

FREE SURFACE FLOW OVER AN OBSTACLE. THEORETICAL STUDY OF THE FLUVIAL CASE

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The two-dimensional stationary flow of a fluid over an obstacle lying on the bottom of a stream is discussed. We take into account the gravity and we neglect the effects of the surface tension. An existence theory for the solution of this problem is established by the implicit function theorem, for small obstacles and Froude numbers in an interval included in $]0, 1[$.

1. Introduction

This paper considers the problem of determining the free surface flow of an ideal fluid down an infinitely long channel of uniform width. We suppose that the bottom of the channel is perturbed by an obstacle represented by a regular function with a compact support. Free surface flows of an ideal fluid are nonlinear problems; the nonlinearity is essentially due to the dynamic condition written at the free surface.

The practical applications of this type of flow arise both in the hydraulic engineering of fast flow in a channel or a river. The numerical study of this problem has been treated by various authors. Bouhadeb [3] has given a numerical study of the problem in the fluvial and torrential case. King and Bloor [6] have given a generalization of the Schwarz-Christoffel transformation to formulate the problem of free streamline jet flow over curved wall as a pair of coupled equations for the tangential angles onto the free surface and the wall shape. Linearized solutions and nonlinear numerical solutions are presented for a variety of wall shapes. But no theoretical result has been given. However several authors have given theoretical results of this problem in the linear case. Abergel and Bona [1] have considered a steady, two-dimensional of an incompressible, Newtonian fluid flowing under gravity down an inclined channel; they have established an

existence theory for steady, highly viscous flow. In this paper, we want to establish a theoretical result of existence and unicity of a solution which decrease exponentially at infinity. The plan of this paper is as follows.

In [Section 2](#), we formulate the governing equations of the problem in dimensionless form. In [Section 3](#), we introduce the stream function in these equations. In [Section 4](#), the resolution of the dynamic equation written at the free surface boils down to a fixed point problem and here we apply the implicit function theorem. We achieve this work by a conclusion given in [Section 5](#).

2. The governing equations

We consider a steady two-dimensional flow of an ideal fluid in a channel in which an obstacle described by the equation $y = b(x)$ has been placed. We denote by Ω_b^γ the domain occupied by the fluid, where b is the equation of the obstacle and γ is the perturbation of the free surface. We put

$$\Omega_b^\gamma = \{(x, y) \in \mathbb{R}^2 \mid -\infty < x < +\infty, b(x) < y < y_0 + \gamma(x)\}, \quad (2.1)$$

(x, y) is a coordinate system in which x and y , respectively, the horizontal and positive vertical directions, see [Figure 2.1](#).

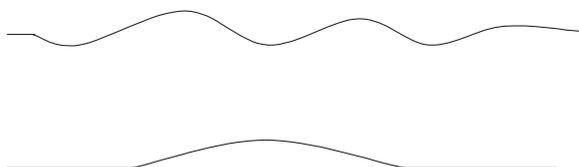


Figure 2.1

The function $b(x)$ verifies $0 \leq b(x) < y_0$ and is regular, with a compact support. The problem is formulated as follows: given a bottom configuration b , find a function $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ (free boundary) and a vector field \vec{u} (velocity of the fluid) such that:

Governing equations in Ω_b^γ

$$\operatorname{div} \vec{u} = 0 \quad \text{in } \Omega_b^\gamma, \quad (2.2)$$

$$\operatorname{curl} \vec{u} = 0 \quad \text{in } \Omega_b^\gamma. \quad (2.3)$$

Equation (2.2) expresses the incompressibility of the fluid, (2.3) is given by the irrotationality of the flow.

Boundary conditions

$$\vec{u} \cdot \vec{\nu} = 0 \quad \text{for } y = b(x), \tag{2.4}$$

$$\vec{u} \cdot \vec{\nu} = 0 \quad \text{for } y = y_0 + \gamma(x), \tag{2.5}$$

where $\vec{\nu}$ is the exterior normal to the boundary of Ω_b^γ . Equations (2.4) and (2.5) describe the impermeability of the flow at the boundary of the domain Ω_b^γ .

Conditions at infinity. We suppose that the flow is asymptotically uniform and horizontal far upstream and downstream of the obstacle. We then write

$$\lim_{|x| \rightarrow \infty} \vec{u}(x, y) = (u_0, 0) \tag{2.6}$$

hence

$$\lim_{|x| \rightarrow \infty} \gamma(x) = 0. \tag{2.7}$$

Condition across the free surface. The dynamic condition of continuity of the pressure across the free surface is given by the Bernoulli equation

$$\frac{\rho}{2} |\vec{u}|^2 + \rho g y = c, \tag{2.8}$$

where ρ is the density of the fluid, g is the downward acceleration due to the gravity, and c is a constant.

Dimensionless equations. Dimensionless variables are defined by referring all lengths to the quantity y_0 , and all velocities to u_0 . We put

$$\vec{u} = u_0 \vec{u}^*, \quad x = y_0 x^*, \quad y = y_0 y^*. \tag{2.9}$$

Systems (2.2), (2.3), (2.4), and (2.5) become

$$\begin{aligned} \operatorname{div} u^* &= 0, & \text{in } \Omega_{b^*}^{\gamma^*}, \\ \operatorname{curl} u^* &= 0 & \text{in } \Omega_{b^*}^{\gamma^*}, \\ u^* \cdot \vec{\nu} &= 0 & \text{for } y^* = b^*(x^*), \\ u^* \cdot \vec{\nu} &= 0 & \text{for } y^* = 1 + \gamma^*(x^*), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} \Omega_{b^*}^{\gamma^*} &= \{(x^*, y^*) \in \mathbb{R}^2 \mid -\infty < x^* < +\infty, b^*(x^*) < y^* < 1 + \gamma^*(x^*)\}, \\ b^*(x^*) &= \frac{1}{y_0} b(y_0 x^*), \quad \gamma^*(x^*) = \frac{1}{y_0} \gamma(y_0 y^*). \end{aligned} \tag{2.11}$$

The conditions at infinity become

$$\lim_{|x| \rightarrow \infty} \vec{u}^*(x^*, y^*) = (1, 0). \tag{2.12}$$

The Bernoulli equation takes the form

$$\frac{F^2}{2} |\vec{u}^*|^2 + 1 + \gamma^*(x^*) = c, \quad (2.13)$$

where $F = u_0/\sqrt{gy_0}$ is the Froude number of the flow.

3. Formulation of the problem in stream function

In what follows, we write all the variables without the symbol *.

The irrotationality and the incompressibility of the fluid lead us to define a harmonic stream function Ψ such that

$$\vec{u} = \begin{pmatrix} \frac{\partial \Psi}{\partial y} \\ -\frac{\partial \Psi}{\partial x} \end{pmatrix}. \quad (3.1)$$

Equation (2.4) will be written as

$$\begin{pmatrix} \frac{\partial \Psi}{\partial y} \\ -\frac{\partial \Psi}{\partial x} \end{pmatrix} \cdot \begin{pmatrix} b'(x) \\ -1 \end{pmatrix} = 0 \quad (3.2)$$

and becomes

$$b'(x) \frac{\partial \Psi}{\partial y} + \frac{\partial \Psi}{\partial x} = 0 \quad \text{in } y = b(x). \quad (3.3)$$

This is equivalent to

$$\frac{\partial \Psi}{\partial \tau} = 0 \quad \text{in } y = b(x). \quad (3.4)$$

In the same way, (2.5) gives

$$\frac{\partial \Psi}{\partial \tau} = 0 \quad \text{in } y = 1 + \gamma(x). \quad (3.5)$$

We deduce that Ψ is constant in $y = b(x)$ and $y = 1 + \gamma(x)$.

Thanks to the condition at infinity, we evaluate the constant which appears in (2.13) and the values of Ψ at the bound of Ω_b^y . In fact, at infinity we have

$$\lim_{|x| \rightarrow \infty} \Psi(x, y) = y + k. \quad (3.6)$$

The function Ψ is a stream function then we can choose $k = 0$. Hence

$$\lim_{|x| \rightarrow \infty} \Psi(x, y) = y. \quad (3.7)$$

Replacing these limits in (2.13) we obtain

$$\frac{F^2}{2} + 1 = c. \tag{3.8}$$

Moreover, we deduce from (3.7) that

$$\begin{aligned} \Psi &= 0 && \text{in } y = b(x), \\ \Psi &= 1 && \text{in } y = 1 + \gamma(x). \end{aligned} \tag{3.9}$$

Then the stream function Ψ verifies

$$\begin{aligned} \Delta \Psi &= 0 && \text{in } \Omega'_b, \\ \Psi &= 0 && \text{in } y = b(x), \end{aligned} \tag{3.10}$$

$$\begin{aligned} \Psi &= 1 && \text{in } y = 1 + \gamma(x), \\ \lim_{|x| \rightarrow \infty} \Psi(x, y) &= y, \end{aligned} \tag{3.11}$$

$$\frac{F^2}{2} |\nabla \Psi|^2(x, 1 + \gamma(x)) + \gamma(x) = \frac{F^2}{2}. \tag{3.12}$$

Taking into account condition (3.11), we can write

$$\Psi = y + \psi, \tag{3.13}$$

where ψ is the perturbation of the stream function.

Problems (3.10), (3.11), and (3.12) will be written as

$$\begin{aligned} \Delta \psi &= 0 && \text{in } \Omega'_b, \\ \psi &= -b(x) && \text{in } y = b(x), \end{aligned} \tag{3.14}$$

$$\begin{aligned} \psi &= -\gamma(x) && \text{in } y = 1 + \gamma(x), \\ \lim_{|x| \rightarrow \infty} \psi(x, y) &= 0, \end{aligned} \tag{3.15}$$

$$\frac{F^2}{2} \left(|\nabla \psi|^2 + 2 \frac{\partial \psi}{\partial y} + 1 \right) + \gamma(x) = \frac{F^2}{2} \quad \text{in } y = 1 + \gamma(x). \tag{3.16}$$

4. Solution of the free surface problem

We transform (3.16) in

$$\gamma(x) = -\frac{F^2}{2} \left[|\nabla \psi|^2(x, 1 + \gamma(x)) + 2 \frac{\partial \psi}{\partial y}(x, 1 + \gamma(x)) \right] \tag{4.1}$$

and we put

$$T(b, \gamma) = -\frac{F^2}{2} \left[|\nabla \psi|^2(x, 1 + \gamma(x)) + 2 \frac{\partial \psi}{\partial y}(x, 1 + \gamma(x)) \right]. \tag{4.2}$$

The problem can be formulated as follows: given a function $y = b(x)$ which represents an obstacle, find a function $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ (free surface) such that $T(b, \gamma) = \gamma$ with ψ solution of problems (3.14) and (3.15).

This is equivalent to solving the equation

$$T_1(b, \gamma) = \gamma - T(b, \gamma) = 0 \quad \text{for a fixed } b. \quad (4.3)$$

For $b = \gamma = 0$, $\psi = 0$ verifies (3.14), (3.15), and (3.16). So $T(0, 0) = 0$ and $T_1(0, 0) = 0$. To solve $T_1(b, \gamma) = 0$, we use the implicit function theorem at a neighbourhood of $(b, \gamma) = (0, 0)$.

Consider the change of variables

$$\tilde{x} = x, \quad \tilde{y} = \frac{y - b(x)}{1 + \gamma(x) - b(x)} \quad (4.4)$$

we transform the domain Ω_b^γ in the following infinite strip Q :

$$Q = \{(x, y) \in \mathbb{R}^2 \mid -\infty < x < +\infty, 0 < y < 1\}. \quad (4.5)$$

We put $\psi(x, y) = \tilde{\psi}(\tilde{x}, \tilde{y})$ then $\tilde{\psi}$ verifies

$$\begin{aligned} \Delta \tilde{\psi} + \mathcal{P}_b^\gamma \tilde{\psi} &= 0 \quad \text{in } Q, \\ \tilde{\psi}(\tilde{x}, 0) &= -b(\tilde{x}), \quad \tilde{\psi}(\tilde{x}, 1) = -\gamma(\tilde{x}), \quad \tilde{x} \in \mathbb{R} \end{aligned} \quad (4.6)$$

\mathcal{P}_b^γ is an operator defined by

$$\mathcal{P}_b^\gamma = a_1 \frac{\partial^2}{\partial \tilde{x} \partial \tilde{y}} + a_2 \frac{\partial^2}{\partial \tilde{y}^2} + a_3 \frac{\partial}{\partial \tilde{y}}, \quad (4.7)$$

where

$$\begin{aligned} a_1 &= \frac{\tilde{y}(b' - \gamma') - b'}{1 + \gamma - b}, \quad a_2 = \left(\frac{a_1}{2}\right)^2 - 1 + \frac{1}{(1 + \gamma - b)^2}, \\ a_3 &= \frac{-1}{1 + \gamma - b} [b'' + \tilde{y}(\gamma'' - b'')] + \frac{2}{(1 + \gamma - b)^2} (\gamma' - b') [b' + \tilde{y}(\gamma' - b')]. \end{aligned} \quad (4.8)$$

The symbols ' and '' denote, respectively, the first and second derivative.

The gradient operator becomes

$$\tilde{\nabla}_{b,\gamma} = \begin{pmatrix} \frac{\partial}{\partial \tilde{x}} + \frac{-b' - \tilde{y}(\gamma' - b')}{(1 + \gamma - b)^2} \frac{\partial}{\partial \tilde{y}} \\ \frac{1}{1 + \gamma - b} \frac{\partial}{\partial \tilde{y}} \end{pmatrix}. \quad (4.9)$$

Equation (4.1) will be written

$$\gamma(\tilde{x}) = \frac{-F^2}{2} \left[|\tilde{\nabla}_{b,\gamma} \tilde{\psi}|^2(\tilde{x}, 1) + \frac{2}{1 + \gamma - b} \frac{\partial \tilde{\psi}}{\partial \tilde{y}}(\tilde{x}, 1) \right], \quad (4.10)$$

where b is given and γ is searched in the space

$$B_c^{2,\lambda}(\mathbb{R}) = \left\{ v \in C^{2,\lambda}(\mathbb{R}) \sum_{0 \leq k \leq 2} \sup_{x \in \mathbb{R}} e^{c|x|} |D_x^k v(x)| < \infty \right\} \tag{4.11}$$

and $\tilde{\psi}$ in the space

$$B_c^{2,\lambda}(\bar{Q}) = \left\{ v \in C^{2,\lambda}(\bar{Q}) \sup_{k+l \leq 2} \sup_{(\tilde{x}, \tilde{y}) \in Q} e^{c|\tilde{x}|} |D_{\tilde{x}}^k D_{\tilde{y}}^l v| < \infty \right\}, \tag{4.12}$$

where $0 < \lambda < 1$ and $c > 0$ [1].

The choice of these spaces will appear evident later.

Remark 4.1. (i) The space $B_c^{m,\lambda}(\bar{Q})$ defined by

$$B_c^{m,\lambda}(\bar{Q}) = \left\{ v \in C^{m,\lambda}(\bar{Q}) \sup_{k+l \leq m} \sup_{(\tilde{x}, \tilde{y}) \in Q} e^{c|\tilde{x}|} |D_{\tilde{x}}^k D_{\tilde{y}}^l v| < \infty \right\} \tag{4.13}$$

equipped with the norm

$$\begin{aligned} \|v\|_{m,c,\lambda} = & \sum_{k+l \leq m} \sup_{(\tilde{x}, \tilde{y}) \in Q} e^{c|\tilde{x}|} |D_{\tilde{x}}^k D_{\tilde{y}}^l v| \\ & + \sup_{k+l=m} \sup_{(\tilde{x}, \tilde{y}) \neq (\tilde{x}', \tilde{y}')} \frac{|D_{\tilde{x}}^k D_{\tilde{y}}^l v(\tilde{x}, \tilde{y}) - D_{\tilde{x}}^k D_{\tilde{y}}^l v(\tilde{x}', \tilde{y}')|}{[(\tilde{x} - \tilde{x}')^2 + (\tilde{y} - \tilde{y}')^2]^{\lambda/2}} \end{aligned} \tag{4.14}$$

is a Banach algebra.

(ii) The space $B_c^{m,\lambda}(\mathbb{R})$ defined by

$$B_c^{m,\lambda}(\mathbb{R}) = \left\{ v \in C^{m,\lambda}(\mathbb{R}) \sum_{0 \leq k \leq m} \sup_{x \in \mathbb{R}} e^{c|x|} |D_x^k v(x)| < \infty \right\} \tag{4.15}$$

equipped with the norm

$$\|v\|_{m,c,\lambda} = \sum_{0 \leq k \leq m} \sup_{x \in \mathbb{R}} e^{c|x|} |D_x^k v| + \sup_{\substack{(x,x') \in \mathbb{R}^2 \\ x \neq x'}} \frac{|D_x^m v(x) - D_x^m v(x')|}{|x - x'|^\lambda} \tag{4.16}$$

is also a Banach algebra.

Now we are able to state the main result of this section.

THEOREM 4.2. *There exists $\tilde{c} > 0$, $K < 1$ such that for all $\lambda, 0 < \lambda < 1$, and all $c, 0 < c < \tilde{c}$, there exists a neighbourhood \mathcal{V} of zero in $B_c^{2,\lambda}(\mathbb{R})$ such that for all $F \in]0, K[$ problems (3.14), (3.15), and (3.16) have a unique solution ψ such that $\tilde{\psi}$ belongs to $B_c^{2,\lambda}(\bar{Q})$, and there exists a mapping g of class \mathcal{C}^1 such that $\gamma = g(b)$.*

This theorem is equivalent to the following one.

THEOREM 4.3. *There exists $\tilde{c} > 0$, $K < 1$, and an open ball \mathfrak{B} of radius r_0 centered at the origin of $B_c^{2,\lambda}(\mathbb{R}) \times B_c^{2,\lambda}(\mathbb{R})$ where $c \in]0, \tilde{c}[$, and $0 < \lambda < 1$, there exists a neighbourhood \mathcal{V}_b of zero in $B_c^{2,\lambda}(\mathbb{R})$, there exists a mapping*

$$g : \mathcal{V}_b \longrightarrow B_c^{2,\lambda}(\mathbb{R}) \quad (4.17)$$

of class \mathcal{C}^1 , such that for all $F \in]0, K[$, $\{ \text{for all } (b, \gamma) \in \mathfrak{B}, T_1(b, \gamma) = 0 \} \Leftrightarrow \{ b \in \mathcal{V}_b, \gamma = g(b) \}$.

Proof. In the next subsections, we will verify the hypothesis of the implicit function theorem. \square

4.1. Differentiability of the operator T_1 with respect to (b, γ) . We have $T_1(b, \gamma) = \gamma - T(b, \gamma)$; to show the differentiability of T_1 with respect to b and γ , it suffices to study the differentiability of T with respect to b and γ . For this we use the following results.

THEOREM 4.4. *There exists $\tilde{c} > 0$ such that for all $c \in]0, \tilde{c}[$ and $\lambda \in]0, 1[$, there exists an open ball \mathfrak{B} of radius $r_0 > 0$, centered at the origin in $B_c^{2,\lambda}(\mathbb{R}) \times B_c^{2,\lambda}(\mathbb{R})$ such that whenever $(b, \gamma) \in \mathfrak{B}$, the following statements hold:*

(a) *the problem*

$$\begin{aligned} \Delta \psi &= 0 \quad \text{in } \Omega_b^\gamma, \\ \psi(x, 1 + \gamma(x)) &= -\gamma(x), \quad \psi(x, b(x)) = -b(x), \quad x \in \mathbb{R} \end{aligned} \quad (4.18)$$

has a unique solution ψ such that $\tilde{\psi}$, the transform of ψ by (4.4) is in $B_c^{2,\lambda}(\bar{Q})$;

(b) *the mapping $S : (b, \gamma) \mapsto \tilde{\psi}$ is continuously differentiable from \mathfrak{B} into $B_c^{2,\lambda}(\bar{Q})$.*

To prove this theorem, we use the following proposition which is proved in the annex.

PROPOSITION 4.5. *Let the boundary value problem*

$$\begin{aligned} \Delta v &= b_1 \quad \text{in } Q, \\ v(\tilde{x}, 1) &= b_2(\tilde{x}), \quad v(\tilde{x}, 0) = b_3(\tilde{x}), \quad \tilde{x} \in \mathbb{R}, \end{aligned} \quad (4.19)$$

where $(b_1, b_2, b_3) \in B_c^{0,\lambda}(\bar{Q}) \times B_c^{2,\lambda}(\mathbb{R}) \times B_c^{2,\lambda}(\mathbb{R})$, then there exists $\tilde{c} > 0$ such that whenever $0 < c < \tilde{c}$, problem (4.19) has a unique solution $v \in B_c^{2,\lambda}(\bar{Q})$. Furthermore the solution map is a topological isomorphism between the corresponding spaces.

Proof of Theorem 4.4

Proof of (a). Denote by \mathcal{A}_b^γ the linear operator defined by

$$\begin{aligned} \mathcal{A}_b^\gamma : B_c^{2,\lambda}(\bar{Q}) &\longrightarrow B_c^{0,\lambda}(\bar{Q}) \times B_c^{2,\lambda}(\mathbb{R}) \times B_c^{2,\lambda}(\mathbb{R}) = \mathfrak{W}, \\ v &\longmapsto (\Delta v + \mathcal{P}_b^\gamma v, v(\cdot, 0), v(\cdot, 1)), \end{aligned} \tag{4.20}$$

and $\mathcal{A} = \mathcal{A}_0^0$. We will verify that

$$\|(\mathcal{A} - \mathcal{A}_b^\gamma)v\|_{\mathfrak{W}} \leq L\left(\|b\|_{B_c^{2,\lambda}(\mathbb{R})}, \|\gamma\|_{B_c^{2,\lambda}(\mathbb{R})}\right) \cdot \|v\|_{B_c^{2,\lambda}(\bar{Q})}, \tag{4.21}$$

where $L(\cdot, \cdot)$ is a continuous function on \mathbb{R}^2 verifying $L(0, 0) = 0$. We have $(\mathcal{A} - \mathcal{A}_b^\gamma)v = (-\mathcal{P}_b^\gamma v, 0, 0)$.

Then

$$\begin{aligned} \|(\mathcal{A} - \mathcal{A}_b^\gamma)v\|_{\mathfrak{W}} &= \|\mathcal{P}_b^\gamma v\|_{B_c^{0,\lambda}(\bar{Q})} \\ &= \left\| a_1 \frac{\partial^2}{\partial \tilde{x} \partial \tilde{y}} + a_2 \frac{\partial^2}{\partial \tilde{y}^2} + a_3 \frac{\partial}{\partial \tilde{y}} \right\|_{B_c^{0,\lambda}(\bar{Q})}. \end{aligned} \tag{4.22}$$

We need the following lemma which is evident to prove.

LEMMA 4.6. *Let $(b, \gamma) \in B_c^{2,\lambda}(\mathbb{R}) \times B_c^{2,\lambda}(\mathbb{R})$. We have $(a_1, a_2, a_3) \in (B_c^{0,\lambda}(\bar{Q}))^3$ furthermore, $\|a_i\|_{B_c^{0,\lambda}(\bar{Q})} \leq L_i(\|b\|_{B_c^{2,\lambda}(\mathbb{R})}, \|\gamma\|_{B_c^{2,\lambda}(\mathbb{R})})$, $1 \leq i \leq 3$ where $L_i(\cdot, \cdot)$ is a continuous function verifying $L_i(0, 0) = 0$.*

Then we have

$$\begin{aligned} \|\mathcal{P}_b^\gamma v\|_{B_c^{0,\lambda}(\bar{Q})} &\leq \left\| a_1 \frac{\partial^2 v}{\partial \tilde{x} \partial \tilde{y}} \right\|_{B_c^{0,\lambda}(\bar{Q})} + \left\| a_2 \frac{\partial^2 v}{\partial \tilde{y}^2} \right\|_{B_c^{0,\lambda}(\bar{Q})} + \left\| a_3 \frac{\partial v}{\partial \tilde{y}} \right\|_{B_c^{0,\lambda}(\bar{Q})} \\ &\leq \left(\|a_1\|_{B_c^{0,\lambda}(\bar{Q})} + \|a_2\|_{B_c^{0,\lambda}(\bar{Q})} + \|a_3\|_{B_c^{0,\lambda}(\bar{Q})} \right) \|v\|_{B_c^{2,\lambda}(\bar{Q})} \end{aligned} \tag{4.23}$$

and using the last lemma we obtain

$$\begin{aligned} \|\mathcal{P}_b^\gamma v\|_{B_c^{0,\lambda}(\bar{Q})} &\leq \left(L_1\left(\|b\|_{B_c^{2,\lambda}(\mathbb{R})}, \|\gamma\|_{B_c^{2,\lambda}(\mathbb{R})}\right) + L_2\left(\|b\|_{B_c^{2,\lambda}(\mathbb{R})}, \|\gamma\|_{B_c^{2,\lambda}(\mathbb{R})}\right) \right. \\ &\quad \left. + L_3\left(\|b\|_{B_c^{2,\lambda}(\mathbb{R})}, \|\gamma\|_{B_c^{2,\lambda}(\mathbb{R})}\right) \right) \|v\|_{B_c^{2,\lambda}(\bar{Q})}. \end{aligned} \tag{4.24}$$

So

$$\|(\mathcal{A} - \mathcal{A}_b^\gamma)v\|_{\mathfrak{W}} \leq L\left(\|b\|_{B_c^{2,\lambda}(\mathbb{R})}, \|\gamma\|_{B_c^{2,\lambda}(\mathbb{R})}\right) \|v\|_{B_c^{2,\lambda}(\bar{Q})}, \tag{4.25}$$

where $L(\cdot, \cdot)$ is a continuous function verifying $L(0, 0) = 0$.

The operator \mathcal{A} being an isomorphism, this shows that \mathcal{A}_b^γ is also an isomorphism for small b and γ .

Thus the problem

$$\begin{aligned} \Delta \tilde{\psi} + \mathcal{P}_b^\gamma \tilde{\psi} &= 0 \quad \text{in } Q, \\ \tilde{\psi}(\tilde{x}, 0) &= -b(\tilde{x}), \quad \tilde{\psi}(\tilde{x}, 1) = -\gamma(\tilde{x}), \quad \tilde{x} \in \mathbb{R}, \end{aligned} \quad (4.26)$$

has a unique solution in $B_c^{2,\lambda}(\bar{Q})$. This gives the proof of [Theorem 4.4\(a\)](#). \square

Now we will prove (b), that is, the application

$$\begin{aligned} S : \mathcal{B}(0, r_0) \subset B_c^{2,\lambda}(\mathbb{R}) \times B_c^{2,\lambda}(\mathbb{R}) &\longrightarrow B_c^{2,\lambda}(\bar{Q}) \\ (b, \gamma) &\longmapsto \tilde{\psi} \end{aligned} \quad (4.27)$$

is continuously differentiable.

We define S_1 and S_2 as follows:

$$\begin{aligned} S_1 : B(0, r_0) &\longrightarrow \mathcal{L}(B_c^{2,\lambda}(\bar{Q}), \mathfrak{Y}), \\ (b, \gamma) &\longmapsto \mathcal{A}_b^\gamma \\ S_2 : \text{Isom}(B_c^{2,\lambda}(\bar{Q}), \mathfrak{Y}) &\longrightarrow \text{Isom}(\mathfrak{Y}, B_c^{2,\lambda}(\bar{Q})), \\ L &\longmapsto L^{-1} \end{aligned} \quad (4.28)$$

and we put

$$\mathcal{F}(b, \gamma) = (0, -b(\tilde{x}), -\gamma(\tilde{x})) = \mathcal{A}_b^\gamma \tilde{\psi} \quad \text{in } \mathfrak{Y}. \quad (4.29)$$

We have

$$\begin{aligned} S_2 \circ S_1(b, \gamma) &= (\mathcal{A}_b^\gamma)^{-1} \quad \text{with } (b, \gamma) \in B(0, r_0), \\ S(b, \gamma) &= (S_2 \circ S_1(b, \gamma)) \mathcal{F}(b, \gamma) = (\mathcal{A}_b^\gamma)^{-1} (\mathcal{A}_b^\gamma \tilde{\psi}) = \tilde{\psi}. \end{aligned} \quad (4.30)$$

The differentiability of $\tilde{\psi}$ is given by the differentiability of S_1 , S_2 , and $\mathcal{F}(b, \gamma)$. It is evident that $\mathcal{F}(b, \gamma)$ is continuously differentiable with respect to b and γ . S_2 is a \mathcal{C}^∞ operator. It remains to prove the continuous differentiability of S_1 .

We have

$$S_1 : (b, \gamma) \longrightarrow \mathcal{A}_b^\gamma = \Delta + a_1 \frac{\partial^2}{\partial \tilde{x} \partial \tilde{y}} + a_2 \frac{\partial^2}{\partial \tilde{y}^2} + a_3 \frac{\partial}{\partial \tilde{y}}. \quad (4.31)$$

It is sufficient to prove that a_i , $1 \leq i \leq 3$, which are rational functions in b , γ , b' , γ' , b'' , γ'' , are continuously differentiable with respect to b and γ . For this we use the next lemma which is evident to prove [1].

LEMMA 4.7. *Let p be a rational function of k variables which is devoid of poles in a neighbourhood of the origin in \mathbb{R}^k (k a positive integer) such that $p(0) = 0$. Then the mapping*

$$\begin{aligned}
 P : \prod_{1 \leq i \leq k} B_c^{n_i, \lambda}(\mathbb{R}) &\longrightarrow B_c^{n_0, \lambda}(\mathbb{R}), \\
 (g_1, \dots, g_k) &\longmapsto P(g_1, \dots, g_k),
 \end{aligned}
 \tag{4.32}$$

where $n_i \in \mathbb{N}$, $1 \leq i \leq k$, $n_0 = \min\{n_i, 1 \leq i \leq k\}$ is continuously differentiable in a neighbourhood of the origin in $\prod_{1 \leq i \leq k} B_c^{n_i, \lambda}(\mathbb{R})$.

The coefficients a_1, a_2 , and a_3 verify the hypothesis of Lemma 4.7, then they are continuously differentiable with respect to b and γ . This gives continuous differentiability of S_1 with respect to b and γ . Then Theorem 4.4 is proved.

In the new variables \tilde{x}, \tilde{y} , the operator $T(b, \gamma)$ takes the form

$$\begin{aligned}
 T(b, \gamma) &= -\frac{F^2}{2} \left\{ \left[\frac{\partial \tilde{\psi}}{\partial \tilde{x}}(\tilde{x}, 1) - \frac{\gamma'}{(1+\gamma-b)^2} \frac{\partial \tilde{\psi}}{\partial \tilde{y}}(\tilde{x}, 1) \right]^2 \right. \\
 &\quad \left. + \frac{1}{(1+\gamma-b)^2} \left(\frac{\partial \tilde{\psi}}{\partial \tilde{y}}(\tilde{x}, 1) \right)^2 + \frac{2}{1+\gamma-b} \frac{\partial \tilde{\psi}}{\partial \tilde{y}}(\tilde{x}, 1) \right\} \\
 &= -\frac{F^2}{2} \left\{ \left[\frac{\partial \tilde{\psi}}{\partial \tilde{x}}(\tilde{x}, 1) + \lambda_1(b, \gamma) \frac{\partial \tilde{\psi}}{\partial \tilde{y}}(\tilde{x}, 1) \right]^2 \right. \\
 &\quad \left. + \lambda_2^2(b, \gamma) \left(\frac{\partial \tilde{\psi}}{\partial \tilde{y}}(\tilde{x}, 1) \right)^2 + 2\lambda_2(b, \gamma) \frac{\partial \tilde{\psi}}{\partial \tilde{y}}(\tilde{x}, 1) \right\}
 \end{aligned}
 \tag{4.33}$$

with

$$\lambda_1(b, \gamma) = -\frac{\gamma'}{(1+\gamma-b)^2}; \quad \lambda_2(b, \gamma) = \frac{1}{1+\gamma-b}.
 \tag{4.34}$$

THEOREM 4.8. *Under the hypothesis of Theorem 4.4 the operator T is continuously Gâteaux differentiable on B .*

Proof. We have shown that $\tilde{\psi}$ is continuously differentiable with respect to b and γ . Moreover, it is evident that $\lambda_1(b, \gamma)$ and $\lambda_2(b, \gamma)$ are continuously differentiable with respect to b and γ . We deduce that $T(b, \gamma)$ is continuously Gâteaux differentiable with respect to b and γ . Then T_1 is continuously differentiable on B . □

4.2. Expression of $(\partial T_1 / \partial \gamma)(0, 0)$. In the last subsection, we have seen that $\tilde{\psi}$ is the solution of the problem

$$\begin{aligned}
 \Delta \tilde{\psi} + \mathcal{P}_b^\gamma \tilde{\psi} &= 0 \quad \text{in } Q, \\
 \tilde{\psi}(\tilde{x}, 0) &= -b(\tilde{x}), \quad \tilde{\psi}(\tilde{x}, 1) = -\gamma(\tilde{x}), \quad \tilde{x} \in \mathbb{R}.
 \end{aligned}
 \tag{4.35}$$

We have $\tilde{\psi}|_{b=\gamma=0} = 0$ in Q and $\mathcal{P}_0^0 \tilde{\psi} = 0$.

Let $h \in B_c^{2,\lambda}(\mathbb{R})$. We put $b = 0$ in system (4.35), we derive with respect to γ in the direction h and we evaluate the derivative at $\gamma = 0$. We put

$$w = \frac{\partial \tilde{\psi}}{\partial \gamma} \Big|_{b=\gamma=0}. \quad (4.36)$$

We obtain

$$\Delta w + \mathcal{P}_0^0 w + \frac{\partial}{\partial \gamma} (\mathcal{P}_0^\gamma) \Big|_{\gamma=0} \tilde{\psi}|_{b=\gamma=0} \cdot h = 0 \quad \text{in } Q, \quad (4.37)$$

$$w(\tilde{x}, 0) = 0, \quad w(\tilde{x}, 1) = -h(\tilde{x}), \quad \tilde{x} \in \mathbb{R}$$

and we get the following result.

THEOREM 4.9. *Let $h \in B_c^{2,\lambda}(\mathbb{R})$ and $w = w(h) = (\partial \tilde{\psi} / \partial \gamma)(\cdot, 1)|_{b=\gamma=0} \cdot h$. Then $w(h)$ is the unique solution of the problem*

$$\Delta w = 0 \quad \text{in } Q, \quad (4.38)$$

$$w(\tilde{x}, 0) = 0, \quad w(\tilde{x}, 1) = -h(\tilde{x}), \quad \tilde{x} \in \mathbb{R}$$

and satisfies

$$\|w(h)\|_{B_c^{2,\lambda}(\bar{Q})} \leq k \|h\|_{B_c^{2,\lambda}(\mathbb{R})}, \quad k > 1. \quad (4.39)$$

Proof. For the existence and uniqueness, we use [Proposition 4.5](#). Now to prove inequality (4.39), we need the next lemma proved in the annex. \square

LEMMA 4.10. *Let h be in the space $B_c^{2,\lambda}(\mathbb{R})$. Then the function $(\tilde{x}, \tilde{y}) \mapsto h(\tilde{x})\tilde{y}$ is in the space $B_c^{2,\lambda}(\bar{Q})$ and verify $\|h(\tilde{x})\tilde{y}\|_{B_c^{2,\lambda}(\bar{Q})} \leq k \|h\|_{B_c^{2,\lambda}(\mathbb{R})}$, where $k > 1$.*

Now we are able to establish relation (4.39). We can write

$$w(\tilde{x}, \tilde{y}) = u(\tilde{x}, \tilde{y}) - h(\tilde{x})\tilde{y}, \quad (4.40)$$

where u is the solution of the problem

$$\begin{aligned} \Delta u &= h''(\tilde{x})\tilde{y} \quad \text{in } Q, \\ u(\tilde{x}, 0) &= 0, \quad u(\tilde{x}, 1) = 0, \quad \tilde{x} \in \mathbb{R}. \end{aligned} \quad (4.41)$$

From [Proposition 4.5](#) we have

$$\begin{aligned} \|u\|_{B_c^{2,\lambda}(\Sigma)} &\leq k_1 \|\Delta u\|_{B_c^{0,\lambda}(\Sigma)}, \quad k_1 > 1 \text{ (evident)} \\ &\leq k_1 \|h''(\tilde{x})\tilde{y}\|_{B_c^{0,\lambda}(\bar{Q})} \\ &\leq k_1 k_2 \|h\|_{B_c^{2,\lambda}(\mathbb{R})}, \quad k_2 > 1 \text{ (Lemma 4.10)} \\ &\leq k_3 \|h\|_{B_c^{2,\lambda}(\mathbb{R})}, \quad \text{where } k_3 = k_1 k_2. \end{aligned} \quad (4.42)$$

Using Lemma 4.10 we obtain

$$\begin{aligned} \|w(h)\|_{B_c^{2,\lambda}(\bar{Q})} &\leq k_3 \|h\|_{B_c^{2,\lambda}(\mathbb{R})} + k_4 \|h\|_{B_c^{2,\lambda}(\mathbb{R})} \\ &\leq k \|h\|_{B_c^{2,\lambda}(\mathbb{R})}, \quad k > 1, \end{aligned} \tag{4.43}$$

where $k = k_3 + k_4$. Now we can evaluate $(\partial T_1 / \partial \gamma)(0, 0)$. We have

$$\frac{\partial T_1}{\partial \gamma}(0, 0) \cdot h = \left[\text{Id} - \frac{\partial T}{\partial \gamma}(0, 0) \right] \cdot h, \quad h \in B_c^{2,\lambda}(\mathbb{R}), \tag{4.44}$$

where Id is the identity mapping of $B_c^{2,\lambda}(\mathbb{R})$.

We calculate $(\partial T / \partial \gamma)(0, 0) \cdot h$

$$\begin{aligned} T(0, \gamma) = &-\frac{F^2}{2} \left\{ \left[\frac{\partial \tilde{\psi}}{\partial \tilde{x}}(\cdot, 1) - \frac{\gamma'}{(1+\gamma-b)^2} \frac{\partial \tilde{\psi}}{\partial \tilde{y}}(\cdot, 1) \right]^2 \right. \\ &\left. + \frac{1}{(1+\gamma-b)^2} \left(\frac{\partial \tilde{\psi}}{\partial \tilde{y}}(\cdot, 1) \right)^2 + \frac{2}{1+\gamma-b} \frac{\partial \tilde{\psi}}{\partial \tilde{y}}(\cdot, 1) \right\} \Big|_{b=0}. \end{aligned} \tag{4.45}$$

We derive with respect to γ in the direction h at $\gamma = 0$

$$\begin{aligned} \frac{\partial T}{\partial \gamma}(0, 0) \cdot h = &-\frac{F^2}{2} \left\{ 2 \left[\frac{\partial \tilde{\psi}}{\partial \tilde{x}}(\cdot, 1) - \frac{\gamma'}{(1+\gamma)^2} \frac{\partial \tilde{\psi}}{\partial \tilde{y}}(\cdot, 1) \right] \Big|_{b=\gamma=0} \right. \\ &\times \frac{\partial}{\partial \gamma} \left[\frac{\partial \tilde{\psi}}{\partial \tilde{x}}(\cdot, 1) - \frac{\gamma'}{(1+\gamma)^2} \frac{\partial \tilde{\psi}}{\partial \tilde{y}}(\cdot, 1) \right] \cdot h \\ &+ \frac{\partial}{\partial \gamma} \left(\frac{\partial \tilde{\psi}}{\partial \tilde{y}} \right)^2 \Big|_{b=\gamma=0} \cdot h - 2 \left[\frac{\partial \tilde{\psi}}{\partial \tilde{y}} \Big|_{b=\gamma=0}(\cdot, 1) \right]^2 \cdot h \\ &\left. - 2 \left[\frac{\partial \tilde{\psi}}{\partial \tilde{y}} \Big|_{b=\gamma=0}(\cdot, 1) \right] \cdot h + 2 \frac{\partial}{\partial \gamma} \left(\frac{\partial \tilde{\psi}}{\partial \tilde{y}}(\cdot, 1) \Big|_{b=\gamma=0} \right) \cdot h \right\} \end{aligned} \tag{4.46}$$

which gives

$$\begin{aligned} \frac{\partial T}{\partial \gamma}(0, 0) \cdot h &= -F^2 \left\{ \frac{\partial}{\partial \gamma} \left(\frac{\partial \tilde{\psi}}{\partial \tilde{y}}(\cdot, 1) \Big|_{b=\gamma=0} \right) \cdot h \right\} \\ &= -F^2 \left\{ \frac{\partial}{\partial \tilde{y}} \left(\frac{\partial \tilde{\psi}}{\partial \gamma}(\cdot, 1) \Big|_{b=\gamma=0} \right) \cdot h \right\} \\ &= -F^2 \frac{\partial}{\partial \tilde{y}}(w(h)). \end{aligned} \tag{4.47}$$

We replace $(\partial T / \partial \gamma)(0, 0) \cdot h$ by its expression in (4.44) and we find

$$\frac{\partial T_1}{\partial \gamma}(0, 0) \cdot h = \left[\text{Id} + F^2 \frac{\partial w}{\partial \tilde{y}} \right] \cdot h. \tag{4.48}$$

4.3. Inversibility of $(\partial T_1/\partial \gamma)(0, 0)$. To prove the inversibility of $(\partial T_1/\partial \gamma)(0, 0) = \text{Id} + F^2(\partial w/\partial \tilde{y})$, it suffices to show that

$$\left\| F^2 \frac{\partial w}{\partial \tilde{y}} \right\|_{\mathcal{L}(B_c^{2,\lambda}(\mathbb{R}), B_c^{1,\lambda}(\mathbb{R}))} < 1. \tag{4.49}$$

We have

$$\begin{aligned} \left\| F^2 \frac{\partial w}{\partial \tilde{y}} \right\|_{\mathcal{L}(B_c^{2,\lambda}(\mathbb{R}), B_c^{1,\lambda}(\mathbb{R}))} &= F^2 \sup_{h \neq 0} \frac{\|\partial w/\partial \tilde{y}\|_{B_c^{1,\lambda}(\mathbb{R})}}{\|h\|_{B_c^{2,\lambda}(\mathbb{R})}} \\ &\leq F^2 \sup_{h \neq 0} \frac{\|w(h)\|_{B_c^{2,\lambda}(\mathbb{R})}}{\|h\|_{B_c^{2,\lambda}(\mathbb{R})}} \\ &\leq F^2 \sup_{h \neq 0} \frac{k \|h\|_{B_c^{2,\lambda}(\mathbb{R})}}{\|h\|_{B_c^{2,\lambda}(\mathbb{R})}} \|h\|_{B_c^{2,\lambda}(\mathbb{R})} \quad (\text{see Theorem 4.9}) \\ &\leq F^2 k. \end{aligned} \tag{4.50}$$

It suffices that $F^2 k < 1$ for the inversibility of $(\partial T_1/\partial \gamma)(0, 0)$. Then the inversibility of $(\partial T_1/\partial \gamma)(0, 0)$ is obtained for $F \in]0, 1/\sqrt{k}[$. These results achieve the proof of [Theorem 4.3](#) and then the proof of [Theorem 4.2](#).

Remark 4.11. We have a result of existence and uniqueness for Froude numbers $F < 1$.

5. Conclusion

The conclusion of this work is that we have established a local result of existence and uniqueness of the solution for the values of Froude number in $]0, k[\subset]0, 1[$. Precisely, for a given small obstacle b in $B_c^{2,\lambda}(\mathbb{R})$ there exists a unique function γ in $B_c^{2,\lambda}(\mathbb{R})$ which describes the perturbation of the free surface. The result established here does not exclude the existence of solutions in other spaces. In [\[5\]](#), it is said that there is no uniqueness of the solution when $F < 1$. This result has been confirmed numerically in [\[5\]](#) where moreover the existence and uniqueness of the solution are established after linearization of the equations. In our paper, without linearization, we have solved the problem of existence and uniqueness using the implicit function theorem in subspaces of Hölder spaces.

Annex

Proof of [Proposition 4.5](#).

Consider the homogeneous problem

$$\begin{aligned} \Delta v &= f_1 \quad \text{in } Q, \\ v(\cdot, 1) &= v(\cdot, 0) = 0 \quad \text{in } \mathbb{R}. \end{aligned} \tag{5.1}$$

First of all, f_1 belongs to $B_c^{0,\lambda}(\bar{Q})$ then f_1 obviously belongs to $L^2(Q)$. So there exists a weak solution $v \in H_0^1(Q)$. In fact $v \in H^2(Q) \cap H_0^1(Q)$. Because $f_1 \in B_c^{0,\lambda}(\bar{Q})$ we conclude that v is a classical solution in $\mathcal{C}^{2,\lambda}(\bar{Q})$ (see [4]).

To prove that $v \in B_c^{2,\lambda}(\bar{Q})$, we must show that

$$\sup_{k+l \leq 2} \sup_Q e^{c|x|} |D_x^k D_y^l v| < \infty. \tag{5.2}$$

For this we use the result established in [2] where we replace the operator L by $-\Delta$. We conclude that there exists $\tilde{c} > 0$ such that for each $c \in]0, \tilde{c}[$, problem (5.1) has a unique solution in $B_c^{2,\lambda}(\bar{Q})$ when f_1 is given in $B_c^{0,\lambda}(\bar{Q})$.

We return to the nonhomogeneous problem

$$\Delta v = b_1 \quad \text{in } Q, \tag{5.3}$$

$$v(x, 1) = b_2(x), \quad v(x, 0) = b_3(x), \quad x \in \mathbb{R}.$$

We put $v(x, y) = v_1(x, y) + (1 - y)b_3(x) + yb_2(x)$ where v_1 verifies

$$\Delta v_1 = b_1 - \Delta((1 - y)b_3(x)) - \Delta(yb_2(x)) = F(x, y) \quad \text{in } Q, \tag{5.4}$$

$$v_1(x, 1) = 0, \quad v_1(x, 0) = 0, \quad x \in \mathbb{R}.$$

It is evident that $F \in B_c^{0,\lambda}(\bar{Q})$ so $v_1 \in B_c^{2,\lambda}(\bar{Q})$.

We deduce that $v \in B_c^{2,\lambda}(\bar{Q})$. □

Proof of Lemma 4.10. We put

$$u(x, y) = h(x)y. \tag{5.5}$$

The functions $(x, y) \mapsto h(x)$ and $(x, y) \mapsto y$ are in $\mathcal{C}^{2,\lambda}(\bar{Q})$. Then $(x, y) \mapsto h(x)y$ is in $\mathcal{C}^{2,\lambda}(\bar{Q})$.

We also have

$$\sup_{(x,y) \in Q} \sup_{k+l \leq 2} e^{c|x|} |D_x^k D_y^l u(x, y)| < \infty. \tag{5.6}$$

Then

$$u(x, y) = h(x)y \in B_c^{2,\lambda}(\bar{Q}). \tag{5.7}$$

Now we show that

$$\begin{aligned}
& \|h(x)y\|_{B_c^{2,\lambda}(\bar{Q})} \leq k\|h\|_{B_c^{2,\lambda}(\mathbb{R})}, \\
& \|h(x)y\|_{B_c^{2,\lambda}(\bar{Q})} \\
&= \sum_{k+l \leq 2} \sup_{(x,y) \in \bar{Q}} e^{c|x|} |D_x^k D_y^l (h(x)y)| \\
&+ \sup_{k+l=2} \sup_{(x,y) \neq (x',y')} \frac{|D_x^k D_y^l (h(x)y)(x,y) - D_x^k D_y^l (h(x)y)(x',y')|}{[(x-x')^2 + (y-y')^2]^{\lambda/2}} \\
&= \sup_{(x,y) \in \bar{Q}} e^{c|x|} |h(x)y| + \sup_{(x,y) \in \bar{Q}} e^{c|x|} |h'(x)y| + \sup_{(x,y) \in \bar{Q}} e^{c|x|} |h''(x)y| \\
&+ \sup_{(x,y) \in \bar{Q}} e^{c|x|} |h(x)| + \max \left\{ \sup_{(x,y) \neq (x',y')} \frac{|h''(x)y - h''(x')y'|}{[(x-x')^2 + (y-y')^2]^{\lambda/2}}, \right. \\
&\quad \left. \times \sup_{(x,y) \neq (x',y')} \frac{|h'(x) - h'(x')|}{[(x-x')^2 + (y-y')^2]^{\lambda/2}} \right\}, \\
&\|h(x)y\|_{B_c^{2,\lambda}(\bar{Q})} \leq 2\|h\|_{B_c^{2,\lambda}(\mathbb{R})} + \|h\|_{B_c^{1,\lambda}(\mathbb{R})} \\
&\quad + \sup_{(x,y) \neq (x',y')} \frac{|h''(x)y - h''(x')y'|}{[(x-x')^2 + (y-y')^2]^{\lambda/2}}. \tag{5.8}
\end{aligned}$$

We have

$$\begin{aligned}
& \sup_{(x,y) \neq (x',y')} \frac{|h''(x)y - h''(x')y'|}{[(x-x')^2 + (y-y')^2]^{\lambda/2}} \\
&\leq \sup_{(x,y) \neq (x',y')} \frac{|h''(x)y - h''(x')y + h''(x')y - h''(x')y'|}{[(x-x')^2 + (y-y')^2]^{\lambda/2}} \\
&\leq \sup_{x \neq x'} \frac{|h''(x) - h''(x')|}{|x-x'|^\lambda} + \sup_{(x,y) \neq (x',y')} \frac{|h''(x')||y-y'|}{|y-y'|^\lambda} \\
&\leq \|h\|_{B_c^{2,\lambda}(\mathbb{R})} + |y-y'|^{1-\lambda} \|h\|_{B_c^{2,\lambda}(\mathbb{R})} \\
&\leq 2\|h\|_{B_c^{2,\lambda}(\mathbb{R})}. \tag{5.9}
\end{aligned}$$

Then we obtain

$$\|h(x)y\|_{B_c^{2,\lambda}(\bar{Q})} \leq (4+c)\|h\|_{B_c^{2,\lambda}(\mathbb{R})} \tag{5.10}$$

and we have

$$\|h(x)y\|_{B_c^{2,\lambda}(\bar{Q})} \leq k\|h\|_{B_c^{2,\lambda}(\mathbb{R})}, \quad (5.11)$$

where $k > 1$. □

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