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Research Article

On the Sixth Power Mean Value of the Generalized Three-Term Exponential Sums

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The main purpose of this paper is using the estimate for trigonometric sums and the properties of the congruence equations to study the computational problem of one kind sixth power mean value of the generalized three-term exponential sums and give an exact computational formula for it.

1. Introduction

Let $q \geq 3$ be a positive integer and let χ be any Dirichlet character mod q. For any integers m and n, the generalized three-term exponential sum $C(m, n, k, \chi; q)$ is defined as follows:

$$C(m, n, k, \chi; q) = \sum_{a=1}^{q-1} \chi(a) e\left(\frac{a^k + ma^2 + na}{q}\right), \quad (1)$$

where $k \geq 3$ is a fixed integer and $e(y) = e^{2\pi i y}$.

Many authors have studied this and related exponential sums and obtained a series of results; some related contents can be found in [1–9]. For example, Du and Han [5] proved that, for any integer $k \geq 3$, we have the identity

$$\sum_{m=1}^{p}\sum_{n=1}^{p}\left|\sum_{a=1}^{p-1} e\left(\frac{ma^k + na^2 + a}{p}\right)\right|^4 = 2p^4 - 3p^3 - p^2 \cdot C(k, p), \quad (2)$$

where the constant $C(k, p)$ is defined as follows:

$$C(k, p) = \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{\substack{c=1 \\ a^k + b^k \equiv c^k + 1 \bmod p \\ a^2 + b^2 \equiv c^2 + 1 \bmod p}}^{p-1} 1. \quad (3)$$

In particular, if $k = 6$, then we have the identity

$$\sum_{m=1}^{p}\sum_{n=1}^{p}\left|\sum_{a=1}^{p-1} e\left(\frac{ma^6 + na^2 + a}{p}\right)\right|^4 = \begin{cases} 2p^4 - 11p^3 + 16p^2, & \text{if } p \equiv 3 \bmod 4, \\ 2p^4 - 15p^3 + 36p^2, & \text{if } p \equiv 1 \bmod 4. \end{cases} \quad (4)$$

It seems that no one has studied the sixth power mean of the generalized three-term exponential sums

$$\sum_{m=1}^{p}\sum_{n=1}^{p}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1} \chi(a) e\left(\frac{a^k + ma^2 + na}{p}\right)\right|^6, \quad (5)$$

where $\chi \bmod p$ denotes the summation over all characters $\chi \bmod p$. The problem is interesting, because it can reflect more or less the upper bound estimates of $C(m, n, k, \chi; p)$. It is easy to see that mean value (2) is the best possible. So, we have reason to believe that (5) and (2) have similar asymptotic properties. In fact, we can use the analytic method and the properties of the congruence equation to give an exact computational formula for (5). That is, we will prove the following.

Theorem 1. *Let $p > 3$ be a prime. Then, for any integer $k \geq 3$ with $(k, p-1) = 1$, we have the identity*

$$\sum_{m=1}^{p}\sum_{n=1}^{p}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{a^k+ma^2+na}{p}\right)\right|^6 = p^2\cdot(p-1)^2\cdot\left(6p^2-21p+19\right). \tag{6}$$

It is very strange that the mean value in our theorem is independent of the size of k; it depends only on whether $(k, p-1) = 1$ or $(k, p-1) > 1$. For any Dirichlet character χ mod p, whether there exists a computational formula for the sixth power mean value

$$\sum_{m=1}^{p}\sum_{n=1}^{p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{a^k+ma^2+na}{p}\right)\right|^6 \tag{7}$$

or $2h$-th ($h \geq 4$) power mean value

$$\sum_{m=1}^{p}\sum_{n=1}^{p}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{a^k+ma^2+na}{p}\right)\right|^{2h} \tag{8}$$

is two open problems, which we will further study.

2. Proof of Theorem 1

In this section, we will give the proof of our theorem directly. Hereinafter, we will use many properties of trigonometric sums and congruence equation, all of which can be found in [6, 10], so they will not be repeated here. Note that $(k, p-1) = 1$, from the trigonometric identity

$$\sum_{a=0}^{p-1}e\left(\frac{na}{p}\right) = \begin{cases} p, & \text{if } (n,p)=p, \\ 0, & \text{if } (n,p)=1. \end{cases} \tag{9}$$

Regarding the properties of reduced residue system mod p and the orthogonality relation for characters mod p, we have

$$\begin{aligned}
&\sum_{m=1}^{p}\sum_{n=1}^{p}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{a^k+ma^2+na}{p}\right)\right|^6 \\
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\sum_{f=1}^{p-1}\sum_{\chi \bmod p}\chi\left(abc\overline{d}\overline{e}\overline{f}\right)e\left(\frac{a^k+b^k+c^k-d^k-e^k-f^k}{p}\right) \\
&\quad\times\sum_{m=1}^{p}\sum_{n=1}^{p}e\left(\frac{m\left(a^2+b^2+c^2-d^2-e^2-f^2\right)+n\left(a+b+c-d-e-f\right)}{p}\right) \\
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\sum_{f=1}^{p-1}\sum_{\chi \bmod p}\chi\left(abc\overline{d}\overline{e}\right)e\left(\frac{f^k\left(a^k+b^k+c^k-d^k-e^k-1\right)}{p}\right) \\
&\quad\times\sum_{m=1}^{p}\sum_{n=1}^{p}e\left(\frac{mf^2\left(a^2+b^2+c^2-d^2-e^2-1\right)+nf\left(a+b+c-d-e-1\right)}{p}\right) \\
&= p^2\sum_{\substack{a=1\\ a+b+c\equiv d+e+1 \bmod p\\ a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\sum_{\chi \bmod p}\chi\left(abc\overline{d}\overline{e}\right)\sum_{f=1}^{p-1}e\left(\frac{f\left(a^k+b^k+c^k-d^k-e^k-1\right)}{p}\right) \\
&= p^3(p-1)\sum_{\substack{a=1\\ a+b+c\equiv d+e+1 \bmod p\\ a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p\\ a^k+b^k+c^k\equiv d^k+e^k+1 \bmod p\\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}1 - p^2(p-1)\sum_{\substack{a=1\\ a+b+c\equiv d+e+1 \bmod p\\ a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p\\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}1 \equiv p^3(p-1)U - p^2(p-1)V.
\end{aligned} \tag{10}$$

Now, we compute the values of U and V in (10), respectively. It is clear that the value of V is equal to the number of the solutions of the system of the congruence equations:

$$\begin{aligned} a+b+c &\equiv d+e+1 \bmod p, \\ a^2+b^2+c^2 &\equiv d^2+e^2+1 \bmod p, \\ abc &\equiv de \bmod p, \end{aligned} \tag{11}$$

where $1 \leq a, b, c, d, e \leq p-1$.

The system of congruence equation (11) is equivalent to the system of the congruence equations:

$$\begin{gathered} a+b+c\equiv d+e+1 \bmod p,\\ ab+bc+ca\equiv de+d+e \bmod p,\\ a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p,\\ abc\equiv de \bmod p. \end{gathered} \tag{12}$$

That is equivalent to the system of the congruence equations as follows:

$$\begin{gathered} abc+a+b+c-1\equiv ab+bc+ca \bmod p,\\ a+b+c\equiv d+e+1 \bmod p,\\ a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p, \end{gathered} \tag{13}$$

$$\begin{gathered} (a-1)(b-1)(c-1)\equiv 0 \bmod p,\\ a+b+c\equiv d+e+1 \bmod p,\\ a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p. \end{gathered} \tag{14}$$

For all integers $1 \le a,b,c,d,e \le p-1$, we compute the number of the solutions of (14). We separate the solutions of (14) into three cases as follows:

(A) $a=1, 2\le b,c\le p-1$; $b=1, 2\le a,c\le p-1$; $c=1$, $2\le a,b\le p-1$,

(B) $a=b=1, 2\le c\le p-1$; $a=c=1, 2\le b\le p-1$; $b=c=1, 2\le a\le p-1$,

(C) $a=b=c=1$.

In case (C), (14) becomes $d+e\equiv 2 \bmod p$ and $d^2+e^2\equiv 2 \bmod p$, so $d=e=1$. That is, in this case, (14) has only one solution $(a,b,c,d,e)=(1,1,1,1,1)$.

In case (B), if $a=b=1$ and $2\le c\le p-1$, then (14) becomes $2\le c\le p-1$, $d+e\equiv c+1 \bmod p$, $d^2+e^2\equiv c^2+1 \bmod p$, and $de\equiv c \bmod p$ or $(d-1)(e-1)\equiv 0 \bmod p$ and $(d^2-1)(e^2-1)\equiv 0 \bmod p$, $de\equiv c \bmod p$ with $2\le c\le p-1$. In this case, the number of the solutions of the congruence equation is $2(p-2)$. So, in case (B), the number of all solutions of congruence equation (14) is $3\times 2(p-2)=6(p-2)$.

It is clear that the number of the solutions of congruence equation (14) in case (A) is three times the number of the solutions of the congruence equation $d+e\equiv b+c \bmod p$, $d^2+e^2\equiv b^2+c^2 \bmod p$ with $2\le b,c\le p-1$ and $1\le d,e\le p-1$. While the number of the solutions of the latter congruence equation is $(p-1)(2p-3)-4(p-2)-1$. In fact, the congruence equation $d+e\equiv b+c \bmod p$, $d^2+e^2\equiv b^2+c^2 \bmod p$ with $1\le b,c,d,e\le p-1$ is equivalent to congruence equation $dc+ec\equiv bc+c \bmod p$, $d^2c^2+e^2c^2\equiv b^2c^2+c^2 \bmod p$, $1\le b,c,d,e\le p-1$. So, from (B), we know that the number of the solutions is $(p-1)(2p-3)$. From (B) and (C), we know that the number of the solutions of congruence equation (14) in case (A) is $3\times[(p-1)(2p-3)-4(p-2)-1]$.

Combining three cases (A), (B), and (C), we deduce that the number of all solutions of (14) is

$$\begin{gathered} 3\times\left[(p-1)(2p-3)-4(p-2)-1\right]\\ +6(p-2)+1=6p^2-21p+19. \end{gathered} \tag{15}$$

Note that for all integers $1\le a,b,c,d,e\le p-1$, the number of all solutions of the congruence equation

$$\begin{gathered} a+b+c\equiv d+e+1 \bmod p,\\ a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p,\\ a^k+b^k+c^k\equiv d^k+e^k+1 \bmod p,\\ abc\equiv de \bmod p \end{gathered} \tag{16}$$

is also $6p^2-21p+19$. In fact, all solutions of (16) are the solutions of (11). Thus, they are also the solutions of (14). On the other hand, any solution in (14) must belong to case (A), (B), or (C). From the computational process of the solutions in these three cases, we can see that any solution must satisfy (16). So, the number of the solutions of congruence equation (16) is $6p^2-21p+19$.

Now, from (10), (11), (14), (15), and (16), we may immediately deduce the identity

$$\begin{aligned} &\sum_{m=1}^{p}\sum_{n=1}^{p}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{a^k+ma^2+na}{p}\right)\right|^6\\ &\quad= p^2\cdot(p-1)^2\cdot\left(6p^2-21p+19\right). \end{aligned} \tag{17}$$

This completes the proof of our theorem.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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