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Research Article

An Iterative Shrinking Metric f-Projection Method for Finding a Common Fixed Point of a Closed and Quasi-Strict f-Pseudocontraction and a Countable Family of Firmly Nonexpansive Mappings and Applications in Hilbert Spaces

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We create some new ideas of mappings called quasi-strict f-pseudocontractions. Moreover, we also find the significant inequality related to such mappings and firmly nonexpansive mappings within the framework of Hilbert spaces. By using the ideas of metric f-projection, we propose an iterative shrinking metric f-projection method for finding a common fixed point of a quasi-strict f-pseudocontraction and a countable family of firmly nonexpansive mappings. In addition, we provide some applications of the main theorem to find a common solution of fixed point problems and generalized mixed equilibrium problems as well as other related results.

1. Introduction

It is well known that the metric projection operators in Hilbert spaces and Banach spaces play an important role in various fields of mathematics such as functional analysis, optimization theory, fixed point theory, nonlinear programming, game theory, variational inequality, and complementarity problem (see, e.g., [1, 2]). In 1994, Alber [3] introduced and studied the generalized projections from Hilbert spaces to uniformly convex and uniformly smooth Banach spaces. Moreover, Alber [1] presented some applications of the generalized projections to approximately solve variational inequalities and von Neumann intersection problem in Banach spaces. In 2005, Li [2] extended the generalized projection operator from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces and studied some properties of the generalized projection operator with applications to solve the variational inequality in Banach spaces. Later, Wu and Huang [4] introduced a new generalized f-projection operator in Banach spaces. They extended the definition of the generalized projection operators introduced by [3] and proved some properties of the generalized f-projection operator. Fan et al. [5] presented some basic results for the generalized f-projection operator and discussed the existence of solutions and approximation of the solutions for generalized variational inequalities in noncompact subsets of Banach spaces.

Let H be a real Hilbert space; a mapping T with domain $D(T)$ and range $R(T)$ in H is called firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \quad \forall x, y \in D(T), \tag{1}$$

nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in D(T). \tag{2}$$

Throughout this paper, I stands for an identity mapping. The mapping T is said to be a strict pseudocontraction if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in D(T). \tag{3}$$

In this case, T may be called a k-strict pseudocontraction. We use $F(T)$ to denote the set of fixed points of T (i.e. $F(T) = \{x \in D(T) : Tx = x\}$). T is said to be a quasi-strict pseudocontraction if the set of fixed point $F(T)$ is nonempty and if there exists a constant $0 \le k < 1$ such that

$$\|Tx - p\|^2 \le \|x - p\|^2 + k\|x - Tx\|^2, \quad \forall x \in D(T), \quad p \in F(T). \tag{4}$$

Construction of fixed points of nonexpansive mappings via Mann's algorithm [6] has extensively been investigated in the literature; see, for example, [6–10] and references therein. However, we note that Mann's iterations have only weak convergence even in a Hilbert space (see, e.g., [11]). Nakajo and Takahashi [12] modified the Mann iteration method so that strong convergence is guaranteed, later well known as a hybrid projection iteration method. Since then, the hybrid method has received rapid developments. For the details, the readers are referred to papers [13–26] and the references therein.

On the other hand, for a real Banach space E and the dual E^*, let C be a nonempty closed convex subset of E. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction, let $\varphi : C \to \mathbb{R}$ be a real-valued function, and let $A : C \to E^*$ be a nonlinear mapping. The generalized mixed equilibrium problem is to find $x \in C$ such that

$$\Theta(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \ge 0, \quad \forall y \in C. \tag{5}$$

The solution set of (5) is denoted by $GMEP(\Theta, A, \varphi)$; that is,

$$GMEP(\Theta, A, \varphi) = \{x \in C : \Theta(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \ge 0, \forall y \in C\}. \tag{6}$$

If $A = 0$, the problem (5) reduces to the mixed equilibrium problem for Θ, denoted by $\mathrm{MEP}(\Theta, \varphi)$, which is to find that $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \ge 0, \quad \forall y \in C. \tag{7}$$

If $\Theta = 0$, the problem (5) reduces to the mixed variational inequality of Browder type, denoted by $\mathrm{VI}(C, A, \varphi)$, which is to find that $x \in C$ such that

$$\langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \ge 0, \quad \forall y \in C. \tag{8}$$

If $A = 0$ and $\varphi = 0$, the problem (5) reduces to the equilibrium problem for Θ (for short, EP), denoted by $\mathrm{EP}(\Theta)$, which is to find that $x \in C$ such that

$$\Theta(x, y) \ge 0, \quad \forall y \in C. \tag{9}$$

If $\Theta = 0$ and $A = 0$, the problem (5) reduces to the minimization problem for φ, denoted by $\mathrm{Arg\ min}(\varphi)$, which is to find that $x \in C$ such that

$$\varphi(x) \le \varphi(y), \quad \forall y \in C. \tag{10}$$

The previous formulation, (8), was shown in [27] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, vector equilibrium problems, and Nash equilibria in noncooperative games. In addition, there are several other problems, for example, the complementarity problem, fixed point problem, and the optimization problem, which can also be written in the form of (9). However, (5) is very general; it covers the problems mentioned above as special cases.

In 2007, S. T. Takahashi and W. T. Takahashi [28] and Tada and Takahashi [29, 30] proved weak and strong convergence theorems for finding a common element of the set of solutions of the equilibrium problem (9) and the set of fixed points of a nonexpansive mapping in a Hilbert space. Takahashi et al. [22] studied a strong convergence theorem by the hybrid method for a family of nonexpansive mappings in Hilbert spaces as follows: $x_0 \in H$, $C_1 = C$, and $x_1 = P_{C_1}x_0$, and let

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n) T_n x_n, \\ C_{n+1} &= z \in C_n : \|y_n - z\| \le \|x_n - z\|, \\ x_{n+1} &= P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{aligned} \tag{11}$$

where $0 \le \alpha_n \le a < 1$, for all $n \in \mathbb{N}$, and $\{T_n\}$ is a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \ne \phi$. They proved that if $\{T_n\}$ satisfies some appropriate conditions, then $\{x_n\}$ converges strongly to $P_{\bigcap_{n=1}^{\infty} F(T_n)} x_0$.

Motivated by Takahashi et al. [22], Takahashi and Zembayashi [31] (see also [32]) introduced and proved a hybrid projection algorithm for solving equilibrium problems and fixed point problems of a relatively nonexpansive mapping within the framework of a uniformly smooth and uniformly convex Banach space.

In 2011, Saewan and Kumam [33] introduced a new hybrid projection method based on the modified Mann iterative scheme by the generalized f-projection operator for a countable family of relatively quasi-nonexpansive mappings and the solutions of the system of generalized mixed equilibrium problems. Later, they [34] also studied the new hybrid Ishikawa iteration process by the generalized f-projection operator for finding a common element of the fixed point set for two countable families of weak relatively nonexpansive mappings and the set of solutions of the system of generalized Ky Fan inequalities in a uniformly convex and uniformly smooth Banach space.

Recently, Li et al. [35] have studied the following hybrid iterative scheme for a relatively nonexpansive mapping by using the generalized f-projection operator in Banach spaces:

$$\begin{aligned} &x_0 \in C, \qquad C_0 = C, \\ &y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T x_n), \\ &C_{n+1} = \{w \in C_n : G(w, J y_n) \le G(w, J x_n)\}, \\ &x_{n+1} = \prod\nolimits_{C_{n+1}}^{f} x_0, \quad n \ge 1. \end{aligned} \tag{12}$$

Under some appropriate assumptions, they obtained strong convergence theorems in Banach spaces.

Motivated and inspired by the work mentioned above, in this paper, we are interested to study our theorems within the framework of a real Hibert space, and we create some new ideas of mappings called quasi-strict f-pseudocontractions. Moreover, we also find the significant inequality related to such mappings and firmly nonexpansive mappings within the framework of Hilbert spaces. By using the ideas of metric f-projection, we propose an iterative shrinking metric f-projection method for find a common fixed point of an quasi-strict f-pseudocontraction and a countable family of firmly nonexpansive mappings. In addition, we provide some applications of the main theorem to finding a common solution of fixed point problems and generalized mixed equilibrium problems as well as other related results.

2. Preliminaries

In this section, some definitions are provided, and some relevant lemmas which are useful to prove in the following section are collected. Most of them are known, and others are not hard to find or understand their proofs. Throughout this paper, we will use the notation $\rightharpoonup$ for weak convergence and $\rightarrow$ for strong convergence.

Lemma 1 (see Takahashi [36]). *Let $\{a_n\}$ be a sequence of real numbers. Then,* $\lim_{n\to\infty} a_n = 0$ *if and only if, for any subsequence $\{a_{n_i}\}$ of $\{a_n\}$, there exists a subsequence $\{a_{n_{i_j}}\}$ of $\{a_{n_i}\}$ such that* $\lim_{j\to\infty} a_{n_{i_j}} = 0$.

Definition 2 (see [37–40]). Let C be a nonempty, closed, and convex subset of a Banach space E, and let $\{T_n\}$ be a sequence of mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then, $\{T_n\}$ is said to satisfy the NST-condition if, for each bounded sequence $\{z_n\} \subset C$,

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0 \tag{13}$$

implies that $\omega_w(z_n) \subset \bigcap_{n=1}^{\infty} F(T_n)$, where $\omega_w(z_n)$ is the set of all weak cluster points of $\{z_n\}$ (i.e., $\omega_w(z_n) = \{z \mid \exists \{z_{n_i}\} \subset \{z_n\}$ such that $z_{n_i} \rightharpoonup z\}$).

Let H be a real Hilbert space, and let C be nonempty, closed, and convex subset of H. Let $(\cdot,\cdot)_f : C \times H \to (-\infty, +\infty]$ be a functional defined as follows (see Li et al. [35] (see also [4])):

$$\begin{aligned}(y,x)_f &:= \|y\|^2 - 2\langle y, x\rangle + \|x\|^2 \\ &\quad + 2\rho f(y) = \|y-x\|^2 + 2\rho f(y),\end{aligned} \tag{14}$$

where $y \in C$, $x \in H$, ρ is positive number, and $f : C \to (-\infty, +\infty]$ is proper, convex, and lower semicontinuous. From the definitions of $(\cdot,\cdot)_f$ and f, it is easy to see the following properties:

(i) $(y,x)_f$ is convex and continuous with respect to x when y is fixed;

(ii) $(y,x)_f$ is convex and lower semicontinuous with respect to y when x is fixed.

Definition 3 (see Li et al. [35] (see also [4])). Let H be a real Hilbert space, and let C be nonempty, closed, and convex subset of H. We say that $P_C^f : H \to 2^C$ is a metric f-projection operator if

$$P_C^f x = \left\{ u \in C \mid (u,x)_f = \inf_{\xi\in C} (\xi, x)_f \right\}, \quad \forall x \in H. \tag{15}$$

Lemma 4 (see Li et al. [35, Lemma 3.1(ii)]). *Let H be a real Hilbert space, and let $\phi \neq C \subset H$. Then, for every $x \in H$, $\hat{x} = P_C^f x$ if and only if*

$$\langle \hat{x} - y, x - \hat{x}\rangle + \rho f(y) - \rho f(\hat{x}) \ge 0, \quad \forall y \in C. \tag{16}$$

Lemma 5 (see Li et al. [35, Lemma 3.2]). *Let H be a real Hilbert space, let C be a nonempty, closed, and convex subset of H, and let $x \in H$, $\hat{x} = P_C^f x$. Then,*

$$\|y - \hat{x}\|^2 + (\hat{x}, x)_f \leqslant (y, x)_f, \quad \forall y \in C. \tag{17}$$

Lemma 6 (see Deimling [41]). *Let H be a real Hilbert space, and $f : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex functional. Then, there exist $z \in H$ and $\alpha \in \mathbb{R}$ such that*

$$f(x) \ge \langle x, z\rangle + \alpha, \quad \forall x \in H. \tag{18}$$

Due to the properties of f, we have the motivation and ideas to create a new type of mappings which is general and covers a quasi-strict pseud-contraction as follows.

Definition 7. Let H be a real Hilbert space, and a mapping T with domain $D(T)$ and range $R(T)$ in H is called *quasi-strict f-pseudocontraction* if the set of fixed points $F(T)$ is nonempty and if there exists a constant $0 \le k < 1$ such that, for each $p \in F(T)$,

$$(p, Tx)_f \leqslant (p, x)_f + k\left((x, Tx)_f - 2\rho f(p)\right), \quad \forall x \in D(T). \tag{19}$$

It is obvious from the previous definition that (19) is equivalent to

$$\begin{aligned}\|p - Tx\|^2 &\le \|p - x\|^2 + k\|x - Tx\|^2 \\ &\quad + 2k\rho\left(f(x) - f(p)\right), \quad \forall x \in C,\ p \in F(T).\end{aligned} \tag{20}$$

Definition 8. A mapping $T : C \to C$ is said to be *closed* if, for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, $x = y$.

Example 9. Let $T : H \to H$ be a mapping defined by $Tx = (3/2)x$, for all $x \in H$. Then, it is easy to see that $F(T) = \{x \in H : Tx = x\} = \{0\}$. Moreover, it is found that

$$\begin{aligned}\|0 - Tx\|^2 &= \left\|\frac{3}{2}x\right\|^2 \\ &= \frac{9}{4}\|x\|^2 = \|x\|^2 + \frac{5}{4}\|x\|^2 \\ &\le \|x\|^2 + \frac{3}{2}\|x\|^2 = \|x\|^2 + \left(\frac{1}{6} + \frac{8}{6}\right)\|x\|^2\end{aligned}$$

$$= \|0 - x\|^2 + \frac{2}{3}\left\|x - \frac{3}{2}x\right\|^2 + 2\left(\frac{2}{3}\right)(1)\left(\|x\|^2 - \|0\|^2\right) = \|0 - x\|^2 + k\|x - Tx\|^2 + 2k\rho\left(f(x) - f(0)\right), \quad (21)$$

for all $x \in H$, where $k = 2/3$, $\rho = 1$, and $f = \|\cdot\|^2$. Furthermore, if $\{x_n\} \subset H$ such that $x_n \to x$, then we have $Tx_n = (3/2)x_n \to (3/2)x$ and $Tx = (3/2)x$. This means that T is closed and quasi-strict f-pseudocontraction.

For solving the equilibrium problem for a bifunction $\Theta : C \times C \to \mathbb{R}$, let us assume that Θ satisfies the following conditions:

(A1) $\Theta(x, x) = 0$, for all $x \in C$;

(A2) Θ is monotone; that is, $\Theta(x, y) + \Theta(y, x) \leq 0$, for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0}\Theta(tz+(1-t)x, y) \leq \Theta(x, y)$;

(A4) for each $x \in C$, $y \to \Theta(x, y)$ is convex and lower semicontinuous.

Lemma 10 (see [27]). *Let C be a nonempty closed convex subset of H, and let Θ be a bifunction of $C\times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$\Theta(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \quad (22)$$

Lemma 11 (see [32]). *Let C be a closed convex subset of H and let Θ be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r x = \left\{z \in C : \Theta(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}, \quad (23)$$

for all $x \in C$. Then, the following hold:

(i) *T_r is single-valued;*

(ii) *T_r is firmly nonexpansive-type mapping; that is, for any $x, y \in H$, $\langle T_r x - T_r y, T_r x - T_r y\rangle \leq \langle T_r x - T_r y, x - y\rangle$;*

(iii) *$F(T_r) = EP(\Theta)$;*

(iv) *$EP(\Theta)$ is closed and convex;*

(v) *$\|p - T_r x\|^2 + \|T_r x - x\|^2 \leq \|p - x\|^2$, for all $x \in H$, $p \in F(T_r)$.*

Lemma 12 (see [42]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Hilbert space H. Let $A : C \to C$ be a continuous and monotone mapping, let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function, and let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). For $r > 0$ and $x \in E$, then, there exists $u \in C$ such that*

$$\Theta(u, y) + \langle Au, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0, \quad \forall y \in C. \quad (24)$$

Define a mapping $K_r : C \to C$ as follows:

$$K_r(x) = \left\{u \in C : \Theta(u, y) + \langle Au, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0, \forall y \in C\right\}, \quad (25)$$

for all $x \in C$. Then, the following conclusions hold:

(1) *K_r is single-valued;*

(2) *K_r is firmly nonexpansive type; that is, for all $x, y \in E$, $\langle K_r x - K_r y, K_r x - K_r y\rangle \leq \langle K_r x - K_r y, x - y\rangle$;*

(3) *$F(K_r) = GMEP(\Theta, A, \varphi)$;*

(4) *$GMEP(\Theta, A, \varphi)$ is closed and convex;*

(5) *$\|p - K_r x\|^2 + \|K_r x - x\|^2 \leq \|p - x\|^2$, for all $x \in H$, $p \in F(K_r)$.*

Lemma 13. *Let H be a real Hilbert space, and let $K : D(K) \to R(K)$ be a mapping. Then, the following are equivalent:*

(i) *K is firmly nonexpansive (i.e., $\|Kx - Ky\|^2 \leq \langle Kx - Ky, x - y\rangle$, for all $x, y \in D(K)$);*

(ii) *$\|Kx - Ky\|^2 \leq \|x - y\|^2 - \|(I - K)x - (I - K)y\|^2$, for all $x, y \in D(K)$.*

Proof. For each $x, y \in D(K)$, we notice that

$$\begin{aligned}
&\|Kx - Ky\|^2 \\
&\leq \langle Kx - Ky, x - y\rangle \\
&\Longleftrightarrow \|Kx - Ky\|^2 \leq \|x - y\|^2 \\
&\quad - \|x - y\|^2 + \langle x - y, Kx - Ky\rangle \\
&\quad + \langle Kx - Ky, x - y\rangle - \|Kx - Ky\|^2 \\
&\Longleftrightarrow \|Kx - Ky\|^2 \leq \|x - y\|^2 \\
&\quad - \langle x - y, (I - K)x - (I - K)y\rangle \\
&\quad + \langle Kx - Ky, (I - K)x - (I - K)y\rangle \\
&\Longleftrightarrow \|Kx - Ky\|^2 \leq \|x - y\|^2 \\
&\quad - \langle (I - K)x - (I - K)y, (I - K)x - (I - K)y\rangle \\
&\Longleftrightarrow \|Kx - Ky\|^2 \leq \|x - y\|^2 \\
&\quad - \|(I - K)x - (I - K)y\|^2.
\end{aligned} \quad (26)$$

The proof is complete. □

The following lemma is important since it provides the significant inequality related to quasi-strict f-pseudocontractions and firmly nonexpansive mappings within the framework of Hilbert spaces.

Lemma 14. *Let C be a nonempty, closed, convex subset of real Hilbert spaces H. Let $T : C \to C$ be a quasi-strict*

f-pseudocontraction, and let $K : C \to C$ be a firmly nonexpansive mapping such that $\Omega := F(T) \cap F(K) \neq \emptyset$. Then,

$$\begin{aligned}
&\|x - KTx\|^2 + \|KTx - Tx\|^2 \\
&\quad \le \frac{2}{1-k}\langle x - p, x - Tx\rangle + 2\langle x - p, Tx - KTx\rangle \\
&\quad + \frac{2k\rho}{1-k}(f(x) - f(p)),
\end{aligned} \tag{27}$$

for all $x \in C$ and $p \in \Omega$.

Proof. Let $x \in C$ and $p \in \Omega$. By the quasi-strict f-pseudocontractility of T, we have that

$$\begin{aligned}
&(p, Tx)_f \leqslant (p, x)_f + k\left((x, Tx)_f - 2\rho f(p)\right) \\
&\Longleftrightarrow \|p - Tx\|^2 \le \|p - x\|^2 \\
&\quad + k\|x - Tx\|^2 + 2k\rho(f(x) - f(p)) \\
&\Longleftrightarrow \|p - x\|^2 + 2\langle p - x, x - Tx\rangle + \|x - Tx\|^2 \\
&\le \|p - x\|^2 + k\|x - Tx\|^2 + 2k\rho(f(x) - f(p)) \\
&\Longleftrightarrow (1-k)\|x - Tx\|^2 \le 2\langle x - p, x - Tx\rangle \\
&\quad + 2k\rho(f(x) - f(p)) \\
&\Longleftrightarrow \|x - Tx\|^2 \le \frac{2}{1-k}\langle x - p, x - Tx\rangle \\
&\quad + \frac{2k\rho}{1-k}(f(x) - f(p)).
\end{aligned} \tag{28}$$

It follows from Lemma 13 and (28) that

$$\begin{aligned}
&\|p - x\|^2 + \|x - KTx\|^2 + 2\langle p - x, x - KTx\rangle \\
&\quad = \|p - KTx\|^2 \\
&\quad \le \|p - Tx\|^2 - \|KTx - Tx\|^2 \\
&\quad = (p, Tx)_f - 2\rho f(p) - \|KTx - Tx\|^2 \\
&\quad \le (p, x)_f + k\left((x, Tx)_f - 2\rho f(p)\right) \\
&\qquad - 2\rho f(p) - \|KTx - Tx\|^2 \\
&\quad = \left((p, x)_f - 2\rho f(p)\right) + k\|x - Tx\|^2 \\
&\qquad + 2k\rho(f(x) - f(p)) - \|KTx - Tx\|^2 \\
&\quad \le \|p - x\|^2 \\
&\qquad + k\left(\frac{2}{1-k}\langle x - p, x - Tx\rangle \right. \\
&\qquad\qquad \left. + \frac{2k\rho}{1-k}(f(x) - f(p))\right) \\
&\qquad + 2k\rho(f(x) - f(p)) - \|KTx - Tx\|^2 \\
&\quad \le \|p - x\|^2 + \frac{2k}{1-k}\langle x - p, x - Tx\rangle \\
&\qquad + \frac{2k^2\rho}{1-k}(f(x) - f(p)) \\
&\qquad + 2k\rho(f(x) - f(p)) - \|KTx - Tx\|^2,
\end{aligned} \tag{29}$$

and, then,

$$\begin{aligned}
&\|x - KTx\|^2 + \|KTx - Tx\|^2 \\
&\quad \le \frac{2k}{1-k}\langle x - p, x - Tx\rangle + 2\langle x - p, x - KTx\rangle \\
&\qquad + \left[\frac{2k^2\rho}{1-k} + 2k\rho\right](f(x) - f(p)) \\
&\quad = \frac{2k}{1-k}\langle x - p, x - Tx\rangle + 2\langle x - p, x - Tx\rangle \\
&\qquad + 2\langle x - p, Tx - KTx\rangle \\
&\qquad + 2k\rho\left[\frac{k}{1-k} + 1\right](f(x) - f(p)) \\
&\quad = \frac{2}{1-k}\langle x - p, x - Tx\rangle + 2\langle x - p, Tx - KTx\rangle \\
&\qquad + \frac{2k\rho}{1-k}(f(x) - f(p)).
\end{aligned} \tag{30}$$

This completes the proof. □

3. Main Result

In this section, some available properties of a quasi-strict f-pseudocontraction T are used to prove that the set of fixed points $F(T)$ is closed and convex. An iterative shrinking metric f-projection method is provided in order to find a common fixed point of a quasi-strict f-pseudocontraction and a countable family of firmly nonexpansive mappings.

Lemma 15. *Let C be a nonempty, closed, convex subset of a real Hilbert space H, and let $T : C \to C$ be a quasi-strict f-pseudocontraction. Then, the fixed point set $F(T)$ of T is closed and convex.*

Proof. Let $\{p_n\}$ be a sequence in $F(T)$ such that $p_n \to p \in C$ as $n \to \infty$. It follows from (28) that

$$\begin{aligned}
&(p_n, Tp)_f \leqslant (p_n, p)_f + k\left((p, Tp)_f - 2\rho f(p_n)\right) \\
&\qquad \Longleftrightarrow \|p - Tp\|^2 \le \frac{2}{1-k}\langle p - p_n, p - Tp\rangle \\
&\qquad\quad + \frac{2k\rho}{1-k}(f(p) - f(p_n)).
\end{aligned} \tag{31}$$

Taking $\limsup_{n\to\infty}$ on both sides of (31), so we have that

$$\begin{aligned}
\|p - Tp\|^2 &= \limsup_{n\to\infty}\|p - Tp\|^2 \\
&\le \limsup_{n\to\infty}\left(\frac{2}{1-k}\langle p - p_n, p - Tp\rangle \right. \\
&\qquad \left. + \frac{2k\rho}{1-k}(f(p) - f(p_n))\right) \\
&\le \frac{2}{1-k}\limsup_{n\to\infty}\langle p - p_n, p - Tp\rangle \\
&\quad + \frac{2k\rho}{1-k}\limsup_{n\to\infty}(f(p) - f(p_n))
\end{aligned}$$

$$\leq \frac{2k\rho}{1-k}\left(\limsup_{n\to\infty} f(p) + \limsup_{n\to\infty}(-f(p_n))\right)$$
$$= \frac{2k\rho}{1-k}\left(f(p) - \liminf_{n\to\infty} f(p_n)\right) \leq 0. \tag{32}$$

This means that $p = Tp$.

We next show that $F(T)$ is convex. For arbitrary $p_1, p_2 \in F(T)$ and $t \in (0,1)$, we let $p_t = tp_1 + (1-t)p_2$. By the definition of T, we have that

$$\begin{aligned}(p_1, Tp_t)_f &\leq (p_1, p_t)_f + k\left((p_t, Tp_t)_f - 2\rho f(p_1)\right),\\ (p_2, Tp_t)_f &\leq (p_2, p_t)_f + k\left((p_t, Tp_t)_f - 2\rho f(p_2)\right).\end{aligned} \tag{33}$$

By (28), it is easy to see that (33) are equivalent to

$$\|p_t - Tp_t\|^2 \leq \frac{2}{1-k}\langle p_t - p_1, p_t - Tp_t\rangle + \frac{2k\rho}{1-k}(f(p_t) - f(p_1)), \tag{34}$$

$$\|p_t - Tp_t\|^2 \leq \frac{2}{1-k}\langle p_t - p_2, p_t - Tp_t\rangle + \frac{2k\rho}{1-k}(f(p_t) - f(p_2)), \tag{35}$$

respectively. Multiplying by t and $(1-t)$ on both sides of (34) and (35), respectively, and adding the two inequalities, we have that

$$\begin{aligned}\|p_t - Tp_t\|^2 &\leq \frac{2}{1-k}\langle p_t - (tp_1 + (1-t)p_2), p_t - Tp_t\rangle\\ &\quad + \frac{2k\rho}{1-k}(f(p_t) - tf(p_1) - (1-t)f(p_2))\\ &= \frac{2}{1-k}\langle p_t - p_t, p_t - Tp_t\rangle\\ &\quad + \frac{2k\rho}{1-k}(f(tp_1 + (1-t)p_2)\\ &\qquad - tf(p_1) - (1-t)f(p_2))\\ &\leq \frac{2k\rho}{1-k}(tf(p_1) + (1-t)f(p_2)\\ &\qquad - tf(p_1) - (1-t)f(p_2)) = 0.\end{aligned} \tag{36}$$

Hence, $Tp_t = p_t$. This complete the proof. □

Theorem 16. *Let H be a real Hilbert space, C a nonempty, closed, be convex subset of E, let T be a closed and quasi-strict f-pseudocontraction from C into itself, and let $\{K_n\}_{n=1}^{\infty}$ be a countable family of firmly nonexpansive mappings from C into itself which satisfies the NST-condition such that $\Omega := F(T) \cap \bigcap_{n=1}^{\infty} F(K_n) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{aligned}&x_0 \in H, \quad \text{chosen arbitrarily},\\ &C_1 = C,\\ &x_1 = P_{C_1}^f x_0,\\ &C_{n+1} = \Big\{z \in C_n \mid \|x_n - K_nTx_n\|^2 + \|K_nTx_n - Tx_n\|^2\\ &\qquad \leq \frac{2}{1-k}\langle x_n - z, x_n - Tx_n\rangle + 2\langle x_n - z, Tx_n - K_nTx_n\rangle\\ &\qquad + \frac{2k\rho}{1-k}(f(x_n) - f(z))\Big\}\\ &x_{n+1} = P_{C_{n+1}}^f x_0.\end{aligned} \tag{37}$$

Then, the sequence $\{x_n\}$ converges strongly to $P_{\Omega}^f x_0$.

Proof. The proof is divided into six steps.

Step 1. Show that C_n is closed and convex, for all $n \geq 1$.

For $n = 1$, $C_1 = C$ is closed and convex. Assume that C_i is closed and convex for some $i \in \mathbb{N}$. For $z \in C_{i+1}$, we have that

$$\begin{aligned}&\|x_i - K_iTx_i\|^2 + \|K_iTx_i - Tx_i\|^2\\ &\quad \leq \frac{2}{1-k}\langle x_i - z, x_i - Tx_i\rangle + 2\langle x_i - z, Tx_i - KTx_i\rangle\\ &\quad + \frac{2k\rho}{1-k}(f(x_i) - f(z)).\end{aligned} \tag{38}$$

It is not hard to see that the continuity and linearity of $\langle\cdot, x_i - Tx_i\rangle$ and $\langle\cdot, Tx_i - KTx_i\rangle$ together with the lower semicontinuity and convexity of f allow C_{i+1} to be closed and convex. Then, for all $n \geq 1$, C_n is closed and convex.

Step 2. Show that $\Omega \subset \bigcap_{n=1}^{\infty} C_n := D$.

It is obvious that $\Omega := F(T) \cap \bigcap_{n=1}^{\infty} F(K_n) \subset C = C_1$. Suppose that $\Omega \subset C_i$ for some $i \in \mathbb{N}$. For any $p \in \Omega$, we have $p \in C_i$, and by Lemma 14 we obtain that

$$\begin{aligned}&\|x_i - K_iTx_i\|^2 + \|K_iTx_i - Tx_i\|^2\\ &\quad \leq \frac{2}{1-k}\langle x_i - p, x_i - Tx_i\rangle + 2\langle x_i - p, Tx_i - KTx_i\rangle\\ &\quad + \frac{2k\rho}{1-k}(f(x_i) - f(p)).\end{aligned} \tag{39}$$

This means that $p \in C_{i+1}$. By mathematical induction, $\Omega \subset C_n$ for all $n \geq 1$. Therefore, $\Omega \subset \bigcap_{n=1}^{\infty} C_n := D \neq \emptyset$.

Step 3. Show that $\{x_n\}$ is bounded and that the $\lim_{n\to\infty}(x_n, x_0)_f$ exists.

Since $f : X \to \mathbb{R}$ is a convex and lower semicontinuous mapping, applying Lemma 6, we see that there exist $z \in H$ and $\alpha \in \mathbb{R}$ such that

$$f(y) \geq \langle y, z\rangle + \alpha, \quad \forall y \in H. \tag{40}$$

It follows that

$$\begin{aligned}(x_n, x_0)_f &= \|x_n\|^2 - 2\langle x_n, x_0\rangle + \|x_0\|^2 + 2\rho f(x_n) \\ &\geq \|x_n\|^2 - 2\langle x_n, x_0\rangle + \|x_0\|^2 + 2\rho\langle x_n, z\rangle + 2\rho\alpha \\ &= \|x_n\|^2 - 2\langle x_n, x_0 - \rho z\rangle + \|x_0\|^2 + 2\rho\alpha \\ &\geq \|x_n\|^2 - 2\|x_0 - \rho z\|\|x_n\| + \|x_0\|^2 + 2\rho\alpha \\ &= (\|x_n\| - \|x_0 - \rho z\|)^2 + \|x_0\|^2 - \|x_0 - \rho z\|^2 + 2\rho\alpha.\end{aligned} \tag{41}$$

Since $x_n = P^f_{C_n} x_0$, it follows from (41) that

$$\begin{aligned}&\|x_0\|^2 - \|x_0 - \rho z\| + 2\rho\alpha \\ &\quad\leq (\|x_n\| - \|x_0 - \rho z\|)^2 + \|x_0\|^2 \\ &\qquad - \|x_0 - \rho z\| + 2\rho\alpha \\ &\quad\leq (x_n, x_0)_f = \left(P^f_{C_n}(x_0), x_0\right)_f \\ &\quad= \inf_{\xi\in C_n}(\xi, x_0)_f \leq (u, x_0)_f\end{aligned} \tag{42}$$

for each $u \in \Omega$. This implies that $\{x_n\}$ and $(x_n, x_0)_f$ are bounded. By the fact that $x_{n+1} \in C_{n+1} \subset C_n$ and Lemma 5, we obtain that

$$\|x_{n+1} - x_n\|^2 + (x_n, x_0)_f \leq (x_{n+1}, x_0)_f. \tag{43}$$

Since $\|x_{n+1} - x_n\|^2 \geq 0$, $\{(x_n, x_0)_f\}$ is nondecreasing. Therefore, the limit of $\{(x_n, x_0)_f\}$ exists.

Step 4. Show that $x_n \to p$ as $n \to \infty$, where $p = P^f_D x_0$.

Let $\{x_{n_i}\} \subset \{x_n\}$. From the boundedness of $\{x_{n_i}\}$, there exists $\{x_{n_{i_j}}\} \subset \{x_{n_i}\}$ such that

$$x_{n_{i_j}} \rightharpoonup p \quad \text{as } j \longrightarrow \infty. \tag{44}$$

Writing $\tilde{x}_j := x_{n_{i_j}}$, it is easy to see that $p \in \widetilde{C}_j$, where $\widetilde{C}_j := C_{n_{i_j}}$. Note that

$$\left(\tilde{x}_j, x_0\right)_f = \left(P^f_{\widetilde{C}_j}(x_0), x_0\right)_f = \min_{\xi\in\widetilde{C}_j}(\xi, x_0)_f \leq (p, x_0)_f. \tag{45}$$

On the other hand, since $\tilde{x}_j \rightharpoonup p$, so $\tilde{x}_j - x_0 \rightharpoonup p - x_0$, and, then, by weakly lower semicontinuity of $\|\cdot\|^2$ and f, we obtain that

$$\|p - x_0\|^2 \leq \liminf_{j\to\infty} \left\|\tilde{x}_j - x_0\right\|^2, \tag{46}$$

$$f(p) \leq \liminf_{j\to\infty} f\left(\tilde{x}_j\right). \tag{47}$$

Combining (46) and (47), we obtain that

$$\begin{aligned}(p, x_0)_f &= \|p - x_0\|^2 + 2\rho f(p) \\ &\leq \liminf_{j\to\infty} \left\|\tilde{x}_j - x_0\right\|^2 + 2\rho \liminf_{j\to\infty} f\left(\tilde{x}_j\right) \\ &\leq \liminf_{j\to\infty} \left(\left\|\tilde{x}_j - x_0\right\|^2 + 2\rho f\left(\tilde{x}_j\right)\right) \\ &= \liminf_{j\to\infty}\left(\tilde{x}_j, x_0\right)_f.\end{aligned} \tag{48}$$

It follows from (45) and (48), that

$$(p, x_0)_f \leq \liminf_{j\to\infty}\left(\tilde{x}_j, x_0\right)_f \leq \limsup_{j\to\infty}\left(\tilde{x}_j, x_0\right)_f \leq (p, x_0)_f, \tag{49}$$

and, then,

$$\lim_{j\to\infty}\left(\tilde{x}_j, x_0\right)_f = (p, x_0)_f. \tag{50}$$

Next, we consider that

$$\begin{aligned}\limsup_{j\to\infty} \left\|\tilde{x}_j - x_0\right\|^2 &= \limsup_{j\to\infty} \left(\left(\tilde{x}_j, x_0\right)_f - 2\rho f\left(\tilde{x}_j\right)\right) \\ &\leq \limsup_{j\to\infty}\left(\tilde{x}_j, x_0\right)_f + \limsup_{j\to\infty}\left(-2\rho f\left(\tilde{x}_j\right)\right) \\ &= (p, x_0)_f - 2\rho \liminf_{j\to\infty} f\left(\tilde{x}_j\right) \\ &\leq (p, x_0)_f - 2\rho f(p) = \|p - x_0\|^2.\end{aligned} \tag{51}$$

Combining (46) and (51), we obtain that

$$\begin{aligned}\|p - x_0\|^2 &\leq \liminf_{j\to\infty} \left\|\tilde{x}_j - x_0\right\|^2 \\ &\leq \limsup_{j\to\infty} \left\|\tilde{x}_j - x_0\right\|^2 \\ &\leq \|p - x_0\|^2,\end{aligned} \tag{52}$$

and, then,

$$\lim_{j\to\infty} \left\|\tilde{x}_j - x_0\right\|^2 = \|p - x_0\|^2. \tag{53}$$

Note that

$$f\left(\tilde{x}_j\right) = \frac{1}{2\rho}\left(\left(\tilde{x}_j, x_0\right)_f - \left\|\tilde{x}_j - x_0\right\|^2\right). \tag{54}$$

Then, we have that

$$\begin{aligned}&\limsup_{j\to\infty} f\left(\tilde{x}_j\right) \\ &\quad= \frac{1}{2\rho}\limsup_{j\to\infty}\left(\left(\tilde{x}_j, x_0\right)_f - \left\|\tilde{x}_j - x_0\right\|^2\right) \\ &\quad= \frac{1}{2\rho}\left((p, x_0)_f - \|p - x_0\|^2\right) \\ &\quad= f(p).\end{aligned} \tag{55}$$

Combining (47) and (55), we obtain that

$$f(p) \leq \liminf_{j\to\infty} f\left(\tilde{x}_j\right) \leq \limsup_{j\to\infty} f\left(\tilde{x}_j\right) = f(p), \tag{56}$$

and, then,

$$\lim_{j\to\infty} f\left(x_{n_{i_j}}\right) = \lim_{j\to\infty} f\left(\tilde{x}_j\right) = f(p). \tag{57}$$

By Lemma 1, this implies that

$$\lim_{n\to\infty} f(x_n) = f(p). \tag{58}$$

On the other hand, we note that

$$\begin{aligned}\left\|\tilde{x}_j - p\right\|^2 &= \left\|\left(\tilde{x}_j - x_0\right) - \left(p - x_0\right)\right\|^2 \\ &= \left\|\tilde{x}_j - x_0\right\|^2 - 2\left\langle \tilde{x}_j - x_0, p - x_0\right\rangle \\ &\quad + \left\|p - x_0\right\|^2.\end{aligned} \tag{59}$$

It follows from (44) and (53) that

$$\begin{aligned}&\lim_{j\to\infty}\left\|\tilde{x}_j - p\right\|^2 \\ &= \lim_{j\to\infty}\Big(\left\|\tilde{x}_j - x_0\right\|^2 - 2\left\langle \tilde{x}_j - x_0, p - x_0\right\rangle \\ &\qquad + \left\|p - x_0\right\|^2\Big) \\ &= \lim_{j\to\infty}\left\|\tilde{x}_j - x_0\right\|^2 - 2\lim_{j\to\infty}\left\langle \tilde{x}_j - x_0, p - x_0\right\rangle \\ &\quad + \left\|p - x_0\right\|^2 \\ &= \left\|p - x_0\right\|^2 - 2\left\langle p - x_0, p - x_0\right\rangle + \left\|p - x_0\right\|^2 \\ &= 0.\end{aligned} \tag{60}$$

Therefore, $x_{n_{i_j}} = \tilde{x}_j \to p$ as $j \to \infty$. This implies by Lemma 1 that

$$x_n \longrightarrow p \quad \text{as } n \longrightarrow \infty. \tag{61}$$

Thus,

$$\omega_w(x_n) = \{p\}. \tag{62}$$

It is easy to show that $p \in C_n$, for all $n \geq 1$. Hence, $p \in \bigcap_{n=1}^{\infty} C_n =: D$. Since $x_n = P_{C_n}^{f} x_0$, so, by Lemma 4, we have that

$$\langle x_n - y, x_0 - x_n\rangle + \rho f(y) - \rho f(x_n) \geq 0, \quad \forall y \in D. \tag{63}$$

Letting $n \to \infty$, so we obtain that

$$\langle p - y, x_0 - p\rangle + \rho f(y) - \rho f(p) \geq 0, \quad \forall y \in D, \tag{64}$$

which implies that $p = P_D^f x_0$.

Step 5. Show that $p \in \Omega$.

Firstly, we prove that $\{Tx_n\}$ and $\{K_n T x_n\}$ are bounded. Indeed, taking $v \in \Omega = F(T) \cap \bigcap_{n=1}^{\infty} F(K_n)$ and then by (28), we have that

$$\begin{aligned}&\|v\|^2 - 2\|v\|\left\|Tx_n\right\| + \left\|Tx_n\right\|^2 \\ &\quad = \left(\|v\| - \left\|Tx_n\right\|\right)^2 \leq \left\|v - Tx_n\right\|^2 \\ &\quad \leq \left\|v - x_n\right\|^2 + k\left\|x_n - Tx_n\right\|^2 \\ &\qquad + 2k\rho\left(f(x_n) - f(v)\right) \\ &\quad \leq \left\|v - x_n\right\|^2 \\ &\qquad + k\left(\frac{2}{1-k}\left\langle x_n - v, x_n - Tx_n\right\rangle \right. \\ &\qquad\qquad \left. + \frac{2k\rho}{1-k}\left(f(x_n) - f(v)\right)\right) \\ &\qquad + 2k\rho\left(f(x_n) - f(v)\right) \\ &\quad \leq \left\|v - x_n\right\|^2 + \frac{2k}{1-k}\left\langle x_n - v, x_n - Tx_n\right\rangle \\ &\qquad + 2k\rho\left(\frac{k}{1-k} + 1\right)\left(f(x_n) - f(v)\right) \\ &\quad \leq \left\|v - x_n\right\|^2 + \frac{2k}{1-k}\left\|x_n - v\right\|\left\|x_n\right\| \\ &\qquad + \frac{2k}{1-k}\left\|x_n - v\right\|\left\|Tx_n\right\| \\ &\qquad + \frac{2k\rho}{1-k}\left(f(x_n) - f(v)\right).\end{aligned} \tag{65}$$

By a simple calculation, we get that

$$\begin{aligned}\left\|Tx_n\right\|^2 &\leq \Big(\left\|v - x_n\right\|^2 + \frac{2k}{1-k}\left\|x_n - v\right\|\left\|x_n\right\| \\ &\qquad - \|v\|^2 + \frac{2k\rho}{1-k}\left(f(x_n) - f(v)\right)\Big) \\ &\quad + \left(\frac{2k}{1-k}\left\|x_n - v\right\| + 2\|v\|\right)\left\|Tx_n\right\| \\ &\leq M + \widetilde{M}\left\|Tx_n\right\| = M + \frac{1}{2}\left(2\widetilde{M}\left\|Tx_n\right\|\right) \\ &\leq M + \frac{1}{2}\left(\widetilde{M}^2 + \left\|Tx_n\right\|^2\right) \\ &= M + \frac{1}{2}\widetilde{M}^2 + \frac{1}{2}\left\|Tx_n\right\|^2,\end{aligned} \tag{66}$$

where $M := \sup\{\|v - x_n\|^2 + (2k/1-k)\|x_n - v\| \times \|x_n\| - \|v\|^2 + (2k\rho/1-k)(f(x_n) - f(v)) \mid n \in \mathbb{N}\}$ and $\widetilde{M} := \sup\{(2k/1-k)\|x_n - v\| + 2\|v\| \mid n \in \mathbb{N}\}$. So we have that

$$\left\|Tx_n\right\|^2 \leq 2M + \widetilde{M}^2, \tag{67}$$

for all $n \in \mathbb{N}$. Therefore, $\{Tx_n\}$ is bounded. Notice that, for each $v \in \Omega$,

$$\left\|v - K_n T x_n\right\| \leq \left\|v - Tx_n\right\|, \tag{68}$$

for all $n \in \mathbb{N}$. Therefore, $\{K_n T x_n\}$ is also bounded. Moreover, we note that

$$\left\|x_{n+1} - x_n\right\| \leq \left\|x_{n+1} - p\right\| + \left\|p - x_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{69}$$

Thus, by the fact that $x_{n+1} = P_{C_{n+1}}^{f} x_0 \in C_{n+1}$ and by (58), we obtain that

$$\begin{aligned}&\left\|x_n - K_n T x_n\right\|^2 + \left\|K_n T x_n - Tx_n\right\|^2 \\ &\quad \leq \frac{2}{1-k}\left\langle x_n - x_{n+1}, x_n - Tx_n\right\rangle \\ &\qquad + 2\left\langle x_n - x_{n+1}, Tx_n - K_n T x_n\right\rangle \\ &\qquad + \frac{2k\rho}{1-k}\left(f(x_n) - f(x_{n+1})\right) \longrightarrow 0 \\ &\qquad\qquad \text{as } n \longrightarrow \infty.\end{aligned} \tag{70}$$

This means that

$$\left\|x_n - K_n T x_n\right\| \longrightarrow 0, \quad \left\|K_n T x_n - Tx_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{71}$$

For this reason, we have that

$$\begin{aligned}\|x_n - Tx_n\| &= \|x_n - K_nTx_n + K_nTx_n - Tx_n\| \\ &\le \|x_n - K_nTx_n\| \\ &\quad + \|K_nTx_n - Tx_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{72}$$

Next, we have that

$$\|Tx_n - p\| \le \|Tx_n - x_n\| + \|x_n - p\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{73}$$

That is, $Tx_n \to p$ as $n \to \infty$. It follows from the closed mapping of T, that $Tp = p$; thus, $p \in F(T)$.

On the other hand, let us consider that

$$\begin{aligned}\|x_n - K_nx_n\| &\le \|x_n - K_nTx_n\| + \|K_nTx_n - K_nx_n\| \\ &\le \|x_n - K_nTx_n\| + \|Tx_n - x_n\| \longrightarrow 0 \\ &\qquad \text{as } n \longrightarrow \infty.\end{aligned} \tag{74}$$

It follows from the NST-condition of $\{K_n\}$ and (62), that $p \in \bigcap_{n=1}^{\infty} F(K_n)$. Therefore, $x_n \to p \in F(T) \cap \bigcap_{n=1}^{\infty} F(K_n) = \Omega$.

Step 6. Show that $p = P_{\Omega}^{f} x_0$.

Notice by Step 2 that $\Omega \subset D$; so we have that $P_{\Omega}^{f}x_0 \in D$, and then by Step 5 it yields that

$$\begin{aligned}(p, x_0)_f &= \left(P_D^f x_0, x_0\right)_f = \inf_{\xi \in D} (\xi, x_0)_f \\ &\le \left(P_{\Omega}^f x_0, x_0\right)_f = \inf_{\zeta \in \Omega} (\zeta, x_0)_f \\ &\le (p, x_0)_f.\end{aligned} \tag{75}$$

This shows that $(P_{\Omega}^f x_0, x_0)_f = (p, x_0)_f$. It follows from the uniqueness $p = P_{\Omega}^f x_0$. Then, $\{x_n\}$ converges strongly to $p = P_{\Omega}^f x_0$. This completes the proof. □

4. Deduced Theorems and Applications

In this section, some applications of the main theorem are provided in order to find some common solutions of problems in a Hilbert space.

If $f = \|\cdot\|^2$, then $(x, y)_{\|\cdot\|^2} = \|x - y\|^2 + 2\rho\|x\|^2$, for all $(x, y) \in C \times H$, and $P_{C_n}^f x_0 = P_{C_n}^{\|\cdot\|^2} x_0$, for all $n \in \mathbb{N}$; then, by Theorem 16, we obtain the following corollary.

Corollary 17. *Let C, H, $\{K_n\}$ be the same as in Theorem 16, and T be a closed and quasi-strict $\|\cdot\|^2$-pseudocontraction from C into itself such that $\Omega := F(T) \cap \bigcap_{n=1}^{\infty} F(K_n) \ne \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{aligned}&x_0 \in H, \quad \text{chosen arbitrarily}, \\ &C_1 = C, \\ &x_1 = P_{C_1}^{\|\cdot\|^2}(x_0), \\ &C_{n+1} = \Big\{z \in C_n \mid \|x_n - K_nTx_n\|^2 + \|K_nTx_n - Tx_n\|^2 \\ &\quad \le \frac{2}{1-k}\langle x_n - z, x_n - Tx_n\rangle \\ &\quad + 2\langle x_n - z, Tx_n - K_nTx_n\rangle \\ &\quad + \frac{2k\rho}{1-k}\left(\|x_n\|^2 - \|z\|^2\right)\Big\}, \\ &x_{n+1} = P_{C_{n+1}}^{\|\cdot\|^2}(x_0).\end{aligned} \tag{76}$$

Then, the sequence $\{x_n\}$ converges strongly to $P_{\Omega}^{\|\cdot\|^2} x_0$.

If f is a constant function, saying that $f = a \in \mathbb{R}$, then $(x, y)_a = \|x - y\|^2 + 2\rho a$, and it is not hard to see that T coincides with a quasi-strict pseudocontraction. Thus, by Theorem 16, we obtain the following corollary.

Corollary 18. *Let C, H, $\{K_n\}$ be the same as in Theorem 16, and let T be a closed and quasi-strict pseudocontraction from C into itself such that $\Omega := F(T) \cap \bigcap_{n=1}^{\infty} F(K_n) \ne \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{aligned}&x_0 \in H, \quad \text{chosen arbitrarily}, \\ &C_1 = C, \\ &x_1 = P_{C_1}^{a}(x_0), \\ &C_{n+1} = \Big\{z \in C_n \mid \|x_n - K_nTx_n\|^2 + \|K_nTx_n - Tx_n\|^2 \\ &\quad \le \frac{2}{1-k}\langle x_n - z, x_n - Tx_n\rangle \\ &\quad + 2\langle x_n - z, Tx_n - K_nTx_n\rangle\Big\}, \\ &x_{n+1} = P_{C_{n+1}}^{a}(x_0).\end{aligned} \tag{77}$$

Then, the sequence $\{x_n\}$ converges strongly to $P_{\Omega}^{a} x_0$.

Let $K : C \to C$ be a firmly nonexpansive mapping and $K_n = K$ for all $n \in \mathbb{N}$. Then, it is not hard to verify that a family $\{K_n\}$ satisfies NST-condition. Therefore, if $f = a = 0$, then Corollary 18 reduces to the following corollary.

Corollary 19. *Let C, H, T be the same as in Corollary 18, and let K be a firmly nonexpansive mapping from C into itself such that $\Omega := F(T) \cap F(K) \ne \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{aligned}&x_0 \in H, \quad \text{chosen arbitrarily}, \\ &C_1 = C, \\ &x_1 = P_{C_1}(x_0), \\ &C_{n+1} = \Big\{z \in C_n \mid \|x_n - KTx_n\|^2 + \|KTx_n - Tx_n\|^2 \\ &\quad \le \frac{2}{1-k}\langle x_n - z, x_n - Tx_n\rangle \\ &\quad + 2\langle x_n - z, Tx_n - KTx_n\rangle\Big\}, \\ &x_{n+1} = P_{C_{n+1}}(x_0).\end{aligned} \tag{78}$$

Then, the sequence $\{x_n\}$ converges strongly to $P_{\Omega} x_0$.

Finally, we provide some applications of the main theorem to find a common solution of fixed point problems of a closed and quasi-strict f-pseudocontraction and generalized mixed equilibrium problems via an iterative shrinking metric f-projection method within the framework of Hilbert spaces.

Theorem 20. *Let C, H, T be the same as in Theorem 16, let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), let φ be a lower semicontinuous and convex function, and let $A : C \to H$ be a continuous and monotone mapping such that $\Omega := F(T) \cap GMEP(\Theta, A, \varphi) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{gathered} x_0 \in H, \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P^f_{C_1} x_0, \\ u_n \in C \text{ such that} \\ \Theta(u_n, y) + \langle Au_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - Tx_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \Big\{ z \in C_n \mid \|x_n - u_n\|^2 + \|u_n - Tx_n\|^2 \leq \frac{2}{1-k}\langle x_n - z, x_n - Tx_n\rangle + 2\langle x_n - z, Tx_n - u_n\rangle + \frac{2k\rho}{1-k}\left(f(x_n) - f(z)\right)\Big\}, \\ x_{n+1} = P^f_{C_{n+1}} x_0, \end{gathered} \tag{79}$$

where $k \in [0,1)$ and $r_n > 0$, for all $n \in \mathbb{N}$, with $\liminf_{n\to\infty} r_n > 0$. *Then, $\{x_n\}$ converges strongly to $P^f_{\Omega}(x_0)$.*

Proof. By Lemma 12(3) it is not hard to see that $\bigcap_{n=1}^{\infty} F(K_{r_n}) = GMEP(\Theta, A, \varphi)$. From the definition of $\{x_n\}$, it is easy to see that $u_n = K_{r_n} Tx_n$, and by (62) we have that $\omega_w(x_n) = \{p\}$. Next, we will show that a countable family $\{K_{r_n}\}$ satisfies the NST-condition. It is sufficient to show that $p \in GMEP(\Theta, A, \varphi)$. It follows from (61) and (71) that $u_n = K_{r_n} Tx_n \to p$ as $n \to \infty$. Define $\Phi : C \times C \to \mathbb{R}$ by $\Phi(x, y) = \Theta(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x)$, for all $x, y \in C$. It is not hard to verify that Φ satisfies conditions (A1)–(A4). By (A2), we have that

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \Phi(y, u_n), \quad \forall y \in C. \tag{80}$$

By using (A4) and $\liminf_{n\to\infty} r_n > 0$, we obtain, that $0 \geq \Phi(y, p)$, for all $y \in C$. For $t \in (0,1]$ and $y \in C$, let $y_t = ty + (1-t)p$. So, from (A1) and (A4), we have that

$$\begin{aligned} 0 = \Phi(y_t, y_t) &= \Phi(y_t, ty + (1-t)p) \\ &\leq t\Phi(y_t, y) + (1-t)\Phi(y_t, p) \leq t\Phi(y_t, y). \end{aligned} \tag{81}$$

Dividing by t, we have that

$$\Phi(y_t, y) \geq 0, \quad \forall y \in C. \tag{82}$$

From (A3), we have that $0 \leq \lim_{t\to 0}\Phi(y_t, y) = \lim_{t\to 0}\Phi(ty + (1-t)p, y) \leq \Phi(p, y)$, for all $y \in C$, and, hence, $p \in GMEP(\Theta, A, \varphi) = \bigcap_{n=1}^{\infty} F(K_{r_n})$, so $\omega_w(x_n) \subset \bigcap_{n=1}^{\infty} F(K_{r_n})$. Therefore, $\{K_{r_n}\}_{n=1}^{\infty}$ satisfies NST-condition. Applying Theorem 16, we conclude that $x_n \to P^f_{\Omega} x_0$. □

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