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Attractors for a Class of Kirchhoff Models with p-Laplacian and Time Delay

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Abstract. This paper is concerned with a class of Kirchhoff models with time delay and perturbation of p-Laplacian type

$$u_{tt}(x,t) + \Delta^2 u(x,t) - \Delta_p u(x,t) - a_0 \Delta u_t(x,t) + a_1 u_t(x,t-\tau) + f(u(x,t)) = g(x),$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the usual p-Laplacian operator. Many researchers have studied well-posedness and decay rates of energy for these equations without delay effects. But, there are not many studies on attractors for other delayed systems. Thus we establish the existence of global attractors and the finite dimensionality of the attractors by establishing some functionals which are related to the norm of the phase space to our problem.

1. Introduction

We consider the following Kirchhoff models with time delay and perturbation of p-Laplacian type

$$u_{tt} + \Delta^2 u - \Delta_p u - a_0 \Delta u_t + a_1 u_t(x,t-\tau) + f(u) = g(x) \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1.1}$$

$$u = \Delta u = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \tag{1.2}$$

$$u(0) = u_0, \quad u_t(0) = u_1 \quad \text{on } \Omega, \tag{1.3}$$

$$u_t(x,t) = j_0(x,t) \quad \text{for } (x,t) \in \Omega \times (-\tau, 0), \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. $a_0 > 0$, $a_1 \in \mathbb{R}$, $\tau > 0$ is time delay, and $g \in L^2(\Omega)$.

This model can be regarded as a fourth order viscoelastic plate equation with a lower order perturbation of p-Laplacian type and is related to one-dimensional nonlinear equation of elastoplastic microstructure flows given by

$$u_{tt} + u_{xxxx} - a(u_x^2)_x = 0. \tag{1.5}$$

As a general form of (1.5), many authors [8,14,16,17] studied the following equation with appropriate boundary and initial conditions:

$$u_{tt} + \alpha \Delta^2 u - \Delta_p u - \Delta u_t + h(u_t) + f(u) = g(x). \tag{1.6}$$

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Ma and Soriano [8] proved the global existence and decay of the solutions when $\alpha = h(u_t) = 0$ in (1.6). Park et al. [14] improved the results in [8] by generalizing the assumptions on h, that is, the function h is assumed to be a discontinuous and nonlinear multi-valued function.

With respect to plate equations with memory of the form

$$u_{tt} + \Delta^2 u - \Delta_p u + \int_0^t \mu(t-s)\Delta u^2(s)\, ds - \Delta u_t = 0,$$

the authors of [1] proved the existence result and established the exponential decay rate when the relaxation function μ decays at the same rate. The interaction of the memory term with p-Laplacian operator was first considered by them. Later, Park [12] obtained general decay rate by weakening the conditions of kernel function μ.

It is well known that the strong damping $-\Delta u_t$ plays an important role to obtain global well-posedness and uniqueness of solutions due to the presence of the p-Laplacian. Most recently, Jorge silva et al. [6] studied the nonlinear viscoelastic Kirchhoff plate equation of the form

$$u_{tt} - \Delta u_{tt} + \alpha\Delta^2 u - \operatorname{div}(F(|\nabla u|^2)\nabla u) - \int_0^t \mu(t-s)\Delta^2 u(s)\, ds = 0, \tag{1.7}$$

where $F\colon \mathbb{R}^N \to \mathbb{R}^N$ is a vector field satisfying some conditions, here $|\nabla u|^{p-2}$ is an example of $F(|\nabla u|)$. Dropping the strong damping, they proved the existence and uniqueness of stronger solutions in the presence of the rotational inertia term $-\Delta u_{tt}$, which gives the regularity of solutions and hence enables to control the the p-Laplacian term, and showed the memory component is enough to decay the energy of solutions. But, the energy of solutions is stationary when $\mu = 0$ in (1.7). So, it will be needed additional dissipation to investigate energy decay rates or the long time dynamics in terms of attractors.

The aim of this work is to investigate the existence of attractors and finite dimensionality of the attractors for the Kirchhoff equations (1.1)–(1.4) with p-Laplacian and time-delay and without the memory.

Since time delays arise in many applications depending not only on the present state but also on some past occurrences and the presence of delay may be a source of instability, partial differential equations with time delay effects have become an active area of research (see [3, 9–11] and references therein). Nicaise and Pignotti [10] investigated the wave equation with time delay

$$u_{tt}(x,t) - \Delta u(x,t) + a_0 u_t(x,t) + a_1 u_t(x, t-\tau) = 0.$$

They proved that the energy of the problem decays exponentially under the condition $0 < a_1 < a_0$. And then they extended the result to the time varying delay case in [11].

For the related works of equations with time delay, we also refer [7,13,16] and references therein.

It is worth mentioning that there are not much literature on attractors for delayed systems. Furthermore, as far as we are concerned, this is the first work in the literature that takes into account the global attractors for Kirchhoff models with p-Laplacian and time delay. To obtain our desired results, we establish some functionals which are related to the norm of the phase space to problem (1.1)–(1.4).

The outline of this paper is as follows. In Section 2, we give some notations and material needed for our work. In Section 3, we prove the existence of attractors for problem (1.1)–(1.4). Finally, in Section 4, we examine the finite dimensionality of the attractors.

2. Preliminaries

We denote the inner product in $L^2(\Omega)$ by $(\cdot\,,\cdot)$ and the usual norm of $L^p(\Omega)$ by $\|\cdot\|_p$. For simplicity, we denote $\|\cdot\|_2$ by $\|\cdot\|$. For a Banach space X, $\|\cdot\|_X$ denotes the norm of X. Let λ and $\widetilde{\lambda}$ be the best constants satisfying $\lambda\|u\|^2 \leq \|\Delta u\|^2$ for $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\widetilde{\lambda}\|u\|^2 \leq \|\nabla u\|^2$ for $u \in H_0^1(\Omega)$, respectively.

We present the precise hypotheses to problem (1.1)–(1.4).

(H1) We assume that p satisfy

$$2 \leq p \leq \frac{2N-2}{N-2} \ \text{ if } N \geq 3 \quad \text{and} \quad p \geq 2 \ \text{ if } N = 1,2.$$

This condition guarantees

$$H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow W_0^{1,2(p-1)}(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega).$$

(H2) The forcing term $f\colon \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(u) - f(\widetilde{u})| \leq l(1 + |u|^m + |\widetilde{u}|^m)|u - \widetilde{u}| \quad \text{for } u, \widetilde{u} \in \mathbb{R}, \tag{2.1}$$

$$-l_0 \leq F(u) \leq f(u)u \quad \text{for } u \in \mathbb{R}, \tag{2.2}$$

here $F(u) = \int_0^u f(s)\,ds$, $l > 0$, $l_0 > 0$, and

$$0 < m \leq \frac{4}{N-4} \ \text{ if } N \geq 5 \quad \text{and} \quad m > 0 \ \text{ if } 1 \leq N \leq 4. \tag{2.3}$$

The condition (2.3) ensures that $H^2(\Omega) \hookrightarrow L^{2(m+1)}(\Omega)$.

(H3) $g \in L^2(\Omega)$ and $j_0 \in L^2(\Omega \times (-\tau, 0))$.

(H4) The coefficients a_0 and a_1 satisfy

$$0 < |a_1| < a_0\widetilde{\lambda}.$$

2.1. Well-posedness

As in [10], we define the function z as

$$z(x,\rho,t)=u_t(x,t-\rho\tau) \quad \text{for } (x,\rho,t)\in\Omega\times(0,1)\times(0,\infty).$$

Then problem (1.1)–(1.4) is equivalent to

$$u_{tt}+\Delta^2 u-\Delta_p u-a_0\Delta u_t+a_1 z(x,1,t)+f(u)=g(x) \quad \text{on } \Omega\times\mathbb{R}^+, \tag{2.4}$$

$$\tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0 \quad \text{for } (x,\rho,t)\in\Omega\times(0,1)\times(0,\infty), \tag{2.5}$$

$$u(x,t)=\Delta u(x,t)=0 \quad \text{for } (x,t)\in\partial\Omega\times\mathbb{R}^+, \tag{2.6}$$

$$u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x) \quad \text{for } x\in\Omega, \tag{2.7}$$

$$z(x,\rho,0)=j_0(x,-\rho\tau):=z_0(x,\rho) \quad \text{for } (x,\rho)\in\Omega\times(0,1). \tag{2.8}$$

Let

$$V_0=L^2(\Omega), \quad V_1=H_0^1(\Omega), \quad V_2=H^2(\Omega)\cap H_0^1(\Omega)$$

and define the phase space

$$\mathcal{H}=V_2\times V_0\times L^2(\Omega\times(0,1))$$

equipped with the norm

$$\|(u,v,z)\|_{\mathcal{H}}^2=\|\Delta u\|^2+\|v\|^2+\|z\|_{L^2(\Omega\times(0,1))}^2.$$

We now state the well-posedness result which can be established by combining the arguments of [4, 15].

Theorem 2.1. *Assume that* (H1), (H2), (H3) *hold. Then we have:*

(i) *For every* $(u_0,u_1,z_0)\in\mathcal{H}$ *and* $T>0$, *there exists a weak solution* (u,u_t,z) *of problem* (2.4)–(2.8) *in the class*

$$u\in L^\infty(0,T;V_2), \quad u_t\in L^\infty(0,T;V_0)\cap L^2(0,T;V_1), \quad z\in L^\infty(0,T;L^2(\Omega\times(0,1)))$$

satisfying $(u,u_t,z)\in C([0,T];\mathcal{H})$. *Moreover, the solution is unique and depends continuously on the initial data* $(u_0,u_1,z_0)\in\mathcal{H}$ *and* $g\in L^2(\Omega)$.

(ii) *Let* (u,u_t,z) *and* $(\widetilde{u},\widetilde{u}_t,\widetilde{z})$ *be two weak solutions of problem* (2.4)–(2.8) *corresponding to initial data* (u_0,u_1,z_0) *and* $(\widetilde{u}_0,\widetilde{u}_1,\widetilde{z}_0)$, *respectively. Then one gets*

$$\|(u,u_t,z)-(\widetilde{u},\widetilde{u}_t,\widetilde{z})\|_{\mathcal{H}}\le e^{ct}\|(u_0,u_1,z_0)-(\widetilde{u}_0,\widetilde{u}_1,\widetilde{z}_0))\|_{\mathcal{H}} \quad \textit{for some } c>0.$$

2.2. A short overview on infinite-dimensional dynamical systems

To study the existence of attractors for problem (2.4)–(2.8) and the finite dimensionality of the attractors, we present some basic concepts and abstract results on dynamical systems given by the Chueshov and Lasiecka's book [2].

Let $\mathcal{F}$ be a Banach space and B be a bounded subset of $\mathcal{F}$. We call a function $\phi(\,\cdot\,,\cdot\,)$ which defined on $\mathcal{F}\times\mathcal{F}$ is *a contractive function on* $B\times B$ if for any sequence $\{x_n\}_{n=1}^{\infty}\subset B$, there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}\subset\{x_n\}_{n=1}^{\infty}$ such that

$$\liminf_{k\to\infty}\liminf_{l\to\infty}\phi(x_{n_k},x_{n_l})=0.$$

Theorem 2.2. [2, Theorem 7.1.11] *Let $\{S(t)\}_{t\geq 0}$ be a semigroup on a Banach space $(\mathcal{F},\|\cdot\|_{\mathcal{F}})$ and have a bounded absorbing set B_0. Assume that for any $\epsilon>0$ there exist $T=T(B_0,\epsilon)$ and a contractive function $\phi_T(\,\cdot\,,\cdot\,)$ on $B_0\times B_0$ such that*

$$\|S(T)x-S(T)y\|_{\mathcal{F}}\leq\epsilon+\phi_T(x,y)\quad\textit{for all } x,y\in B_0,$$

where ϕ_T depends on T. Then $S(t)$ is asymptotically smooth in $\mathcal{F}$.

Theorem 2.3. [2, Theorem 7.2.4] *A dissipative dynamical system $(S(t),\mathcal{F})$ has a compact global attractor if and only if it is asymptotically smooth.*

Let X, Y, Z be three reflexive Banach spaces with X compactly embedded in Y, $\mathcal{F}=X\times Y\times Z$, and $(S(t),\mathcal{F})$ a dynamical system given by an evolution operator

$$S(t)x=(u(t),u_t(t),z(t))\quad\text{for } x=(u_0,u_1,z_0)\in\mathcal{F},\tag{2.9}$$

where the functions u has regularity

$$u\in C(\mathbb{R}^+;X)\cap C(\mathbb{R}^+;Y),\quad z\in C(\mathbb{R}^+;Z).\tag{2.10}$$

We call the dynamical system $(S(t),\mathcal{F})$ is *quasi-stable on* $B\subset\mathcal{F}$ if there exists a compact seminorm n_X on X and nonnegative scalar function $a(t)$ and $c(t)$, locally bounded in $[0,\infty)$, and $b(t)\in L^1(\mathbb{R}^+)$ with $\lim_{t\to\infty}b(t)=0$ such that

$$\|S(t)x-S(t)y\|_{\mathcal{F}}^2\leq a(t)\|x-y\|_{\mathcal{F}}^2\tag{2.11}$$

and

$$\|S(t)x-S(t)y\|_{\mathcal{F}}^2\leq b(t)\|x-y\|_{\mathcal{F}}^2+c(t)\sup_{0<s<t}[n_X(u(s)-\widetilde{u}(s))]^2,\tag{2.12}$$

where $S(t)x=(u(t),u_t(t),z(t))$, $S(t)y=(\widetilde{u}(t),\widetilde{u}_t(t),\widetilde{z}(t))$ and $x,y\in B$.

Theorem 2.4. [2, Theorem 7.9.6] *Let $(S(t),\mathcal{F})$ be given by (2.9) and satisfy (2.10). If $(S(t),\mathcal{F})$ has a compact global attractor $\mathcal{A}$ and is quasi-stable on $\mathcal{A}$, then the attractor $\mathcal{A}$ has finite fractional dimension.*

3. Existence of attractors

In this section we prove the existence of global attractors for problem (2.4)–(2.8) by applying Theorem 2.3. For this, let us define a map $S(t)\colon \mathcal{H} \to \mathcal{H}$ by

$$S(t)(u_0, u_1, z_0) = (u(t), u_t(t), z(t)), \tag{3.1}$$

where $(u(t), u_t(t), z(t))$ is the unique weak solution of system (2.4)–(2.8) corresponding to initial data (u_0, u_1, z_0). Then, by Theorem 2.1, $\{S(t)\}_{t\geq 0}$ is a C^0-semigroup on $\mathcal{H}$.

To obtain our desired result, we need to show that the dynamical system given in (3.1) is dissipative and asymptotically smooth. Inspired by [11], let us define the energy of problem (2.4)–(2.8) by

$$\begin{aligned} E(t) = {} & \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2 + \frac{1}{p}\|\nabla u(t)\|_p^p + \int_\Omega F(u(t))\,dx \\ & - (g, u(t)) + \frac{\xi}{2}\int_{t-\tau}^{t} e^{\theta(s-t)}\|u_t(s)\|^2\,ds, \end{aligned} \tag{3.2}$$

where

$$|a_1| < \xi < 2a_0\widetilde{\lambda} - |a_1| \quad \text{and} \quad 0 < \theta < \frac{1}{\tau}\ln\frac{\xi}{|a_1|}. \tag{3.3}$$

The lemma below states the relation between the norm of phase space and the energy, which plays an important role in the process of our work.

Lemma 3.1. *There exists a positive constant c_0 such that*

$$\|(u, u_t, z)\|_{\mathcal{H}}^2 \leq c_0\left(E(t) + l_0|\Omega| + \frac{\|g\|^2}{\lambda}\right). \tag{3.4}$$

Proof. Integration by substitution $s = t - \rho\tau$ implies that

$$\begin{aligned} \frac{\xi}{2}\int_{t-\tau}^{t} e^{\theta(s-t)}\|u_t(s)\|^2\,ds &= -\frac{\xi\tau}{2}\int_1^0\int_\Omega e^{-\theta\rho\tau}u_t^2(x, t-\rho\tau)\,dxd\rho \\ &= \frac{\xi\tau}{2}\int_0^1\int_\Omega e^{-\theta\rho\tau}z^2(x,\rho,t)\,dxd\rho. \end{aligned} \tag{3.5}$$

Young's inequality and (2.2) give

$$\int_\Omega F(u)\,dx - (g, u) \geq -l_0|\Omega| - \frac{1}{4}\|\Delta u\|^2 - \frac{\|g\|^2}{\lambda}. \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.2), one sees that

$$E(t) \geq \frac{1}{2}\|u_t\|^2 + \frac{1}{4}\|\Delta u\|^2 + \frac{1}{p}\|\nabla u\|_p^p - l_0|\Omega| - \frac{\|g\|^2}{\lambda} + \frac{\xi\tau}{2}\int_0^1\int_\Omega e^{-\theta\tau\rho}z^2(x,\rho,t)\,dxd\rho$$

$$
\begin{aligned}
&\geq \frac{1}{2}\|u_t\|^2 + \frac{1}{4}\|\Delta u\|^2 + \frac{1}{p}\|\nabla u\|_p^p - l_0|\Omega| - \frac{\|g\|^2}{\lambda} + \frac{\xi\tau e^{-\theta\tau}}{2}\int_0^1\int_\Omega z^2(x,\rho,t)\,dxd\rho \\
&\geq \frac{1}{c_0}\left(\|u_t\|^2 + \|\Delta u\|^2 + \|z\|^2_{L^2(\Omega\times(0,1))}\right) - l_0|\Omega| - \frac{\|g\|^2}{\lambda} \\
&= \frac{1}{c_0}\|(u,u_t,z)\|_{\mathcal{H}} - l_0|\Omega| - \frac{\|g\|^2}{\lambda},
\end{aligned}
$$

where $1/c_0 := \min\{1/4, \xi\tau e^{-\theta\tau}/2\}$. □

Lemma 3.2. *Assume that* (H1)–(H4) *hold. Then there exists positive constants c_1 and c_2 satisfying*

$$
E'(t) \leq -c_1\|\nabla u_t(t)\|^2 - c_2\|z(1,t)\|^2 - \frac{\theta\xi}{2}\int_{t-\tau}^t e^{\theta(s-t)}\|u_t(s)\|^2\,ds.
$$

Proof. Multiplying (2.4) by u_t integrating over $x \in \Omega$, we get

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2 + \frac{1}{p}\|\nabla u(t)\|_p^p + \int_\Omega F(u(t))\,dx - (g,u(t))\right) \\
&= -a_0\|\nabla u_t(t)\|^2 - a_1(z(1,t),u_t(t)).
\end{aligned}
$$

From this, (3.2), and the following estimate

$$
\frac{d}{dt}\int_{t-\tau}^t e^{\theta(s-t)}\|u_t(s)\|^2\,ds = -\theta\int_{t-\tau}^t e^{\theta(s-t)}\|u_t(s)\|^2\,ds + \|u_t(t)\|^2 - e^{-\theta\tau}\|u_t(t-\tau)\|^2,
$$

we see that

$$
\begin{aligned}
E'(t) = &-a_0\|\nabla u_t(t)\|^2 - a_1(z(1,t),u_t(t)) - \frac{\theta\xi}{2}\int_{t-\tau}^t e^{\theta(s-t)}\|u_t(s)\|^2\,ds \\
&+ \frac{\xi}{2}\|u_t(t)\|^2 - \frac{\xi e^{-\theta\tau}}{2}\|u_t(t-\tau)\|^2.
\end{aligned}
$$

This and Young's inequality give

$$
\begin{aligned}
E'(t) \leq &-\left(a_0 - \frac{|a_1|}{2\widetilde{\lambda}} - \frac{\xi}{2\widetilde{\lambda}}\right)\|\nabla u_t(t)\|^2 - \left(\frac{\xi e^{-\theta\tau}}{2} - \frac{|a_1|}{2}\right)\|z(1,t)\|^2 \\
&- \frac{\theta\xi}{2}\int_{t-\tau}^t e^{\theta(s-t)}\|u_t(s)\|^2\,ds.
\end{aligned}
\tag{3.7}
$$

Letting $c_1 := a_0 - |a_1|/(2\widetilde{\lambda}) - \xi/(2\widetilde{\lambda})$ and $c_2 := \xi e^{-\theta\tau}/2 - |a_1|/2$, which are positive owing to (3.3), we complete the proof. □

Now, define the perturbed functional by

$$
L(t) = E(t) + \epsilon\Phi(t),
$$

where $\Phi(t) = (u_t(t), u(t))$, then one sees from Young's inequality and (3.4) that

$$|\Phi(t)| \le \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2\lambda}\|\Delta u(t)\|^2 \le c_0 \max\left\{\frac{1}{2}, \frac{1}{2\lambda}\right\}\left(E(t) + l_0|\Omega| + \frac{\|g\|^2}{\lambda}\right).$$

So, it holds that

$$|L(t) - E(t)| \le \epsilon c\left(E(t) + l_0|\Omega| + \frac{\|g\|^2}{\lambda}\right),$$

here and in the sequel c denotes a generic positive constant different from line to line or even in the same line. Choosing $\epsilon > 0$ small, we deduce that

$$\alpha_1 E(t) - c_3\left(l_0|\Omega| + \frac{\|g\|^2}{\lambda}\right) \le L(t) \le \alpha_2 E(t) + c_3\left(l_0|\Omega| + \frac{\|g\|^2}{\lambda}\right) \quad \text{for } t \ge 0, \tag{3.8}$$

where $\alpha_1 = 1 - \epsilon c$, $\alpha_2 = 1 + \epsilon c$, and $c_3 = \epsilon c$.

Lemma 3.3. *Assume the conditions* (H1)–(H4) *hold. Then, the semigroup* $\{S(t)\}_{t\ge0}$ *defined by* (3.1) *has a bounded absorbing set in* $\mathcal{H}$.

Proof. We have from (2.4) that

$$\begin{aligned}\Phi'(t) &= \|u_t(t)\|^2 - \|\Delta u(t)\|^2 - \|\nabla u(t)\|_p^p + a_0(u_t(t), \Delta u(t)) \\ &\quad - a_1(u(t), z(1,t)) - (f(u(t)), u(t)) + (g, u(t)).\end{aligned} \tag{3.9}$$

Young's inequality gives that

$$a_0(u_t(t), \Delta u(t)) \le a_0^2\|u_t(t)\|^2 + \frac{1}{4}\|\Delta u(t)\|^2,$$
$$-a_1(u(t), z(1,t)) \le \frac{1}{4}\|\Delta u(t)\|^2 + \frac{a_1^2}{\lambda}\|z(1,t)\|^2.$$

Substituting these into (3.9) and applying (2.2), we find

$$\Phi'(t) \le c_4\|u_t(t)\|^2 - \frac{1}{2}\|\Delta u(t)\|^2 - \|\nabla u(t)\|_p^p + c_5\|z(1,t)\|^2 - \int_\Omega F(u)\,dx + (g, u(t)), \tag{3.10}$$

where $c_4 = 1 + a_0^2$ and $c_5 = a_1^2/\lambda$. Thanks to Lemma 3.2 and (3.10), we see that

$$\begin{aligned}L'(t) &\le -c_1\|\nabla u_t(t)\|^2 - c_2\|z(1,t)\|^2 - \frac{\theta\xi}{2}\int_{t-\tau}^t e^{\theta(s-t)}\|u_t(s)\|^2\,ds + \epsilon c_4\|u_t(t)\|^2 \\ &\quad - \frac{\epsilon}{2}\|\Delta u(t)\|^2 - \epsilon\|\nabla u(t)\|_p^p + \epsilon c_5\|z(1,t)\|^2 - \epsilon\int_\Omega F(u)\,dx + \epsilon(g, u(t)) \\ &\le -\left(c_1 - \frac{2\epsilon c_4}{\widetilde{\lambda}}\right)\|\nabla u_t(t)\|^2 - \epsilon c_4\|u_t(t)\|^2 - \frac{\epsilon}{2}\|\Delta u(t)\|^2 - \epsilon\|\nabla u(t)\|_p^p \\ &\quad - \epsilon\int_\Omega F(u)\,dx + \epsilon(g, u(t)) - \frac{\theta\xi}{2}\int_{t-\tau}^t e^{\theta(s-t)}\|u_t(s)\|^2\,ds - (c_2 - \epsilon c_5)\|z(1,t)\|^2.\end{aligned}$$

Choosing $\epsilon > 0$ small enough such that $c_1 - 2\epsilon c_4/\widetilde{\lambda} > 0$, $c_2 - \epsilon c_5 > 0$, we deduce that

$$L'(t) \leq -\alpha_3 E(t) \quad \text{for some } \alpha_3 > 0.$$

From this, (3.8), and (3.4) we infer that

$$\|(u(t), u_t(t), z)\|_{\mathcal{H}}^2 \leq \frac{c_0\alpha_2}{\alpha_1} E(0)e^{-\alpha_3 t/\alpha_2} + c_0\left(\frac{2c_3}{\alpha_1} + 1\right)\left(l_0|\Omega| + \frac{\|g\|^2}{\lambda}\right).$$

This shows that any closed ball $B_0 = \overline{B}(0, R)$ with $R > \sqrt{c_0(2c_3/\alpha_1 + 1)(l_0|\Omega| + \|g\|^2/\lambda)}$ is a bounded absorbing set of $(S(t), \mathcal{H})$. □

Lemma 3.4. *Assume the conditions* (H1)–(H4) *hold. Let B_0 be a bounded absorbing set obtained in Lemma* 3.3*, $S(t)y_0 = (u, u_t, z)$ and $S(t)\widetilde{y_0} = (\widetilde{u}, \widetilde{u}_t, \widetilde{z})$ be two weak solutions of problem* (2.4)–(2.8) *corresponding to initial data $y_0 = (u_0, u_1, z_0) \in B_0$ and $\widetilde{y_0} = (\widetilde{u}_0, \widetilde{u}_1, \widetilde{z}_0) \in B_0$, respectively. Then,*

$$\begin{aligned} &\|S(t)y_0 - S(t)\widetilde{y_0}\|_{\mathcal{H}}^2 \\ &\leq ce^{-\omega t}\|y_0 - \widetilde{y_0}\|_{\mathcal{H}}^2 \\ &\quad + C(B_0)\int_0^t e^{-\omega(t-s)}\left(\|\nabla u(s) - \nabla\widetilde{u}(s)\|_{2(p-1)}^2 + \|u(s) - \widetilde{u}(s)\|_{2(m+1)}^2\right) ds, \end{aligned} \tag{3.11}$$

where $c > 0$, $\omega > 0$, and $C(B_0)$ is a constant depending on the size of B_0.

Proof. Let $w(t) = u(t) - \widetilde{u}(t)$, $q(x, \rho, t) = z(x, \rho, t) - \widetilde{z}(x, \rho, t)$. Then from (2.4)–(2.8), w and q satisfy

$$\begin{aligned} &w_{tt} + \Delta^2 w - (\Delta_p u - \Delta_p\widetilde{u}) - a_0\Delta w_t + a_1 q(x, 1, t) + f(u) - f(\widetilde{u}) = 0 \quad \text{on } \Omega \times \mathbb{R}^+, \\ &\tau q_t(x, \rho, t) + q(x, \rho, t) = 0 \quad \text{for } (x, \rho, t) \in \Omega \times (0, 1) \times (0, \infty), \\ &w = \Delta w = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ &w(0) = u_0 - \widetilde{u}_0, \quad w_t(0) = u_1 - \widetilde{u}_1 \quad \text{on } \Omega, \\ &q(x, \rho, 0) = z_0(x, \rho) - \widetilde{z}_0(x, \rho) := q_0 \quad \text{for } (x, \rho) \in \Omega \times (0, 1). \end{aligned} \tag{3.12}$$

The similar calculation to that of (3.7) yields

$$\begin{aligned} E_w'(t) \leq &-\left(a_0 - \frac{|a_1|}{2\widetilde{\lambda}} - \frac{\xi}{2\widetilde{\lambda}}\right)\|\nabla w_t(t)\|^2 - \left(\frac{\xi e^{-\theta\tau}}{2} - \frac{|a_1|}{2}\right)\|q(1, t)\|^2 \\ &- \frac{\theta\xi}{2}\int_{t-\tau}^t e^{\theta(s-t)}\|w_t(s)\|^2\, ds - (\Delta_p u(t) - \Delta_p\widetilde{u}(t), w_t(t)) \\ &- (f(u(t)) - f(\widetilde{u}(t)), w_t(t)), \end{aligned} \tag{3.13}$$

where

$$E_w(t) = \frac{1}{2}\|w_t(t)\|^2 + \frac{1}{2}\|\Delta w(t)\|^2 + \frac{\xi}{2}\int_{t-\tau}^t e^{\theta(s-t)}\|w_t(s)\|^2\, ds,$$

here θ and ξ are as given in (3.3). In what follows, we shall estimate the last two terms on the right had side of (3.13). From the Hölder inequality with $\frac{p-2}{2(p-1)}+\frac{1}{2(p-1)}+\frac{1}{2}=1$, we see that

$$\begin{aligned}
-(\Delta_p u-\Delta_p\widetilde{u},w_t) &= \int_\Omega (|\nabla u|^{p-2}\nabla u-|\nabla\widetilde{u}|^{p-2}\nabla\widetilde{u})\nabla w_t\,dx \\
&\le c\int_\Omega (|\nabla u|^{p-2}+|\nabla\widetilde{u}|^{p-2})|\nabla w||\nabla w_t|\,dx \\
&\le c(\|\nabla u\|^{p-2}_{2(p-1)}+\|\nabla\widetilde{u}\|^{p-2}_{2(p-1)})\|\nabla w\|_{2(p-1)}\|\nabla w_t\| \\
&\le \frac{C(B_0)}{2\eta}\|\nabla w\|^2_{2(p-1)}+\frac{\eta}{2}\|\nabla w_t\|^2.
\end{aligned}\tag{3.14}$$

Using (2.1) and $\frac{m}{2(m+1)}+\frac{1}{2(m+1)}+\frac{1}{2}=1$, we also obtain

$$\begin{aligned}
-(f(u)-f(\widetilde{u}),w_t) &\le lc\left(|\Omega|^{\frac{m}{2(m+1)}}+\|u\|^m_{2(m+1)}+\|\widetilde{u}\|^m_{2(m+1)}\right)\|w\|_{2(m+1)}\|w_t\| \\
&\le C(B_0)\|w\|_{2(m+1)}\|\nabla w_t\| \\
&\le \frac{C(B_0)}{2\eta}\|w\|^2_{2(m+1)}+\frac{\eta}{2}\|\nabla w_t\|^2.
\end{aligned}\tag{3.15}$$

Applying these to (3.13), we observe

$$\begin{aligned}
E'_w(t) \le &-\left(a_0-\frac{|a_1|}{2\widetilde{\lambda}}-\frac{\xi}{2\widetilde{\lambda}}-\eta\right)\|\nabla w_t(t)\|^2-\left(\frac{\xi e^{-\theta\tau}}{2}-\frac{|a_1|}{2}\right)\|q(1,t)\|^2 \\
&-\frac{\theta\xi}{2}\int_{t-\tau}^t e^{\theta(s-t)}\|w_t(s)\|^2\,ds+\frac{C(B_0)}{2\eta}\|\nabla w(t)\|^2_{2(p-1)} \\
&+\frac{C(B_0)}{2\eta}\|w(t)\|^2_{2(m+1)}.
\end{aligned}\tag{3.16}$$

On the other hand, it can be observed that E_w is equivalent to $\|(w,w_t,q)\|_{\mathcal{H}}$. Indeed, integration by substitution $s=t-\rho\tau$ gives

$$\begin{aligned}
E_w(t) &= \frac{1}{2}\|w_t(t)\|^2+\frac{1}{2}\|\Delta w(t)\|^2+\frac{\xi\tau}{2}\int_0^1 e^{-\theta\rho\tau}\|w_t(t-\rho\tau)\|^2\,d\rho \\
&\ge \frac{1}{2}\|w_t(t)\|^2+\frac{1}{2}\|\Delta w(t)\|^2+\frac{\xi\tau e^{-\theta\tau}}{2}\int_0^1\int_\Omega q^2(x,\rho,t)\,dxd\rho \\
&\ge \min\left\{\frac{1}{2},\frac{\xi\tau e^{-\theta\tau}}{2}\right\}\|(w,w_t,q)\|_{\mathcal{H}}
\end{aligned}$$

and it holds that

$$\begin{aligned}
E_w(t) &= \frac{1}{2}\|w_t(t)\|^2+\frac{1}{2}\|\Delta w(t)\|^2+\frac{\xi\tau}{2}\int_0^1 e^{-\theta\rho\tau}\|w_t(t-\rho\tau)\|^2\,d\rho \\
&\le \frac{1}{2}\|w_t(t)\|^2+\frac{1}{2}\|\Delta w(t)\|^2+\frac{\xi\tau}{2}\int_0^1\int_\Omega q^2(x,\rho,t)\,dxd\rho \\
&= \max\left\{\frac{1}{2},\frac{\xi\tau}{2}\right\}\|(w,w_t,q)\|_{\mathcal{H}}.
\end{aligned}$$

Next, let us define

$$L_w(t) = E_w(t) + \varepsilon\phi(t),$$

where $\phi(t) = (w_t(t), w(t))$. It can be easily shown that for appropriately small $\varepsilon > 0$ there exist positive constants α_6 and α_7 satisfying

$$\alpha_6 E_w(t) \le L_w(t) \le \alpha_7 E_w(t). \tag{3.17}$$

From (3.12), it follows

$$\begin{aligned} \phi'(t) = \|w_t(t)\|^2 - \|\Delta w(t)\|^2 - (\Delta_p u(t) - \Delta_p \widetilde{u}(t), w(t)) + a_0(w_t(t), \Delta w(t)) \\ - a_1(q(1,t), w(t)) - (f(u(t)) - f(\widetilde{u}(t)), w(t)). \end{aligned} \tag{3.18}$$

By Young's inequality, we know that

$$a_0(w_t, \Delta w) \le \frac{\eta}{2}\|\Delta w\|^2 + \frac{a_0^2}{2\eta}\|w_t\|^2 \quad \text{and} \quad - a_1(q(1,t), w) \le \frac{\eta}{2}\|\Delta w\|^2 + \frac{a_1^2}{2\eta\lambda}\|q(1,t)\|^2.$$

By the same arguments of (3.14) and (3.15), we also see that

$$-(\Delta_p u - \Delta_p \widetilde{u}, w) \le C(B_0)\|\nabla w\|_{2(p-1)}\|\nabla w\| \le C(B_0)\|\nabla w\|_{2(p-1)}^2,$$

where we used the embedding $W_0^{1,2(p-1)}(\Omega) \hookrightarrow H_0^1(\Omega)$, and

$$-(f(u) - f(\widetilde{u}), w) \le C(B_0)\|w\|_{2(m+1)}\|w\| \le \frac{C(B_0)}{2\eta\lambda}\|w\|_{2(m+1)}^2 + \frac{\eta}{2}\|\Delta w\|^2.$$

Substituting these into (3.18), we deduce

$$\begin{aligned} \phi'(t) &\le \left(1 + \frac{a_0^2}{2\eta}\right)\|w_t(t)\|^2 - \left(1 - \frac{3\eta}{2}\right)\|\Delta w(t)\|^2 \\ &\quad + \frac{a_1^2}{2\eta\lambda}\|q(1,t)\|^2 + C(B_0)\|\nabla w(t)\|_{2(p-1)}^2 + \frac{C(B_0)}{2\eta\lambda}\|w(t)\|_{2(m+1)}^2. \end{aligned} \tag{3.19}$$

Combining (3.16) with (3.19), we have

$$\begin{aligned} L_w'(t) &\le -\left\{a_0 - \frac{|a_1|}{2\widetilde{\lambda}} - \frac{\xi}{2\widetilde{\lambda}} - \eta - 2\varepsilon\left(1 + \frac{a_0^2}{2\eta}\right)\right\}\|\nabla w_t(t)\|^2 - \varepsilon\left(1 + \frac{a_0^2}{2\eta}\right)\|w_t(t)\|^2 \\ &\quad - \varepsilon\left(1 - \frac{3\eta}{2}\right)\|\Delta w(t)\|^2 - \frac{\theta\xi}{2}\int_{t-\tau}^{t} e^{\theta(s-t)}\|w_t(s)\|^2\,ds \\ &\quad - \left(\frac{\xi e^{-\theta\tau}}{2} - \frac{|a_1|}{2} - \varepsilon\frac{a_1^2}{2\eta\lambda}\right)\|q(1,t)\|^2 + \left(\frac{C(B_0)}{2\eta} + \varepsilon C(B_0)\right)\|\nabla w(t)\|_{2(p-1)}^2 \\ &\quad + \left(\frac{C(B_0)}{2\eta} + \frac{\varepsilon C(B_0)}{2\eta\lambda}\right)\|w(t)\|_{2(m+1)}^2. \end{aligned}$$

Choosing $\eta > 0$ and $\varepsilon > 0$ small enough, we infer

$$L_w'(t) \le -cE_w(t) + C(B_0)(\|\nabla w(t)\|_{2(p-1)}^2 + \|w(t)\|_{2(m+1)}^2).$$

This and (3.17) yield that

$$L_w'(t) \le -\omega L_w(t) + C(B_0)(\|\nabla w(t)\|_{2(p-1)}^2 + \|w(t)\|_{2(m+1)}^2) \quad \text{for some } \omega > 0.$$

Thus, we have from this and (3.17) that

$$E_w(t) \le ce^{-\omega t}E_w(0) + C(B_0)\int_0^t e^{-\omega(t-s)}(\|\nabla w(s)\|_{2(p-1)}^2 + \|w(s)\|_{2(m+1)}^2)\, ds.$$

Owing to $E_w \sim \|(w, w_t, q)\|_{\mathcal{H}}$, we complete the proof. □

Lemma 3.5. *Assume* (H1)–(H4) *hold. Then, the semigroup* $\{S(t)\}_{t\ge0}$ *defined by* (3.1) *is asymptotically smooth in* $\mathcal{H}$.

Proof. The process of the proof is the same as that of Lemma 4.3 in [5]. So, we omit the details here. □

Our main result of this section reads as:

Theorem 3.6. *Under the conditions* (H1)–(H4), *the semigroup* $\{S(t)\}_{t\ge0}$ *corresponding to problem* (2.4)–(2.8) *has a global attractor in* $\mathcal{H}$.

Proof. Lemmas 3.3, 3.5 and Theorem 2.3 guarantee the existence of a global attractor. □

4. Finite-dimensional attractor

In this section we prove the finite dimensionality of the attractors given in Theorem 3.6 making use of Theorem 2.4.

Lemma 4.1. *Let the conditions* (H1)–(H4) *hold. If we assume subcritical conditions*

$$2 \le p < \frac{2N-2}{N-2} \ \text{ if } N \ge 3 \quad \text{and} \quad 0 < m < \frac{4}{N-4} \ \text{ if } N \ge 5,$$

then the dynamical system $(S(t), \mathcal{H})$ *defined by* (3.1) *is quasi-stable on any bounded positively invariant set* $B \subset \mathcal{H}$.

Proof. Theorem 2.1(i) ensures that the dynamical system $(S(t), \mathcal{H})$ satisfies (2.9) and (2.10) by considering $X = V_2$, $Y = V_0$, and $Z = L^2(\Omega \times (0,1))$. Furthermore, we observe from Theorem 2.1(ii) that $(S(t), \mathcal{H})$ satisfies (2.11). Now, it remains to show that $(S(t), \mathcal{H})$ satisfies (2.12). Let $B_0 \subset \mathcal{H}$ be a bounded set positively invariant with respect to $S(t)$. Let $S(t)y_0 = (u, u_t, z)$ and $S(t)\widetilde{y_0} = (\widetilde{u}, \widetilde{u}_t, \widetilde{z})$ for $y_0 \in B_0$ and $\widetilde{y_0} \in B_0$, respectively. Define the seminorm

$$n_X(u) = \|\nabla u\|_{2(p-1)} + \|u\|_{2(m+1)},$$

then $n_X(\cdot)$ is a compact seminorm on X because the embeddings $V_2 \hookrightarrow W_0^{1,2(p-1)}(\Omega)$ and $V_2 \hookrightarrow L^{2(m+1)}(\Omega)$ are compact. Hence, (3.11) can be rewritten as

$$\|S(t)y_0 - S(t)\widetilde{y_0}\|_{\mathcal{H}}^2 \leq b(t)\|y_0 - \widetilde{y_0}\|_{\mathcal{H}}^2 + c(t) \sup_{0<s<t} (n_X(u(s) - \widetilde{u}(s)))^2,$$

where $b(t) = ce^{-\omega t}$ and $c(t) = C(B_0)\int_0^t e^{-\omega(t-s)}\,ds$. Moreover we see that $b \in L^1(\mathbb{R}^+)$, $\lim_{t\to\infty} b(t) = 0$, and $c(t)$ is locally bounded on $[0,\infty)$ because B_0 is bounded. $\square$

Our desired result of this section is the following:

Theorem 4.2. *Let the conditions of Lemma* 4.1 *hold. Then the global attractor $\mathcal{A}$ given in Theorem* 3.6 *has finite fractal dimension.*

Proof. Since the global attractor $\mathcal{A}$ given in Theorem 3.6 is a bounded positively invariant set of $\mathcal{H}$, Lemma 4.1 yields that the dynamical system $(S(t), \mathcal{H})$ defined (3.1) is quasi-stable on $\mathcal{A}$. Thus, Theorem 2.4 implies that $\mathcal{A}$ has finite fractal dimension. $\square$

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