathbb{R}^p$ is denoted by $\mathbb{B}_p$.

Given $A, B \subset \mathbb{R}^p$. We denote algebraic sum of A and B by $A+B := \{a+b \mid a \in A, b \in B\}$, algebraic difference of A and B by $A - B := \{a - b \mid a \in A, b \in B\}$ and Minkowski (Pontryagin) difference of A and B by $A \dot{-} B := \{x \in X \mid x + B \subset A\} = \bigcap_{b \in B}(A - b)$ [25]. If A is closed, then $A \dot{-} B$ is also closed.

Now, we recall the definition of first partial order relation on family of sets.

Definition 2.1. [14] Let $A, B \in \mathcal{P}_0(\mathbb{R}^p)$. m_1 order relation $\preceq_C^{m_1}$ is defined by

$$A \preceq_C^{m_1} B \iff (B \dot{-} A) \cap C \neq \emptyset.$$

Remark 2.2. $\preceq_C^{m_1}$ is a pre-order relation on $\mathcal{P}_0(\mathbb{R}^p)$ and it is a partial order relation on $\mathcal{B}^*(\mathbb{R}^p)$ [14].

Definition 2.3. [14] Let $A, B \in \mathcal{P}_0(\mathbb{R}^p)$. Strict m_1 order relation $\prec_C^{m_1}$ is defined by

$$A \prec_C^{m_1} B \quad \Longleftrightarrow \quad (B \dot{-} A) \cap \operatorname{int}(C) \neq \emptyset.$$

Since $\preceq_C^{m_1}$ is a partial order relation on $\mathcal{B}^*(\mathbb{R}^p)$ definitions of efficient sets are given in the following way.

Definition 2.4. [14] Let $\mathcal{S} \subset \mathcal{B}^*(\mathbb{R}^p)$ and $A \in \mathcal{S}$. Then,

(i) A is m_1-minimal (m_1-maximal) set of $\mathcal{S}$ if there isn't any $B \in \mathcal{S}$ such that $B \preceq_C^{m_1} A$ ($A \preceq_C^{m_1} B$) and $A \neq B$,

(ii) A is weakly m_1-minimal (m_1-maximal) set of $\mathcal{S}$ if there isn't any $B \in \mathcal{S}$ such that $B \prec_C^{m_1} A$ ($A \prec_C^{m_1} B$).

Let $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set-valued map. Domain of the set-valued map F is $\operatorname{dom}(F) := \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}$. If $F(x)$ is a compact (convex) set for all $x \in \mathbb{R}^n$, we say that F is compact (convex) valued. Set-valued optimization problem is defined by

$$\text{(SOP)} \qquad \min\ (\max)\ F(x)\ \text{ s.t. }\ x \in \operatorname{dom}(F).$$

In particular, if we consider (SOP) with respect to $\preceq_C^{m_1}$, we denote it by $(m_1 - \mathrm{SOP})$.

We say that x_0 is a solution of $(m_1 - \mathrm{SOP})$ if $F(x_0)$ is an m_1-minimal (m_1-maximal) set of the image family $\mathcal{F}(\mathbb{R}^n) = \{F(x) \mid x \in \mathbb{R}^n\}$. Similarly, we say that x_0 is a weakly solution of $(m_1 - \mathrm{SOP})$ if $F(x_0)$ is a weakly m_1-minimal (weakly m_1-maximal) set of $\mathcal{F}(\mathbb{R}^n)$. $x_0 \in \mathbb{R}^n$ is called a local minimal (maximal) solution of $(m_1 - \mathrm{SOP})$ if there exists an $\varepsilon > 0$ such that x_0 is a solution of the problem

$$\min\ (\max)\ F(x)\ \text{ s.t. }\ x \in B_n(x_0, \varepsilon) \cap \operatorname{dom}(F).$$

$x_0 \in \mathbb{R}^n$ is called a strictly minimal (maximal) solution of $(m_1 - \mathrm{SOP})$ if $F(x_0) \prec_C^{m_1} F(x)$ ($F(x) \prec_C^{m_1} F(x_0)$) for all $x \in \operatorname{dom}(F) \setminus \{x_0\}$ and $x_0 \in \mathbb{R}^n$ is called a strict local minimal (maximal) solution of $(m_1 - \mathrm{SOP})$ if there exists an $\varepsilon > 0$ such that $F(x_0) \prec_C^{m_1} F(x)$ ($F(x) \prec_C^{m_1} F(x_0)$) for all $x \in B_n(x_0, \varepsilon) \setminus \{x_0\}$.

3. M-directional derivative

In this section, a directional derivative (M-directional derivative) is defined to obtain optimality conditions for $(m_1 - \mathrm{SOP})$. In this definition we use Minkowski difference because $\preceq_C^{m_1}$ is defined in terms of Minkowski difference of sets. Also, we show that the directional derivative is positively homogeneous. Some existence theorems are presented. Moreover, the directional derivative and a directional derivative given by Pilecka [26] are compared.

Definition 3.1. Let $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set-valued map, $\overline{x} \in \operatorname{int}(\operatorname{dom}(F))$ and $d \in \mathbb{R}^n$. The limit

$$D_M F(\overline{x}, d) := \limsup_{t \to 0^+} \frac{F(\overline{x} + td) \dot{-} F(\overline{x})}{t}$$

is called M-directional derivative of F at $\overline{x}$ in direction d where $\limsup_{x' \to x} F(x') := \{y \in \mathbb{R}^p \mid \liminf_{x' \to x} d(y, F(x')) = 0\}$ denotes the upper limit of F at x. If $D_M F(\overline{x}, d) \neq \emptyset$ for $\overline{x} \in \operatorname{int}(\operatorname{dom}(F))$ and for all $d \in \mathbb{R}^n$, then F is called M-directionally differentiable at $\overline{x}$.

If F is a vector valued function, M-directional derivative and upper Dini directional derivative of a vector valued function [3] defined as

$$F_D(\overline{x}; d) := \limsup_{t \to 0^+} \frac{F(\overline{x} + td) - F(\overline{x})}{t}$$

coincides. So, M-directional derivative is a generalization of the upper directional derivative of a vector valued function.

The following example shows how to calculate the M-directional derivative of a set-valued map.

Example 3.2. Let a set-valued map $F\colon \mathbb{R} \rightrightarrows \mathbb{R}^2$ be defined as

$$F(x) = [x, 2x] \times [x, 2x]$$

for all $x \in \mathbb{R}$. Some image sets of F are given in Figure 3.1.

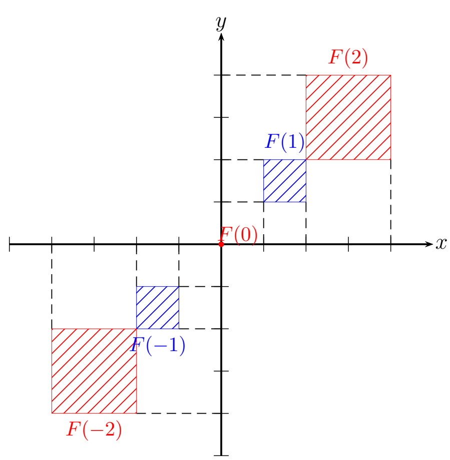

Figure 3.1: Some image sets of $F(x) = [x, 2x] \times [x, 2x]$.

Let's start with the calculation of M-directional derivative of F at $x = 0$ and $d > 0$:

$$\begin{aligned}
D_M F(0, d) &= \limsup_{t\to 0^+} \frac{F(0 + td) \dot{-} F(0)}{t} = \limsup_{t\to 0^+} \frac{F(td) \dot{-} F(0)}{t} \\
&= \limsup_{t\to 0^+} \frac{[td, 2td] \times [td, 2td]}{t} = \limsup_{t\to 0^+} [d, 2d] \times [d, 2d] \\
&= [d, 2d] \times [d, 2d].
\end{aligned}$$

In the case of $x = 0$ and $d < 0$:

$$\begin{aligned}
D_M F(0, d) &= \limsup_{t\to 0^+} \frac{F(0 + td) \dot{-} F(0)}{t} = \limsup_{t\to 0^+} \frac{F(td) \dot{-} F(0)}{t} \\
&= \limsup_{t\to 0^+} \frac{[2td, td] \times [2td, td]}{t} = \limsup_{t\to 0^+} [2d, d] \times [2d, d] \\
&= [2d, d] \times [2d, d].
\end{aligned}$$

Now, we will calculate $D_M F(x, d)$ for $x \in \mathbb{R} \setminus \{0\}$. Let $x > 0$ and $d > 0$. Then,

$$\begin{aligned}
D_M F(x, d) &= \limsup_{t\to 0^+} \frac{F(x + td) \dot{-} F(x)}{t} = \limsup_{t\to 0^+} \frac{[td, 2td] \times [td, 2td]}{t} \\
&= \limsup_{t\to 0^+} [d, 2d] \times [d, 2d] = [d, 2d] \times [d, 2d].
\end{aligned}$$

Let $x > 0$ and $d < 0$. Then,

$$D_M F(x, d) = \limsup_{t\to 0^+} \frac{F(x + td) \dot{-} F(x)}{t} = \limsup_{t\to 0^+} \emptyset = \emptyset.$$

Let $x < 0$ and $d < 0$. Then,

$$\begin{aligned}
D_M F(x, d) &= \limsup_{t\to 0^+} \frac{F(x + td) \dot{-} F(x)}{t} = \limsup_{t\to 0^+} \frac{[td, 2td] \times [td, 2td]}{t} \\
&= \limsup_{t\to 0^+} [d, 2d] \times [d, 2d] = [2d, d] \times [2d, d].
\end{aligned}$$

Let $x < 0$ and $d > 0$. Then,

$$D_M F(x, d) = \limsup_{t\to 0^+} \frac{F(x + td) \dot{-} F(x)}{t} = \limsup_{t\to 0^+} \emptyset = \emptyset.$$

Therefore,

$$D_M F(x, d) = \begin{cases} [d, 2d] \times [d, 2d] & \text{if } x \geq 0,\ d > 0, \\ [2d, d] \times [2d, d] & \text{if } x \leq 0,\ d < 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

So, the set-valued map F is M-directionally differentiable only at 0.

The following proposition gives positive homogeneity of M-directional derivative.

Proposition 3.3. *Let $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set-valued map. If F is M-directionally differentiable at $\overline{x} \in \operatorname{int}(\operatorname{dom}(F))$, then $D_M F(\overline{x}, \lambda d) = \lambda D_M F(\overline{x}, d)$ for all $d \in \mathbb{R}^n$ and for all $\lambda > 0$.*

Proof. Let $d \in \mathbb{R}^n$ and $\lambda > 0$. Then, we have

$$\begin{aligned} D_M F(\overline{x}, \lambda d) &= \limsup_{t \to 0^+} \frac{F(\overline{x} + \lambda t d) \dot{-} F(\overline{x})}{t} = \limsup_{t \to 0^+} \lambda \frac{F(\overline{x} + \lambda t d) \dot{-} F(\overline{x})}{\lambda t} \\ &= \lambda \limsup_{s \to 0^+} \frac{F(\overline{x} + s d) \dot{-} F(\overline{x})}{s} = \lambda D_M F(\overline{x}, d). \end{aligned}$$ □

Now, we give an existence theorem of M-directional derivative.

Theorem 3.4. *Let $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be compact valued map and $\overline{x} \in \operatorname{int}(\operatorname{dom}(F))$. If there exist a positive constant L and an $\varepsilon > 0$ such that*

$$\big(F(x) \dot{-} F(\overline{x})\big) \cap L\|x - \overline{x}\|\mathbb{B}_p \neq \emptyset \tag{3.1}$$

for all $x \in B_n(\overline{x}, \varepsilon)$, then F is M-directionally differentiable at $\overline{x}$ and $D_M F(\overline{x}, d) \cap L\|d\|\overline{\mathbb{B}}_p \neq \emptyset$ for all $d \in \mathbb{R}^n$.

Proof. Let $d \in \mathbb{R}^p$. There exists a $t_0 > 0$ such that $t_0 d \in B_n(0, \varepsilon)$. If we take $x = \overline{x} + td$, then we get $\big(F(\overline{x} + td) \dot{-} F(\overline{x})\big) \cap Lt\|d\|\mathbb{B}_p \neq \emptyset$ for all $t \in (0, t_0)$ from (3.1). Hence, we obtain

$$\frac{F(\overline{x} + td) \dot{-} F(\overline{x})}{t} \cap L\|d\|\mathbb{B}_p \neq \emptyset$$

for all $t \in (0, t_0)$. Let's consider a net $\{z_t\}_{t \in (0, t_0)}$ such that $z_t \in \frac{F(\overline{x}+td) \dot{-} F(\overline{x})}{t} \cap L\|d\|\mathbb{B}_p$ for all $t \in (0, t_0)$. Because $z_t \in L\|d\|\mathbb{B}_p$ and $L\|d\|\mathbb{B}_p$ is bounded for all $t \in (0, t_0)$, there exists an accumulation point of $\{z_t\}_{t \in (0, t_0)}$, say $\overline{z}$. Since the upper limit of set-valued map contains accumulation points, we get $\overline{z} \in \limsup_{t \to 0^+} \frac{F(\overline{x}+td) \dot{-} F(\overline{x})}{t} = D_M F(\overline{x}, d)$. So, we have $D_M F(\overline{x}, d) \neq \emptyset$. Also, we get $D_M F(\overline{x}, d) \cap L\|d\|\overline{\mathbb{B}}_p \neq \emptyset$. □

The following definition, a generalization of local Lipschitzness, is used to obtain an existence theorem of M-directional derivative.

Definition 3.5. Let $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a compact valued map and $\overline{x} \in \operatorname{int}(\operatorname{dom}(F))$. If there exist an $\varepsilon > 0$ and a positive constant L such that

$$F(x) \dot{-} F(\overline{x}) \subset L\|x - \overline{x}\|\mathbb{B}_p$$

for all $x \in B_n(\overline{x}, \varepsilon)$, then F is called locally upper M-Lipschitz at $\overline{x}$.

Theorem 3.6. *Let $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a compact valued map which is locally upper M-Lipschitz at $\overline{x} \in \operatorname{int}(\operatorname{dom}(F))$. Let $B_n(\overline{x}, \varepsilon)$ be the neighborhood that F is locally upper M-Lipschitz on. If $F(x) \dot{-} F(\overline{x}) \neq \emptyset$ for all $x \in B_n(\overline{x}, \varepsilon)$, then F is M-directionally differentiable at $\overline{x}$.*

Proof. The proof is similar to that of Theorem 3.4. We omit it here. □

Theorem 3.7. *Let $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a compact valued map which is locally upper M-Lipschitz at $\overline{x} \in \operatorname{int}(\operatorname{dom}(F))$. If there exists a $\overline{t} > 0$ such that*

$$F(\overline{x}) \preceq_C^{m_1} F(\overline{x} + td) \tag{3.2}$$

for all $t \in (0, \overline{t})$ and for all $d \in \mathbb{R}^n$, then F is M-directionally differentiable at $\overline{x}$ and $\{0\} \preceq_C^{m_1} D_M F(\overline{x}, d)$.

Proof. Let us choose an arbitrary direction $d \in \mathbb{R}^n$. Since F is M-Lipschitz at $\overline{x}$, there exists a $t_0 > 0$ such that $F(\overline{x} + td) \dot{-} F(\overline{x}) \subset Lt\|d\|\mathbb{B}_p$ for all $t \in (0, t_0)$. Then,

$$\frac{F(\overline{x} + td) \dot{-} F(\overline{x})}{t} \subset L\|d\|\mathbb{B}_p$$

for all $t \in (0, t_0)$. From (3.2) we get $\big(F(\overline{x} + td) \dot{-} F(\overline{x})\big) \cap C \neq \emptyset$ for all $t \in (0, \overline{t})$. Let $t' := \min\{\overline{t}, t_0\}$. Then, we have $\frac{F(\overline{x}+td) \dot{-} F(\overline{x})}{t} \cap C \subset L\|d\|\mathbb{B}_p$ for all $t \in (0, t')$. Consider a net $\{z_t\}_{t \in (0,t')}$ such that $z_t \in \frac{F(\overline{x}+td) \dot{-} F(\overline{x})}{t} \cap C \subset L\|d\|\mathbb{B}_p$ for all $t \in (0, t')$. Since $L\|d\|\mathbb{B}_p$ is bounded, there exists an accumulation point of $\{z_t\}_{t \in (0,t')}$. Assume that $\overline{z}$ is an accumulation point of $\{z_t\}_{t \in (0,t_0)}$. Because the upper limit of set-valued map contains accumulation points and C is closed, we have

$$\overline{z} \in \limsup_{t \to 0^+} \frac{F(\overline{x} + td) \dot{-} F(\overline{x})}{t} \cap C = D_M F(\overline{x}, d) \cap C.$$

Hence, $D_M F(\overline{x}, d) \neq \emptyset$ and $\{0\} \preceq_C^{m_1} D_M F(\overline{x}, d)$ for $d \in \mathbb{R}^n$. Because $d \in \mathbb{R}^n$ is arbitrary, $D_M F(\overline{x}, d) \neq \emptyset$ for all $d \in \mathbb{R}^n$. So F is M-directionally differentiable at $\overline{x}$. □

Remark 3.8. Let $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ compact valued map be locally upper M-Lipschitz at $\overline{x} \in \operatorname{int}(\operatorname{dom}(F))$. If

$$F(\overline{x} + t_1 d) \preceq_C^{m_1} F(\overline{x} + t_2 d)$$

for all $d \in \mathbb{R}^n$ and $t_1, t_2 \in \mathbb{R}$ such that $0 \leq t_1 < t_2$, then F is M-directionally differentiable at $\overline{x}$ and $\{0\} \preceq_C^{m_1} D_M F(\overline{x}, d)$.

Pilecka [26] defined a directional derivative by using ℓ difference in order to obtain optimality conditions of set-valued optimization problem with respect to lower set less order relation. Now, we give relation between this directional derivative and M-directional derivative.

Definition 3.9. [26] Assume that $A, B \subset \mathbb{R}^n$. Then ℓ difference is defined as follows:

$$A \ominus_\ell B := \{x \in \mathbb{R}^n \mid x + B \subset A + C\} = (A + C) \dot{-} B.$$

Remark 3.10. There are following relations between ℓ difference and Minkowski difference:

(i) An ordering cone is required to define ℓ difference. However, Minkowski difference is defined without using any cone.

(ii) $A \dot{-} B \subset A \ominus_\ell B$.

(iii) If A, B are compact, convex and nonempty (convex bodies), we have $(A \dot{-} B) + C \subset A \ominus_\ell B$. But $A \ominus_\ell B \not\subset (A \dot{-} B) + C$. For example, let $A = \{(0,0)\}$ and $B = [0,2] \times [0,2]$. Then, we obtain $C = A \ominus_\ell B \not\subset (A \dot{-} B) + C = \emptyset$.

(iv) Although Minkowski difference of two bounded sets is bounded, ℓ difference of two sets are either empty or unbounded. Also if ℓ difference of sets is empty from (ii) Minkowski difference is empty.

Definition 3.11. [26] Let set valued map $F \colon \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be compact, convex valued. The directional derivative of F at $x \in \operatorname{int}(\operatorname{dom}(F))$ in the direction $d \in \mathbb{R}^n$ is defined as

$$DF(x,d) := \limsup_{t \to 0^+} \frac{F(x+td) \ominus_\ell F(x)}{t}.$$

Remark 3.12. Convexity and an order cone are necessary to obtain directional derivative via ℓ difference from Remark 3.10(i). Also, this directional derivative is larger than M-directional derivative i.e., $D_M F(x,d) \subset DF(x,d)$ from Remark 3.10(ii).

Now, we calculate M-directional derivative and a directional derivative given by Pilecka [26] of a set-valued map on an example.

Example 3.13. Let $C = \mathbb{R}^2_+$ and a set-valued map $F \colon \mathbb{R} \rightrightarrows \mathbb{R}^2$ be defined as

$$F(x) = \operatorname{conv}\{(x,x), (x+1,x)\}$$

for all $x \in \mathbb{R}$. Some image sets are seen in Figure 3.2. Then, we have

$$\begin{aligned}
D_M F(x,d) &= \limsup_{t \to 0^+} \frac{F(x+td) \dot{-} F(x)}{t} = \limsup_{t \to 0^+} \frac{\{(td,td)\}}{t} \\
&= \limsup_{t \to 0^+} \{(d,d)\} = \{(d,d)\} = \{d(1,1)\}, \\
DF(x,d) &= \limsup_{t \to 0^+} \frac{F(x+td) \ominus_\ell F(x)}{t} = \limsup_{t \to 0^+} \frac{\{(td,td)\} + C}{t} \\
&= \limsup_{t \to 0^+} \{(d,d)\} + C = \{(d,d)\} + C = [d,\infty] \times [d,\infty].
\end{aligned}$$

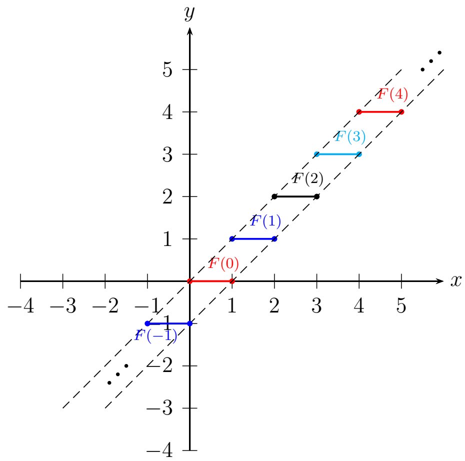

Figure 3.2: Some image sets of $F(x) = \operatorname{conv}\{(x, x), (x + 1, x)\}$.

4. Optimality conditions via M-directional derivative

In this section, we obtain necessary and sufficient optimality conditions for $(m_1 - \mathrm{SOP})$ via M-directional derivative.

A sufficient condition for strictly minimal solutions of $(m_1 - \mathrm{SOP})$ is given in the following theorem.

Theorem 4.1. *Let $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a compact valued map, $\overline{x} \in \operatorname{int}(\operatorname{dom}(F))$. If there exist an $\varepsilon > 0$ and a positive constant L such that*

$$L\|x - \overline{x}\|\mathbb{B}_p \subset F(x) \dot{-} F(\overline{x}) \tag{4.1}$$

for all $x \in B_n(\overline{x}, \varepsilon) \setminus \{\overline{x}\}$, then $\overline{x}$ is strictly minimal solution of $(m_1 - \mathrm{SOP})$ and $\{0\} \prec_C^{m_1} D_M F(\overline{x}, d)$ for all $d \in \mathbb{R}^n \setminus \{0\}$.

Proof. Since $L\|x-\overline{x}\|\mathbb{B}_p \subset F(x) \dot{-} F(\overline{x})$ for all $x \in B_n(\overline{x}, \varepsilon)\setminus\{\overline{x}\}$, we have $\big(F(x) \dot{-} F(\overline{x})\big) \cap \operatorname{int}(C) \neq \emptyset$ for all $x \in B_n(\overline{x}, \varepsilon) \setminus \{\overline{x}\}$. Then, $F(\overline{x}) \prec_C^{m_1} F(x)$ for all $x \in B_n(\overline{x}, \varepsilon) \setminus \{\overline{x}\}$. Therefore, $\overline{x}$ is strictly minimal solution of $(m_1 - \mathrm{SOP})$.

Let us choose an arbitrary $d \in \mathbb{R}^n$. From (4.1), there exists a $t_0 > 0$ such that

$$Lt\|d\|\mathbb{B}_p \subset F(\overline{x} + td) \dot{-} F(\overline{x}) \tag{4.2}$$

for all $t \in (0, t_0)$. From (4.2) we have

$$L\|d\|\mathbb{B}_p \subset \frac{F(\overline{x} + td) \dot{-} F(\overline{x})}{t}. \tag{4.3}$$

Let $z \in L\|d\|\mathbb{B}_p$ be an arbitrary element. If we set $z_t = z$ for all $t \in (0, t_0)$ from (4.3) we get $z \in D_M F(\overline{x}, d)$. So, we have $L\|d\|\mathbb{B}_p \subset D_M F(\overline{x}, d)$. Thus we obtain $D_M F(\overline{x}, d) \cap \operatorname{int}(C) \neq \emptyset$ and $\{0\} \prec_C^{m_1} D_M F(\overline{x}, d)$. □

Now, we give an illustrative example for Theorem 4.1.

Example 4.2. Let a set-valued map $F\colon \mathbb{R} \rightrightarrows \mathbb{R}^2$ be defined as

$$F(x) = \{(a,b) \in \mathbb{R}^2 \mid a^2 + b^2 \leq x^2 + 1\}$$

for all $x \in \mathbb{R}$ and consider set-valued optimization problem

$$(m_1\text{– SOP}) \qquad \min F(x), \quad x \in \mathbb{R}.$$

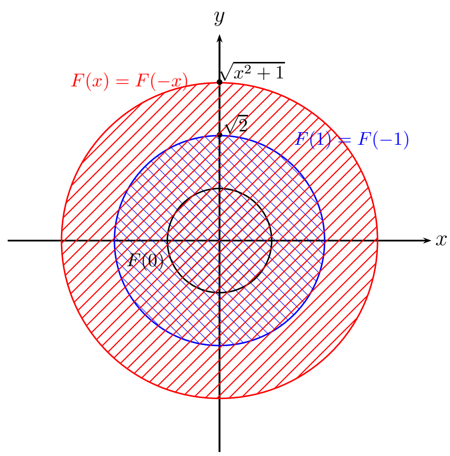

Figure 4.1: Some image sets of $F(x) = \{(a,b) \in \mathbb{R}^2 \mid a^2 + b^2 \leq x^2 + 1\}$.

As seen in Figure 4.1, $F(x)$ is compact valued for all $x \in \mathbb{R}$ and $0 \in \operatorname{int}(\operatorname{dom}(F))$. For $L \in (0,1]$ and for all $\varepsilon > 0$ we have

$$L\|x\|\mathbb{B}_2 \subset F(x)$$

for all $x \in B(0,\varepsilon) \setminus \{0\}$. Hence, the conditions of Theorem 4.1 are satisfied. Then, 0 is strictly minimal solution of $(m_1\text{– SOP})$. Also, we can see $\{(0,0)\} \prec_C^{m_1} D_M F(0,d)$ for all $d \in \mathbb{R} \setminus \{0\}$:

$$\begin{aligned} D_M F(0,d) &= \limsup_{t \to 0^+} \frac{F(0+td) \dot{-} F(0)}{t} \\ &= \limsup_{t \to 0^+} \frac{F(td)}{t} \\ &= \limsup_{t \to 0^+} \frac{\{(a,b) \in \mathbb{R}^2 \mid a^2 + b^2 \leq (td)^2\}}{t} \\ &= \limsup_{t \to 0^+} \{(a,b) \in \mathbb{R}^2 \mid a^2 + b^2 \leq d^2\} \\ &= B_2((0,0),|d|) \end{aligned}$$

for $d \in \mathbb{R} \setminus \{0\}$. Then, we get $D_M F(0,d) \cap \operatorname{int}(C) \neq \emptyset$.

A necessary condition for local maximal solution of $(m_1-\mathrm{SOP})$ is given in the following theorem.

Theorem 4.3. *Let a set-valued map* $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ *be compact valued and* $\overline{x} \in \operatorname{int}(\operatorname{dom}(F))$. *If* $\overline{x}$ *is a local maximal solution of* $(m_1-\mathrm{SOP})$ *and* F *is* M*-directionally differentiable at* $\overline{x}$, *then* $\{0\} \not\prec_C^{m_1} D_M F(\overline{x}, d)$ *for all* $d \in \mathbb{R}^n$.

Proof. Because $\overline{x}$ is a local maximal solution of $(m_1-\mathrm{SOP})$, there exists an $\varepsilon > 0$ such that $F(\overline{x}) \not\preceq_C^{m_1} F(x)$ for all $x \in B_n(\overline{x}, \varepsilon) \setminus \{\overline{x}\}$. Let us choose $d \in \mathbb{R}^n$. Then, there exists a $t_0 > 0$ such that $\overline{x} + td \in B_n(\overline{x}, \varepsilon)$ for all $t \in (0, t_0)$. Hence, we have $F(\overline{x}) \not\preceq_C^{m_1} F(\overline{x} + td)$ for all $t \in (0, t_0)$. We get $\big(F(\overline{x}+td) \dot{-} F(\overline{x})\big) \cap C = \emptyset$ from the definition of $\preceq_C^{m_1}$. Because C is cone, we obtain

$$\left(\frac{F(\overline{x}+td) \dot{-} F(\overline{x})}{t}\right) \cap C = \emptyset$$

and $\limsup_{t\to 0^+} \frac{F(\overline{x}+td) \dot{-} F(\overline{x})}{t} \cap \operatorname{int}(C) = \emptyset$. Then we have $D_M F(\overline{x}, d) \cap \operatorname{int}(C) = \emptyset$ i.e., $\big(D_M F(\overline{x}, d) \dot{-} \{0\}\big) \cap \operatorname{int}(C) = \emptyset$. So, $\{0\} \not\prec_C^{m_1} D_M F(\overline{x}, d)$. □

A sufficient condition for local maximal solution of $(m_1-\mathrm{SOP})$ is given in the following theorem.

Theorem 4.4. *Let a set-valued map* $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ *be compact valued and* M*-directionally differentiable at* $\overline{x} \in \operatorname{int}(\operatorname{dom}(F))$. *Let* $d \in \mathbb{R}^n$ *satisfy*

$$\emptyset \neq \limsup_{k\to\infty} \frac{F(\overline{x}+t_k d_k) \dot{-} F(\overline{x})}{t_k} \cap C \subset D_M F(\overline{x}, d), \tag{4.4}$$

where d_k *and* t_k *are sequence such that* $d_k \to d$ *and* $t_k \to 0^+$. *If* $\{0\} \not\preceq_C^{m_1} D_M F(\overline{x}, d)$ *for all* $d \in \mathbb{R}^n$, *then* $\overline{x}$ *is a local maximal solution of* $(m_1-\mathrm{SOP})$.

Proof. Since $\{0\} \not\preceq_C^{m_1} D_M F(\overline{x}, d)$ for all $d \in \mathbb{R}^n$, we have $\big(D_M F(\overline{x}, d) \dot{-} \{0\}\big) \cap C = \emptyset$, i.e.,

$$D_M F(\overline{x}, d) \cap C = \emptyset. \tag{4.5}$$

Assume that $\overline{x}$ isn't a local maximal solution of $(m_1-\mathrm{SOP})$. Then, there exists a sequence $x_k \in B_n(\overline{x}, 1/k)$ such that $F(\overline{x}) \preceq_C^{m_1} F(x_k)$ for all $k \in \mathbb{N}$. Also $x_k \to \overline{x}$. So, there exists a $\{d_k\} \subset \mathbb{R}^n$ with $\|d_k\| = 1$ and $t_k \to 0^+$ with $x_k = \overline{x} + t_k d_k$ for all $k \in \mathbb{N}$. Because $\{d_k\}$ is bounded, it has a convergent subsequence. Without loss of generality say $d_k \to \overline{d}$. We obtain $F(\overline{x}) \preceq_C^{m_1} F(\overline{x} + t_k d_k)$ i.e.,

$$\left(\frac{F(\overline{x}+t_k d_k) \dot{-} F(\overline{x})}{t_k}\right) \cap C \neq \emptyset$$

for all $k \in \mathbb{N}$. Therefore, we get $D_M F(\overline{x}, \overline{d}) \cap C \neq \emptyset$ from (4.4). This contradicts to (4.5). Thus, $\overline{x}$ is local maximal solution of $(m_1-\mathrm{SOP})$. □

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