TAIWANESE JOURNAL OF MATHEMATICS
Vol. 22, No. 4, pp. 881–899, August 2018
DOI: 10.11650/tjm/171205

Reconsidering Hocking's Edge Condition: Solution of the Problem of Forced Capillary-Gravity Waves in a Finite Depth Setting

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Abstract. The paper constructs the solution of a forced capillary-gravity waves problem outside a cylindrical wavemaker of finite depth under Hocking's edge condition. After introducing the setting of the problem, it obtains the solution and proves its uniqueness. Possible applications of the result for future researches, including the application of the model to earthquakes, are also discussed at the end.

1. Introduction

In 1924, a paper was published by Havelock [3] on the problem of forced surface waves under gravity generated by a plane wavemaker. It was solved by using an expansion method without considering surface tension or edge condition. Then Evans [1, 2] studied the problem of a heaving circular cylinder in a fluid with the effect of surface tension being taken into consideration and proposed an edge condition. Later another dynamic edge condition was considered by Hocking [4]. He proposed that at a contact line the time derivative of the free surface is proportional to the slope of the free surface. Of these edge conditions, both of which have been widely studied for their contributions to the solution of related problems, Hocking's model [4] is generally considered more physically plausible than that of Evans's, because Hocking's provided relations more than simply sinusoidal condition. However, Miles [5] argued that Hocking's settings was not practical when considered in the case of a heaving cylinder with stick/slip edge condition.

Ting and Perlin [9] did a thorough study on edge condition using modern equipment to record edge condition and confirmed Miles's observation on the case of a stick/slip condition. However, this result doesn't influence the edge condition in capillary-gravity waves generated by a wavemaker which oscillates horizontally without stick/slip condition. Recently Yeh [11] pointed out that in fact both Ting and Perlin's study and Hocking's edge condition agree on the point that the contact angle between free surface and oscillator/wavemaker is the key factor in determining edge condition, and this agreement further

Received May 18, 2017; Accepted December 19, 2017.
Communicated by Cheng-Hsiung Hsu.
2010 *Mathematics Subject Classification*. 76B45, 35Q92, 76D45.
Key words and phrases. capillary-gravity waves, cylindrical wavemaker, Hocking's edge condition, earthquakes.

confirms the physical plausibility of Hocking's edge condition in the study of capillary-gravity waves.

On the other hand, Rhodes-Robinson [6] studied the problems of forced capillary-gravity waves generated by a plane or cylindrical wavemaker under Evans's edge condition. Some very unconventional expansion theorems for arbitrary functions were introduced in his solving process, and the wave motion was considered axisymmetrical in the case that the waves are generated outside of the cylindrical wavemaker. Yeh [10] rigorously proved the expansion theorems and offered further interpretation. Besides, Shen and Yeh [7, 8] successfully obtained the unique solution of forced capillary-gravity waves inside a circular basin under Hocking's edge condition using Green's function method. However, the solutions exterior to the cylinder under Hocking's edge condition and proper interpretation of the theorems were not yet considered in either of these papers.

Hence this study incorporates one of the expansion theorems to find the desired solution of asymmetrical finite depth problem under Hocking's edge condition, and use the idea of Green's function to prove its uniqueness.

A rigorous process of solution-finding and the proof of its uniqueness will be given in Section 2, and a model of earthquakes using modified formulation from the result will be discussed in Section 3. This study will conclude with Section 4, summarizing the theoretical significance of the solution and pointing out its applicability for future researches. Original formulation and proof of the related expansion theorem are provided in Sections 5 and 6 respectively for references.

2. Solution for the finite depth problem

The *governing equations* of the problem considered are as follows:

(2.1)
$$\mathcal{L}_2\varphi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} + \frac{\partial^2}{\partial z^2}\right)\varphi = 0 \quad \text{on } V = (a,\infty)\times(-1,0) \text{ for some } a>0,$$

(2.2)
$$-\omega^2\varphi + \varphi_z = \mathrm{T}\mathcal{L}_1\varphi_z = \mathrm{T}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2}\right)\varphi_z \quad \text{on } r>a,\ z=0,$$

(2.3)
$$\varphi_z = 0 \quad \text{on } z=-1,$$

(2.4)
$$\varphi_r = f(z) \quad \text{on } r=a,$$

(2.5)
$$\varphi \to C_0\cosh(k_0(1+z))H_m^{(1)}(k_0 r) \quad \text{as } r\to\infty,$$

and the edge condition

(2.6)
$$\varphi_{rz} = \mathrm{i}\omega\delta\varphi_z \quad \text{at } z=0,\ r=a,$$

where

$$\mathcal{L}_2 = \mathcal{L}_1 + \frac{\partial^2}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{m^2}{r^2} + \frac{\partial^2}{\partial z^2}.$$

The original process of deriving these equations is shown in Section 5, and the equation (2.6) is *Hocking's Edge Condition.* Notice that φ is the potential function, T is the surface tension coefficient, ω is the angular frequency, m is the azimuthal number, δ is some real constant, f is an arbitrary smooth function, C_0 is a constant, k_0 is the positive real root of $\alpha(\mathrm{T}\alpha^2+1)\sinh\alpha - \omega^2\cosh\alpha = 0$, and $H_m^{(1)}(\cdot)$ is the Hankel's function of the first kind with order m. The following is an expansion theorem presented by Rhodes-Robinson [5, p. 334] and proved by Yeh (see [7], quoted in Section 6) when f is an arbitrary smooth function defined on $(-1,0)$:

Theorem 2.1. *An arbitrary smooth function $u(z)$ for $-1 < z < 0$ possesses a series expansion in the following form*

$$u(z) = -4\pi\frac{k_0A_0^*(\cosh k_0)(\cosh(k_0(1+z)))}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)\sinh 2k_0} - 4\pi\sum_{n=1}^{\infty}\frac{k_nA_n^*(\cos k_n)(\cos(k_n(1+z)))}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)\sin 2k_n},$$

where $\pm k_0, \pm \mathrm{i}k_1, \pm \mathrm{i}k_2, \ldots, \pm \mathrm{i}k_n, \ldots$ are zeros of

$$\Delta(\alpha) = \alpha(\mathrm{T}\alpha^2+1)\sinh\alpha - \omega^2\cosh\alpha = 0, \tag{2.7}$$

$k_0 > 0$, $0 < k_1 < k_2 < \cdots < k_n < \cdots$,

$$A_0^* = -\frac{1+\mathrm{T}k_0^2}{\pi(\cosh k_0)}\int_{-1}^{0} u(\xi)\cosh(k_0(1+\xi))\,d\xi + \mathrm{T}\mu, \tag{2.8}$$

$$A_n^* = -\frac{1-\mathrm{T}k_n^2}{\pi(\cos k_n)}\int_{-1}^{0} u(\xi)\cos(k_n(1+\xi))\,d\xi + \mathrm{T}\mu, \quad n = 1,2,3,\ldots \tag{2.9}$$

and μ is an arbitrary parameter.

For simplicity, let

$$\beta_0 = 2k_0(1+\mathrm{T}k_0^2) + (1+3\mathrm{T}k_0^2)\sinh 2k_0, \tag{2.10}$$

$$\beta_n = 2k_n(1-\mathrm{T}k_n^2) + (1-3\mathrm{T}k_n^2)\sin 2k_n, \quad n = 1,2,3,\ldots \tag{2.11}$$

and rewrite the expansion as

$$\begin{aligned} u(z) = 2\Bigg[&\frac{2k_0(1+\mathrm{T}k_0^2)(\cosh(k_0(1+z)))\int_{-1}^{0} u\cosh(k_0(1+\xi))\,\mathrm{d}\xi}{\beta_0} \\ &+ \sum_{n=1}^{\infty}\frac{2k_n(1-\mathrm{T}k_n^2)(\cos(k_n(1+z)))\int_{-1}^{0} u\cos(k_n(1+\xi))\,\mathrm{d}\xi}{\beta_n}\Bigg] \\ &- 2\pi\mathrm{T}\mu\left[\frac{2k_0(\cosh k_0)(\cosh(k_0(1+z)))}{\beta_0} + \sum_{n=1}^{\infty}\frac{2k_n(\cos k_n)(\cos(k_n(1+z)))}{\beta_n}\right]. \end{aligned}$$

Then let

$$u(z) = 2u_1(z) - 2\pi \mathrm{T}\mu u_2(z), \tag{2.12}$$

where

$$\begin{aligned} u_1(z) = {}& \frac{1}{\beta_0}\left[2k_0(1+\mathrm{T}k_0^2)(\cosh(k_0(1+z)))\int_{-1}^{0} u(\xi)\cosh(k_0(1+\xi))\,\mathrm{d}\xi\right] \\ &+ 2\sum_{n=1}^{\infty}\frac{1}{\beta_n}\left[k_n(1-\mathrm{T}k_n^2)(\cos(k_n(1+z)))\int_{-1}^{0} u(\xi)\cos(k_n(1+\xi))\,\mathrm{d}\xi\right], \end{aligned}$$

$$u_2(z) = \frac{1}{\beta_0}\left[2k_0(\cosh k_0)(\cosh(k_0(1+z)))\right] + 2\sum_{n=1}^{\infty}\left[\frac{k_n}{\beta_n}(\cos k_n)(\cos(k_n(1+z)))\right]. \tag{2.13}$$

There is a zero term $-2\pi\mathrm{T}\mu u_2(z)$ in (2.12). However, the presence of zero term and the independent parameter μ have not been explained. It will become clear after the solution being found. Simply note that μ will be determined by the edge condition.

The next theorem is the main result of this paper, which is to construct solution of the governing equations by using Theorem 2.1. Uniqueness of the solution will be presented in Theorem 2.6.

Theorem 2.2. *The solution of the governing equations is*

$$\begin{aligned} \varphi(r,z) = {}& -\frac{1+\mathrm{T}k_0^2}{\pi(\cosh k_0)}(-4\pi)\int_{-1}^{0} f(\xi)\cosh(k_0(1+\xi))\,\mathrm{d}\xi \\ &\times \frac{(\cosh k_0)(\cosh(k_0(1+z)))}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)(\sinh 2k_0)} \times \frac{H_m^{(1)}(k_0 r)}{H_m^{(1)'}(k_0 a)} \\ &- \sum_{n=1}^{\infty}\frac{1-\mathrm{T}k_n^2}{\pi(\cos k_n)}(-4\pi)\int_{-1}^{0} f(\xi)\cos(k_n(1+\xi))\,\mathrm{d}\xi \\ &\times \frac{(\cos k_n)(\cos(k_n(1+z)))}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)(\sin 2k_n)} \times \frac{K_m(k_n r)}{K_m'(k_n a)} \\ &+ \mathrm{T}\mu\left[(-4\pi)\frac{(\cosh k_0)(\cosh(k_0(1+z)))}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)\sinh 2k_0} \times \frac{H_m^{(1)}(k_0 r)}{H_m^{(1)'}(k_0 a)}\right. \\ &\left. + (-4\pi)\sum_{n=1}^{\infty}\frac{(\cos k_n)(\cos(k_n(1+z)))}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)\sin 2k_n} \times \frac{K_m(k_n r)}{K_m'(k_n a)}\right]. \end{aligned} \tag{2.14}$$

where

$$\mu = \frac{-\mathrm{i}\omega\delta\widetilde{\varphi_{1z}}(a^+,0)}{\pi + \mathrm{i}\mathrm{T}\omega\delta G_{0z}(a^+,0)},$$

$K_m(\cdot)$ are modified Bessel's functions of the second kind of order m, and

$$\begin{aligned}\widetilde{\varphi_{1z}}(a^+,0) = 4\omega^2 \Bigg\{ &\frac{(\cosh k_0)H_m^{(1)}(k_0a)}{\beta_0 H_m^{(1)'}(k_0a)} \int_{-1}^{0} f(\xi)\cosh(k_0(1+\xi))\,\mathrm{d}\xi \\ &+ \sum_{n=1}^{\infty} \frac{(\cos k_n)K_m(k_na)}{\beta_n K_m'(k_na)} \int_{-1}^{0} f(\xi)\cos(k_n(1+\xi))\,\mathrm{d}\xi \Bigg\}, \\ G_{0z}(a^+,0) = -4\pi \Bigg\{ &\frac{k_0(\cosh k_0)(\sinh k_0)H_m^{(1)}(k_0a)}{\beta_0 H_m^{(1)'}(k_0a)} \\ &- \sum_{n=1}^{\infty} \frac{k_n(\cos k_n)(\sin k_n)K_m(k_na)}{\beta_n K_m'(k_na)} \Bigg\},\end{aligned}$$

where β_0 and β_n are the same as in (2.10) *and* (2.11)*, respectively.*

Proof. By the conjecture of Yeh [10, pp. 6–7] and by Theorem 2.1, assume that the solution has the following form:

$$\varphi(r,z) = \sum_{n=0}^{\infty} f_n(z)\varphi_n(r),$$

where

$$f(z) = \sum_{n=0}^{\infty} f_n(z),$$

$$f_0(z) = -4\pi \frac{k_0A_0^*(\cosh k_0)\cosh(k_0(1+z))}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)(\sinh 2k_0)}, \tag{2.15}$$

$$f_n(z) = -4\pi \frac{k_nA_n^*(\cos k_n)\cos(k_n(1+z))}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)(\sin 2k_n)}, \quad n = 1,2,\ldots, \tag{2.16}$$

A_0^* and A_n^*, $n = 1,2,3,\ldots$, are expressed as in (2.8) and (2.9), respectively. Here $\varphi_n(r)$ ($n = 0,1,2,\ldots$) are radial functions of which forms will be solved later. Furthermore,

$$f_n(z) = 2P_n(z) - 2\pi\mathrm{T}\mu Q_n(z),$$

where

$$\begin{aligned}P_0(z) &= \frac{1}{\beta_0}\left[2k_0(1+\mathrm{T}k_0^2)(\cosh(k_0(1+z)))\int_{-1}^{0} f(\xi)\cosh(k_0(1+\xi))\,\mathrm{d}\xi\right], \\ Q_0(z) &= \frac{1}{\beta_0}\left[2k_0(\cosh k_0)(\cosh(k_0(1+z)))\right], \\ P_n(z) &= \frac{1}{\beta_n}\left[2k_n(1-\mathrm{T}k_n^2)(\cos(k_n(1+z)))\int_{-1}^{0} f(\xi)\cos(k_n(1+\xi))\,\mathrm{d}\xi\right], \quad n = 1,2,3,\ldots, \\ Q_n(z) &= \frac{1}{\beta_n}\left[2k_n(\cos k_n)(\cos(k_n(1+z)))\right], \quad n = 1,2,3,\ldots,\end{aligned}$$

$\mathrm{T}\mu\big[\sum_{n=0}^{\infty} Q_n(z)\big] = 0$ in terms of this expression, and μ is a parameter to be determined. Since $\mathcal{L}_2\varphi = 0$, let's assume that $\mathcal{L}_2$ can be taken inside of $\sum$ to obtain

$$\mathcal{L}_2(f_n(z)\varphi_n(r)) = 0 \quad \text{for all } n. \tag{2.17}$$

Notice that $\mathcal{L}_2\varphi = 0$ doesn't necessarily imply (2.17) would hold. Plausibility of this assumption will be verified in Lemma 2.3 after the solution is constructed through the setting. Again, suppose that $\mathcal{L}_2$ may be taken into the series, then the following two cases should hold:

Case 1: $n = 0$. (2.15) can be written as

$$f_0(z) = F_0 \cosh(k_0(1+z)), \tag{2.18}$$

$$\begin{aligned} \mathcal{L}_2[f_0(z)\varphi_0(r)] &= f_0(z)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left(\frac{m}{r}\right)^2\right)\varphi_0(r) \\ &\quad + \varphi_0(r)\frac{\partial^2}{\partial z^2} F_0 \cosh(k_0(1+z)) \\ &= f_0(z)(\mathcal{L}_1 + k_0^2 I)\varphi_0(r) = 0. \end{aligned} \tag{2.19}$$

That is,

$$(\mathcal{L}_1 + k_0^2 I)\varphi_0 = 0,$$

therefore

$$\varphi_0(r) = C_0^1 H_m^{(1)}(k_0 r) + C_0^2 H_m^{(2)}(k_0 r),$$

where $H_m^{(1)}(\cdot)$ and $H_m^{(2)}(\cdot)$ are Hankel's functions of the first kind and second kind of order m, respectively.

Case 2: $n \geq 1$.

$$f_n(z) = F_n \cos(k_n(1+z)), \tag{2.20}$$

$$\mathcal{L}_2[f_n(z)\varphi_n(r)] = f_n(z)(\mathcal{L}_1 - k_n^2 I)\varphi_n(r) = 0. \tag{2.21}$$

That is,

$$(\mathcal{L}_1 - k_n^2 I)\varphi_n = 0,$$

and therefore

$$\varphi_n(r) = C_n^1 K_m(k_n r) + C_n^2 I_m(k_n r),$$

where $K_m(\cdot)$ and $I_m(\cdot)$ are modified Bessel's functions of the second kind and first kind of order m, respectively.

By radiation condition (2.5) and the fact that the fluid domain extends to $r \to \infty$, it is clear that

$$C_0^2 = 0 = C_n^2,$$

and that

$$\varphi(r,z)=\sum_{n=1}^{\infty}C_n^1F_n\cos(k_n(1+z))K_m(k_nr)+C_0^1F_0\cosh(k_0(1+z))H_m^{(1)}(k_0r).$$

From (2.4), $f(z)$ can be found as

$$\begin{aligned}f(z)&=\sum_{n=1}^{\infty}C_n^1F_nk_nK_m'(k_na)\cos(k_n(1+z))\\&\quad+C_0^1F_0k_0H_m^{(1)'}(k_0a)\cosh(k_0(1+z))\\&=\sum_{n=1}^{\infty}F_n\cos(k_n(1+z))+F_0\cosh(k_0(1+z)),\end{aligned}\tag{2.22}$$

which indicates that

$$\begin{aligned}C_0^1&=\frac{1}{k_0H_m^{(1)'}(k_0a)},\\C_n^1&=\frac{1}{k_nK_m'(k_na)},\quad n\geq 1.\end{aligned}$$

By using the expressions in (2.8), (2.9), (2.15), (2.16), (2.18) and (2.20), φ can be written as

$$\begin{aligned}\varphi(r,z)=&-\frac{1+\mathrm{T}k_0^2}{\pi(\cosh k_0)}(-4\pi)\int_{-1}^{0}f(\xi)\cosh(k_0(1+\xi))\,\mathrm{d}\xi\\&\times\frac{(\cosh k_0)(\cosh(k_0(1+z)))}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)(\sinh 2k_0)}\times\frac{H_m^{(1)}(k_0r)}{H_m^{(1)'}(k_0a)}\\&-\sum_{n=1}^{\infty}\frac{1-\mathrm{T}k_n^2}{\pi(\cos k_n)}(-4\pi)\int_{-1}^{0}f(\xi)\cos(k_n(1+\xi))\,\mathrm{d}\xi\\&\times\frac{(\cos k_n)(\cos(k_n(1+z)))}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)(\sin 2k_n)}\times\frac{K_m(k_nr)}{K_m'(k_na)}\\&+\mathrm{T}\mu\Bigg[(-4\pi)\frac{(\cosh k_0)(\cosh(k_0(1+z)))}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)\sinh 2k_0}\times\frac{H_m^{(1)}(k_0r)}{H_m^{(1)'}(k_0a)}\\&\qquad+(-4\pi)\sum_{n=1}^{\infty}\frac{(\cos k_n)(\cos(k_n(1+z)))}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)\sin 2k_n}\times\frac{K_m(k_nr)}{K_m'(k_na)}\Bigg].\end{aligned}\tag{2.23}$$

To determine μ, edge condition (2.6) and the expansion in (2.23) should be used. Set

$$\varphi(r,z)\equiv\widetilde{\varphi_1}(r,z)+\mathrm{T}\mu G_0(r,z),\tag{2.24}$$

where

$$\widetilde{\varphi_1}(r,z) = 4\Bigg\{ \frac{(1+\mathrm{T}k_0^2)H_m^{(1)}(k_0 r)}{\beta_0 H_m^{(1)'}(k_0 a)} \cosh(k_0(1+z)) \int_{-1}^{0} f(\xi)\cosh(k_0(1+\xi))\,\mathrm{d}\xi \tag{2.25}$$

$$+ \sum_{n=1}^{\infty} \frac{(1-\mathrm{T}k_n^2)K_m(k_n r)}{\beta_n K_m'(k_n a)} \cos(k_n(1+z))$$

$$\times \int_{-1}^{0} f(\xi)\cos(k_n(1+\xi))\,\mathrm{d}\xi \Bigg\},$$

$$G_0(r,z) = -4\pi \left\{ \frac{(\cosh k_0)H_m^{(1)}(k_0 r)}{\beta_0 H_m^{(1)'}(k_0 a)} \cosh(k_0(1+z)) \right. \tag{2.26}$$

$$\left. + \sum_{n=1}^{\infty} \frac{(\cos k_n)(K_m(k_n r))}{\beta_n K_m'(k_n a)} \cos(k_n(1+z)) \right\},$$

and β_0 and β_n are the same as in (2.10) and (2.11), respectively. Note that part of the series with independent parameter μ is in $\mathrm{T}\mu G_0$ and the other part of the series is without μ in $\widetilde{\varphi_1}(r,z)$.

Now the parameter μ shall be determined. Look at $\widetilde{\varphi_{1rz}}(a^+,0)$ first:

$$\widetilde{\varphi_{1rz}}(a^+,0) = 4\left\{ \frac{k_0^2}{\beta_0}(1+\mathrm{T}k_0^2)(\sinh k_0) \int_{-1}^{0} f(\xi)\cosh(k_0(1+\xi))\,\mathrm{d}\xi \right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{k_n^2}{\beta_n}(1-\mathrm{T}k_n^2)(\sin k_n) \int_{-1}^{0} f(\xi)\cos(k_n(1+\xi))\,\mathrm{d}\xi \right\}$$

$$= 4\omega^2 \left\{ \frac{k_0}{\beta_0}(\cosh k_0) \int_{-1}^{0} f(\xi)\cosh(k_0(1+\xi))\,\mathrm{d}\xi \right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{k_n}{\beta_n}(\cos k_n) \int_{-1}^{0} f(\xi)\cos(k_n(1+\xi))\,\mathrm{d}\xi \right\},$$

as a result of applying the fact that $\Delta(\pm k_0) = \Delta(\pm \mathrm{i}k_n) = 0$ in (2.7), $n = 1, 2, 3, \ldots$. Also,

$$G_{0rz}(a^+,0) = 4\pi \left\{ -\frac{k_0^2}{\beta_0}(\cosh k_0)(\sinh k_0) + \sum_{n=1}^{\infty} \frac{k_n^2}{\beta_n}(\cos k_n)(\sin k_n) \right\}$$

$$= -4\pi\omega^2 \left\{ \frac{k_0}{\beta_0}\left(\frac{\cosh^2 k_0}{1+\mathrm{T}k_0^2}\right) + \sum_{n=1}^{\infty} \frac{k_n}{\beta_n}\left(\frac{\cos^2 k_n}{1-\mathrm{T}k_n^2}\right) \right\}.$$

Then one may find that

$$\widetilde{\varphi_{1rz}}(a^+,0) = 2\omega^2 \int_{-1}^{0} f(\xi) \int_{\mathcal{C}} \frac{\alpha\cosh(\alpha(1+\xi))}{\Delta(\alpha)}\,\mathrm{d}\alpha\mathrm{d}\xi = 0, \tag{2.27}$$

and that

$$G_{0rz}(a^+,0) = -2\pi\omega^2 \left(\frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}} \frac{\alpha\cosh\alpha}{\Delta(\alpha)(1+\mathrm{T}\alpha^2)}\,\mathrm{d}\alpha - \operatorname*{Res}_{\alpha=\mathrm{i}/\sqrt{\mathrm{T}}} \frac{\alpha\cosh\alpha}{\Delta(\alpha)(1+\mathrm{T}\alpha^2)} \right) \tag{2.28}$$

$$= -\frac{\pi}{\mathrm{T}},$$

where $\mathcal{C}$ is the contour from $-\infty$ to ∞, which is indented above $-k_0$ and below k_0 (see, Yeh [7].) Finally, edge condition (2.6), (2.24), (2.27) and (2.28) are used to obtain

$$\varphi_{rz}(a^+,0) = \mathrm{i}\omega\delta \left[\widetilde{\varphi_{1z}}(a^+,0) + \mathrm{T}\mu G_{0z}(a^+,0)\right] = -\pi\mu.$$

That is,

$$\mu = \frac{-\mathrm{i}\omega\delta\widetilde{\varphi_{1z}}(a^+,0)}{\pi + \mathrm{iT}\omega\delta G_{0z}(a^+,0)},$$

which confirms the description of μ in Theorem 2. In addition, by applying $\Delta(\pm k_0) = \Delta(\pm \mathrm{i}k_n) = 0$, $\forall\, n \in \mathbb{N}$, $\widetilde{\varphi_{1z}}(a^+,0)$ is found and $G_{0z}(a^+,0)$ are easily obtained by differentiating them in terms of z in (2.25) and (2.26), respectively. $\square$

It is natural to ask if the expansion method could be applied to the interior problem. The answer is not affirmative. The expansion for $f(z)$ leads to a singularity in series expansion of the solution, which can be regularized as in the Green's function method. Yet the limiting solution does not satisfy the edge condition. The question whether a new expansion for $f(z)$ inside the cylinder should be constructed is left to a subsequent study. However, the reason why (2.4) can be written as (2.22) and why (2.17), (2.19) and (2.21) would hold remains unresolved. Assumptions in (2.4) and (2.5) are also related to the same reason. Therefore demonstrating their plausibility in the following lemma is necessary.

Lemma 2.3. *The differential operators $\mathcal{L}_1$, $\mathcal{L}_2$, $\partial/\partial r$, $\partial/\partial z$, $\partial^2/\partial r^2$, $\partial^2/\partial z^2$ and $\partial^2/(\partial z\partial r)$ can be taken into the sum over n, i.e.,* (2.14) *indeed satisfies governing equations.*

The lemma is a verified result [12]. Thus, $\varphi(r,z)$ is indeed the solution of governing equations.

Physical meaning of the zero term (2.13) and the independent parameter μ in Theorem 2.1 is stated in the following two remarks.

Remark 2.4. The z-component of G_0 is exactly the zero term in equation (2.13). Therefore the zero term (2.13) in Theorem 2.1 is essential for obtaining the independent parameter μ at contact line.

Remark 2.5. Theorem 2.1 is always valid (see Section 6); however, the expansion itself is not unique because of its zero term, which suggests various possibilities for different settings of edge conditions.

Finally, the uniqueness of the solution is shown below.

Theorem 2.6. *The solution $\varphi(r,z)$ of governing equations is unique.*

Proof. To show the uniqueness, all one has to do is just consider the solution of homogeneous equations. By Green's identity, the solution φ for Hocking's edge condition can be expressed as

$$\begin{aligned}
&\iiint_{V'} (G\mathcal{L}_2'\varphi - \varphi\mathcal{L}_2'G)\gamma \,\mathrm{d}\gamma\mathrm{d}\xi\mathrm{d}\eta = 2\pi\varphi(r,z)\\
&= \iint_{\gamma=a} Gf\,\mathrm{d}\xi\mathrm{d}\eta + \frac{\mathrm{T}}{\omega^2}\iint_{S'}(G'_\xi\mathcal{L}_1\varphi_\xi - \varphi'_\xi\mathcal{L}_1 G_\xi)\gamma\,\mathrm{d}\gamma\mathrm{d}\eta\\
&= 2\pi\int_{-1}^{0} G|_{\gamma=a} f\,\mathrm{d}\xi + \frac{2\pi\mathrm{T}a}{\omega^2}(\mathrm{i}\omega\delta)(G_\xi\varphi_\xi)|_{\Gamma'}\\
&= 2\pi\left[\int_{-1}^{0} G|_{\gamma=a} f\,\mathrm{d}\xi + \frac{\mathrm{iT}a\delta}{\omega}(G_\xi\varphi_\xi)|_{\Gamma'}\right],
\end{aligned}$$

where $V' = [a,\infty)\times[0,-1]\times[0,2\pi)$, $S' = \{(\gamma,\xi)\mid \gamma\in[a,\infty),\xi=0\}$, and $\Gamma' = \{(\gamma,\xi)\mid \gamma=a,\xi=0\}$. Note that $G = G(r,z;\gamma,\xi)$ satisfies

$$\mathcal{L}_2'G = \left[\frac{\partial^2}{\partial\gamma^2} + \frac{1}{\gamma}\frac{\partial}{\partial\gamma} - \left(\frac{m}{r}\right)^2 + \frac{\partial^2}{\partial\xi^2}\right]G = -\frac{1}{\gamma}\delta(\gamma-r)\delta(\xi-z) \quad \text{in } V', \tag{2.29}$$

$$-\omega^2 G + G_\xi - \mathrm{T}\mathcal{L}_1' G_\xi = 0 \quad \text{on } S', \tag{2.30}$$

$$G_\gamma = 0 \quad \text{on } M', \tag{2.31}$$

$$G_\xi = 0 \quad \text{on } B', \tag{2.32}$$

$$G \to C_1 H_m^{(1)}(k_0\gamma)\cosh(k_0(1+\xi)) \quad \text{as } \gamma\to\infty, \tag{2.33}$$

$$G_{\gamma\xi} = 0 \quad \text{at } \Gamma',$$

where $M' = \{(\gamma,\xi)\mid \gamma=a,\xi\in[0,-1]\}$, $B' = \{(\gamma,\xi)\mid \gamma\in[a,\infty),\xi=-1\}$, and C_1 is a constant, $\eta\in[0,2\pi)$. Note that G could be constructed by the image method. In fact a solution may be obtained in a different way by comparing different forms of the solution. Since

$$\varphi(r,z) = \int_{-1}^{0} G|_{\gamma=a} f\,\mathrm{d}\xi + \frac{\mathrm{iT}a\delta}{\omega}(G_\xi\varphi_\xi)|_{\Gamma'}, \tag{2.34}$$

by comparing with the result in (2.23) to (2.26), one can see that

$$\widetilde{\varphi_1} = \int_{-1}^{0} G|_{\gamma=a} f\,\mathrm{d}\xi, \tag{2.35}$$

$$\mathrm{T}\mu G_0 = \frac{\mathrm{iT}a\delta}{\omega}\varphi_\xi|_{\Gamma'}G_\xi|_{\Gamma'}. \tag{2.36}$$

Again, it is easy to see that by splitting terms with and without the independent parameter μ, the above two equations are valid. By comparing (2.35) and (2.25), $G|_{\gamma=a}$ becomes obvious. Also because

$$\varphi_\xi|_{\Gamma'} = \frac{\int_{-1}^{0} G_\xi|_{\gamma=a,\Gamma} f\,\mathrm{d}\xi}{1 - \frac{\mathrm{iT}a\delta}{\omega}G_{\xi z}|_{\Gamma'\Gamma}},$$

and (2.36) suggests

$$\begin{aligned}
\mathrm{T}\mu G_0 &= \frac{-\mathrm{i}\omega\delta\mathrm{T}\varphi_{1z}(a^+,0)}{\pi+\mathrm{i}\omega\delta\mathrm{T}G_{0z}(a^+,0)}G_0(r,z) = \frac{\frac{\mathrm{iT}a\delta}{\omega}\int_{-1}^{0} G_z|_{\gamma=a,\Gamma} f\,\mathrm{d}\xi}{1-\frac{\mathrm{iT}a\delta}{\omega}G_{z\xi}|_{\Gamma\Gamma'}}G_\xi|_{\Gamma'}\\
&= \frac{-\frac{\mathrm{i}\omega\delta\mathrm{T}}{\pi}\left(\int_{-1}^{0} G_z|_{\gamma=a,\Gamma} f\,\mathrm{d}\xi\right)}{1+\frac{\mathrm{i}\omega\delta\mathrm{T}}{\pi}G_{0z}(a^+,0)}G_0(r,z),
\end{aligned}$$

the result

$$-\frac{\omega^2}{\pi a}G_0(r,z) = G_\xi|_{\Gamma'} \tag{2.37}$$

is obtained. Since $G_0(r,z)$ is given by (2.26), $G_\xi|_{\Gamma'}$ is thus determined by (2.37). It follows from (2.34) that the solution φ appears to be

$$\begin{aligned}
\varphi(r,z) &= \int_{-1}^{0} G|_{\gamma=a} f\,\mathrm{d}\xi + \frac{-\frac{\mathrm{i}\omega\delta\mathrm{T}}{\pi}\left(\int_{-1}^{0} G_z|_{\gamma=a,\Gamma} f\,\mathrm{d}\xi\right)}{1+\frac{\mathrm{i}\omega\delta\mathrm{T}}{\pi}G_{0z}(a^+,0)}G_0(r,z)\\
&= \int_{-1}^{0}\left\{G|_{\gamma=a} - \frac{\mathrm{i}\omega\delta\mathrm{T}G_0(r,z)G_z|_{\gamma=a,\Gamma}}{\pi+\mathrm{i}\omega\delta\mathrm{T}G_{0z}(a^+,0)}\right\} f\,\mathrm{d}\xi\\
&= \int_{-1}^{0} G^H(r,z;a,\xi)f(\xi)\,\mathrm{d}\xi,
\end{aligned} \tag{2.38}$$

where $G^H(r,z;\gamma,\xi)$ satisfies (2.29) to (2.33) but $G^H_{\gamma\xi} = G^H_\xi$ at Γ', and that $G^H(r,z;a,\xi)$ can be easily obtained from (2.38). Then the homogeneous solution φ^h becomes 0 because when $f=0$,

$$\varphi^h(r,z) = \int_{-1}^{0} G^H(r,z;a,\xi)\times 0\,\mathrm{d}\xi \equiv 0.$$

Therefore, a solution of the problem is unique has been proved. □

3. Discussion

The possibility mentioned here was proposed in the earlier paper (see, Yeh [11]). This fluid model could be applied to earthquakes with certain modifications. Assume the amplitude of pressure (vertical) waves on land being far less than its wave length, and think of the pressure waves caused by earthquakes on land satisfying potential function

$$\Phi+\eta,$$

where Φ is the potential described in Section 5, η is the damping effect,

$$\eta = \rho(\Phi_r)^2,$$

ρ is the factor to be determined by actual data, and $r>a$ is the proper distance away from the center of earthquakes. Therefore, the pressure waves of earthquakes on land are considered being gauge invariant or equivalent to this model. However, a different boundary condition should be posed when $r \to \infty$:

$$\Phi + \eta \to C_1 H_m^{(1)}(k_0 r) e^{(-r/\Lambda + k_0 z) + \mathrm{i}(\omega t \pm m\theta)},$$

the number Λ should be determined by data fitting. In reality, finite radial length and depth should be considered, therefore

$$\Phi + \eta = 0 \quad \text{on } z = -l$$

for some depth l. As for the "edge condition" under such a formulation, it can either be taken off or become an additional factor for adjustment. Further research on the model and related problems shall be conducted in different papers.

4. Conclusion

This study has constructed the solution and proved its uniqueness by using expansion theorem and a special assumption. It also considers the asymmetric case, which has not been studied in the past. Although the problem is based on cylindrical wavemaker under Hocking's edge condition, technique used in this paper is not restricted to a specific edge condition or cylindrical wavemaker, and can be applied to any other linearized edge condition as well as plane wavemaker. Furthermore, it is reported that liquid Helium, an important medium used in high temperature superconductivity experiment in case of neutral superfluid below transition in the absence of topological defects, manifests such properties even though ideal fluid is rarely seen in nature. Thus, application of the model in the study to this field of liquid Helium may be possible.

From the view of asymptotic expansion, this solution can be considered as the first term of an asymptotic expansion of the "complete" solution when all non-linear conditions are included. Besides, from series form of the solution, it may be interpreted that the finite depth problem has discrete eigenvalues or spectrums, which is different from infinite depth problem. Furthermore, in the problem of capillary-gravity waves generated inside cylindrical wavemaker of finite depth, complex eigenvalues may be found, as indicated by the exact solution derived by Shen and Yeh [7, 8]. Thus eigenvalues of interior problem require further study. Finally, possible simulation model on earthquakes could be derived as indicated in the previous section.

5. Appendix A: Formulations

We shall consider capillary-gravity waves generated by a cylindrical wavemaker in an incompressible, inviscid fluid, and assume that the fluid motion is irrotational. Use a cylindrical coordinate system in which the z-axis is pointing vertically upwards, so that $z = 0$, $r > a$ is the undisturbed state of the fluid. The fluid region is exterior ($r > a$) to the wave maker. At equilibrium it is of uniform depth h. We may describe the fluid motion by a velocity potential $\Phi(r, \theta, z, t)$. The linearized equations governing the fluid motion are:

$$
\begin{aligned}
&\nabla_3^2 \Phi = 0 \quad \text{in the fluid region} \quad (V), \\
&\Phi_z = Z_t, \\
&\Phi_t + \mathrm{g}Z = \mathrm{T}\nabla_2^2 Z \quad \text{on } z = 0,\ r > a \quad (S),
\end{aligned}
\tag{5.1}
$$

where ∇_3^2, ∇_2^2 denote three-dimensional and two-dimensional cylindrical Laplacians respectively, g is the gravitational constant, $\rho\mathrm{T}$ is the surface tension constant, ρ is the fluid density, and Z represents free surface of the fluid.

$$\Phi_r = f(z)e^{\mathrm{i}(\omega t \pm m\theta)} \quad \text{on } r = a \quad (M),$$

where ω is the angular frequency, m is the azimuthal number (i.e., the waves are generated asymmetrically), and f is an arbitrary smooth function. The bottom condition is given by

$$\Phi_z = 0 \quad \text{on } z = -h \quad (B).$$

A radiation condition is prescribed as follows:

$$\Phi \to C_0 \cosh(k_0(h+z)) H_m^{(1)}(k_0 r) e^{\mathrm{i}(\omega t \pm m\theta)} \quad \text{as } r \to \infty,$$

where $\alpha = k_0$ is the unique positive zero of equation

$$\Delta(\alpha) = \alpha(\mathrm{T}\alpha^2 + 1)\sinh \alpha h - \omega^2 \cosh \alpha h = 0,$$

$H_m^{(1)}(\cdot)$ is the Hankel's function of the first kind with order m, and C_0 is an unknown constant.

The edge condition prescribed for our problem here is the Hocking's edge condition, and is given by

$$Z_t = \lambda Z_r, \quad \left(\lambda \equiv \frac{1}{\delta}\right) \quad \text{at } r = a,\ z = 0 \quad (\Gamma). \tag{5.2}$$

Since the above equations are all linear, we may time-reduce and θ-reduce the problem and assume that

$$\Phi(r, \theta, z, t) = \varphi(r, z) e^{\mathrm{i}(\omega t \pm m\theta)}, \tag{5.3}$$

$$Z(r, \theta, t) = \widehat{\zeta}(r) e^{\mathrm{i}(\omega t \pm m\theta)}. \tag{5.4}$$

Let's measure r, z, Z and $\widehat{\zeta}$ in units of h, t in units of $(h/\mathrm{g})^{1/2}$, Φ and φ in units of $\mathrm{g}h^{3/2}$, ω in units of $(\mathrm{g}/h)^{1/2}$, T in units of $\mathrm{g}h^2$, f in units of $(\mathrm{g}h)^{1/2}$ and λ in units of $(\mathrm{g}/h)^{1/2}$. After writing down the equations for the linearized and time- and θ-reduced problem, the governing equations from (2.1) to (2.5) and (2.7) are obtained. As for the derivation of edge condition (2.6), since by (5.2) to (5.4),

$$Z_t = \mathrm{i}\omega\widehat{\zeta}(r)e^{\mathrm{i}(\omega t\pm m\theta)} = \lambda\widehat{\zeta}_r(r)e^{\mathrm{i}(\omega t\pm m\theta)} = \lambda Z_r,$$

or

$$\widehat{\zeta}_r = \mathrm{i}\omega\delta\widehat{\zeta} \quad \text{at } r=a,\ \lambda \equiv \frac{1}{\delta}. \tag{5.5}$$

Also (5.1) shows that $\Phi_z = Z_t$ on S, then it is clear that $\widehat{\zeta} = \varphi_z/(\mathrm{i}\omega)$ on S by (5.3) and (5.4). Hence (5.5) becomes

$$\frac{\varphi_{rz}}{\mathrm{i}\omega} = \mathrm{i}\omega\delta\frac{\varphi_z}{\mathrm{i}\omega} \quad \Longleftrightarrow \quad \varphi_{rz} = \mathrm{i}\omega\delta\varphi_z$$

at $r=a$, $z=0$, which is (2.6) exactly.

6. Appendix B: Proof of Expansion Theorem

This part is the proof of Theorem 2.1 and is quoted from Yeh [10, pp. 7–11].

Proof. First of all, set

$$u(y) = 2u_1(y) - 2\pi\mu u_2(y), \tag{6.1}$$

where

$$\begin{aligned} u_1(y) = &\frac{1}{\beta_0}\left[2k_0(1+\mathrm{T}k_0^2)(\cosh(k_0(1+y)))\int_{-1}^{0} u\cdot(\cosh(k_0(1+\eta)))\,\mathrm{d}\eta\right] \\ &+ 2\sum_{n=1}^{\infty}\left[\frac{1}{\beta_n}k_n(1-\mathrm{T}k_n^2)(\cos k_n(1+y))\int_{-1}^{0} u(\cos k_n(1+\eta))\,\mathrm{d}\eta\right], \\ u_2(y) = &\frac{1}{\beta_0}\left[2k_0(\cosh k_0)(\cosh(k_0(1+y)))\right] + 2\sum_{n=1}^{\infty}\left[\frac{k_n}{\beta_n}(\cos k_n)(\cos k_n(1+y))\right]. \end{aligned}$$

Consider the integral

$$\mathbf{I} = \frac{1}{2\pi\mathrm{i}}\int_C \frac{\alpha}{\Delta(\alpha)}(\cosh(\alpha(1+y)))\,\mathrm{d}\alpha,$$

where C is the contour from $-\infty$ to ∞, indented above $-k_0$ and below k_0. This integral is identically 0, since the integrand is an odd function. By residue theorem, $\mathbf{I}$ can be written as

$$\mathbf{I} = 2\pi\mathrm{i}\cdot\frac{1}{2\pi\mathrm{i}}\left\{\sum_{n=1}^{\infty}\lim_{\alpha\to\mathrm{i}k_n}\frac{\alpha(\alpha-\mathrm{i}k_n)}{\Delta(\alpha)}\left[\cosh(\alpha(1+y)) + \lim_{\alpha\to k_0}\frac{\alpha(\alpha-k_0)}{\Delta(\alpha)}\cosh(\alpha(1+y))\right]\right\}$$

$$
\begin{aligned}
&= \frac{k_0(\cosh k_0)(\cosh(k_0(1+y)))}{k_0(1+\mathrm{T}k_0^2)(\cosh^2 k_0) + (1+3\mathrm{T}k_0^2)(\sinh k_0)(\cosh k_0) - \omega^2(\sinh k_0)(\cosh k_0)} \\
&\quad + \sum_{n=1}^{\infty} \frac{k_n(\cos k_n)(\cos(k_n(1+y)))}{k_n(1-\mathrm{T}k_n^2)\cos^2 k_n + (1-3\mathrm{T}k_n^2)(\sin k_n)(\cos k_n) - \omega^2(\sin k_n)(\cos k_n)} \\
&= \frac{2k_0(\cosh k_0)(\cosh(k_0(1+y)))}{2k_0(1+\mathrm{T}k_0^2) + (1+3\mathrm{T}k_0^2)\sinh 2k_0} + \sum_{n=1}^{\infty} \frac{2k_n(\cos k_n)(\cos(k_n(1+y)))}{2k_n(1-\mathrm{T}k_n^2) + (1-3\mathrm{T}k_n^2)\sin 2k_n} \\
&= u_2(y),
\end{aligned}
$$

i.e.,

$$\mathbf{I} = u_2(y) = 0.$$

It follows that

$$u(y) = 2u_1(y).$$

By making use of the relations described above,

$$
\begin{aligned}
u_1(y) &= 2\omega^2 \Bigg\{ \frac{1}{\beta_0} \left[(\cosh(k_0(1+y)))(\coth k_0) \int_{-1}^{0} u \cosh(k_0(1+\eta))\,\mathrm{d}\eta \right] \\
&\qquad + \sum_{n=1}^{\infty} \frac{1}{\beta_n} \left[(\cos(k_n(1+y)))(\cot k_n) \int_{-1}^{0} u \cos(k_n(1+\eta))\,\mathrm{d}\eta \right] \Bigg\} \\
&= 2\omega^2 \Bigg\{ \frac{(\cosh(k_0(1+y)))(\cosh k_0) \int_{-1}^{0} u \cosh(k_0(1+\eta))\,\mathrm{d}\eta}{2k_0(1+\mathrm{T}k_0^2)\sinh k_0 + (1+3\mathrm{T}k_0^2)\cdot 2\cosh k_0} \\
&\qquad + \sum_{n=1}^{\infty} \frac{(\cos(k_n(1+y)))(\cos k_n) \int_{-1}^{0} u \cos(k_n(1+\eta))\,\mathrm{d}\eta}{2k_n(1-\mathrm{T}k_n^2)\sin k_n + (1-3\mathrm{T}k_n^2)\cdot 2\cos k_n} \Bigg\}.
\end{aligned}
$$

Notice that for sufficiently large n,

$$\frac{1}{2}\left| \frac{(\cos(k_n(1+y)))(\cos(k_n(1+\eta)))\cos k_n}{k_n(1-\mathrm{T}k_n^2)\sin k_n + (1-3\mathrm{T}k_n^2)\cos k_n} \right| \le \frac{1}{6\mathrm{T}k_n^2},$$

and the series

$$\frac{1}{2}\sum_{n=1}^{\infty} \frac{(\cos(k_n(1+y)))(\cos(k_n(1+\eta)))\cos k_n}{k_n(1-\mathrm{T}k_n^2)\sin k_n + (1-3\mathrm{T}k_n^2)\cos k_n},$$

converges uniformly, according to Weierstrass M-test. Hence $u_1(y)$ can be written as

$$
\begin{aligned}
u_1(y) &= \int_{-1}^{0} u(\eta) \Bigg\{ \frac{1}{\beta_0} \left[2k_0(1+\mathrm{T}k_0^2)(\cosh(k_0(1+y)))(\cosh(k_0(1+\eta))) \right] \\
&\qquad + \sum_{n=1}^{\infty} \frac{1}{\beta_n} \left[2k_n(1-\mathrm{T}k_n^2)(\cos(k_n(1+y)))(\cos(k_n(1+\eta))) \right] \Bigg\}\,\mathrm{d}\eta \\
&= \int_{1}^{0} u(\eta) U(y,\eta)\,\mathrm{d}\eta,
\end{aligned}
$$

where

$$U(y,\eta) = \frac{1}{\beta_0}\left[2k_0(1+\mathrm{T}k_0^2)(\cosh(k_0(1+y)))(\cosh(k_0(1+\eta)))\right] + \sum_{n=1}^{\infty}\frac{1}{\beta_n}\left[2k_n(1-\mathrm{T}k_n^2)(\cos(k_n(1+y)))(\cos(k_n(1+\eta)))\right].$$

Note that when $y \neq 0$, U can be written as the following:

$$U = \frac{1}{2\pi\mathrm{i}}\int_C \frac{\alpha(\mathrm{T}\alpha^2+1)(\cosh(\alpha(1+y)))(\cosh(\alpha(1+\eta)))}{\Delta(\alpha)(\cosh\alpha)}\,\mathrm{d}\alpha - \sum_{\{\alpha|\cosh\alpha=0\}}\mathrm{Res}[\mathbf{P}(\alpha)],$$

where C is the contour described above,

$$\mathbf{P} \equiv \frac{\alpha(\mathrm{T}\alpha^2+1)(\cosh(\alpha(1+y)))(\cosh(\alpha(1+\eta)))}{\Delta(\alpha)(\cosh\alpha)}$$

is an odd function, and the zeros of $\cosh\alpha$ are

$$\alpha = (n-1/2)\pi\mathrm{i}, \quad n = 1,2,3,\ldots.$$

For a fixed n,

$$\begin{aligned}
&\mathrm{Res}\left[\mathbf{P}\left(\alpha=(n-1/2)\pi\mathrm{i}\right)\right]\\
&= \lim_{\alpha\to(n-1/2)\pi\mathrm{i}}\frac{\alpha-(n-1/2)\pi\mathrm{i}}{\Delta(\alpha)(\cosh\alpha)}\times\left[\alpha(\mathrm{T}\alpha^2+1)(\cosh(\alpha(1+y)))(\cosh(\alpha(1+\eta)))\right]\\
&= \frac{\cos\left[(n-1/2)\pi(1+y)\right]}{\mathrm{i}\sin(n-1/2)\pi}\cdot\frac{\cos\left[(n-1/2)\pi(1+\eta)\right]}{\mathrm{i}\sin(n-1/2)\pi}\\
&= (-1)^{2n+1}\left\{\cos\left[(n-1/2)\pi(1+y)\right]\right\}\left\{\cos\left[(n-1/2)\pi(1+\eta)\right]\right\}\\
&= -(-1)^{n+1}\left[\sin\left((n-1/2)\pi y\right)\right](-1)^{n+1}\left[\sin\left((n-1/2)\pi\eta\right)\right]\\
&= -\left[\sin\left((n-1/2)\pi y\right)\right]\left[\sin\left((n-1/2)\pi\eta\right)\right].
\end{aligned}$$

Again, the integral part of U is zero due to an odd integrand, we find

$$\begin{aligned}
U &= -\sum_{n=1}^{\infty}\mathrm{Res}\left[\mathbf{P}\left(\alpha=(n-1/2)\pi\mathrm{i}\right)\right]\\
&= \sum_{n=1}^{\infty}\left[\sin\left((n-1/2)\pi y\right)\right]\left[\sin\left((n-1/2)\pi\eta\right)\right],
\end{aligned}\tag{6.2}$$

which implies that

$$\begin{aligned}
u(y) &= 2u_1(y) = 2\int_{-1}^{0} u(\eta)U(y,\eta)\,\mathrm{d}\eta \\
&= 2\int_{-1}^{0}\sum_{n=1}^{\infty} u(\eta)\left[\sin\left((n-1/2)\pi y\right)\right]\left[\sin\left((n-1/2)\pi\eta\right)\right]\,\mathrm{d}\eta \\
&= 2\sum_{n=1}^{\infty}\left[\int_{-1}^{0} u(\eta)\sin\left((n-1/2)\pi\eta\right)\,\mathrm{d}\eta\right]\sin\left((n-1/2)\pi y\right).
\end{aligned}$$

Let us write

$$u(y) = \sum_{n=1}^{\infty} a_n\sqrt{2}\sin\left((n-1/2)\pi y\right), \tag{6.3}$$

where

$$a_n \equiv \sqrt{2}\int_{-1}^{0} u(\eta)\sin\left((n-1/2)\pi\eta\right)\,\mathrm{d}\eta.$$

Since $\left\{\sqrt{2}\sin((n-1/2)\pi y)\right\}_{n=1}^{\infty}$ is an infinite set of eigenfunctions of a Sturm-Liouville problem, the set then forms a complete and othonormal basis for $L^2[-1,0)$ (see, Coddington and Levinson [1955]). Thus we conclude that for all $u \in L^2[-1,0)$, the series (6.3) for u is always valid, which is equivalent to the expansion (6.1).

Note that when $y = 0$, U becomes

$$U = \frac{1}{2\pi\mathrm{i}}\int_C \frac{\alpha(\mathrm{T}\alpha^2+1)(\cosh(\alpha(1+\eta)))}{\Delta(\alpha)}\,\mathrm{d}\alpha,$$

which is zero, since the integrand is an odd function. The series representing U when $y \neq 0$ is (6.2), which is also zero for $y = 0$. Hence we may use (6.2) to represent U for all y. For all $u \in L^2[-1,0]$, (6.1) is equivalent to (6.3).

However, as one may observe, this does not explain the presence of zero term $-2\pi\mathrm{T}\mu u_2(y)$ in (6.1) and the parameter μ. It will become clear after we find the solution in the next section. We only note that μ will be determined by the edge condition.

Consider the solution of the finite depth plane wavemaker problem can be written as

$$\psi(x,y) = \sum_{n=1}^{\infty}\chi_n(x)\widetilde{u}_n(y),$$

where

$$u(y) = \sum_{n=1}^{\infty}\widetilde{u}_n(y)$$

is the expansion described as in (6.1), and the solution of the finite depth fluid outside a cylindrical wavemaker problem has the form

$$\varphi(r,z) = \sum_{n=1}^{\infty}\chi_n(r)\widetilde{u}_n(z),$$

where

$$u(z) = \sum_{n=1}^{\infty} \widetilde{u}_n(z)$$

is the series expansion in (6.1). This concludes the proof of Rhodes-Robinson's expansion theorem for finite depth. □

Acknowledgments

The author wishes to express gratitude to the National Science Council of Taiwan for its partial support of this research.

References

[1] D. V. Evans, *The influence of surface tension on the reflection of water waves by a plane vertical barrier*, Math. Proc. Cambridge Philos. Soc. **64** (1968), no. 3, 795–810.

[2] ______, *The effect of surface tension on the waves produced by a heaving circular cylinder*, Math. Proc. Cambridge Philos. Soc. **64** (1968), no. 3, 833–847.

[3] T. H. Havelock, *LIX. Forced surface-waves on water*, Philos. Mag. **8** (1929), no. 51, 569–576.

[4] L. M. Hocking, *The damping of capillary-gravity waves at a rigid boundary*, J. Fluid Mech. **179** (1987), 253–266.

[5] J. Miles, *Capillary-viscous forcing of surface waves*, J. Fluid Mech. **219** (1990), 635–646.

[6] P. F. Rhodes-Robinson, *On the forced surface waves due to a vertical wavemaker in the presence of surface tension*, Proc. Cambridge Philos. Soc. **70** (1971), 323–337.

[7] M. C. Shen and N. S. Yeh, *Exact solution for forced capillary-gravity waves in a circular basin under Hocking's edge condition*, Wave Motion **26** (1997), no. 2, 117–126.

[8] ______, *On a special case of forced capillary-gravity waves in a circular basin under Hocking's edge condition*, Wave Motion **30** (1999), no. 1, 91–96.

[9] C.-L. Ting and M. Perlin, *Boundary conditions in the vicinity of the contact line at a vertically oscillating upright plate: an experimental investigation*, J. Fluid Mech. **295** (1995), 263–300.

[10] N.-S. Yeh, *Rigorous proofs of the expansion theorems for the problems of forced capillary-gravity waves*, Fu-Jen Studies **35** (2001), 1–14.

[11] ______, *Hocking's edge condition reconsidered: solution of the problem of capillary-gravity waves in a special setting*, Adv. Appl. Fluid Mech. **17** (2015), no. 2, 241–264.

[12] ______, *A proof of interchangeability between differential operators and series involving the problem of forced capillary-gravity waves in a finite depth setting*, Fu-Jen Studies, submitted.

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