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Research Article

Connections between Some Concepts of Polynomial Trichotomy for Noninvertible Evolution Operators in Banach Spaces

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The present paper treats three concepts of nonuniform polynomial trichotomies for noninvertible evolution operators acting on Banach spaces. The connections between these concepts are established through numerous examples and counterexamples for systems defined on the Banach space of square-summable sequences.

1. Introduction

In the theory of asymptotic behavior of first-order differential equations, one of the main problems is to decompose the state space into a direct sum of subspaces on which the solutions of the given system have prescribed behavior. One of these behaviors can be modelled by the notion of exponential dichotomy, in which the state space is decomposed into a direct sum of two subspaces (the stable and unstable subspace) such that on the stable subspace the norm of the solution tends to zero (exponentially, polynomially, or with the aid of a general function) and on the unstable subspace the norm of the solution tends to infinity (usually with the same type of growth rate—exponential, polynomial, etc.—as the stable one). The notion of exponential dichotomy has its origins from the work of Perron in 1930 [1]. This field has seen a rich development in the last decades, as it can be seen from [2–11].

Another behavior given by the above-mentioned problem is the decomposition of the state space into three subspaces: a stable subspace, an unstable subspace, and a central manifold. The behavior on the stable and unstable subspaces is dichotomic, and, in addition, the solution of the system must be bounded (or have a growth property). This behavior is known in the literature as the trichotomy property. The trichotomy property was first defined by Sacker and Sell in [12], and, later on, the study was widely spread and many results were obtained (see [13–18] and the references therein).

This paper extends the above-mentioned study of the property of trichotomy in the case in which the decay, expansion, and growth on the stable, unstable, and central manifold, respectively, are described by a polynomial behavior. We study three concepts of polynomial trichotomy (both uniform and nonuniform) defined in the general case of noninvertible evolution operators: polynomial trichotomy, strong polynomial trichotomy, and weak polynomial trichotomy. We establish the connections between the three concepts and, with the aid of the examples and counterexamples from Section 5, on one hand we point out the existence of systems which possess the above-defined properties, and, on the other hand, we delimit the behaviors presented in this paper.

2. Supplementary Families of Projections

Throughout this paper, we will consider the following framework:

(i) $l^2(\mathbb{N}, \mathbb{R})$ will be the Banach space of all real valued sequences $x = (x_n)_{n\geq 0}$ satisfying

$$\sum_{n=0}^{\infty} |x_n|^2 < \infty \tag{1}$$

endowed with the norm $\|x\|_2 = (\sum_{n=0}^{\infty} |x_n|^2)^{1/2}$.

(ii) X will be be a real or complex Banach space and $\mathscr{B}(X)$ will be the Banach space of all bounded linear operators on X.

(iii) The norms on X and $\mathscr{B}(X)$ will be denoted by $\|\cdot\|$.

(iv) The identity operator on X is denoted by I.

(v) Δ will be the set defined by $\Delta = \{(t,s) \in \mathbb{R}_+^2 : t \geq s \geq 0\}$.

Definition 1. A mapping $P : \mathbb{R}_+ \to \mathscr{B}(X)$ is called a *family of projections* on X if

$$P(t)P(t) = P(t), \quad \forall t \geq 0. \tag{2}$$

Definition 2. A family of projections $P : \mathbb{R}_+ \to \mathscr{B}(X)$ is called

(i) *polynomially bounded* if there exist $M \geq 1$ and $\gamma \geq 0$ such that

$$\|P(t)\| \leq M(t+1)^\gamma, \quad \forall t \geq 0; \tag{3}$$

(ii) *bounded* if there exists $M \geq 1$ such that

$$\|P(t)\| \leq M, \quad \forall t \geq 0. \tag{4}$$

Definition 3. Three families of projections $P, Q, R : \mathbb{R}_+ \to \mathscr{B}(X)$ are called *supplementary* if for all $t \geq 0$ one has that

$$P(t) + Q(t) + R(t) = I. \tag{5}$$

In what follows, we present two leading examples of families of projections which will be used in Section 5.

Example 4. Let $X = l^2(\mathbb{N}, \mathbb{R})$ and $p : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function. For every $t \geq 0$ we define $P_1(t) : l^2(\mathbb{N}, \mathbb{R}) \to l^2(\mathbb{N}, \mathbb{R})$ by

$$P_1(t)x = (y_n(t))_{n\geq 0}, \tag{6}$$

where

$$\begin{aligned} y_{3n}(t) &= x_{3n} + p(t)\cdot x_{3n+1}, \\ y_{3n+1}(t) &= 0, \\ y_{3n+2}(t) &= 0, \\ &\qquad n \in \mathbb{N}. \end{aligned} \tag{7}$$

Let $t \geq 0$. One can see that $P_1(t)$ is linear and if $x = (x_n)_{n\geq 0} \in l^2(\mathbb{N}, \mathbb{R})$, we have that

$$\begin{aligned} \|P_1(t)x\|_2^2 &= \sum_{n=0}^{\infty} |y_{3n}(t)|^2 \\ &\leq \left[\left(\sum_{n=0}^{\infty}|x_n|^2\right)^{1/2} + p(t)\left(\sum_{n=0}^{\infty}|x_n|^2\right)^{1/2}\right]^2 \\ &= [(1+p(t))]^2 \cdot \|x\|_2^2 \end{aligned} \tag{8}$$

from where it follows that $P_1(t) \in \mathscr{B}(l^2(\mathbb{N}, \mathbb{R}))$ and $\|P_1(t)\| \leq 1 + p(t)$.

Moreover, let $t \geq 0$ and $\tilde{x} = (\tilde{x}_n)_{n\geq 0}$ given by

$$\begin{aligned} \tilde{x}_{3n} &= \tilde{x}_{3n+2} = 0, \\ \tilde{x}_{3n+1} &= \frac{1}{3n+2}, \\ &\qquad n \in \mathbb{N}. \end{aligned} \tag{9}$$

From

$$\begin{aligned} \|P_1(t)\tilde{x}\|_2 &= \left(\sum_{n=0}^{\infty}|y_{3n}(t)|^2\right)^{1/2} \\ &= p(t)\cdot\left(\sum_{n=0}^{\infty}\frac{1}{(3n+2)^2}\right)^{1/2} = p(t)\cdot\|\tilde{x}\|_2 \end{aligned} \tag{10}$$

it follows that $\|P_1(t)\| \geq p(t)$. From here we get that

$$\max\{1, p(t)\} \leq \|P_1(t)\| \leq 1 + p(t) \quad \forall t \in \mathbb{R}_+. \tag{11}$$

Moreover, for $(t, x) \in \mathbb{R}_+ \times l^2(\mathbb{N}, \mathbb{R})$ we define the family of projections $Q_1 : \mathbb{R}_+ \to \mathscr{B}(X)$ by $Q_1(t)x = (z_n(t))_{n\geq 0}$, where

$$\begin{aligned} z_{3n}(t) &= -p(t)\,x_{3n+1}, \\ z_{3n+1}(t) &= x_{3n+1}, \\ z_{3n+2}(t) &= 0, \\ &\qquad n \in \mathbb{N},\ t \geq 0. \end{aligned} \tag{12}$$

Moreover, for $(t, s, x) \in \Delta \times l^2(\mathbb{N}, \mathbb{R})$ one can see that

$$\begin{aligned} \|Q_1(s)x\|_2^2 &= \sum_{n=0}^{\infty}|z_n(s)|^2 \\ &= \sum_{n=0}^{\infty}|z_{3n}(s)|^2 + \sum_{n=0}^{\infty}|z_{3n+1}(s)|^2 \\ &= \left(1 + p(s)^2\right)\sum_{n=0}^{\infty}|x_{3n+1}|^2 \\ &\leq \left(1 + p(t)^2\right)\sum_{n=0}^{\infty}|x_{3n+1}|^2 = \|Q_1(t)x\|_2^2; \end{aligned} \tag{13}$$

hence

$$\begin{aligned} &\|Q_1(s)x\|_2 \leq \|Q_1(t)x\|_2, \\ &\|Q_1(t)x\|_2 = \sqrt{1 + p(t)^2}\cdot\left(\sum_{n=0}^{\infty}|x_{3n+1}|^2\right)^{1/2}. \end{aligned} \tag{14}$$

Finally, define $R_1 : \mathbb{R}_+ \to \mathscr{B}(X)$ by $R_1(t) = (w_n(t))_{n\geq 0}$, where

$$\begin{aligned} w_{3n}(t) &= 0, \\ w_{3n+1}(t) &= 0, \\ w_{3n+2}(t) &= x_{3n+2}, \\ &\qquad n \in \mathbb{N},\ t \geq 0. \end{aligned} \tag{15}$$

We have that R_1 is bounded with

$$\|R_1(t)\| = 1, \quad \forall t \geq 0 \tag{16}$$

and moreover the families of projections P_1, Q_1, and R_1 are supplementary.

Example 5. Let $X = l^2(\mathbb{N}, \mathbb{R})$ and define $P_2, Q_2, R_2 : \mathbb{R}_+ \to \mathscr{B}(l^2(\mathbb{N}, \mathbb{R}))$ by $P_2(t)x = (y_n(t))_{n\geq 0}$, $Q_2(t)x = (z_n(t))_{t\geq 0}$, and $R_2(t)x = (w_n(t))_{n\geq 0}$, where, for $n \in \mathbb{N}$,

$$\begin{aligned}
y_{4n}(t) &= x_{4n},\\
y_{4n+1}(t) &= 0,\\
y_{4n+2}(t) &= 0,\\
y_{4n+3}(t) &= 0,\\
z_{4n}(t) &= 0,\\
z_{4n+1}(t) &= x_{4n+1},\\
z_{4n+2}(t) &= x_{4n+2},\\
z_{4n+3}(t) &= 0,\\
w_{4n}(t) &= 0,\\
w_{4n+1}(t) &= 0,\\
w_{4n+2}(t) &= 0,\\
w_{4n+3}(t) &= x_{4n+3}.
\end{aligned} \tag{17}$$

We have that P_2, Q_2, and R_2 are three supplementary families of projections with

$$\|P_2(t)\| = \|Q_2(t)\| = \|R_2(t)\| = 1 \quad \forall t \geq 0. \tag{18}$$

3. Evolution Operators

Definition 6. A mapping $\Phi : \Delta \to \mathscr{B}(X)$ is called an *evolution operator* on X if

(e_1) $\Phi(t,t) = I, \forall t \geq 0$;

(e_2) $\Phi(t,s)\Phi(s,t_0) = \Phi(t,t_0), \forall (t,s), (s,t_0) \in \Delta$.

Definition 7. A family of projections $P : \mathbb{R}_+ \to \mathscr{B}(X)$ is said to be *invariant* for the evolution operator $\Phi : \Delta \to \mathscr{B}(X)$ if

$$\Phi(t,s)P(s) = P(t)\Phi(t,s) \quad \forall (t,s) \in \Delta. \tag{19}$$

Given three supplementary families of projections P, Q, and R which are invariant for a given evolution operator Φ, we will name the quadruple (Φ, P, Q, R) a *trichotomy quadruple*.

Two important examples of trichotomy quadruples are given below, which will serve as a milestone in our examples and counterexamples.

Example 8. On $X = l^2(\mathbb{N}, \mathbb{R})$ consider the families of projections P_1, Q_1, and R_1 from Example 4. Consider $\varphi : \mathbb{R}_+ \to (0, \infty)$ and $\Phi_1 : \Delta \to \mathscr{B}(l^2(\mathbb{N}, \mathbb{R}))$ given by

$$\Phi_1(t,s) = \frac{\varphi(s)}{\varphi(t)} \cdot P_1(s) + \frac{\varphi(t)}{\varphi(s)} \cdot Q_1(t) + R_1(s) \tag{20}$$

for all $(t,s) \in \Delta$.

Taking into account that for all $t, s \in \mathbb{R}_+$ the following relations hold,

$$\begin{aligned}
P_1(t)P_1(s) &= P_1(s),\\
Q_1(t)Q_1(s) &= Q_1(t),
\end{aligned} \tag{21}$$

it follows that Φ_1 is an evolution operator. It is easy to check that P_1, Q_1, and R_1 are invariant for Φ_1; hence (Φ_1, P_1, Q_1, R_1) is a trichotomy quadruple. Moreover we have that

$$\begin{aligned}
\Phi_1(t,s)P_1(s) &= \frac{\varphi(s)}{\varphi(t)}P_1(s),\\
\Phi_1(t,s)Q_1(s) &= \frac{\varphi(t)}{\varphi(s)}Q_1(t),\\
\Phi_1(t,s)R_1(s) &= R_1(s)
\end{aligned} \tag{22}$$

for all $(t,s) \in \Delta$.

Example 9. On $X = l^2(\mathbb{N}, \mathbb{R})$ consider P_2, Q_2, and R_2 to be the families of projections defined in Example 5. For $\psi : \mathbb{R}_+ \to (0, \infty)$ define $\Phi_2 : \Delta \to \mathscr{B}(l^2(\mathbb{N}, \mathbb{R}))$ by

$$\Phi_2(t,s)x = \begin{cases} (y_n(t,s))_{n\geq 0} & \text{if } t > s \\ x, & \text{if } t = s, \end{cases} \tag{23}$$

where

$$\begin{aligned}
y_{4n}(t,s) &= \frac{\psi(s)}{\psi(t)}x_{4n},\\
y_{4n+1}(t,s) &= \frac{\psi(t)}{\psi(s)}x_{4n+1},\\
y_{4n+2}(t,s) &= 0,\\
y_{4n+3}(t,s) &= x_{4n+3},\\
&\qquad n \in \mathbb{N}
\end{aligned} \tag{24}$$

for all $(t,s,x) \in \Delta \times l^2(\mathbb{N}, \mathbb{R})$.

It is easy to see that (Φ_2, P_2, Q_2, R_2) is a trichotomy quadruple and for $(t,s,x) \in \Delta \times l^2(\mathbb{N}, \mathbb{R})$ one has that

$$\begin{aligned}
\Phi_2(t,s)P_2(s)x &= (p_n(t,s))_{n\geq 0},\\
p_{4n}(t,s) &= \frac{\psi(s)}{\psi(t)}x_{4n},\\
p_{4n+1}(t,s) &= p_{4n+2}(t,s) = p_{4n+3}(t,s) = 0.
\end{aligned} \tag{25}$$

$$\Phi_2(t,s)Q_2(s)x = \begin{cases} (q_n(t,s))_{n\geq 0}, & t > s \\ (\rho_n(t,s))_{n\geq 0}, & t = s \end{cases}$$

which is given by

$$\begin{gathered}q_{4n}(t,s)=q_{4n+3}(t,s)=0,\\ q_{4n+1}(t,s)=\frac{\psi(t)}{\psi(s)}x_{4n+1},\\ q_{4n+2}(t,s)=0,\\ \rho_{4n}(t,s)=\rho_{4n+3}(t,s)=0,\\ \rho_{4n+1}(t,s)=x_{4n+1},\\ \rho_{4n+2}(t,s)=x_{4n+2},\\ n\in\mathbb{N}\end{gathered}\tag{26}$$

and $\Phi_2(t,s)R_2(s)x=(r_n(t,s))_{n\geq0}$, where

$$\begin{gathered}r_{4n}(t,s)=r_{4n+1}(t,s)=r_{4n+2}(t,s)=0,\\ r_{4n+3}(t,s)=x_{4n+3}.\end{gathered}\tag{27}$$

In what follows, we will present the main trichotomy concepts that will be studied in the present paper.

4. Polynomial Trichotomies

Definition 10. A trichotomy quadruple (Φ,P,Q,R) is said to be *polynomially trichotomic* (p.t.) if there exist $N\geq1$, $\alpha>0$, and $\beta\geq0$ such that

(pt_1) $(t+1)^\alpha\|\Phi(t,s)P(s)\|\leq N(s+1)^{\alpha+\beta}$;

(pt_2) $(t+1)^\alpha\leq N(t+1)^\beta(s+1)^\alpha\|\Phi(t,s)Q(s)\|$;

(pt_3) $(s+1)^\alpha\|\Phi(t,s)R(s)\|\leq N(t+1)^\alpha(s+1)^\beta$;

(pt_4) $(s+1)^\alpha\leq N(t+1)^{\alpha+\beta}\|\Phi(t,s)R(s)\|$

for all $(t,s)\in\Delta$.

If β from the above definition is equal to 0, then we say that (Φ,P,Q,R) is *uniformly polynomially trichotomic* (u.p.t.).

Remark 11. The following assertions hold:

(i) If a trichotomy quadruple (Φ,P,Q,R) is (p.t.) then P and R are polynomially bounded, and hence Q is also polynomially bounded.

(ii) If a trichotomy quadruple (Φ,P,Q,R) is (u.p.t.) then P and R are bounded, and hence Q is also bounded.

In other words, if (Φ,P,Q,R) is (p.t.) with constants N, α, and β then

$$\max\{\|P(t)\|,\|Q(t)\|,\|R(t)\|\}\leq3N(t+1)^\beta,\quad \forall t\geq0.\tag{28}$$

Remark 12. If (Φ,P,Q,R) is (u.p.t.) then it is (p.t.). The converse is not generally true. Take, for example, the trichotomy quadruple (Φ_1,P_1,Q_1,R_1) from Example 8 with $p(t)=\varphi(t)=t+1$. It is easy to check that (Φ_1,P_1,Q_1,R_1) is (p.t.), but it cannot be (u.p.t.), because P is not bounded.

Definition 13. A trichotomy quadruple (Φ,P,Q,R) is said to be *strongly polynomially trichotomic* (s.p.t.) if there exist $N\geq1$, $\alpha>0$, and $\beta\geq0$ such that

(spt_1) $(t+1)^\alpha\|\Phi(t,s)P(s)x\|\leq N(s+1)^{\alpha+\beta}\|P(s)x\|$;

(spt_2) $(t+1)^\alpha\|Q(s)x\|\leq N(t+1)^\beta(s+1)^\alpha\|\Phi(t,s)Q(s)x\|$;

(spt_3) $(s+1)^\alpha\|\Phi(t,s)R(s)x\|\leq N(t+1)^\alpha(s+1)^\beta\|R(s)x\|$;

(spt_4) $(s+1)^\alpha\|R(s)x\|\leq N(t+1)^{\alpha+\beta}\|\Phi(t,s)R(s)x\|$

for all $(t,s,x)\in\Delta\times X$.

If β from the above definition is equal to 0, then we say that (Φ,P,Q,R) is *uniformly strongly polynomially trichotomic* (u.s.p.t.).

Remark 14. If (Φ,P,Q,R) is (u.s.p.t.) then it is (s.p.t.). The converse is not generally true, as shown in Example 1.

Remark 15. If (Φ,P,Q,R) is (s.p.t.) then the following condition holds:

$$\operatorname{Range}Q(s)\cap\operatorname{Ker}\Phi(t,s)=\{0\}\quad\forall(t,s)\in\Delta.\tag{29}$$

In other words, for all $(t,s)\in\Delta$,

$$\begin{gathered}x\in\operatorname{Range}Q(s),\\ \Phi(t,s)x=0\Longrightarrow x=0.\end{gathered}\tag{30}$$

Remark 16. Under the same assumption as in Remark 15, we also have that

$$\operatorname{Range}R(s)\cap\operatorname{Ker}\Phi(t,s)=\{0\}\quad\forall(t,s)\in\Delta.\tag{31}$$

Definition 17. A trichotomy quadruple (Φ,P,Q,R) is said to be *weakly polynomially trichotomic* (w.p.t.) if there exist $N\geq1$, $\alpha>0$, and $\beta\geq0$ such that

(wpt_1) $(t+1)^\alpha\|\Phi(t,s)P(s)\|\leq N(s+1)^{\alpha+\beta}\|P(s)\|$;

(wpt_2) $(t+1)^\alpha\|Q(s)\|\leq N(t+1)^\beta(s+1)^\alpha\|\Phi(t,s)Q(s)\|$;

(wpt_3) $(s+1)^\alpha\|\Phi(t,s)R(s)\|\leq N(t+1)^\alpha(s+1)^\beta\|R(s)\|$;

(wpt_4) $(s+1)^\alpha\|R(s)\|\leq N(t+1)^{\alpha+\beta}\|\Phi(t,s)R(s)\|$

for all $(t,s)\in\Delta$.

If $\beta=0$ then we say that (Φ,P,Q,R) is *uniformly weakly polynomially trichotomic* (u.w.p.t.).

Remark 18. If (Φ,P,Q,R) is (u.w.p.t.) then it is (w.p.t.). The converse is not generally true, as shown in Example 2.

In what follows we will study the connections between these three trichotomy concepts.

Remark 19. If a trichotomy quadruple (Φ,P,Q,R) is (s.p.t.) then it is also (w.p.t.). Moreover, if (Φ,P,Q,R) is (u.s.p.t.), then it is (u.w.p.t.).

Proposition 20. *Let (Φ,P,Q,R) be a trichotomy quadruple. If (Φ,P,Q,R) is (p.t.) then it is also (w.p.t.).*

Proof. Let $N \geq 1$, $\alpha > 0$, and $\beta \geq 0$ be given by Definition 10. By Remark 11, we have that

$$1 \leq \max\{\|P(t)\|, \|Q(t)\|, \|R(t)\|\} \leq 3N(t+1)^{\beta} \quad \forall t \geq 0. \tag{32}$$

Let now $(t,s) \in \Delta$. From the estimations

$$\begin{aligned}
&(t+1)^{\alpha}\|\Phi(t,s)P(s)\| \leq N(s+1)^{\alpha+\beta} \\
&\quad \leq 3N^2(s+1)^{\alpha+2\beta}\|P(s)\|, \\
&(t+1)^{\alpha}\|Q(s)\| \\
&\quad \leq N(t+1)^{\beta}(s+1)^{\alpha}\|\Phi(t,s)Q(s)\| \cdot \|Q(s)\| \\
&\quad \leq 3N^2(s+1)^{\alpha}(t+1)^{2\beta}\|\Phi(t,s)Q(s)\| \\
&(s+1)^{\alpha}\|\Phi(t,s)R(s)\| \leq N(t+1)^{\alpha}(s+1)^{\beta} \\
&\quad \leq N(t+1)^{\alpha}(s+1)^{\beta}\|R(s)\| \\
&\quad \leq 3N^2(t+1)^{\alpha}(s+1)^{2\beta}\|R(s)\|, \\
&(s+1)^{\alpha}\|R(s)\| \leq N(t+1)^{\alpha+\beta}\|\Phi(t,s)R(s)\| \cdot \|R(s)\| \\
&\quad \leq 3N^2(t+1)^{\alpha+\beta}(s+1)^{\beta}\|\Phi(t,s)R(s)\| \\
&\quad \leq 3N^2(t+1)^{\alpha+2\beta}\|\Phi(t,s)R(s)\|,
\end{aligned} \tag{33}$$

it follows that (Φ, P, Q, R) is (w.p.t.) with constants $3N^2 \geq 1$, $\alpha > 0$, and $2\beta \geq 0$. □

Remark 21. From the proof of the above proposition, we can easily see that, by setting $\beta = 0$, we obtain the implication (u.p.t.) $\Rightarrow$ (u.w.p.t.).

Other connections are given by the following.

Remark 22. (i) (s.p.t.) does not imply (p.t.) and (u.s.p.t.) does not imply (u.p.t.) as shown by Example 3.

(ii) The concepts of (p.t.) and (w.p.t.) do not coincide, as we can see from Example 4.

(iii) (p.t.) does not imply (s.p.t.) and (u.p.t.) does not imply (u.s.p.t.), as shown by Example 5.

(iv) (w.p.t.) does not imply (s.p.t.) and (u.w.p.t.) does not imply (u.s.p.t.) as shown in Example 6.

Remark 23. The connection between the presented concepts is given by the following diagram:

$$\begin{array}{ccccccc}
\boxed{\text{u.p.t.}} & \overset{\nLeftarrow}{\nRightarrow} & \boxed{\text{u.s.p.t.}} & \overset{\nLeftarrow}{\Rightarrow} & \boxed{\text{u.w.p.t.}} & \overset{\Leftarrow}{\nRightarrow} & \boxed{\text{u.p.t.}} \\
\not\Uparrow\Downarrow & & \not\Uparrow\Downarrow & & \not\Uparrow\Downarrow & & \not\Uparrow\Downarrow \\
\boxed{\text{p.t.}} & \overset{\nLeftarrow}{\nRightarrow} & \boxed{\text{s.p.t.}} & \overset{\nLeftarrow}{\Rightarrow} & \boxed{\text{w.p.t.}} & \overset{\Leftarrow}{\nRightarrow} & \boxed{\text{p.t.}}
\end{array} \tag{34}$$

5. Examples and Counterexamples

Example 1 (trichotomy quadruple that is (s.p.t.) but not (u.s.p.t.)). Let (Φ_1, P_1, Q_1, R_1) be the trichotomy quadruple from Example 8 with $\varphi(t) = (t+1)^{3-\cos\ln(t+1)}$ and $p(t) = 0$, $t \geq 0$. Then we have that

$$\begin{aligned}
&(t+1)^2\left\|\Phi_1(t,s)P_1(s)x\right\|_2 \leq (s+1)^4\left\|P_1(s)x\right\|_2, \\
&(t+1)^2\left\|Q_1(s)x\right\|_2 \leq (s+1)^4\left\|\Phi_1(t,s)Q_1(s)x\right\|_2, \\
&\|\Phi(t,s)R(s)x\|_2 = \|R(s)x\|_2
\end{aligned} \tag{35}$$

for all $(t,s,x) \in \Delta \times X$; hence (Φ_1, P_1, Q_1, R_1) is (s.p.t.).

Assume, by a contradiction, that (Φ, P, Q, R) is (u.s.p.t.). Then there exist $N \geq 1$ and $\alpha > 0$ such that for all $n \in \mathbb{N}$ and for $t_n = e^{2n\pi} - 1$ and $s_n = e^{2n\pi-\pi/2} - 1$ we have, from (spt_1), that

$$\frac{\left(e^{2n\pi-\pi/2}\right)^4}{e^{4n\pi}} \leq N\left(\frac{e^{2n\pi-\pi/2}}{2^{2n\pi}}\right)^{\alpha} \tag{36}$$

which leads us to the contradiction

$$e^{2n\pi} \leq Ne^{-\pi/2}, \quad \forall n \in \mathbb{N}. \tag{37}$$

Example 2 (trichotomy quadruple that is (w.p.t.) but not (u.w.p.t.)). Let (Φ_1, P_1, Q_1, R_1) be as in Example 1. By Remark 19 we have that (Φ_1, P_1, Q_1, R_1) is (w.p.t.). The same contradiction is obtained as in Example 1, by assuming that (Φ_1, P_1, Q_1, R_1) is (u.w.p.t.).

Example 3 (trichotomy quadruple which is (s.p.t.) but fails to be (p.t.)). Let (Φ_1, P_1, Q_1, R_1) be the trichotomy quadruple from Example 8 with $p(t) = (t+1)^{t+1}$ and $\varphi(t) = t+1$.

From

$$\begin{aligned}
&(t+1)\left\|\Phi_1(t,s)P_1(s)x\right\|_2 = (s+1)\left\|P_1(s)x\right\|_2, \\
&(t+1)\left\|Q_1(s)x\right\|_2 \leq \varphi(t)\left\|Q_1(t)x\right\|_2 \\
&\quad = (s+1)\left\|\Phi_1(t,s)Q_1(s)x\right\|_2, \\
&(s+1)\|\Phi(t,s)R(s)x\|_2 \leq (s+1)(t+1)\|R(s)x\|_2, \\
&(s+1)\|R(s)x\|_2 \leq N(t+1)^2\|\Phi(t,s)R(s)x\|_2
\end{aligned} \tag{38}$$

for all $(t,s,x) \in \Delta \times X$, we can see that (Φ_1, P_1, Q_1, R_1) is (u.s.p.t.) and hence (s.p.t.).

Assume, by a contradiction, that (Φ_1, P_1, Q_1, R_1) is (p.t.). Then, by Remark 11, we have that there exist $M \geq 1$, $\gamma \geq 0$, such that

$$\left\|P_1(t)\right\| \leq M(t+1)^{\gamma} \quad \forall t \geq 0. \tag{39}$$

This leads us to the contradiction

$$(t+1)^{t+1} = p(t) \leq \left\|P_1(t)\right\| \leq M(t+1)^{\gamma} \quad \forall t \geq 0. \tag{40}$$

It follows that (Φ_1, P_1, Q_1, R_1) is not (p.t.) and hence not (u.p.t.).

Example 4 (trichotomy quadruple which is (w.p.t.) but not (p.t.)). Let (Φ_1, P_1, Q_1, R_1) the trichotomy quadruple from Example 3. By Remark 19, we have that (Φ_1, P_1, Q_1, R_1) is (u.w.p.t.) and hence (w.p.t.). But, by Example 3, it is not (p.t.) and hence not (u.p.t.).

Example 5 (trichotomy quadruple which is (p.t.) but fails to be (s.p.t.)). Let (Φ_2, P_2, Q_2, R_2) be the trichotomy quadruple from Example 9 with $\psi(t) = t + 1$. First of all we will show that (Φ_2, P_2, Q_2, R_2) is (u.p.t.). Let $(t, s, x) \in \Delta \times l^2(\mathbb{N}, \mathbb{R})$. We have that

$$\begin{aligned}(t+1)^2 \left\|\Phi_2(t,s) P_2(s) x\right\|_2^2 &= \sum_{n=0}^{\infty} \left|p_n(t,s)\right|^2 \\ &= (s+1)^2 \sum_{n=0}^{\infty} \left|x_{4n}\right|^2 = (s+1)^2 \left\|P_2(s) x\right\|_2^2 ;\end{aligned} \tag{41}$$

hence

$$(t+1) \left\|\Phi_2(t,s) P_2(s)\right\| \le s+1. \tag{42}$$

If $t > s$, consider $x = (x_n)_{n\ge0}$ given by

$$x_n = \begin{cases} \dfrac{1}{n}, & n = 4k+1 \\ 0, & \text{otherwise.} \end{cases} \tag{43}$$

We have that $\|x\|_2 = (\sum_{n=0}^{\infty}(1/(4n+1)^2))^{1/2}$ and

$$\begin{aligned}(s+1) \left\|\Phi_2(t,s) Q_2(s) x\right\|_2 &= \left(\sum_{n=0}^{\infty} \left|q_n(t,s)\right|^2\right)^{1/2} \\ &= (t+1) \|x\|_2 ;\end{aligned} \tag{44}$$

hence

$$t+1 \le (s+1) \left\|\Phi_2(t,s) Q_2(s)\right\|. \tag{45}$$

Having in mind that $\|\Phi_2(t,s)R(s)x\|_2 = \|R(s)x\|_2$, it follows that (spt_3) and (spt_4) hold for $(t, s, x) \in \Delta \times X$ with $t > s$. The case in which $t = s$ obviously leads us to the above estimation, and so the conclusion follows.

In what follows, we will show that (Φ_2, P_2, Q_2, R_2) is not (s.p.t.), and hence it is not (u.s.p.t.). Assume by a contradiction that (Φ_2, P_2, Q_2, R_2) is (s.p.t.). We will disprove the result from Remark 15. Let $x = (x_n)_{n\ge0}$ given by

$$\begin{aligned} x_{4n+2} &= \frac{1}{4n+2}, \\ x_{4n+3} &= x_{4n+1} = x_{4n} = 0 \\ & \qquad n \in \mathbb{N}. \end{aligned} \tag{46}$$

Obviously $x \in l^2(\mathbb{N}, \mathbb{R})$ and denote, for every $s \ge 0$, $Q_2(s)x = (z_n(s))_{n\ge0}$, where

$$\begin{aligned} z_{4n}(s) &= 0, \\ z_{4n+1}(s) &= x_{4n+1} = 0, \\ z_{4n+2}(s) &= x_{4n+2} = \frac{1}{4n+2}, \\ z_{4n+3}(s) &= 0; \end{aligned} \tag{47}$$

hence $(z_n(s))_{n\ge0}$ is a nonzero sequence.

Consider now $(t, s) \in \Delta$ with $t > s$. By denoting $\Phi_2(t,s)Q_2(s)x = (q_n(t,s))_{n\ge0}$, with

$$\begin{aligned} q_{4n}(t,s) &= 0, \\ q_{4n+1}(t,s) &= \frac{t+1}{s+1} x_{4n+1} = 0, \\ q_{4n+2}(t,s) &= 0, \\ q_{4n+3}(t,s) &= 0, \end{aligned} \tag{48}$$

it follows that $\Phi_2(t,s)Q_2(s)x = 0$, which contradicts Remark 15; hence (Φ_2, P_2, Q_2, R_2) is not (s.p.t.).

Example 6 (trichotomy quadruple that is (w.p.t.) but fails to be (s.p.t.)). Let (Φ_2, P_2, Q_2, R_2) be the trichotomy quadruple from Example 5. Taking into account that, for all $s \ge 0$, $\|P_2(s)\| = \|Q_2(s)\| = \|R_2(s)\| = 1$, it follows that (Φ_2, P_2, Q_2, R_2) is (u.w.p.t.) and hence (w.p.t.).

Again, from Example 5, we get that (Φ_2, P_2, Q_2, R_2) is not (s.p.t.) and hence not (u.s.p.t.).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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