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Research Article

Almost Sure and L^p Convergence of Split-Step Backward Euler Method for Stochastic Delay Differential Equation

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The convergence of the split-step backward Euler (SSBE) method applied to stochastic differential equation with variable delay is proven in L^p-sense. Almost sure convergence is derived from the L^p convergence by Chebyshev's inequality and the Borel-Cantelli lemma; meanwhile, the convergence rate is obtained.

1. Introduction

In probability theory, there are several types of convergence of sequences of random variables such as convergence in pth mean (L^p sense), almost sure, in probability, and in distribution. As we know, the almost sure (a.s.) convergence and the convergence in L^p sense each imply the convergence in probability, and the convergence in probability implies the convergence in distribution. Among them, the almost sure convergence, also known as convergence with probability one, is the convergence concept most closely related to that of nonrandom sequences. The mean-square convergence analysis of numerical schemes for solving stochastic delay differential equation (SDDE) has gained considerable research attention, and we refer here to the papers of Baker and Buckwar [1], Buckwar [2], Hu et al. [3], Liu et al. [4], and Mao and Sabanis [5] just to mention a few of them. In particular, a type of split-step method for stochastic differential equation (SDE) was first introduced by Higham et al. [6] and, subsequently, the method was extended to solve a linear SDDE with constant delay (see [7]) and to solve an SDDE with variable delay (see [8]). However, the almost sure and L^p convergence of a numerical method for an SDDE are rarely investigated in the literature. Until recently, Gyöngy and Sabanis [9] proved the almost sure convergence of Euler approximations for a class of SDDE under local Lipschitz and monotonicity conditions.

In this paper we study the following nonlinear SDDE:

$$\begin{aligned} \mathrm{d}x(t) = {} & f(x(t), x(t-\tau(t)))\,\mathrm{d}t \\ & + g(x(t), x(t-\tau(t)))\,\mathrm{d}W(t), \quad t \ge 0 \end{aligned} \tag{1}$$

with initial data $x(t) = \psi(t)$ for $t \in [-\tau, 0]$. Here the time delay $\tau(t)$ is a real-valued function satisfying $\tau(t) \ge \tau_{\min} > 0$ and $-\tau := \inf\{t - \tau(t) : t \ge 0\}$. Unlike the delay in [9], however, the function $\tau(t)$ will not be limited to an increasing function of t. The drift function f and diffusion function g are all continuous, and $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The SDDE (1) is driven by a scalar Wiener process $W(t)$.

We can write (1) in the following integral form:

$$\begin{aligned} x(t) = {} & x(0) + \int_0^t f(x(s), x(s-\tau(s)))\,\mathrm{d}s \\ & + \int_0^t g(x(s), x(s-\tau(s)))\,\mathrm{d}W(s). \end{aligned} \tag{2}$$

Here we focus mainly on the SSBE method proposed by Wang and Gan [8]. Although the convergence rate in mean-square (L^2) sense has been obtained, the convergence analysis in L^p sense is more difficult; for example, Jensen's inequality cannot be used directly. The aim of this paper is to obtain the almost sure convergence rate together with L^p convergence rate of the SSBE method for SDDE (1). In the next section, we recall the SSBE method and give some assumptions. The main convergence results are given in Section 3.

2. The SSBE Method

First, let us state the numerical method. In the following, we consider a uniform mesh $\mathcal{I}_N = \{t_0, t_1, \ldots, t_N\}$, where the positive integer N is given, the time step size $h_N = T/N$, and $t_n = nh_N$ for $0 \le n \le N$. The numerical approximation of $x(t)$ at time t_n is denoted by Y_n. The SSBE method [8] for SDDE (1) can be written as

$$\begin{aligned} Y_n^* &= Y_n + h_N f\left(Y_n^*, Z_n\right), \\ Y_{n+1} &= Y_n^* + g\left(Y_n^*, Z_n\right)\Delta w_n, \end{aligned} \tag{3}$$

where $\Delta w_n := w(t_{n+1}) - w(t_n)$ and

$$Z_n = \begin{cases} \psi\left(t_n - \tau\left(t_n\right)\right), \\ \quad t_n - \tau\left(t_n\right) < 0, \\ \mu Y_{n-q_n+1} + (1-\mu) Y_{n-q_n}, \\ \quad 0 \le t_n - \tau\left(t_n\right) \in \left[t_{n-q_n}, t_{n-q_n+1}\right), \end{cases} \tag{4}$$

for $0 \le \mu < 1$ and positive integer $q_n \ge 1$.

To prove the almost sure convergence of the SSBE method (3), we need to define its continuous extension. Let us introduce two step processes as follows:

$$\begin{aligned} y^*(s) &:= \sum_{k=0}^{\infty} \mathbf{1}_{\{t_k \le s < t_{k+1}\}} Y_k^*, \\ z(s) &:= \sum_{k=0}^{\infty} \mathbf{1}_{\{t_k \le s < t_{k+1}\}} Z_k, \end{aligned} \tag{5}$$

where $\mathbf{1}_S$ is the indicator function of set S.

The continuous SSBE approximate solution is then defined by

$$y(t) := \begin{cases} \psi(t), & t \in [-\tau, 0], \\ \psi(0) + \displaystyle\int_0^t f\left(y^*(s), z(s)\right) \mathrm{d}s \\ \quad + \displaystyle\int_0^t g\left(y^*(s), z(s)\right) \mathrm{d}W(s), & t \in [0, T]. \end{cases} \tag{6}$$

It is not difficult to see that $y(t_k) = Y_k$ for every $k \ge 0$.

At various points in this paper, we will assume subsets of the following set of conditions [8].

(A1) The SDDE (1) has a unique solution $x(t)$ on $[-\tau, T]$. The functions $f(x, y)$ and $g(x, y)$ are both locally Lipschitz continuous in x and y; that is, there exists a constant L_D such that

$$\begin{aligned} &\left|f(x, y) - f(\bar{x}, \bar{y})\right| \vee \left|g(x, y) - g(\bar{x}, \bar{y})\right| \\ &\quad \le L_D\left(|x - \bar{x}| + |y - \bar{y}|\right), \end{aligned} \tag{7}$$

for all $t \ge 0$ and those $x, y, \bar{x}, \bar{y} \in \mathbb{R}$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \le D$. Here, $\vee$ is the maximal operator.

(A2) The function values $f(0,0)$ and $g(0,0)$ are bounded. The exact solution $x(t)$ and its continuous-time approximation solution $y(t)$ have pth moment bounds; that is, there exist constants $L_A > 0$ and integer $p > 2$ such that

$$\mathrm{E}\left[\sup_{0 \le t \le T} |x(t)|^p\right] \vee \mathrm{E}\left[\sup_{0 \le t \le T} |y(t)|^p\right] \le L_A. \tag{8}$$

(A3) (The Hölder continuity of the initial data) There exist constants $K_1 > 0$ and $K_2 > 0$ such that for all $-\tau \le s < t \le 0$ and positive integer p,

$$\mathrm{E}\left|\psi(t) - \psi(s)\right|^{2p} \le K_1 |t - s|^p, \tag{9}$$

and $\tau(t)$ is a continuous function satisfying

$$|\tau(t) - \tau(s)| \le K_2 |t - s|. \tag{10}$$

3. The Convergence Analysis

In this section, we first give some lemmas for deriving the main theorem. We define three stopping times [8] as follows:

$$\begin{aligned} \rho_D &= \inf\{t \ge 0 : |x(t)| \ge D\}, \\ \theta_D &= \inf\{t \ge 0 : |y(t)| \ge D \text{ or } |y^*(t)| \ge D\}, \end{aligned} \tag{11}$$

and $\sigma_D = \rho_D \wedge \theta_D$, where $\wedge$ is the minimal operator. Further, the infimum of the empty set is set as ∞.

In what follows, constant K is generic, which depends on f, g, the initial data ψ, the interval of integration $[0, T]$, and D, but it is independent of the discretization parameters n and h_N.

Lemma 1. *Let $y(t)$ be the solution of* (6). *Under assumptions* (A1) *and* (A2), *one has*

$$\mathrm{E}\left[\mathbf{1}_{\{t \le \theta_D\}}\left|y^*(t) - y(t_k)\right|^p\right] \le K h_N^p, \tag{12}$$

$$\mathrm{E}\left[\mathbf{1}_{\{t \le \theta_D\}}\left|y(t) - y(t_k)\right|^p\right] \le K h_N^{p/2} \tag{13}$$

for $t \in [t_k, t_{k+1}]$ with $k = 0, 1, \ldots, N-1$, and integer $p > 2$.

Proof. We note that for $|x| \vee |y| \le D$,

$$\begin{aligned} \left|f(x, y)\right|^2 &\le 2\left|f(x, y) - f(0,0)\right|^2 + 2\left|f(0,0)\right|^2 \\ &\le K\left(1 + |x|^2 + |y|^2\right). \end{aligned} \tag{14}$$

Combining (3), (5), (6), and (14), we obtain

$$\begin{aligned} \mathrm{E}\left[\mathbf{1}_{\{t \le \theta_D\}}\left|y^*(t) - y(t_k)\right|^p\right] &\le \mathrm{E}\left|h_N f\left(Y_k^*, Z_k\right)\right|^p \\ &\le K h_N^p \end{aligned} \tag{15}$$

under assumptions (A1) and (A2).

Now we prove (13). From (6), we have

$$\begin{aligned} y(t) = y(t_k) &+ \int_{t_k}^t f\left(y^*(s), z(s)\right) \mathrm{d}s \\ &+ \int_{t_k}^t g\left(y^*(s), z(s)\right) \mathrm{d}W(s), \end{aligned} \tag{16}$$

for $t \in [t_k, t_{k+1}]$. To estimate $\mathrm{E}|y(t) - y(t_k)|^p$, we will first apply the elementary inequality, which states that, for every $r > 0$, it follows that

$$|a+b|^r \leq \left(2^{r-1} \vee 1\right)\left(|a|^r + |b|^r\right). \tag{17}$$

Then, for $p > 2$ and $t_k < \theta_D$,

$$\begin{aligned} &\mathrm{E}\left[\mathbf{1}_{\{t\leq\theta_D\}}\left|y(t) - y(t_k)\right|^p\right] \\ &\quad\leq 2^{p-1}\mathrm{E}\left[\left|\int_{t_k}^{t\wedge\theta_D} f\left(y^*(s), z(s)\right)\mathrm{d}s\right|^p \right. \\ &\qquad\left. + \left|\int_{t_k}^{t\wedge\theta_D} g\left(y^*(s), z(s)\right)\mathrm{d}W(s)\right|^p\right]. \end{aligned} \tag{18}$$

Applying Hölder's inequality to the first integral of (18), as well as Burkholder-Davis-Gundy inequality to the Itô integral of (18), we obtain

$$\begin{aligned} &\mathrm{E}\left[\mathbf{1}_{\{t\leq\theta_D\}}\left|y(t) - y(t_k)\right|^p\right] \\ &\leq 2^{p-1}\left[(t-t_k)^{p-1}\underbrace{\int_{t_k}^{t}\mathrm{E}\left|f\left(y^*(s\wedge\theta_D), z(s\wedge\theta_D)\right)\right|^p\mathrm{d}s}_{J_1}\right. \\ &\qquad + K(t-t_k)^{p/2-1} \\ &\qquad \left.\underbrace{\times\int_{t_k}^{t}\mathrm{E}\left|g\left(y^*(s\wedge\theta_D), z(s\wedge\theta_D)\right)\right|^p\mathrm{d}s}_{J_2}\right]. \end{aligned} \tag{19}$$

We then have

$$\begin{aligned} J_1 &= \int_{t_k}^{t}\mathrm{E}\left(\left|f\left(y^*(s\wedge\theta_D), z(s\wedge\theta_D)\right)\right|^2\right)^{p/2}\mathrm{d}s \\ &\leq \int_{t_k}^{t}\mathrm{E}\left[K_D\left(1 + \left|y^*(s\wedge\theta_D)\right|^2 + \left|z(s\wedge\theta_D)\right|^2\right)\right]^{p/2}\mathrm{d}s \\ &\leq 2^{p/2-1}K_D^{p/2} \\ &\quad\times\int_{t_k}^{t}\mathrm{E}\left[1 + 2^{p/2-1}\left(\left|y^*(s\wedge\theta_D)\right|^p + \left|z(s\wedge\theta_D)\right|^p\right)\right]\mathrm{d}s \\ &\leq 2^{p/2-1}K_D^{p/2} \\ &\quad\times\left[1 + 2^{p/2-1}\mathrm{E}\left|y^*(s\wedge\theta_D)\right|^p + 2^{p/2-1}\mathrm{E}\left|z(s\wedge\theta_D)\right|^p\right] \\ &\quad\times(t-t_k) \\ &= K(t-t_k) \end{aligned} \tag{20}$$

under the assumption (A2).

Similarly, replacing f by g and repeating the previous procedure, we obtain $J_2 \leq K(t-t_k)$. Therefore, by substituting this result and (20) into (19), the proof is complete. □

Lemma 2. *Let $y(t)$ be the solution of* (6). *Under assumptions (A1), (A2), and (A3), one has*

$$\mathrm{E}\left[\mathbf{1}_{\{t\leq\theta_D\}}\left|y(t-\tau(t)) - z(t)\right|^p\right] \leq Kh_N^{p/2} \tag{21}$$

for $t \in [t_n, t_{n+1}]$ with $n = 0, 1, \ldots, N-1$, and integer $p > 2$.

Proof. To show the desired result, let us consider the following four possible cases:

(1) $t_n - \tau(t_n) \geq 0, t - \tau(t) \geq 0$,
(2) $t_n - \tau(t_n) \geq 0, t - \tau(t) < 0$,
(3) $t_n - \tau(t_n) < 0, t - \tau(t) \geq 0$,
(4) $t_n - \tau(t_n) < 0, t - \tau(t) < 0$.

In case (1), without loss of generality, we assume $t - \tau(t) \in [t_i, t_{i+1})$, $t_n - \tau(t_n) = (1-\mu)t_j + \mu t_{j+1} \in [t_j, t_{j+1})$, $i > j \geq 0$. Thus, from (4), (5), and (6), we have

$$\begin{aligned} &y(t-\tau(t)) - z(t) \\ &\quad= \left[t - \tau(t) - t_i\right] f\left(Y_i^*, Z_i\right) + g\left(Y_i^*, Z_i\right)\Delta w_i \\ &\qquad - (1-\mu)\sum_{k=j}^{i-1}\left[h_N f\left(Y_{k+1}^*, Z_{k+1}\right) + g\left(Y_k^*, Z_k\right)\Delta w_k\right] \\ &\qquad - \mu\sum_{k=j+1}^{i-1}\mathbf{1}_{\{i>j+1\}}\left[h_N f\left(Y_{k+1}^*, Z_{k+1}\right) + g\left(Y_k^*, Z_k\right)\Delta w_k\right]. \end{aligned} \tag{22}$$

We note that

$$\mathrm{E}\left|\Delta w_l\right|^p = \frac{\Gamma\left((p+1)/2\right)}{\sqrt{\pi}}\left(2h_N\right)^{p/2} \tag{23}$$

with $l = 0, 1, 2, \ldots, N-1$. Therefore, under assumptions (A1) and (A2), applying (17) and (23), we can derive (21) from (22). Under assumptions (A1)–(A3), the proof of the three other cases follows in a similar manner that of Lemma 2.5 in [8]. Then combining these results gives the required result. □

Lemma 3. *Let $x(t)$ be the solution of SDDE* (1) *and let $y(t)$ be the solution of* (6). *Under assumptions (A1), (A2), and (A3), one obtains*

$$\sup_{t\in[0,T]}\mathrm{E}\left|y(t\wedge\sigma_D) - x(t\wedge\sigma_D)\right|^p \leq Kh_N^{p/2} \tag{24}$$

for integer $p > 2$.

Proof. Suppose that $t \in [t_k, t_{k+1}]$ and $t_k \leq \sigma_D$. From (2) and (6), we have

$$\begin{aligned} y(t) - x(t) &= y(t_k) - x(t_k) \\ &\quad + \int_{t_k}^{t}\left[f\left(y^*(s), z(s)\right)\right. \\ &\qquad\left. - f\left(x(s), x(s-\tau(s))\right)\right]\mathrm{d}s \end{aligned}$$

$$+\int_{t_k}^{t}[g(y^*(s),z(s)) -g(x(s),x(s-\tau(s)))]\,dW_s. \quad (25)$$

Let $\phi(t)$ denote the difference $y(t)-x(t)$, by using Itô's formula to the function $u(\phi)=|\phi|^p$ [10] and then, by taking expectation, we get

$$\begin{aligned}\mathrm{E}\left|y(t\wedge\sigma_D)-x(t\wedge\sigma_D)\right|^p \le{}& \mathrm{E}\left|y(t_k)-x(t_k)\right|^p + p\mathrm{E}\underbrace{\int_{t_k}^{t\wedge\sigma_D}\left|f(y^*(s),z(s))-f(x(s),x(s-\tau(s)))\right|\left|\phi(s)\right|^{p-1}\mathrm{d}s}_{I_1(t)}\\ &+\frac{p(p-1)}{2}\mathrm{E}\underbrace{\int_{t_k}^{t\wedge\sigma_D}\left|g(y^*(s),z(s))-g(x(s),x(s-\tau(s)))\right|^2\left|\phi(s)\right|^{p-2}\mathrm{d}s}_{I_2(t)}\\ &+\underbrace{p\mathrm{E}\int_{t_k}^{t\wedge\sigma_D}\left|g(y^*(s),z(s))-g(x(s),x(s-\tau(s)))\right|\left|\phi(s)\right|^{p-1}\mathrm{d}W(s)}_{I_3(t)}.\end{aligned} \quad (26)$$

Let $V(t)$ denote the left side of (26); that is,

$$V(t):=\mathrm{E}\left|y(t\wedge\sigma_D)-x(t\wedge\sigma_D)\right|^p. \quad (27)$$

As we know, if ϱ is a stopping time and if M_t is a martingale, then $M_{t\wedge\varrho}$ is a martingale too. Therefore, the expectation $I_3(t)$ vanishes because of martingale property for Itô integral; then, the inequality (26) implies that

$$V(t)\le V(t_k)+pI_1(t)+\frac{p(p-1)}{2}I_2(t) \quad (28)$$

for $t\in[t_k,t_{k+1}]$ with $k=0,1,\ldots,N-1$.

By applying the local Lipschitz condition and the elementary *Young inequality*, we obtain

$$\begin{aligned}I_1(t)&\le \mathrm{E}\int_{t_k}^{t\wedge\sigma_D}L_D\left[\left|y^*(s)-x(s)\right|+\left|z(s)-x(s-\tau(s))\right|\right]\times\left|\phi(s)\right|^{p-1}\mathrm{d}s\\ &\le L_D\mathrm{E}\int_{t_k}^{t\wedge\sigma_D}\left[\left|y^*(s)-y(s)\right|+\left|y(s)-x(s)\right|\right]\\ &\qquad+\left[\left|z(s)-y(s-\tau(s))\right|+\left|y(s-\tau(s))-x(s-\tau(s))\right|\right]\times\left|\phi(s)\right|^{p-1}\mathrm{d}s\\ &\le L_D\mathrm{E}\int_{t_k}^{t\wedge\sigma_D}\left[\left|y^*(s)-y(t_k)\right|+\left|y(t_k)-y(s)\right|+\left|y(s)-x(s)\right|\right]\\ &\qquad+\left[\left|z(s)-y(s-\tau(s))\right|+\left|y(s-\tau(s))-x(s-\tau(s))\right|\right]\times\left|\phi(s)\right|^{p-1}\mathrm{d}s\\ &\le L_D\left\{\left(1+\frac{4p}{p-1}\right)\int_{t_k}^{t}V(s)\,\mathrm{d}s+\frac{1}{p}\int_{t_k}^{t}V(s-\tau(s))\,\mathrm{d}s\right.\\ &\qquad+\frac{1}{p}\int_{t_k}^{t}\mathrm{E}\left|y^*(s\wedge\sigma_D)-y(t_k)\right|^p\mathrm{d}s +\frac{1}{p}\int_{t_k}^{t}\mathrm{E}\left|y(s\wedge\sigma_D)-y(t_k)\right|^p\mathrm{d}s\\ &\qquad\left.+\frac{1}{p}\int_{t_k}^{t}\mathrm{E}\left|y(s\wedge\sigma_D-\tau(s\wedge\sigma_D))-z(s\wedge\sigma_D)\right|^p\,\mathrm{d}s\right\},\end{aligned} \quad (29)$$

which implies that

$$I_1(t)\le L_D\left(1+\frac{4p}{p-1}\right)\int_{t_k}^{t}V(s)\,\mathrm{d}s+\frac{L_D}{p}\int_{t_k}^{t}V(s-\tau(s))\,\mathrm{d}s+Kh_N^{p/2+1} \quad (30)$$

by Lemmas 1 and 2.

A similar result can be derived for $I_2(t)$ so that

$$V(t)\le V(t_k)+Kh_N^{p/2+1}+\alpha\int_{t_k}^{t}V(s)\,\mathrm{d}s+\beta\int_{t_k}^{t}V(s-\tau(s))\,\mathrm{d}s \quad (31)$$

from (28), where α and β are generic constants independent of h_N.

Now we will proceed by using an induction argument over consecutive intervals of length $\tau_{\min}$ up to the end of the interval $[0,T]$.

Step 1. Given $t\in[0,\tau_{\min}]$, it is easy to show that $s-\tau(s)\le t-\tau_{\min}\le 0$ for $s\le t$, so

$$\int_{t_k}^{t}V(s-\tau(s))\,\mathrm{d}s=\mathrm{E}\int_{t_k}^{t\wedge\sigma_D}\left|\psi(s-\tau(s))-\psi(s-\tau(s))\right|^p\mathrm{d}s=0. \quad (32)$$

Inserting this into (31) gives

$$V(t) \le V(t_k) + Kh_N^{p/2+1} + \int_{t_k}^{t} \alpha V(s)\,\mathrm{d}s \le \left(V(t_k) + Kh_N^{p/2+1}\right) e^{\alpha(t-t_k)} \quad (33)$$

by applying the Gronwall inequality [11].

Further, from (33), we have

$$\begin{aligned} V(t_k) &\le V(t_{k-1}) e^{\alpha h_N} + Kh_N^{p/2+1} e^{\alpha h_N} \\ &\le V(t_0)\left(e^{\alpha h_N}\right)^k + Kh_N^{p/2+1} \sum_{j=1}^{k} \left(e^{\alpha h_N}\right)^j \\ &\le Kh_N^{p/2}. \end{aligned} \quad (34)$$

Combining (33) and (34) gives $V(t) \le Kh_N^{p/2}$ for $t \in [0, \tau_{\min}]$.

Step 2. Suppose $t \in [m\tau_{\min}, (m+1)\tau_{\min}]$ with $m = 1, 2, \ldots$, which implies $s - \tau(s) \le t - \tau_{\min} \le m\tau_{\min}$ for $s \le t$; then we can obtain

$$\int_{t_k}^{t} V(s - \tau(s))\,\mathrm{d}s \le Kh_N^{p/2+1} \quad (35)$$

from the previous recursive step. Hence, the inequality (31) also gives

$$V(t) \le V(t_k) + Kh_N^{p/2+1} + \int_{t_k}^{t} \alpha V(s)\,\mathrm{d}s. \quad (36)$$

Finally, we can obtain the desired result using an approach similar to Step 1. □

The next lemma shows that the sequence of the approximate solutions $y(t)$ tends to the exact solution $x(t)$ as $h_N \downarrow 0$ in the sense of L^p-norm under local condition.

Lemma 4. *We assume that the conditions of Lemma 3 are fulfilled. Then, for $p > 2$, one has*

$$\mathrm{E}\left[\sup_{t\in[0,T]} \left|x(t\wedge\sigma_D) - y(t\wedge\sigma_D)\right|^p\right] \le Kh_N^{p/2}, \quad \text{as } h_N \downarrow 0. \quad (37)$$

Proof. Using (2), (6), (17), the Burkholder-Davis-Gundy inequality, and Hölder's inequality, we have

$$\begin{aligned} &\mathrm{E}\left[\sup_{t\in[0,T]} \left|x(t\wedge\sigma_D) - y(t\wedge\sigma_D)\right|^p\right] \\ &\le 2^{p-1}\mathrm{E}\left[\sup_{t\in[0,T]} \left|\int_0^{t\wedge\sigma_D} \left(f\left(y^*(s), z(s)\right) - f\left(x(s), x(s-\tau(s))\right)\right) \mathrm{d}s\right|^p\right] \\ &\quad + 2^{p-1}\mathrm{E}\left[\sup_{t\in[0,T]} \left|\int_0^{t\wedge\sigma_D} \left[g\left(y^*(s), z(s)\right) - g\left(x(s), x(s-\tau(s))\right)\right] \mathrm{d}W(s)\right|^p\right] \\ &\le 2^{p-1}(T\vee 1)^{p-1} \int_0^T \mathrm{E}\left|f\left(y^*(s), z(s)\right) - f\left(x(s), x(s-\tau(s))\right)\right|^p \mathrm{d}s \\ &\quad + 2^{p-1}(T\vee 1)^{p/2-1} \int_0^T \mathrm{E}\left|g\left(y^*(s), z(s)\right) - g\left(x(s), x(s-\tau(s))\right)\right|^p \mathrm{d}s. \end{aligned} \quad (38)$$

Combining Lemmas 1, 2, and 3, we get

$$\begin{aligned} &\sup_{s\in[0,T]} \mathrm{E}\left|y^*(s\wedge\sigma_D) - x(s\wedge\sigma_D)\right|^p \le Kh_N^{p/2}, \\ &\sup_{s\in[0,T]} \mathrm{E}\left|z(s\wedge\sigma_D) - x\left(s\wedge\sigma_D - \tau(s\wedge\sigma_D)\right)\right|^p \le Kh_N^{p/2} \end{aligned} \quad (39)$$

by first using Minkowski's inequality.

Applying the local Lipschitz condition (7) to (38), the relation (37) then follows from (39). □

The following two theorems are the main results of this paper. They give, respectively, the L^p error and almost sure error of the SSBE method (3).

Theorem 5. *One assumes that the conditions of Lemma 3 are fulfilled. Then, for $p > 2$ and sufficiently large D, the approximation (3) for SDDE (1) is convergent in L^p sense and*

$$\mathrm{E}\left[\sup_{t\in[0,T]} \left|x(t) - y(t)\right|^p\right] \le Kh_N^{p/2}, \quad \text{as } h_N \downarrow 0. \quad (40)$$

The result can be proved via a similar approach to that in [5, Theorem 2.1] and [6, Theorem 2.2]; see the appendix.

Theorem 6. *One assumes that the conditions of Lemma 3 are fulfilled. Then,*

$$\sup_{t\in[0,T]} \left|x(t) - y(t)\right| = \mathcal{O}\left(h_N^{\gamma}\right) \quad (a.s.) \quad (41)$$

for every positive number $\gamma < 1/2$ and sufficiently large D.

Proof. From Theorem 5, Chebyshev's inequality [11] gives

$$\begin{aligned} &\sum_{N=1}^{\infty} P\left\{\sup_{t\in[0,T]} \left|x(t) - y(t)\right| \ge h_N^{\gamma}\right\} \\ &\le \sum_{N=1}^{\infty} h_N^{-p\gamma} \mathrm{E}\left\{\sup_{t\in[0,T]} \left|x(t) - y(t)\right|^p\right\} \\ &\le K \sum_{N=1}^{\infty} \left(\frac{T}{N}\right)^{p/2-p\gamma} \end{aligned} \quad (42)$$

for any $\gamma > 0$ and $p \ge 3$.

The series $\sum_{N=1}^{\infty}(T/N)^{p/2-p\gamma}$ is convergent for $p/2-p\gamma>1$; that is, $\gamma<1/2-1/p<1/2$. Further, by applying the Borel-Cantelli lemma [11], the inequality (42) implies that

$$P\left\{\lim_{N\to\infty}\left(\sup_{t\in[0,T]}\left|x(t)-y(t)\right|<h_N^{\gamma}\right)\right\}=1. \tag{43}$$

Thus, under assumptions (A1)–(A3), the approximate solution $y(t)$ converges (a.s.) to the exact solution $x(t)$ uniformly on $[0,T]$ as $h_N\to 0$. □

Appendix

Proof of Theorem 5. Obviously, we have

$$\begin{aligned}
&\mathrm{E}\left[\sup_{t\in[0,T]}\left|x(t)-y(t)\right|^p\right]\\
&\quad=\mathrm{E}\left[\sup_{t\in[0,T]}\left|x(t)-y(t)\right|^p\mathbf{1}_{\{\rho_D>T\text{ and }\theta_D>T\}}\right]\\
&\qquad+\mathrm{E}\left[\sup_{t\in[0,T]}\left|x(t)-y(t)\right|^p\mathbf{1}_{\{\rho_D\le T\text{ or }\theta_D\le T\}}\right].
\end{aligned}\tag{A.1}$$

By the improved Young inequality (see, e.g., [5, 6])

$$AB\le\eta\frac{A^{\mu}}{\mu}+\frac{1}{\eta^{(\nu/\mu)}}\frac{B^{\nu}}{\nu}\quad\forall A,B,\eta,\mu>0,\tag{A.2}$$

when $\mu^{-1}+\nu^{-1}=1$, we obtain

$$\begin{aligned}
&\mathrm{E}\left[\sup_{t\in[0,T]}\left|x(t)-y(t)\right|^p\mathbf{1}_{\{\rho_D\le T\text{ or }\theta_D\le T\}}\right]\\
&\quad\le\mathrm{E}\left[\frac{\eta}{\mu}\sup_{t\in[0,T]}\left|x(t)-y(t)\right|^{\mu p}\right]\\
&\qquad+\mathrm{E}\left[\frac{1}{\nu\eta^{(\nu/\mu)}}\left(\mathbf{1}_{\{\rho_D\le T\text{ or }\theta_D\le T\}}\right)^{\nu}\right]\\
&\quad\le\frac{\eta}{\mu}\mathrm{E}\left[\sup_{t\in[0,T]}\left|x(t)-y(t)\right|^{\mu p}\right]\\
&\qquad+\frac{1}{\nu\eta^{(\nu/\mu)}}P\left(\rho_D\le T\text{ or }\theta_D\le T\right).
\end{aligned}\tag{A.3}$$

Note

$$\begin{aligned}
&\mathrm{E}\left[\sup_{t\in[0,T]}\left|x(t)-y(t)\right|^{\mu p}\right]\\
&\quad\le 2^{\mu p-1}\mathrm{E}\left[\sup_{t\in[0,T]}\left|x(t)\right|^{\mu p}+\sup_{t\in[0,T]}\left|y(t)\right|^{\mu p}\right]\\
&\quad\le 2^{\mu p}L_A.
\end{aligned}\tag{A.4}$$

Furthermore,

$$P\left(\rho_D\le T\right)\le\mathrm{E}\left[\mathbf{1}_{\{\rho_D\le T\}}\frac{\left|x(\rho_D)\right|^p}{D^p}\right]\le\frac{L_A}{D^p},\tag{A.5}$$

and, similarly,

$$P\left(\theta_D\le T\right)\le\frac{L_A}{D^p}.\tag{A.6}$$

Therefore,

$$P\left(\rho_D\le T\text{ or }\theta_D\le T\right)\le P\left(\rho_D\le T\right)+P\left(\theta_D\le T\right)\le\frac{2L_A}{D^p}.\tag{A.7}$$

On the other side, Lemma 4 implies that

$$\begin{aligned}
&\mathrm{E}\left[\sup_{t\in[0,T]}\left|x(t)-y(t)\right|^p\mathbf{1}_{\{\rho_D>T\text{ and }\theta_D>T\}}\right]\\
&\quad\le\mathrm{E}\left[\sup_{t\in[0,T]}\left|x(t\wedge\sigma_D)-y(t\wedge\sigma_D)\right|^p\right]\\
&\quad\le Kh_N^{p/2}.
\end{aligned}\tag{A.8}$$

Combining (A.1) and the above inequalities gives

$$\begin{aligned}
&\mathrm{E}\left[\sup_{t\in[0,T]}\left|x(t)-y(t)\right|^p\right]\\
&\quad\le Kh_N^{p/2}+\frac{\eta 2^{\mu p}L_A}{\mu}+\frac{2L_A}{\nu\eta^{(\nu/\mu)}D^p}.
\end{aligned}\tag{A.9}$$

Setting $\eta=h_N^{p/2}$ and choosing $D=h_N^{-(1/2)(1+(\nu/\mu))}$, we obtain (40) for sufficiently large D. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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