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Research Article

Sharp Power Mean Bounds for Sándor Mean

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We prove that the double inequality $M_p(a,b) < X(a,b) < M_q(a,b)$ holds for all $a,b > 0$ with $a \neq b$ if and only if $p \leq 1/3$ and $q \geq \log 2/(1+\log 2) = 0.4093\ldots$, where $X(a,b)$ and $M_r(a,b)$ are the Sándor and rth power means of a and b, respectively.

1. Introduction

Let $p \in \mathbb{R}$ and $a,b > 0$ with $a \neq b$. Then the pth power mean $M_p(a,b)$ of a and b is given by

$$M_p(a,b) = \left(\frac{a^p + b^p}{2}\right)^{1/p} \quad (p \neq 0), \quad M_0(a,b) = \sqrt{ab}. \tag{1}$$

The main properties for the power mean are given in [1]. It is well known that $M_p(a,b)$ is strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a,b > 0$ with $a \neq b$. Many classical means are the special cases of the power mean; for example, $M_{-1}(a,b) = 2ab/(a+b) = H(a,b)$ is the harmonic mean, $M_0(a,b) = \sqrt{ab} = G(a,b)$ is the geometric mean, $M_1(a,b) = (a+b)/2 = A(a,b)$ is the arithmetic mean, and $M_2(a,b) = \sqrt{(a^2+b^2)/2} = Q(a,b)$ is the quadratic mean.

Let $L(a,b) = (a-b)/(\log a - \log b)$, $P(a,b) = (a-b)/[2\arcsin((a-b)/(a+b))]$, $I(a,b) = (a^a/b^b)^{1/(a-b)}/e$, $M(a,b) = (a-b)/[2\sinh^{-1}((a-b)/(a+b))]$, and $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$ be the logarithmic, first Seiffert, identric, Neuman-Sándor, and second Seiffert means of two distinct positive real numbers a and b, respectively. Then it is well known that the inequalities

$$\begin{aligned} H(a,b) &< G(a,b) < L(a,b) < P(a,b) \\ &< I(a,b) < A(a,b) < M(a,b) \\ &< T(a,b) < Q(a,b) \end{aligned} \tag{2}$$

hold for all $a,b > 0$ with $a \neq b$.

Recently, the bounds for certain bivariate means in terms of the power mean have been the subject of intensive research. Seiffert [2] proved that the inequalities

$$\begin{aligned} \frac{2}{\pi}M_1(a,b) &< P(a,b) < M_1(a,b) \\ &< T(a,b) < M_2(a,b) \end{aligned} \tag{3}$$

hold for all $a,b > 0$ with $a \neq b$.

Jagers [3] proved that the double inequality

$$M_{1/2}(a,b) < P(a,b) < M_{2/3}(a,b) \tag{4}$$

holds for all $a,b > 0$ with $a \neq b$.

In [4, 5], Hästö established that

$$\begin{aligned} P(a,b) &> M_{\log 2/\log \pi}(a,b), \\ P(a,b) &> \frac{2\sqrt{2}}{\pi}M_{2/3}(a,b) \end{aligned} \tag{5}$$

for all $a,b > 0$ with $a \neq b$.

Witkowski [6] proved that the double inequality

$$\frac{2\sqrt{2}}{\pi}M_2(a,b) < T(a,b) < \frac{4}{\pi}M_1(a,b) \tag{6}$$

holds for all $a,b > 0$ with $a \neq b$.

In [7], Costin and Toader presented that

$$\begin{aligned} M_{\log 2/(\log \pi - \log 2)}(a,b) &< T(a,b) \\ &< M_{5/3}(a,b) \end{aligned} \tag{7}$$

for all $a,b > 0$ with $a \neq b$.

Chu and Long [8] proved that the double inequality

$$M_p(a,b) < M(a,b) < M_q(a,b) \quad (8)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq \log 2/\log[2\log(1+\sqrt{2})] = 1.224\ldots$ and $q \geq 4/3$.

The following sharp bounds for the logarithmic and identric means in terms of the power means can be found in the literature [9–16]:

$$\begin{aligned} &M_0(a,b) < L(a,b) < M_{1/3}(a,b),\\ &M_{2/3}(a,b) < I(a,b) < M_{\log 2}(a,b),\\ &M_0(a,b) < L^{1/2}(a,b)\,I^{1/2}(a,b) < M_{1/2}(a,b),\\ &M_{\log 2/(1+\log 2)}(a,b) < \frac{L(a,b)+I(a,b)}{2} < M_{1/2}(a,b) \end{aligned} \quad (9)$$

for all $a, b > 0$ with $a \neq b$.

Recently, Sándor [17] introduced the Sándor mean $X(a,b)$ of two positive real numbers a and b, which is given by

$$X(a,b) = A(a,b)\,e^{(G(a,b)/P(a,b))-1}. \quad (10)$$

In [18], Sándor proved that

$$\begin{aligned} &X(a,b) < \frac{P^2(a,b)}{A(a,b)},\\ &\frac{A(a,b)G(a,b)}{P(a,b)} < X(a,b) < \frac{A(a,b)P(a,b)}{2P(a,b)-G(a,b)},\\ &X(a,b) > \frac{A(a,b)L(a,b)}{P(a,b)}e^{(G(a,b)/L(a,b))-1},\\ &X(a,b) > \frac{A(a,b)[P(a,b)+G(a,b)]}{3P(a,b)-G(a,b)},\\ &\frac{A^2(a,b)G(a,b)}{P(a,b)L(a,b)}e^{(L(a,b)/A(a,b))-1} < X(a,b)\\ &\quad < A(a,b)\left[\frac{1}{e} + \left(1-\frac{1}{e}\right)\frac{G(a,b)}{P(a,b)}\right],\\ &A(a,b)+G(a,b)-P(a,b) < X(a,b)\\ &\quad < A^{-1/3}(a,b)\left[\frac{A(a,b)+G(a,b)}{2}\right]^{4/3},\\ &P^{1/(\log\pi-\log 2)}(a,b)\,A^{1-1/(\log\pi-\log 2)}(a,b)\\ &\quad < X(a,b) < P^{-1}(a,b)\left[\frac{A(a,b)+G(a,b)}{2}\right]^2 \end{aligned} \quad (11)$$

for all $a, b > 0$ with $a \neq b$.

In the Introduction we cite only a minor part of the existing literature on the considered means. For example, an important paper on the first Seiffert mean $P(a,b)$ is again due to Sándor [19].

The main purpose of this paper is to present the best possible parameters p and q such that the double inequality $M_p(a,b) < X(a,b) < M_q(a,b)$ holds for all $a, b > 0$ with $a \neq b$.

2. Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

Lemma 1. *Let $g_1 : (0,1)\times\mathbb{R} \to \mathbb{R}$ be defined by*

$$g_1(x,p) = \frac{\sqrt{x}(x-1)\left(x^{p-1}+1\right)}{(x+1)(x^p+1)} - \arcsin\frac{x-1}{x+1}. \quad (12)$$

Then

(1) *$g_1(x,p)$ is strictly decreasing with respect to x on $(0,1)$ if and only if $p \geq 1/2$;*

(2) *$g_1(x,p)$ is strictly increasing with respect to x on $(0,1)$ if and only if $p \leq 1/3$.*

Proof. It follows from (12) that

$$\frac{\partial g_1(x,p)}{\partial x} = \frac{(1-x)\,x^{p-3/2}}{2(x+1)^2(x^p+1)^2}g_2(x,p), \quad (13)$$

where

$$\begin{aligned} g_2(x,p) = &-3x^{1-p} - x^{2-p} + x^p + 3x^{p+1}\\ &+ (2p-1)x^2 - 2p + 1. \end{aligned} \quad (14)$$

(1) If $g_1(x,p)$ is strictly decreasing with respect to x on $(0,1)$, then (13) leads to the conclusion that $g_2(x,p) < 0$ for all $x \in (0,1)$. In particular, we have $g_2(0^+,p) \leq 0$. We assert that $p \geq 1/2$. Indeed, from (14) we clearly see that $g_2(0^+,0) = 2$, $g_2(0^+,p) = +\infty$ if $p < 0$, and $g_2(0^+,p) = 1-2p > 0$ if $0 < p < 1/2$.

If $p \geq 1/2$, then it follows from (14) that

$$\begin{aligned} \frac{\partial g_2(x,p)}{\partial p} = &\left(3x^{p+1} + 3x^{1-p} + x^{2-p} + x^p\right)\log x\\ &- 2\left(1-x^2\right) < 0 \end{aligned} \quad (15)$$

for all $x \in (0,1)$.

Equation (14) and inequality (15) lead to the conclusion that

$$g_2(x,p) \leq g_2\left(x,\frac{1}{2}\right) = -2\sqrt{x}(1-x) < 0 \quad (16)$$

for all $x \in (0,1)$.

Therefore, $g_1(x,p)$ is strictly decreasing with respect to x on $(0,1)$ which follows from (13) and (16).

(2) If $g_1(x,p)$ is strictly increasing with respect to x on $(0,1)$, then (13) leads to the conclusion that $g_2(x,p) > 0$ for all $x \in (0,1)$. In particular, we have

$$\lim_{x\to 1^-}\frac{g_2(x,p)}{1-x} = 4 - 12p \geq 0 \quad (17)$$

and $p \leq 1/3$.

If $p \leq 1/3$, then (14) and (15) lead to the conclusion that

$$g_2(x,p) \geq g_2\left(x,\frac{1}{3}\right) = \frac{1}{3}\left(1+x^{1/3}\right)\left(1+5x^{1/3}+x^{2/3}\right) \times \left(1-x^{1/3}\right)^3 > 0 \tag{18}$$

for all $x \in (0,1)$.

Therefore, $g_1(x,p)$ is strictly increasing with respect to x on $(0,1)$ which follows from (13) and (18). □

Lemma 2. *Let $g_1 : (0,1)\times\mathbb{R} \to \mathbb{R}$ be defined by (12). Then there exists $x_0 \in (0,1)$ such that $g_1(x,p)$ is strictly increasing with respect to x on $(0,x_0]$ and strictly decreasing with respect to x on $[x_0,1)$ if $1/3 < p < 1/2$.*

Proof. Let $p \in (1/3,1/2)$ and $g_2(x,p)$ be defined by (14). Then (14) leads to

$$g_2(0,p) = 1-2p > 0, \qquad g_2(1,p) = 0, \tag{19}$$

$$x^{1-p}\frac{\partial g_2(x,p)}{\partial x} = 3(p-1)x^{1-2p} + (p-2)x^{2-2p} + 2(2p-1)x^{2-p} + 3(p+1)x + p := g_3(x,p), \tag{20}$$

$$g_3(0,p) = p > 0, \qquad g_3(1,p) = 12p-4 > 0, \tag{21}$$

$$x^{2p}\frac{\partial g_3(x,p)}{\partial x} = 3(p+1)x^{2p} - 2(2p-1)(p-2)x^{1+p} - 2(p-1)(p-2)x - 3(2p-1)(p-1) := g_4(x,p), \tag{22}$$

$$g_4(0,p) = -3(1-p)(1-2p) < 0, \quad g_4(1,p) = 4(3p-1)(2-p) > 0, \tag{23}$$

$$\frac{\partial^2 g_4(x,p)}{\partial x^2} = -2p(1-2p)(p+1) \times \left[3+(2-p)x^{1-p}\right]x^{2p-2} < 0 \tag{24}$$

for $x \in (0,1)$.

Inequality (24) implies that $g_4(x,p)$ is strictly convex with respect to x on $(0,1)$. From (22) and (23) together with the strict convexity of $g_4(x,p)$ with respect to x on $(0,1)$ we clearly see that there exists $x_1 \in (0,1)$ such that $g_3(x,p)$ is strictly decreasing with respect to x on $(0,x_1]$ and strictly increasing with respect to x on $[x_1,1)$. We assert that

$$g_3(x_1,p) < 0. \tag{25}$$

Indeed, if $g_3(x_1,p) \geq 0$, then it follows from (20) and the piecewise monotonicity of $g_3(x,p)$ with respect to x on $(0,1)$ that $g_2(x,p)$ is strictly increasing with respect to x on $(0,1)$. Hence, we get $g_2(x,p) < g_2(1,p) = 0$ for all $x \in (0,1)$. This conjunction with Lemma 1 and (13) leads to the conclusion that $p \geq 1/2$, which contradicts with $1/3 < p < 1/2$.

From (20) and (21) together with (25) and the piecewise monotonicity of $g_3(x,p)$ with respect to x on $(0,1)$ we clearly see that there exist $x_{11} \in (0,x_1)$ and $x_{12} \in (x_1,1)$ such that $g_2(x,p)$ is strictly increasing with respect to x on $(0,x_{11}] \cup [x_{12},1)$ and strictly decreasing with respect to x on $[x_{11},x_{12}]$.

Therefore, Lemma 2 follows easily from (13) and (19) together with the piecewise monotonicity of $g_2(x,p)$ with respect to x on $(0,1)$. □

Lemma 3. *Let $g_1 : (0,1)\times\mathbb{R} \to \mathbb{R}$ be defined by (12). Then the following statements are true:*

(1) *$g_1(x,p) > 0$ for all $x \in (0,1)$ if and only if $p \geq 1/2$;*

(2) *$g_1(x,p) < 0$ for all $x \in (0,1)$ if and only if $p \leq 1/3$;*

(3) *if $1/3 < p < 1/2$, then there exists $\mu_0 \in (0,1)$ such that $g_1(\mu_0,p) = 0$, $g_1(x,p) < 0$ for $x \in (0,\mu_0)$, and $g_1(x,p) > 0$ for $x \in (\mu_0,1)$.*

Proof. (1) If $g_1(x,p) > 0$ for all $x \in (0,1)$, then $g_1(0^+,p) \geq 0$. Therefore, $p \geq 1/2$ follows from $g_1(0^+,p) = -\infty$ for $p < 1/2$.

If $p \geq 1/2$, then Lemma 1 (1) leads to the conclusion that $g_1(x,p) > g_1(1,p) = 0$ for all $x \in (0,1)$.

(2) If $g_1(x,p) < 0$ for all $x \in (0,1)$, then by making use of L'Höspital's rules and (12) we get

$$\lim_{x\to 1^-}\frac{g_1(x,p)}{(1-x)^3} = \frac{1}{8}\left(p-\frac{1}{3}\right) \leq 0 \tag{26}$$

and $p \leq 1/3$.

If $p \leq 1/3$, then Lemma 1 (2) leads to the conclusion that $g_1(x,p) < g_1(1,p) = 0$ for all $x \in (0,1)$.

(3) If $1/3 < p < 1/2$, then it follows from (12) that

$$g_1(0^+,p) = -\infty, \qquad g_1(1,p) = 0. \tag{27}$$

Therefore, Lemma 3 (3) follows from Lemma 2 and (27). □

Lemma 4. *Let $g : (0,1)\times(0,\infty) \to \mathbb{R}$ be defined by*

$$g(x,p) = \log\frac{X(1,x)}{M_p(1,x)} = \log\frac{x+1}{2} + \frac{2\sqrt{x}}{1-x}\arcsin\frac{1-x}{1+x} - \frac{1}{p}\log\frac{x^p+1}{2} - 1. \tag{28}$$

Then

(1) *$g(x,p)$ is strictly increasing with respect to x on $(0,1)$ if and only if $p \geq 1/2$;*

(2) *$g(x,p)$ is strictly decreasing with respect to x on $(0,1)$ if and only if $p \leq 1/3$;*

(3) *if $1/3 < p < 1/2$, there exists $\mu_0 \in (0,1)$ such that $g(x,p)$ is strictly decreasing with respect to x on $(0,\mu_0]$ and strictly increasing with respect to x on $[\mu_0,1)$.*

Proof. It follows from (28) that

$$\frac{\partial g(x,p)}{\partial x}=\frac{1+x}{(1-x)^2\sqrt{x}}g_1(x,p), \tag{29}$$

where $g_1(x,p)$ is defined by (12).

Therefore, Lemma 4 follows from Lemma 3 and (29). □

3. Main Results

Theorem 5. *The double inequality*

$$M_p(a,b)<X(a,b)<M_q(a,b) \tag{30}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $p\le 1/3$ and $q\ge \log 2/(1+\log 2)=0.4093\ldots$.

Proof. Since both the Sándor mean $X(a,b)$ and rth power mean $M_r(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a=1$ and $b=x\in(0,1)$.

We first prove that the inequality $X(1,x)>M_p(1,x)$ holds for all $x\in(0,1)$ if and only if $p\le 1/3$.

If $p=1/3$, then from (28) and Lemma 4 (2) we get

$$\log\frac{X(1,x)}{M_{1/3}(1,x)}=g\left(x,\frac{1}{3}\right)>g\left(1^-,\frac{1}{3}\right)=0 \tag{31}$$

for all $x\in(0,1)$.

Therefore, $X(1,x)>M_p(1,x)$ for all $x\in(0,1)$ and $p\le 1/3$ follows from (31) and the monotonicity of the function $p\to M_p(1,x)$.

If $X(1,x)>M_p(1,x)$, then (28) leads to $g(x,p)>0$ for all $x\in(0,1)$. In particular, we have

$$\lim_{x\to 1^-}\frac{g(x,p)}{(1-x)^2}=\frac{1}{8}\left(\frac{1}{3}-p\right)\ge 0 \tag{32}$$

and $p\le 1/3$.

Next, we prove that the inequality $X(1,x)<M_q(1,x)$ holds for all $x\in(0,1)$ if and only if $q\ge\log 2/(1+\log 2)$.

If $X(1,x)<M_q(1,x)$ holds for all $x\in(0,1)$, then (28) leads to $g(x,q)<0$ for all $x\in(0,1)$. In particular, we have

$$g(0,q)=\left(\frac{1}{q}-1\right)\log 2-1\le 0 \tag{33}$$

and $q\ge\log 2/(1+\log 2)$.

If $q=\log 2/(1+\log 2)\in(1/3,1/2)$, then (28) leads to

$$g\left(0,\frac{\log 2}{1+\log 2}\right)=g\left(1,\frac{\log 2}{1+\log 2}\right)=0. \tag{34}$$

It follows from (28) and (34) together with Lemma 4 (3) that

$$\log\frac{X(1,x)}{M_{\log 2/(1+\log 2)}(1,x)}=g\left(x,\frac{\log 2}{1+\log 2}\right)<0 \tag{35}$$

for all $x\in(0,1)$.

Therefore, $X(1,x)<M_q(1,x)$ for all $x\in(0,1)$ and $q\ge\log 2/(1+\log 2)$ follows from (35) and the monotonicity of the function $q\to M_q(1,x)$. □

Theorem 6. *Let $a,b>0$ with $a\neq b$. Then the double inequality*

$$\frac{2}{e}M_{1/2}(a,b)<X(a,b)<\frac{4}{e}M_{1/3}(a,b) \tag{36}$$

holds with the best possible constants $2/e$ and $4/e$.

Proof. Since both the Sándor mean $X(a,b)$ and rth power mean $M_r(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a=1$ and $b=x\in(0,1)$. It follows from Lemma 4 (1) and (2) together with (28) that

$$\begin{aligned}\log\frac{X(1,x)}{M_{1/2}(1,x)}&=g\left(x,\frac{1}{2}\right)>g\left(0,\frac{1}{2}\right)=\log\frac{2}{e},\\ \log\frac{X(1,x)}{M_{1/3}(1,x)}&=g\left(x,\frac{1}{3}\right)<g\left(0,\frac{1}{3}\right)=\log\frac{4}{e}\end{aligned} \tag{37}$$

for all $x\in(0,1)$.

Therefore, $2/eM_{1/2}(1,x)<X(1,x)<4/eM_{1/3}(1,x)$ for all $x\in(0,1)$ follows from (37), and the optimality of the parameters $2/e$ and $4/e$ follows from the monotonicity of the functions $g(x,1/2)$ and $g(x,1/3)$. □

Remark 7. For all $a_1,a_2,b_1,b_2>0$ with $a_1/b_1<a_2/b_2<1$. Then from Lemma 4 (1) and (2) together with (28) we clearly see that the Ky Fan type inequalities

$$\frac{M_p(a_2,b_2)}{M_p(a_1,b_1)}<\frac{X(a_2,b_2)}{X(a_1,b_1)}<\frac{M_q(a_2,b_2)}{M_q(a_1,b_1)} \tag{38}$$

hold if and only if $p\ge 1/2$ and $q\le 1/3$.

Let $p\in\mathbb{R}$ and $L_p(a,b)=(a^{p+1}+b^{p+1})/(a^p+b^p)$ be the pth Lehmer mean of two positive real numbers a and b. Then the function $g_1(x,p)$ defined by (12) can be rewritten as

$$g_1(x,p)=\frac{1}{2}(1-x)\left[\frac{1}{P(1,x)}-\frac{G(1,x)}{A(1,x)L_{p-1}(1,x)}\right]. \tag{39}$$

From Lemma 3 and (39) we get Remark 8 as follows.

Remark 8. The double inequality

$$\frac{A(a,b)}{G(a,b)}L_{p-1}(a,b)<P(a,b)<\frac{A(a,b)}{G(a,b)}L_{q-1}(a,b) \tag{40}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $p\le 1/3$ and $q\ge 1/2$.

From (5) and (9) together with Theorem 5 one has the following.

Remark 9. The inequalities

$$\begin{aligned}L(a,b)&<M_{1/3}(a,b)<X(a,b)<M_{\log 2/(1+\log 2)}(a,b)\\ &<M_{\log 2/\log\pi}(a,b)<P(a,b)\end{aligned} \tag{41}$$

hold for all $a,b>0$ with $a\neq b$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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