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Research Article

On Nonsmooth Semi-Infinite Minimax Programming Problem with (Φ, ρ)-Invexity

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We are interested in a nonsmooth minimax programming Problem (SIP). Firstly, we establish the necessary optimality conditions theorems for Problem (SIP) when using the well-known Caratheodory's theorem. Under the Lipschitz (Φ, ρ)-invexity assumptions, we derive the sufficiency of the necessary optimality conditions for the same problem. We also formulate dual and establish weak, strong, and strict converse duality theorems for Problem (SIP) and its dual. These results extend several known results to a wider class of problems.

1. Introduction

Convexity plays a central role in many aspects of mathematical programming including analysis of stability, sufficient optimality conditions, and duality. Based on convexity assumptions, nonlinear programming problems can be solved efficiently. There have been many attempts to weaken the convexity assumptions in order to treat many practical problems. Therefore, many concepts of generalized convex functions have been introduced and applied to mathematical programming problems in the literature [1]. One of these concepts, invexity, was introduced by Hanson in [2]. Hanson has shown that invexity has a common property in mathematical programming with convexity that Karush-Kuhn-Tucker conditions are sufficient for global optimality of nonlinear programming under the invexity assumptions. Ben-Israel and Mond [3] introduced the concept of preinvex functions which is a special case of invexity. Many other concepts of generalized convexity such as (p, r)-invexity [4], (F, ρ)-convexity [5], (F, α, ρ, d)-convexity [6], (C, α, ρ, d)-convexity [7], and V-r-invexity [8] have also been introduced. With these definitions of generalized invexity on the hand, several authors have been interested recently in the optimality conditions and duality results for different classes of minimax programming problems; see [9–12] for details.

Recently, Antczak and Stasiak [13] generalized the definition of (Φ, ρ)-invexity notion introduced by Caristi et al. and M. V. Ştefănescu and A. Ştefănescu [14, 15] for differentiable optimization problems to the case of mathematical programming problems with locally Lipschitz functions. They proved sufficient optimality conditions and duality results for nondifferentiable optimization problems involving locally Lipschitz (Φ, ρ)-invex functions. Antczak [16] also considered a class of nonsmooth minimax programming problems in which functions involved are locally Lipschitz (Φ, ρ)-invex. We point out that this locally Lipschitz (Φ, ρ)-invexity includes the (C, α, ρ, d)-convexity as a special case and Yuan et al. [7] defined firstly the (C, α, ρ, d)-convexity with a convex functional C.

Due to a growing number of theoretical and practical applications, semi-infinite programming has recently become one of the most substantial research areas in applied mathematics and operations research. For more details on semi-infinite programming we refer to the survey papers [17–19] and for clear understanding of different aspects of semi-infinite programming we refer to [20]. M. V. Ştefănescu and A. Ştefănescu [17] considered differentiable Problem (SIP) with new (Φ, ρ)-invexity. However, the results of this kind of programming can not be used to deal with the concrete nonsmooth semi-infinite minimax programming problem as presenting in Example 9 in Section 3 since the objective function is nondifferentiable at $x = 1$. Therefore, we are interested in dealing with nonsmooth Problem (SIP) with locally Lipschitz (Φ, ρ)-invexity proposed in [13], in this paper.

The rest of the paper is organized as follows. In Section 2, we present concepts regarding Lipschtiz (Φ, ρ)-invexity. In Section 3, we present not only necessary but also sufficient optimality conditions for nonsmooth Problem (SIP). When the necessary optimality conditions and the (Φ, ρ)-invexity concept are utilized, dual Problem (DI) is formulated for the primal (SIP) and duality results between them are presented in Section 4. Section 5 is our conclusions.

2. Notations and Preliminaries

In this section, we provide some definitions and results that we shall use in the sequel. Let X be a subset of $\mathbb{R}^n$ and denote $Q := \{1, 2, \ldots, q\}$, $Q^* := \{1, 2, \ldots, q^*\}$, $M := \{1, 2, \ldots, m\}$, and $M^* := \{1, 2, \ldots, m^*\}$.

Definition 1. A real-valued function $f : X \to \mathbb{R}$ is said to be locally Lipschitz on X if, for any $x \in X$, there exist a neighborhood U of x and a positive constant $T_x > 0$ such that

$$|f(y) - f(z)| \le T_x \|y - z\|, \quad \forall y, z \in U. \tag{1}$$

Definition 2 (see [21]). Let $d \in \mathbb{R}^n$ and $f : X \to \mathbb{R}$. If

$$f^0(x; d) := \lim_{\substack{y \to x \\ \mu \downarrow 0}} \sup \frac{1}{\mu} (f(y + \mu d) - f(y)) \tag{2}$$

exists, then $f^0(x; d)$ is said to be the Clarke derivative of f at x in the direction d. If this limit superior exists for all $d \in \mathbb{R}^n$, then f is called Clarke differentiable at x. The set

$$\partial f(x) = \{\zeta \mid f^0(x; d) \ge \langle \zeta, d \rangle, \forall d \in \mathbb{R}^n\} \tag{3}$$

is called the Clarke subgradient of f at x.

Note that if a function is locally Lipschitz, then its Clarke subgradient must exist.

The definition of the locally Lipschitz (Φ, ρ)-invexity was introduced by Antczak and Stasiak [13]; see also the following Definition 3. This generalized invexity was introduced as a generalization of differentiable (Φ, ρ)-invexity notion defined by Caristi et al. and M. V. Ştefănescu and A. Ştefănescu in [14, 17]. The main tool used in the definition of the locally Lipschitz (Φ, ρ)-invexity notion is the above Clarke generalized subgradient (see Definition 2).

Definition 3. Let $f : X \to \mathbb{R}$ be a real-valued Lipschitz function on X. For fixed $u \in X$, let $\Phi : X \times X \times \mathbb{R}^{n+1} \to \mathbb{R}$ be convex with respect to the third argument on $\mathbb{R}^{n+1}$ such that $\Phi(x, u, (0, a)) \ge 0$ for every $x \in X$ and any $a \ge 0$. If there exists a real-valued function $\rho(\cdot, \cdot) : X \times X \to \mathbb{R}$ such that

$$f(x) - f(u) \ge (>) \, \Phi(x, u, (\xi, \rho(x, u))), \quad \forall \xi \in \partial f(u) \tag{4}$$

holds for all $x \in X$ $(x \ne u)$, then f is said to be (strictly) locally Lipschitz (Φ, ρ)-invex at u on X or shortly (strictly) (Φ, ρ)-invex at u on X. If f is (strictly) locally Lipschitz (Φ, ρ)-invex at any u of X, then f is (strictly) locally Lipschitz (Φ, ρ)-invex on X.

Remark 4. In order to define an analogous class of (strictly) locally Lipschitz (Φ, ρ)-incave functions, the direction of the inequality in the definition of these functions should be changed to the opposite one.

In this paper, we deal with the nonsmooth semi-infinite minimax programming Problem (SIP) with the locally Lipschitz (Φ, ρ)-invexity proposed by Antczak and Stasiak [13]. Here, Problem (SIP) is

$$\begin{array}{ll} \min & \sup_{y \in Y} \phi(x, y) \\ \text{subject to} & \psi(x, z) \le 0, \quad z \in Z, \end{array} \tag{SIP}$$

where Y and Z are compact subsets of some Hausdorff topological spaces, $\phi(\cdot, \cdot) : \mathbb{R}^n \times Y \to \mathbb{R}$, $\psi(\cdot, \cdot) : \mathbb{R}^n \times Z \to \mathbb{R}$. Let E_{SIP} be the set of feasible solutions of Problem (SIP); in other words, $E_{\text{SIP}} = \{x \in \mathbb{R}^n \mid \psi(x, z) \le 0, z \in Z\}$. For convenience, let us define the following sets for every $x \in E_{\text{SIP}}$:

$$\begin{gathered} J(x) = \{z \in Z \mid \psi(x, z) = 0\}, \\ Z(x) = \left\{z^* \in Z \mid \psi(x, z^*) = \sup_{z \in Z} \psi(x, z)\right\}, \\ Y(x) = \left\{y^* \in Y \mid \phi(x, y^*) = \sup_{y \in Y} \phi(x, y)\right\}. \end{gathered} \tag{5}$$

If $x \in E_{\text{SIP}}$, then $J(x)$ represents the index set of the active restrictions at x. Note that $J(x) = Z(x)$ when $J(x)$ is not empty.

Consider the nonlinear programming problem

$$\begin{array}{ll} \min & f(x) \\ \text{subject to} & g(x) \le 0, \end{array} \tag{P}$$

where $f, g : \mathbb{R}^n \to \mathbb{R}$. A particular case of Problem (P) is the minimax problem (SIP) in which the functions f, g are given by

$$f(x) := \sup_{y \in Y} \phi(x, y), \tag{6}$$

$$g(x) := \sup_{z \in Z} \psi(x, z), \tag{7}$$

respectively. Let $x_0 \in E_{\text{SIP}}$. Consider the following unconstrained optimization problem (P'):

$$\min \{h(x) \mid x \in \mathbb{R}^n\}, \tag{P'}$$

where $h(x) := \max\{f(x) - f(x_0), g(x)\}$. Then, the relationship between Problems (P) and (P') is given in the following lemma.

Lemma 5. *If x_0 is a local minimizer for Problem* (P)*, then x_0 is also a local minimizer for Problem* (P').

To deal with the nonsmooth Problem (SIP), we need the following Conditions 1 and 2.

Condition 1. We assume that (a) the sets Y and Z are compact; (b) the function $\phi(x, y)$ is upper semicontinuous in (x, y), and the function $\psi(x, z)$ is upper semicontinuous in (x, z); (c) the function $\phi(x, y)$ is locally Lipschitz in x and uniformly for y in Y, and the function $\psi(x, y)$ is locally Lipschitz in x and uniformly for z in Z; (d) the function $\phi(x, y)$ is regular in x; that is, $\phi_x^\circ(x, y; \cdot) = \phi_x'(x, y; \cdot)$, where the symbol ϕ_x' denotes the derivative with respect to x; also the function $\psi(x, z)$ is regular in x; (e) the set-valued map $\partial\phi_x(x, y)$ is upper semicontinuous in (x, y), and the set-valued map $\partial\psi_x(x, z)$ is upper semicontinuous in (x, z).

Condition 2. For any finite subset $Z' \subset Z(x^*)$, for some $x^* \in E_{\mathrm{SIP}}$, the equality $\sum_{z\in Z'} \eta_z\zeta_z = 0$ with $\eta_z \geq 0$, $\zeta_z \in \partial\psi(x^*, z)$, $z \in Z'$, implies that

$$\eta_z = 0, \quad \forall z \in Z'. \tag{8}$$

Clarke [22, Theorem 2.1] has shown that, under the assumptions (a)–(e) of Condition 1, the maximum function f defined by (6) is locally Lipschitz; $f'(x, d)$ exists and is given by the formula

$$\begin{aligned} f'(x, d) &= f^\circ(x, d) \\ &= \max\{\xi \cdot d \mid \xi \in \partial_x\phi(x, y), y \in Y(x)\}, \end{aligned} \tag{9}$$

where $\xi \cdot d$ denotes the inner product of vectors ξ and d. Moreover, the sup in (6) can be replaced by max and the subgradient $\partial f(x)$ is given by

$$\partial f(x) = \mathrm{co}\left\{\bigcup_{y\in Y(x)} \partial_x\phi(x, y)\right\}. \tag{10}$$

Similarly, the maximum function g defined by (7) is locally Lipschitz; $g'(x, d)$ and $\partial g(x)$ are given by

$$\begin{aligned} g'(x, d) &= g^\circ(x, d) \\ &= \max\{\zeta \cdot d \mid \zeta \in \partial_x\psi(x, z), z \in Z(x)\}, \\ \partial g(x) &= \mathrm{co}\left\{\bigcup_{z\in Z(x)} \partial_x\psi(x, z)\right\}, \end{aligned} \tag{11}$$

respectively.

3. Optimality Conditions

In this section, we establish not only the necessary optimality conditions theorems but also the sufficient optimality conditions theorems for Problem (SIP) with the functions involved being locally Lipschitz with respect to the variable x.

Theorem 6 (necessary optimality conditions). *Let x^* be an optimal solution of* (SIP)*. One also assume that Condition 1 holds. Then there exist nonnegative integers q^* and m^* with $1 \leq q^* + m^* \leq n+1$, vectors $y_i^* \in Y(x^*)$ $(i \in Q^*)$, $z_j^* \in Z(x^*)$ $(j \in M^*)$, and scalars $\lambda_i^* > 0$ $(i \in Q^*)$, $\mu_j^* > 0$ $(j \in M^*)$ such that*

$$0 \in \sum_{i=1}^{q^*} \lambda_i^* \partial_x\phi(x^*, y_i^*) + \sum_{j=1}^{m^*} \mu_j^* \partial_x\psi(x^*, z_j^*), \tag{12}$$

$$\sum_{i=1}^{q^*} \lambda_i^* + \sum_{j=1}^{m^*} \mu_j^* \neq 0. \tag{13}$$

Here, one allow the case, where if $q^* = 0$, then the set Q^* is empty; similarly, if $m^* = 0$, then the set M^* is empty.

Proof. Let x^* be a local minimizer for Problem (SIP). This means that x^* is a local minimizer for Problem (P), where f and g are given by (6) and (7), respectively. Therefore, x^* is a local minimizer for Problem (P').

By Condition 1 and [22, Theorem 2.1], f is locally Lipschitzian and regular at x^*, so the function

$$\tilde{f}(x) := f(x) - f(x^*) \tag{14}$$

has the same properties. Then, using [21, Propositions 2.3.2 and 2.3.12], we obtain

$$\begin{aligned} &0 \in \partial h(x^*) \\ &= \mathrm{co}\{\partial\tilde{f}(x^*) \cup \partial g(x^*)\} = \mathrm{co}\{\partial f(x^*) \cup \partial g(x^*)\} \\ &= \mathrm{co}\left\{\mathrm{co}\left\{\bigcup_{y\in Y(x^*)} \partial_x\phi(x^*, y)\right\} \cup \mathrm{co}\left\{\bigcup_{z\in Z(x^*)} \partial_x\psi(x^*, z)\right\}\right\} \\ &= \mathrm{co}\left\{\bigcup_{y\in Y(x^*)} \partial_x\phi(x^*, y) \cup \bigcup_{z\in Z(x^*)} \partial_x\psi(x^*, z)\right\}; \end{aligned} \tag{15}$$

here the equality

$$\mathrm{co}(\mathrm{co}\, A \cup \mathrm{co}\, B) = \mathrm{co}(A \cup B) \tag{16}$$

is used in the fourth equality. Hence, by Caratheodory's theorem, there exist the nonnegative integers q^* and m^* and the scalars $\lambda_i^* > 0$ $(i \in Q^*)$ and $\mu_j^* > 0$ $(j \in M^*)$ such that

$$\begin{aligned} &1 \leq q^* + m^* \leq n + 1, \\ &0 = \sum_{i=1}^{q^*} \lambda_i\xi_i + \sum_{j=1}^{m^*} \mu_j\zeta_j \end{aligned} \tag{17}$$

for some $\xi_i \in \bigcup_{y\in Y(x^*)}\partial_x\phi(x^*,y)$ and $\zeta_j \in \bigcup_{z\in Z(x^*)}\partial_x\psi(x^*,z)$. Note that, for each $i \in Q^*$, $\xi_i \in \bigcup_{y\in Y(x^*)}\partial_x\phi(x^*,y)$ means that there exists $y \in Y(x^*)$ such that $\xi_i \in \partial_x\phi(x^*,y)$, and denote this y by y_i^*. Similarly, there exists $z_j^* \in Z(x^*)$ such that $\zeta_j \in \partial_x\psi(x^*,z_j^*)$ for each $j \in M^*$. Now the desired inequalities (12) and (13) can be deduced from the above discussion. □

Theorem 7 (necessary optimality conditions). *Let x^* be an optimal solution of* (SIP). *One also assume that Conditions 1 and 2 hold. Then there exist the nonnegative integers $q^* > 0$ and m^* with $1 \le q^* + m^* \le n+1$, the vectors $y_i^* \in Y(x^*)$ ($i \in Q^*$), $z_j^* \in Z(x^*)$ ($j \in M^*$), and the scalars $\lambda_i^* > 0$ ($i \in Q^*$), $\mu_j^* > 0$ ($j \in M^*$) satisfying* (12) *and*

$$\sum_{i=1}^{q^*}\lambda_i^* = 1. \tag{18}$$

Proof. By Theorem 6, we need to prove $q^* \neq 0$, on the contrary, that is, $q^* = 0$, then one obtains from (12) and (13) that

$$0 \in \sum_{j=1}^{m^*}\mu_j^*\partial_x\psi\left(x^*,z_j^*\right), \tag{19}$$

$$\sum_{j=1}^{m^*}\mu_j^* \neq 0, \tag{20}$$

respectively. By (19), there exist $\zeta_j \in \partial_x\psi(x^*,z_j^*)$ for $j \in M^*$ satisfying

$$\sum_{j=1}^{m^*}\mu_j^*\zeta_j = 0. \tag{21}$$

Now one obtains from the assumptions of Condition 2 that $\mu_j^* = 0$ for $j \in M^*$; this contradicts to (20), and we obtain the desired results. □

Next, we derive a sufficient optimality conditions theorem for Problem (SIP) under the assumption of (Φ,ρ)-invexity as defined in Definition 3.

Theorem 8 (sufficient optimality conditions). *Let $(x^*, q^*, m^*, \lambda^*, \mu^*, \overline{y^*}, \overline{z^*})$ satisfy conditions* (12) *and* (18)*, where $\overline{y^*} := (y_1^*,\ldots,y_{q^*}^*)$ and $\overline{z^*} := (z_1^*,\ldots,z_{m^*}^*)$. Assume that $\phi(\cdot,y_i^*)$, $i \in Q^*$, are (Φ,ρ_i^{ϕ})-invex at x^* on X and $\phi(\cdot,z_j^*)$, $j \in M^*$, are (Φ,ρ_j^{ψ})-invex at x^* on X. If $J(x^*) \neq \emptyset$ and*

$$\sum_{i=1}^{q^*}\lambda_i^*\rho_i^{\phi}(x,x^*) + \sum_{j=1}^{m^*}\mu_j^*\rho_j^{\psi}(x,x^*) \ge 0, \quad \forall x \in X, \tag{22}$$

then x^* is an optimal solution to (SIP).

Proof. Suppose, contrary to the result, that x^* is not an optimal solution for Problem (SIP). Hence, there exists $x_0 \in E_{\mathrm{SIP}}$ such that

$$\begin{gathered}\sup_{y\in Y}\phi(x_0,y) < \phi(x^*,y_i^*), \quad i \in Q^*,\\ \psi(x_0,z) \le 0 = \psi\left(x^*,z_j^*\right), \quad j \in M^*.\end{gathered} \tag{23}$$

Thus,

$$\begin{gathered}\phi(x_0,y_i^*) < \phi(x^*,y_i^*), \quad i \in Q^*,\\ \psi\left(x_0,z_j^*\right) \le \psi\left(x^*,z_j^*\right), \quad j \in M^*.\end{gathered} \tag{24}$$

Now, we can write the following statement:

$$\begin{gathered}\sum_{i=1}^{q^*}\lambda_i^*\left(\phi(x_0,y_i^*) - \phi(x^*,y_i^*)\right)\\ + \sum_{j=1}^{m^*}\mu_j^*\left(\psi\left(x_0,z_j^*\right) - \psi\left(x^*,z_j^*\right)\right) < 0.\end{gathered} \tag{25}$$

By the generalized invexity assumptions of $\phi(\cdot,y_i^*)$ and $\psi(\cdot,z_j^*)$, we have

$$\begin{gathered}\phi(x_0,y_i^*) - \phi(x^*,y_i^*)\\ \ge \Phi\left(x_0,x^*,\left(\xi_i,\rho_i^{\phi}(x_0,x^*)\right)\right), \quad \forall\xi_i \in \partial_x\phi(x^*,y_i^*),\\ \psi\left(x_0,z_j^*\right) - \psi\left(x^*,z_j^*\right)\\ \ge \Phi\left(x_0,x^*,\left(\zeta_j,\rho_j^{\psi}(x_0,x^*)\right)\right), \quad \forall\zeta_j \in \partial_x\psi\left(x^*,z_j^*\right).\end{gathered} \tag{26}$$

Employing (26) to (25), we have

$$\begin{gathered}\sum_{i=1}^{q^*}\lambda_i^*\Phi\left(x_0,x^*,\left(\xi_i,\rho_i^{\phi}(x_0,x^*)\right)\right)\\ + \sum_{j=1}^{m^*}\mu_j^*\Phi\left(x_0,x^*,\left(\zeta_j,\rho_j^{\psi}(x_0,x^*)\right)\right) < 0.\end{gathered} \tag{27}$$

By (18) and the convexity of Φ, we deduce that

$$\begin{gathered}\Phi\Bigg(x_0,x^*,\Bigg(\sum_{i=1}^{q^*}\lambda_i^*\xi_i + \sum_{j=1}^{m^*}\mu_j^*\zeta_j, \sum_{i=1}^{q^*}\lambda_i^*\rho_i^{\phi}(x_0,x^*)\\ + \sum_{j=1}^{m^*}\mu_j^*\rho_j^{\psi}(x_0,x^*)\Bigg)\Bigg) < 0.\end{gathered} \tag{28}$$

This, together with (40) and the assumption $\Phi(x_0,x^*,(0,a)) \ge 0$ for any $a \ge 0$, follows that

$$0 \notin \sum_{i=1}^{q^*}\lambda_i^*\partial_x\phi(x^*,y_i^*) + \sum_{j=1}^{m^*}\mu_j^*\partial_x\psi\left(x^*,z_j^*\right). \tag{29}$$

This is a contradiction to condition (12). □

Example 9. Let $Y = [0, 1]$ and $Z = [1, 2]$. Define

$$\phi(x, y) = \begin{cases} x + 2y, & y \geq x, \\ 2x + y, & x < y, \end{cases} \qquad \psi(x, z) = \left| x - \frac{3}{2} \right| - \frac{z}{2}. \tag{30}$$

Then, $\phi(\cdot, y)$ is $(\Phi, 1)$-invex at $x = 1$ for each $y \in Y$, ψ is $(\Phi, 1)$-invex at $x = 1$ for each $z \in Z$, and

$$\begin{aligned} f(x) &:= \sup_{y \in Y} \phi(x, y) = \begin{cases} x + 2, & y \geq x, \\ 2x + 1, & x < y, \end{cases} \\ g(x) &:= \sup_{z \in Z} \psi(x, z) = \left| x - \frac{3}{2} \right| - \frac{1}{2}. \end{aligned} \tag{31}$$

Note that

$$\begin{aligned} \partial_x \phi(x, y) &= \begin{cases} 1, & y \geq x, \\ [1, 2], & x = y, \\ 2, & x < y, \end{cases} \\ \partial_x \psi(x, z) &= \nabla_x \psi(x, z) = 1. \end{aligned} \tag{32}$$

Consider $x_0 = 1$. Since $Y(x_0) = \{1\}$, then we can assume $q = 1$. Therefore,

$$0 \in \lambda \partial_x \phi(1, 1) - \mu \partial_x \psi(1, 1) = \lambda [1, 2] - \mu, \tag{33}$$

where $\lambda = \mu = 1$. Now, from Theorem 8, we can say that $x_0 = 1$ is an optimal solution to (SIP).

4. Duality

Making use of the optimality conditions of the preceding section, we present dual Problem (DI) to the primal one (SIP) and establish weak, strong, and strict converse duality theorems. For convenience, we use the following notations:

$$\begin{aligned} K(x) = \Bigg\{ & (q, m, \lambda, \overline{y}, \overline{z}) \mid 1 \leq q \leq q + m \leq n + 1, \\ & \lambda = (\lambda_1, \ldots, \lambda_q) \text{ with } \sum_{i=1}^{q} \lambda_i = 1, \\ & \lambda_i > 0 \text{ for } i \in Q, \\ & \overline{y} = (y_1, \ldots, y_q) \text{ with } y_i \in Y(x), i \in Q, \\ & \overline{z} = (z_1, \ldots, z_m) \text{ with } z_j \in Z(x), j \in M \Bigg\}. \end{aligned} \tag{34}$$

Our dual problem (DI) can be stated as follows:

$$\max_{(q, m, \lambda, \overline{y}, \overline{z}) \in K(w)} \ \sup_{(w, \mu, \nu) \in H_1(q, m, \lambda, \overline{y}, \overline{z})} \nu, \tag{DI}$$

where $H_1(q, m, \lambda, \overline{y}, \overline{z})$ denotes the set of all (w, μ, ν) satisfying

$$0 \in \sum_{i=1}^{q} \lambda_i \partial_x \phi(w, y_i) + \sum_{j=1}^{m} \mu_j \partial_x \psi(w, z_j), \quad w \in X, \tag{35}$$

$$\phi(w, y_i) \geq \nu \geq 0, \quad i \in Q, \tag{36}$$

$$\mu_j \psi(w, z_j) \geq 0, \quad j \in M, \tag{37}$$

$$\mu = (\mu_1, \ldots, \mu_m) \text{ with } \mu_j > 0 \text{ for } j \in M, \tag{38}$$

$$(q, m, \lambda, \overline{y}, \overline{z}) \in K(w). \tag{39}$$

Note that if $H_1(q, m, \lambda, \overline{y}, \overline{z})$ is empty for some $(q, m, \lambda, \overline{y}, \overline{z}) \in K(w)$, then define $\sup_{(w, \mu, \nu) \in H_1(q, m, \lambda, \overline{y}, \overline{z})} \nu = -\infty$.

Theorem 10 (weak duality). *Let x and $(w, q, m, \lambda, \mu, \nu, \overline{y}, \overline{z})$ be* (SIP)*-feasible and* (DI)*-feasible, respectively. Suppose that $\phi(\cdot, y_i)$, $i \in Q$, are (Φ, ρ_i^{ϕ})-invex at w on X and $\psi(\cdot, z_j)$, $j \in M$, are (Φ, ρ_j^{ψ})-invex at w on X. If*

$$\sum_{i=1}^{q} \lambda_i \rho_i^{\phi}(x, w) + \sum_{j=1}^{m} \mu_j \rho_j^{\psi}(x, w) \geq 0, \tag{40}$$

then

$$\sup_{y \in Y} \phi(x, y) \geqslant \nu. \tag{41}$$

Proof. Suppose to the contrary that $\sup_{y \in Y} \phi(x, y) < \nu$. Therefore, we obtain

$$\phi(x, y) < \nu \leq \phi(w, y_i), \quad y_i \in Y(w), \forall y \in Y, i \in Q. \tag{42}$$

Thus, we obtain

$$\phi(x, y_i) < \phi(w, y_i), \quad y_i \in Y(w), i \in Q. \tag{43}$$

Note that

$$\begin{aligned} &\psi(x, z) \leq 0, \quad z \in Z, \\ &\mu_j \psi(w, z_j) \geq 0, \quad \mu_j > 0, z_j \in Z(w), j \in M. \end{aligned} \tag{44}$$

We obtain that

$$\mu_j \psi(x, z_j) \leq \mu_j \psi(w, z_j), \quad j \in M. \tag{45}$$

Therefore,

$$\begin{aligned} &\sum_{i=1}^{q} \lambda_i (\phi(x, y_i) - \phi(w, y_i)) \\ &\quad + \sum_{j=1}^{m} \mu_j (\psi(x, z_j) - \psi(w, z_j)) < 0. \end{aligned} \tag{46}$$

Similar to the proof of Theorem 8, by (46) and the generalized invexity assumptions of $\phi(\cdot, y_i)$ and $\psi(\cdot, z_j)$, we have

$$\Phi\left(x, w, \left(\sum_{i=1}^{q}\lambda_i\xi_i + \sum_{j=1}^{m}\mu_j\zeta_j, \sum_{i=1}^{q}\lambda_i\rho_i^{\phi}(x, w) + \sum_{j=1}^{m}\mu_j\rho_j^{\psi}(x, w)\right)\right) < 0. \tag{47}$$

This follows that

$$0 \notin \sum_{i=1}^{q}\lambda_i\partial_x\phi(w, y_i) + \sum_{j=1}^{m}\mu_j\partial_x\psi(w, z_j). \tag{48}$$

Thus, we have a contradiction to (35). So $\sup_{y\in Y}\phi(x, y) \geqslant \nu$. □

Theorem 11 (strong duality). *Let Problem* (SIP) *satisfy Conditions 1 and 2; let x^* be an optimal solution of Problem* (SIP). *If the hypothesis of Theorem 10 holds for all* (DI)*-feasible points* $(w, q, m, \lambda, \mu, \nu, \overline{y}, \overline{z})$, *then there exists* $(q^*, m^*, \lambda^*, \overline{y^*}, \overline{z^*}) \in K(x^*)$, $(x^*, \mu^*, \nu^*) \in H_1(q^*, m^*, \lambda^*, \overline{y^*}, \overline{z^*})$ *such that* $(x^*, q^*, m^*, \lambda^*, \mu^*, \nu^*, \overline{y^*}, \overline{z^*})$ *is a* (DI)*-optimal solution, and the two problems* (P) *and* (DI) *have the same optimal values.*

Proof. By Theorem 7, there exists $\nu^* = \phi(x^*, y_i^*)$ $(i = 1, \ldots, q^*)$, satisfying the requirements specified in the theorem, such that $(x^*, q^*, m^*, \lambda^*, \mu^*, \nu^*, \overline{y^*}, \overline{z^*})$ is a (DI)-feasible solution; then the optimality of this feasible solution for (DI) follows from Theorem 10. □

Theorem 12 (strict converse duality). *Let x_0 and* $(w, q, m, \lambda, \mu, \nu, \overline{y}, \overline{z})$ *be optimal solutions of* (SIP) *and* (DI), *respectively. Suppose that $\phi(\cdot, y_i)$ are strictly (Φ, ρ_i^{ϕ})-invex at w for each $i \in Q$ and $\psi(\cdot, z_j)$ are (Φ, ρ_i^{ψ})-invex at w for each $j \in M$. If*

$$\sum_{i=1}^{q}\lambda_i\rho_i^{\phi}(x_0, w) + \sum_{j=1}^{m}\mu_j\rho_j^{\psi}(x_0, w) \geq 0, \tag{49}$$

then $x_0 = w$; that is, w is a (SIP)*-optimal solution, and*

$$\sup_{y\in Y}\phi(x_0, y) = \nu. \tag{50}$$

Proof. Suppose to the contrary that $x_0 \neq w$. By the generalized invexity assumptions of $\phi(\cdot, y_i)$ and $\psi(\cdot, z_j)$, we have

$$\begin{gathered}\phi(x_0, y_i) - \phi(w, y_i) > \Phi(x_0, w, (\xi_i, \rho_i^{\phi}(x_0, w))), \\ \forall\xi_i \in \partial_x\phi(w, y_i), \\ \psi(x_0, z_j) - \psi(w, z_j) \geq \Phi(x_0, w, (\zeta_j, \rho_j^{\psi}(x_0, w))), \\ \forall\zeta_j \in \partial_x\psi(w, z_j).\end{gathered} \tag{51}$$

Therefore, we obtain from (51) and the convexity of Φ that

$$\begin{aligned}&\sum_{i=1}^{q}\lambda_i(\phi(x_0, y_i) - \phi(w, y_i)) + \sum_{j=1}^{m}\mu_j(\psi(x_0, z_j) - \psi(w, z_j)) \\ &\quad > \Phi\left(x_0, w, \left(\sum_{i=1}^{q}\lambda_i\xi_i + \sum_{j=1}^{m}\mu_j\zeta_j, \sum_{i=1}^{q}\lambda_i\rho_i^{\phi}(x_0, w) + \sum_{j=1}^{m}\mu_j\rho_j^{\psi}(x_0, w)\right)\right)\end{aligned} \tag{52}$$

holds for all $\xi_i \in \partial_x\phi(w, y_i)$ and $\zeta_j \in \partial_x\psi(w, z_j)$. This, together with (35), (49), and

$$\Phi(x_0, w, (0, a)) \geq 0 \quad \text{for } a > 0, \tag{53}$$

follows that

$$\sum_{i=1}^{q}\lambda_i(\phi(x_0, y_i) - \phi(w, y_i)) + \sum_{j=1}^{m}\mu_j(\psi(x_0, z_j) - \psi(w, z_j)) > 0 \tag{54}$$

while

$$\sum_{j=1}^{m}\mu_j(\psi(x_0, z_j) - \psi(w, z_j)) \leq 0. \tag{55}$$

Thus

$$\sum_{i=1}^{q}\lambda_i(\phi(x_0, y_i) - \phi(w, y_i)) > 0. \tag{56}$$

From the above inequality, we can conclude that there exists $i_0 \in Q$, such that

$$\phi(x_0, y_{i_0}) - \phi(w, y_{i_0}) > 0 \tag{57}$$

or

$$\phi(x_0, y_{i_0}) > \phi(w, y_{i_0}). \tag{58}$$

It follows that

$$\sup_{y\in Y}\phi(x_0, y) \geq \phi(x_0, y_{i_0}) > \phi(w, y_{i_0}) > \nu. \tag{59}$$

On the other hand, we know from Theorem 10 that

$$\sup_{y\in Y}\phi(x_0, y) = \nu. \tag{60}$$

This contradicts to (59). □

5. Conclusions

In this paper, we have discussed a nonsmooth semi-infinite minimax programming Problem (SIP). We have extended the necessary optimality conditions for Problem (SIP) considered in [17] to the nonsmooth case; we have also extended the sufficient optimality conditions and dual results of Problem (SIP) addressed by M. V. Ştefănescu and A. Ştefănescu in [17] to the nonsmooth case under the Lipschitz (Φ, ρ)-invexity assumptions as defined in [16]. More exactly, we have established the necessary optimality conditions theorems for the Problem (SIP) when using Caratheodory's theorem. Under the Lipschitz (Φ, ρ)-invexity assumptions as defined in [13], we have derived the sufficiency of the necessary optimality conditions for Problem (SIP). In the end, we have constructed a dual model (DI) and derived duality results between Problems (SIP) and (DI). These results extend several known results to a wider class of problems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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