

Research Article

Finite-Time Stability of Fractional-Order BAM Neural Networks with Distributed Delay

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Based on the theory of fractional calculus, the generalized Gronwall inequality and estimates of Mittag-Leffler functions, the finite-time stability of Caputo fractional-order BAM neural networks with distributed delay is investigated in this paper. An illustrative example is also given to demonstrate the effectiveness of the obtained result.

1. Introduction

Fractional calculus (integral and differential operations of noninteger order) was firstly introduced 300 years ago. Due to lack of application background and the complexity, it did not attract much attention for a long time. In recent decades fractional calculus is applied to physics, applied mathematics, and engineering [1–6]. Since the fractional-order derivative is nonlocal and has weakly singular kernels, it provides an excellent instrument for the description of memory and hereditary properties of dynamical processes. Nowadays, study on the complex dynamical behaviors of fractional-order systems has become a very hot research topic.

We know that the next state of a system not only depends upon its current state but also upon its history information. Since a model described by fractional-order equations possesses memory, it is precise to describe the states of neurons. Moreover, the superiority of the Caputo's fractional derivative is that the initial conditions for fractional differential equations with Caputo derivatives take on the similar form as those for integer-order differentiation. Therefore, it is necessary and interesting to study fractional-order neural networks both in theory and in applications.

Recently, fractional-order neural networks have been presented and designed to distinguish the classical integer-order models [7–10]. Currently, some excellent results about

fractional-order neural networks have been investigated, such as Kaslik and Sivasundaram [11, 12], Zhang et al. [13], Delavari et al. [14], and Li et al. [15, 16]. On the other hand, time delay is one of the inevitable problems on the stability of dynamical systems in the real world [17–20]. But till now, there are few results on the problems for fractional-order delayed neural networks; Chen et al. [21] studied the uniform stability for a class of fractional-order neural networks with constant delay by the analytical approach; Wu et al. [22] investigated the finite-time stability of fractional-order neural networks with delay by the generalized Gronwall inequality and estimates of Mittag-Leffler functions; Alofi et al. [23] discussed the finite-time stability of Caputo fractional-order neural networks with distributed delay.

The integer-order bidirectional associative memory (BAM) model known as an extension of the unidirectional autoassociator of Hopfield [24] was first introduced by Kosko [25]. This neural network has been widely studied due to its promising potential for applications in pattern recognition and automatic control. In recent years, integer-order BAM neural networks have been extensively studied [26–29]. However, to the best of our knowledge, there is no effort being made in the literature to study the finite-time stability of fractional-order BAM neural networks so far.

Motivated by the above-mentioned works, we were devoted to establishing the finite-time stability of Caputo fractional-order BAM neural networks with distributed

delay. In this paper, we will apply Laplace transform, generalized Gronwall inequality, and estimates of Mittag-Leffler functions to establish the finite-time stability criterion of fractional-order distributed delayed BAM neural networks.

This paper is organized as follows. In Section 2, some definitions and lemmas of fractional differential and integral calculus are given and the fractional-order BAM neural networks are presented. A criterion for finite-time stability of fractional-order BAM neural networks with distributed delay is obtained in Section 3. Finally, the effectiveness and feasibility of the theoretical result is shown by an example in Section 4.

2. Preliminaries

For the convenience of the reader, we first briefly recall some definitions of fractional calculus; for more details, see [1, 2, 5], for example.

Definition 1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, \infty) \rightarrow R$ is given by

$$I_{0^+}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds \quad (1)$$

provided that the right side is pointwise defined on $(0, \infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2. The Caputo fractional derivative of order $\gamma > 0$ of a function $u : (0, \infty) \rightarrow R$ can be written as

$${}_0^C D_t^\gamma u(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{u^{(n)}(s)}{(t-s)^{\gamma+1-n}} ds, \quad (2)$$

$n-1 < \gamma < n.$

Definition 3. The Mittag-Leffler function in two parameters is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad (3)$$

where $\alpha > 0, \beta > 0$, and $z \in \mathbb{C}$, where \mathbb{C} denotes the complex plane. In particular, for $\beta = 1$, one has

$$E_{\alpha,1}(z) = E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(k\alpha + 1)}. \quad (4)$$

The Laplace transform of Mittag-Leffler function is

$$\mathcal{L}\{t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha)\} = \frac{s^{\alpha-\beta}}{s^\alpha + \lambda}, \quad (5)$$

$(\mathcal{R}(s) > |\lambda|^{1/\alpha}),$

where t and s are, respectively, the variables in the time domain and Laplace domain and $\mathcal{L}\{\cdot\}$ stands for the Laplace transform.

In this paper, we are interested in the finite-time stability of fractional-order BAM neural networks with distributed delay by the following state equations:

$${}_0^C D_t^\alpha x_i(t) = -c_i x_i(t) + \sum_{j=1}^n b_{ij} f_j(y_j(t)) + \sum_{j=1}^n \int_0^\tau r_{ij}(s) f_j(y_j(t-s)) ds + I_i, \quad (6)$$

$t \geq 0,$

$${}_0^C D_t^\beta y_j(t) = -c_j y_j(t) + \sum_{i=1}^n d_{ji} g_i(x_i(t)) + \sum_{i=1}^n \int_0^\tau p_{ji}(s) g_i(x_i(t-s)) ds + \bar{I}_j, \quad (7)$$

$i, j = 1, \dots, n,$

or in the matrix-vector notation

$${}_0^C D_t^\alpha x(t) = -Cx(t) + Bf(y(t)) + \int_0^\tau R(s) f(y(t-s)) ds + I,$$

$${}_0^C D_t^\beta y(t) = -Cy(t) + Dg(x(t)) + \int_0^\tau P(s) g(x(t-s)) ds + \bar{I}, \quad t \geq 0,$$

(7)

where $1 < \alpha, \beta < 2$. The model (6) is made up of two neural fields F_x and F_y , where $x_i(t)$ and $y_j(t)$ are the activations of the i th neuron in F_x and the j th neuron in F_y , respectively;

$$(x(t), y(t)) = (x_1(t), \dots, x_n(t), y_1(t), \dots, y_n(t))^T \in \mathbb{R}^{2n} \quad (8)$$

is the state vector of the network at time t ; the functions

$$f(y(t)) = (f_1(y_1(t)), f_2(y_2(t)), \dots, f_n(y_n(t)))^T, \quad (9)$$

$$g(x(t)) = (g_1(x_1(t)), g_2(x_2(t)), \dots, g_n(x_n(t)))^T$$

are the activation functions of the neurons at time t ; $C = \text{diag}(c_i)$ is a diagonal matrix; $c_i > 0$ represents the rate with which the i th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs; $B = (b_{ij})_{n \times n}$ and $D = (d_{ji})_{n \times n}$ are the feedback matrix; $\tau > 0$ denotes the maximum possible transmission delay from neuron to another; $R = (r_{ij})_{n \times n}$ and $P = (p_{ji})_{n \times n}$ are the delayed feedback matrix; $I = (I_1, \dots, I_n)^T$ and $\bar{I} = (\bar{I}_1, \dots, \bar{I}_n)^T$ are two external bias vectors.

Let $C^1([-\tau, 0], \mathbb{R}^n)$ be the Banach space of all continuously differential functions over a time interval of length τ ,

mapping the interval $[-\tau, 0]$ into \mathbb{R}^n with the norm defined as follows: for every $\varphi \in C^1([-\tau, 0], \mathbb{R}^n)$,

$$\begin{aligned} \|\varphi\|_1 &= \max \{ \|\varphi\|, \|\varphi'\| \} \\ &= \max \left\{ \sup_{\theta \in [-\tau, 0]} |\varphi(\theta)|, \sup_{\theta \in [-\tau, 0]} |\varphi'(\theta)| \right\}. \end{aligned} \tag{10}$$

The initial conditions associated with (6) are given by

$$\begin{aligned} x_i(\theta) &= \varphi_i(\theta), \quad x'_i(\theta) = \varphi'_i(\theta), \quad y_j(\theta) = \psi_j(\theta), \\ y'_j(\theta) &= \psi'_j(\theta), \quad \theta \in [-\tau, 0], \end{aligned} \tag{11}$$

where $\varphi_i, \psi_j \in C^1([-\tau, 0], \mathbb{R})$.

In order to obtain main result, we make the following assumptions.

- (H1) For $i, j = 1, \dots, n$, the functions $r_{ij}(\cdot)$ and $p_{ji}(\cdot)$ are continuous on $[0, \tau]$.
- (H2) The neurons activation functions f_i and g_j ($i, j = 1, \dots, n$) are bounded.
- (H3) The neurons activation functions f_i and g_j are Lipschitz continuous; that is, there exist positive constants h_i, l_j ($i, j = 1, \dots, n$) such that

$$\begin{aligned} |f_i(u) - f_i(v)| &\leq h_i |u - v|, & |g_j(u) - g_j(v)| &\leq l_j |u - v|, \\ & & \forall u, v \in \mathbb{R}. \end{aligned} \tag{12}$$

Since the Caputo's fractional derivative of a constant is equal to zero, the equilibrium point of system (6) is a constant vector $(x^*, y^*) = (x_1^*, x_2^*, \dots, x_n^*, y_1^*, y_2^*, \dots, y_n^*)^T \in \mathbb{R}^{2n}$ which satisfies the system

$$\begin{aligned} c_i x_i^* - \sum_{j=1}^n b_{ij} f_j(y_j^*) - \sum_{j=1}^n \int_0^\tau r_{ij}(s) f_j(y_j^*) ds - I_i &= 0, \\ i &= 1, \dots, n, \\ c_j y_j^* - \sum_{i=1}^n d_{ji} g_i(x_i^*) - \sum_{i=1}^n \int_0^\tau p_{ji}(s) g_i(x_i^*) ds - \bar{I}_j &= 0, \\ j &= 1, \dots, n. \end{aligned} \tag{13}$$

By using the Schauder fixed point theorem and assumptions (H1)–(H3), it is easy to prove that the equilibrium points of system (6) exist. We can shift the equilibrium point of system (6) to the origin. Denoting

$$\begin{aligned} (u(t), v(t)) &= (u_1(t), \dots, u_n(t), v_1(t), \dots, v_n(t))^T \\ &= (x_1(t) - x_1^*, \dots, x_n(t) \\ &\quad - x_n^*, y_1(t) - y_1^*, \dots, y_n(t) - y_n^*)^T, \end{aligned} \tag{14}$$

then system (6) can be written as

$$\begin{aligned} {}^C_0 D_t^\alpha u_i(t) &= -c_i u_i(t) + \sum_{j=1}^n b_{ij} F_j(v_j(t)) \\ &\quad + \sum_{j=1}^n \int_0^\tau r_{ij}(s) F_j(v_j(t-s)) ds, \\ & \quad t \geq 0, \end{aligned} \tag{15}$$

$$\begin{aligned} {}^C_0 D_t^\beta v_j(t) &= -c_j v_j(t) + \sum_{i=1}^n d_{ji} G_i(u_i(t)) \\ &\quad + \sum_{i=1}^n \int_0^\tau p_{ji}(s) G_i(u_i(t-s)) ds, \\ & \quad i, j = 1, \dots, n, \end{aligned}$$

with the initial conditions

$$\begin{aligned} u_i(\theta) &= \bar{\varphi}_i(\theta), \quad u'_i(\theta) = \bar{\varphi}'_i(\theta), \quad v_j(\theta) = \bar{\psi}_j(\theta), \\ v'_j(\theta) &= \bar{\psi}'_j(\theta), \quad \theta \in [-\tau, 0], \end{aligned} \tag{16}$$

where

$$\begin{aligned} F_j(v_j(t)) &= f_j(v_j(t) + y_j^*) - f_j(y_j^*), \\ G_i(u_i(t)) &= g_i(u_i(t) + x_i^*) - g_i(x_i^*), \\ \bar{\varphi}_i(\theta) &= \varphi_i(\theta) - x_i^*, \quad \bar{\psi}_j(\theta) = \psi_j(\theta) - y_j^*, \\ & \quad \theta \in [\tau, 0]. \end{aligned} \tag{17}$$

Similarly, by using the matrix-vector notation, system (15) can be expressed as

$$\begin{aligned} {}^C_0 D_t^\alpha u(t) &= -Cu(t) + BF(v(t)) \\ &\quad + \int_0^\tau R(s) F(v(t-s)) ds, \\ & \quad t \geq 0, \\ {}^C_0 D_t^\beta v(t) &= -Cv(t) + DG(u(t)) \\ &\quad + \int_0^\tau P(s) G(u(t-s)) ds, \\ & \quad t \geq 0, \end{aligned} \tag{18}$$

with the initial condition

$$\begin{aligned} u(\theta) &= \bar{\varphi}(\theta), \quad u'(\theta) = \bar{\varphi}'(\theta), \quad v(\theta) = \bar{\psi}(\theta), \\ v'(\theta) &= \bar{\psi}'(\theta), \quad \theta \in [-\tau, 0], \end{aligned} \tag{19}$$

where

$$\begin{aligned} F(v(t)) &= (F_1(v_1(t)), F_2(v_2(t)), \dots, F_n(v_n(t)))^T, \\ G(u(t)) &= (G_1(u_1(t)), G_2(u_2(t)), \dots, G_n(u_n(t)))^T. \end{aligned} \tag{20}$$

Define the functions as follows:

$$\begin{aligned}
 h_i(t) &= \begin{cases} \frac{F_i(v_i(t))}{v_i(t)}, & v_i(t) \neq 0, \\ 0, & v_i(t) = 0, \end{cases} \\
 l_j(t) &= \begin{cases} \frac{G_j(u_j(t))}{u_j(t)}, & u_j(t) \neq 0, \\ 0, & u_j(t) = 0, \end{cases}
 \end{aligned} \tag{21}$$

where $i, j = 1, \dots, n$. From assumption (H3), we can obtain $|h_i(t)| \leq h_i, |l_j(t)| \leq l_j$. By (21), we have

$$\begin{aligned}
 F_i(v_i(t)) &= h_i(t) v_i(t), & G_j(u_j(t)) &= l_j(t) u_j(t), \\
 i, j &= 1, \dots, n.
 \end{aligned} \tag{22}$$

Thus, system (18) can be further written as the following form:

$$\begin{aligned}
 {}^C_0 D_t^\alpha u(t) &= -Cu(t) + BH(t)v(t) \\
 &+ \int_0^t R(s)H(t-s)v(t-s)ds, \\
 t &\geq 0,
 \end{aligned} \tag{23}$$

$$\begin{aligned}
 {}^C_0 D_t^\beta v(t) &= -Cv(t) + DL(t)u(t) \\
 &+ \int_0^t P(s)L(t-s)u(t-s)ds, \\
 t &\geq 0,
 \end{aligned}$$

where $H(t) = \text{diag}\{h_i(t)\}, L(t) = \text{diag}\{l_j(t)\}$.

Definition 4. System (23) with the initial condition (19) is finite-time stable with respect to $\{\delta, \varepsilon, t_0, J\}, \delta < \varepsilon$, if and only if

$$\|\langle \varphi, \psi \rangle\|_1 := \|\varphi\|_1 + \|\psi\|_1 < \delta \tag{24}$$

implies

$$\|(u(t), v(t))\| = \|u(t)\| + \|v(t)\| < \varepsilon, \quad \forall t \in J, \tag{25}$$

where δ is a positive real number and $\varepsilon > 0, \delta < \varepsilon, t_0$ denotes the initial time of observation of the system, and J denotes time interval $J = [t_0, t_0 + T)$.

A technical result about norm upper-bounding function of the matrix function $E_{\alpha,\beta}$ is given in [30] as follows.

Lemma 5. *If $\alpha \geq 1$, then, for $\beta = 1, 2, \alpha$, one has*

$$\|E_{\alpha,\beta}(At^\alpha)\| \leq \|e^{At^\alpha}\|, \quad t \geq 0. \tag{26}$$

Moreover, if A is a diagonal stability matrix, then

$$\|E_{\alpha,\beta}(At^\alpha)\| \leq e^{-\omega t}, \quad t \geq 0, \tag{27}$$

where $-\omega$ ($\omega > 0$) is the largest eigenvalue of the diagonal stability matrix A .

Lemma 6 (see [31]). *Let $u(t), a(t)$ be nonnegative and local integrable on $[0, T) (T \leq +\infty)$, and let g be a nonnegative, nondecreasing continuous function defined on $[0, T), g(t) \leq M$, and let M be a real constant, $\alpha > 0$, with*

$$u(t) \leq a(t) + g(t) \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad t \in [0, T). \tag{28}$$

Then

$$u(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} a(s) \right] ds, \tag{29}$$

$t \in [0, T).$

Moreover, if $a(t)$ is a nondecreasing function on $[0, T)$, then

$$u(t) \leq a(t) E_{\alpha,1}(g(t)\Gamma(\alpha)t^\alpha), \quad t \in [0, T). \tag{30}$$

3. Main Result

We first give a key lemma in the proof of our main result as follows.

Lemma 7. *Let $u(t), v(t)$ be nonnegative and local integrable on $[0, T) (T \leq +\infty)$, and let $a_1(t), a_2(t)$ be nonnegative, nondecreasing and local integrable on $[0, T)$, and let b_1, b_2 be two positive constants, $\alpha, \beta > 1$, with*

$$u(t) \leq a_1(t) + b_1 \int_0^t (t-s)^{\alpha-1} v(s) ds, \quad t \in [0, T), \tag{31}$$

$$v(t) \leq a_2(t) + b_2 \int_0^t (t-s)^{\beta-1} u(s) ds, \quad t \in [0, T). \tag{32}$$

Then

$$\begin{aligned}
 u(t) &\leq \left(a_1(t) + b_1 \int_0^t (t-s)^{\alpha-1} a_2(s) ds \right) \\
 &\times E_{\alpha+\beta}(b_1 b_2 \Gamma(\alpha) \Gamma(\beta) t^{\alpha+\beta}), \quad t \in [0, T), \\
 v(t) &\leq \left(a_2(t) + b_2 \int_0^t (t-s)^{\beta-1} a_1(s) ds \right) \\
 &\times E_{\alpha+\beta}(b_1 b_2 \Gamma(\alpha) \Gamma(\beta) t^{\alpha+\beta}), \quad t \in [0, T).
 \end{aligned} \tag{33}$$

Proof. Substituting (32) into (31), we obtain

$$\begin{aligned}
 u(t) &\leq a_1(t) + b_1 \int_0^t (t-s)^{\alpha-1} v(s) ds \\
 &\leq a_1(t) + b_1 \int_0^t (t-s)^{\alpha-1} a_2(s) ds \\
 &+ b_1 b_2 \int_0^t (t-s)^{\alpha-1} \int_0^s (s-\xi)^{\beta-1} u(\xi) d\xi.
 \end{aligned} \tag{34}$$

Changing the order of integration in the above double integral, we obtain

$$\begin{aligned}
 u(t) &\leq a_1(t) + b_1 \int_0^t (t-s)^{\alpha-1} a_2(s) ds \\
 &\quad + b_1 b_2 \int_0^t u(\xi) d\xi \int_{\xi}^t (t-s)^{\alpha-1} (s-\xi)^{\beta-1} ds \\
 &= a_1(t) + b_1 \int_0^t (t-s)^{\alpha-1} a_2(s) ds \\
 &\quad + b_1 b_2 \int_0^t \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} (t-\xi)^{\alpha+\beta-1} u(\xi) d\xi.
 \end{aligned} \tag{35}$$

Let $a(t) = a_1(t) + b_1 \int_0^t (t-s)^{\alpha-1} a_2(s) ds$, $g(t) = b_1 b_2 (\Gamma(\alpha)\Gamma(\beta))/\Gamma(\alpha+\beta)$; then $a(t)$ is a nonnegative, non-decreasing, and local integrable function and $g(t)$ is a nonnegative, nondecreasing continuous function. Thus, by Lemma 6 (30), one has

$$\begin{aligned}
 u(t) &\leq \left(a_1(t) + b_1 \int_0^t (t-s)^{\alpha-1} a_2(s) ds \right) \\
 &\quad \times E_{\alpha+\beta} \left(b_1 b_2 \Gamma(\alpha)\Gamma(\beta) t^{\alpha+\beta} \right).
 \end{aligned} \tag{36}$$

Similarly, we get

$$\begin{aligned}
 v(t) &\leq \left(a_2(t) + b_2 \int_0^t (t-s)^{\beta-1} a_1(s) ds \right) \\
 &\quad \times E_{\alpha+\beta} \left(b_1 b_2 \Gamma(\alpha)\Gamma(\beta) t^{\alpha+\beta} \right).
 \end{aligned} \tag{37}$$

For convenience, let

$$\begin{aligned}
 R &= \sup_{0 \leq s \leq \tau} \{ \|R(s)\| \}, & P &= \sup_{0 \leq s \leq \tau} \{ \|P(s)\| \}, \\
 h &= \max_{1 \leq i \leq n} \{ h_i \}, & l &= \max_{1 \leq j \leq n} \{ l_j \}, \\
 \Theta(t) &:= \max \left\{ \frac{h}{\alpha} t^\alpha \left(1 + \frac{t}{\alpha+1} \right) (\mu(B) + R\tau e^{\gamma\tau}), \right. \\
 &\quad \left. \frac{l}{\beta} t^\beta \left(1 + \frac{t}{\beta+1} \right) (\mu(D) + P\tau e^{\gamma\tau}) \right\},
 \end{aligned} \tag{38}$$

where $-\gamma$ is the largest eigenvalue of the diagonal stability matrix $-C$ and $\mu(\cdot)$ denotes the largest singular value of matrix (\cdot) . \square

In the following, sufficient conditions for finite-time stability of fractional-order BAM neural networks with distributed delay are derived.

Theorem 8. *Let $1 < \alpha, \beta < 2$. If system (23) satisfies (H1)–(H3) with the initial condition (19), and*

$$\begin{aligned}
 &e^{-\gamma t} ((1+t) + \Theta(t)) E_{\alpha+\beta} \\
 &\quad \times \left[hl(\mu(B) + R\tau e^{\gamma\tau}) (\mu(D) + P\tau e^{\gamma\tau}) \Gamma(\alpha)\Gamma(\beta) t^{\alpha+\beta} \right] \\
 &\quad < \frac{\varepsilon}{\delta},
 \end{aligned} \tag{39}$$

where $t \in J = [0, T]$, then system (23) is finite-time stable with respect to $\{\delta, \varepsilon, 0, J\}$, $\delta < \varepsilon$.

Proof. By Laplace transform and inverse Laplace transform, system (23) is equivalent to

$$\begin{aligned}
 u(t) &= E_\alpha(-Ct^\alpha) \bar{\varphi}(0) + tE_{\alpha,2}(-Ct^\alpha) \bar{\varphi}'(0) \\
 &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-Ct^\alpha) \\
 &\quad \times \left[BH(s)v(s) \right. \\
 &\quad \left. + \int_0^\tau R(\eta)H(s-\eta)v(s-\eta)d\eta \right] ds,
 \end{aligned} \tag{40}$$

$$\begin{aligned}
 v(t) &= E_\beta(-Ct^\beta) \bar{\psi}(0) + tE_{\beta,2}(-Ct^\beta) \bar{\psi}'(0) \\
 &\quad + \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-Ct^\beta) \\
 &\quad \times \left[DL(s)u(s) \right. \\
 &\quad \left. + \int_0^\tau P(\eta)L(s-\eta)u(s-\eta)d\eta \right] ds.
 \end{aligned} \tag{41}$$

From (40), (41), and Lemma 5, we obtain

$$\begin{aligned}
 \|u(t)\| &\leq \left(\|\bar{\varphi}\| + \|\bar{\varphi}'\| t \right) e^{-\gamma t} \\
 &\quad + \int_0^t (t-s)^{\alpha-1} e^{-\gamma(t-s)} \\
 &\quad \times \left\| BH(s)v(s) \right. \\
 &\quad \left. + \int_0^\tau R(\eta)H(s-\eta)v(s-\eta)d\eta \right\| ds,
 \end{aligned} \tag{42}$$

$$\begin{aligned} \|v(t)\| &\leq (\|\bar{\psi}\| + \|\bar{\psi}'\| t) e^{-\gamma t} \\ &+ \int_0^t (t-s)^{\beta-1} e^{-\gamma(t-s)} \\ &\quad \times \left\| DL(s) u(s) \right. \\ &\quad \left. + \int_0^\tau P(\eta) L(s-\eta) u(s-\eta) d\eta \right\| ds. \end{aligned} \tag{43}$$

Let $U(t) = \sup_{\theta \in [t-\tau, t]} \|u(\theta)\| e^{\gamma\theta}$, and $V(t) = \sup_{\theta \in [t-\tau, t]} \|v(\theta)\| e^{\gamma\theta}$; then

$$\|u(s)\| e^{\gamma s} \leq U(s), \quad \|u(s-\tau)\| e^{\gamma(s-\tau)} \leq U(s), \tag{44}$$

$$\|v(s)\| e^{\gamma s} \leq V(s), \quad \|v(s-\tau)\| e^{\gamma(s-\tau)} \leq V(s). \tag{45}$$

Thus, we have by (42) and (44) that

$$\begin{aligned} \|u(t)\| e^{\gamma t} &\leq \|\bar{\varphi}\| + \|\bar{\varphi}'\| t \\ &+ \int_0^t (t-s)^{\alpha-1} \\ &\quad \times \left[h\mu(B) \|v(s)\| e^{\gamma s} \right. \\ &\quad \left. + \int_0^\tau hR \|v(s-\eta)\| e^{\gamma(s-\eta)} e^{\gamma\eta} d\eta \right] ds \\ &\leq \|\bar{\varphi}\| + \|\bar{\varphi}'\| t \\ &+ \int_0^t (t-s)^{\alpha-1} h [\mu(B) + R\tau e^{\gamma\tau}] V(s) ds, \end{aligned} \tag{46}$$

where $\mu(B)$ denotes the largest singular value of matrix B . Similarly, by (43) and (45), we get

$$\begin{aligned} \|v(t)\| e^{\gamma t} &\leq \|\bar{\psi}\| + \|\bar{\psi}'\| t \\ &+ \int_0^t (t-s)^{\beta-1} \\ &\quad \times \left[l\mu(D) \|x(s)\| e^{\gamma s} \right. \\ &\quad \left. + \int_0^\tau lP \|x(s-\eta)\| e^{\gamma(s-\eta)} e^{\gamma\eta} d\eta \right] ds \\ &\leq \|\bar{\psi}\| + \|\bar{\psi}'\| t \\ &+ \int_0^t (t-s)^{\beta-1} l [\mu(D) + P\tau e^{\gamma\tau}] U(s) ds. \end{aligned} \tag{47}$$

Hence, by (46) and (47), we have

$$\begin{aligned} U(t) &\leq \|\bar{\varphi}\|_1 (1+t) \\ &+ \int_0^t (t-s)^{\alpha-1} h [\mu(B) + R\tau e^{\gamma\tau}] V(s) ds, \\ V(t) &\leq \|\bar{\psi}\|_1 (1+t) \\ &+ \int_0^t (t-s)^{\beta-1} l [\mu(D) + P\tau e^{\gamma\tau}] U(s) ds. \end{aligned} \tag{48}$$

Set

$$\begin{aligned} a_1(t) &= \|\bar{\varphi}\|_1 (1+t), & a_2(t) &= \|\bar{\psi}\|_1 (1+t), \\ b_1 &= h(\mu(B) + R\tau e^{\gamma\tau}), & b_2 &= l(\mu(D) + P\tau e^{\gamma\tau}). \end{aligned} \tag{49}$$

By simple computation, we have

$$\begin{aligned} &\int_0^t (t-s)^{\alpha-1} a_2(s) ds \\ &\leq \|\bar{\psi}\|_1 \int_0^t (t-s)^{\alpha-1} (1+s) ds \\ &= \frac{\|\bar{\psi}\|_1}{\alpha} t^\alpha \left(1 + \frac{t}{\alpha+1}\right), \\ &\int_0^t (t-s)^{\beta-1} a_1(s) ds \\ &\leq \|\bar{\varphi}\|_1 \int_0^t (t-s)^{\beta-1} (1+s) ds \\ &= \frac{\|\bar{\varphi}\|_1}{\beta} t^\beta \left(1 + \frac{t}{\beta+1}\right). \end{aligned} \tag{50}$$

It follows from (48)–(50) and Lemma 7 that

$$\begin{aligned} U(t) &\leq \left[(1+t) \|\bar{\varphi}\|_1 + \frac{\|\bar{\psi}\|_1}{\alpha} t^\alpha \right. \\ &\quad \left. \times \left(1 + \frac{t}{\alpha+1}\right) h(\mu(B) + R\tau e^{\gamma\tau}) \right] \\ &\quad \cdot E_{\alpha+\beta} \left[hl(\mu(B) + R\tau e^{\gamma\tau}) \right. \\ &\quad \left. \times (\mu(D) + P\tau e^{\gamma\tau}) \Gamma(\alpha) \Gamma(\beta) t^{\alpha+\beta} \right], \\ V(t) &\leq \left[(1+t) \|\bar{\psi}\|_1 + \frac{\|\bar{\varphi}\|_1}{\beta} t^\beta \right. \\ &\quad \left. \times \left(1 + \frac{t}{\beta+1}\right) l(\mu(D) + P\tau e^{\gamma\tau}) \right] \\ &\quad \cdot E_{\alpha+\beta} \left[hl(\mu(B) + R\tau e^{\gamma\tau}) \right. \\ &\quad \left. \times (\mu(D) + P\tau e^{\gamma\tau}) \Gamma(\alpha) \Gamma(\beta) t^{\alpha+\beta} \right]. \end{aligned} \tag{51}$$

By (51), we obtain

$$\begin{aligned} & \| (u(t), v(t)) \| \\ &= \| u(t) \| + \| v(t) \| \\ &\leq e^{-\gamma t} \| (\bar{\varphi}, \bar{\psi}) \|_1 ((1+t) + \Theta(t)) \\ &\quad \times E_{\alpha+\beta} [hl(\mu(B) + R\tau e^{\gamma\tau}) \\ &\quad \times (\mu(D) + P\tau e^{\gamma\tau}) \Gamma(\alpha) \Gamma(\beta) t^{\alpha+\beta}]. \end{aligned} \tag{52}$$

Thus, if condition (39) is satisfied and $\|(\bar{\varphi}, \bar{\psi})\|_1 < \delta$, then $\|(u(t), v(t))\| < \varepsilon, t \in J$; that is, system (23) is finite-time stable. This completes the proof. \square

4. An Illustrative Example

In this section, we give an example to illustrate the effectiveness of our main result.

Consider the following two-state Caputo fractional BAM type neural networks model with distributed delay

$$\begin{aligned} {}_0^C D_t^\alpha x_1(t) &= -0.7x_1(t) - 0.2f_1(y_1(t)) + 0.1f_2(y_2(t)) \\ &\quad + \int_0^\tau s^{3/2} f_1(y_1(t-s)) ds \\ &\quad + \int_0^\tau s f_2(y_2(t-s)) ds, \\ {}_0^C D_t^\alpha x_2(t) &= -0.6x_2(t) + 0.3f_1(y_1(t)) + 0.2f_2(y_2(t)) \\ &\quad + \int_0^\tau s f_1(y_1(t-s)) ds \\ &\quad - \int_0^\tau s^{3/2} f_2(y_2(t-s)) ds, \\ {}_0^C D_t^\beta y_1(t) &= -0.7y_1(t) + 0.4g_1(x_1(t)) + 0.2g_2(x_2(t)) \\ &\quad - \int_0^\tau s g_1(x_1(t-s)) ds \\ &\quad + \int_0^\tau s^2 g_2(x_2(t-s)) ds, \\ {}_0^C D_t^\beta y_2(t) &= -0.6y_2(t) + 0.1g_1(x_1(t)) - 0.3g_2(x_2(t)) \\ &\quad + \int_0^\tau s^2 g_1(x_1(t-s)) ds \\ &\quad + \int_0^\tau s g_2(x_2(t-s)) ds \end{aligned} \tag{53}$$

with the initial condition

$$\begin{aligned} x(t) = \varphi(t) &= \frac{1}{15} \sin t, & x'(t) = \varphi'(t) &= \frac{1}{15} \cos t, \\ & & t \in [-\tau, 0], \\ y(t) = \psi(t) &= \frac{1}{15} \cos t, & y'(t) = \psi'(t) &= -\frac{1}{15} \sin t, \\ & & t \in [-\tau, 0], \end{aligned} \tag{54}$$

where $\alpha = 1.2, \beta = 1.3, \tau = 0.2$, and $f_j(x_j) = g_j(x_j) = (1/2)(|x_j + 1| - |x_j - 1|), j = 1, 2$. It is easy to know that $(x_1^*, x_2^*, y_1^*, y_2^*)^T = (0, 0, 0, 0)^T$ is an equilibrium point of system (53). Since $\|(\varphi, \psi)\|_1 = 1/15 < 0.07$, we may let $\delta = 0.07$. Take

$$\begin{aligned} t_0 &= 0, & J &= [0, 30], & \varepsilon &= 1, \\ C &= \begin{bmatrix} -0.7 & 0 \\ 0 & -0.6 \end{bmatrix}, & B &= \begin{bmatrix} -0.2 & 0.1 \\ 0.3 & 0.2 \end{bmatrix}, \\ D &= \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & -0.3 \end{bmatrix}, & R(s) &= \begin{bmatrix} s^{3/2} & s \\ s & -s^{3/2} \end{bmatrix}, \\ P(s) &= \begin{bmatrix} -s & s^2 \\ s^2 & s \end{bmatrix}. \end{aligned} \tag{55}$$

It is easy to check that

$$\begin{aligned} h &= l = 1, & \gamma &= 0.6, & \mu(B) &= 0.3828, \\ \mu(D) &= 0.4515, & R &= 0.2894, & P &= 0.24, \\ \Theta(t) &= \max \{ 0.1697t^{1.2}(t+2.2), 0.1691t^{1.3}(t+2.3) \}, \\ E_{\alpha+\beta} [hl(\mu(B) + R\tau e^{\gamma\tau}) (\mu(D) + P\tau e^{\gamma\tau}) \Gamma(\alpha) \Gamma(\beta) t^{\alpha+\beta}] \\ &= E_{2.5} (0.1867t^{2.5}). \end{aligned} \tag{56}$$

From condition (41) of Theorem 8, we can get

$$\begin{aligned} & e^{-0.6t} [(1+t) \\ & \quad + \max \{ 0.1697t^{1.2}(t+2.2), \\ & \quad \quad 0.1691t^{1.3}(t+2.3) \}] \\ & \quad \times E_{2.5} (0.1867t^{2.5}) < \frac{1}{0.07}. \end{aligned} \tag{57}$$

We can obtain that the estimated time of finite-time stability is $T \approx 23.78$. Hence, system (53) is finite-time stable with respect to $\{0.07, 1, 0, [0, 30]\}$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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