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Research Article

H_∞ Consensus for Multiagent Systems with Heterogeneous Time-Varying Delays

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We apply the linear matrix inequality method to consensus and H_∞ consensus problems of the single integrator multiagent system with heterogeneous delays in directed networks. To overcome the difficulty caused by heterogeneous time-varying delays, we rewrite the multiagent system into a partially reduced-order system and an integral system. As a result, a particular Lyapunov function is constructed to derive sufficient conditions for consensus of multiagent systems with fixed (switched) topologies. We also apply this method to the H_∞ consensus of multiagent systems with disturbances and heterogeneous delays. Numerical examples are given to illustrate the theoretical results.

1. Introduction

In recent years, decentralized coordination of multiagent systems has received many researchers' attention in the areas of system control theory, biology, communication, applied mathematics, computer science, and so forth. In cooperative control of multiagent systems, a critical problem is to design appropriate protocols such that multiple agents in a group can reach consensus. So far, by using the matrix theory, the graph theory, the frequency-domain analysis method, the Lyapunov direct method, and so forth, consensus problems for various kinds of multiagent systems have been studied extensively [1–3].

In the field of systems and control theory, the pioneering work was done by Borkar and Varaiya [4] and Tsitsiklis and Athans [5], where the asynchronous consensus problem with an application in distributed decision-making systems was considered. Later, Vicsek et al. [6] proposed a simple but interesting discrete-time model of multiple agents which can be viewed as a special case of a computer model mimicking animal aggregation. Jadbabaie et al. [7] provided a theoretical explanation of the consensus property of the Vicsek model.

For the case of single integrator multiagent systems, Olfati-Saber and Murray [8] discussed the consensus problem for networks of dynamic agents by defining a disagreement function. Ren and Beard [9] established some more relaxable consensus conditions under dynamically changing interaction topologies. Hui and Haddad [10], Liu et al. [11], and Bauso et al. [12] investigated the consensus problem for nonlinear multiagent systems. Tan and Liu [13] studied consensus of networked multiagent systems via the networked predictive control. Li and Zhang [14] gave the necessary and sufficient condition of mean square average-consensus for multiagent systems with noises. Zheng and Wang [15] studied finite-time consensus of heterogenous multiagent systems with and without velocity measurements. The constrained consensus problem for multiagent systems in unbalanced networks was investigated in Lin and Ren [16, 17]. Liu et al. [18] considered the consensus problem for multiagent systems with inherent nonlinear dynamics under directed topologies. A distributed shortest-distance consensus problem under dynamically changing network topologies was studied by Lin and Ren [19]. Li et al. [20] studied the distributed consensus problem of multiagent systems with general continuous-time dynamics for both the case without and with a leader.

When considering communication delays in the feedback, three types of consensus protocols have been analyzed: (i) both the state of the agent and its neighbors are affected by identical delays [21–27]; (ii) communication delays only affect the state received from neighbors of the agent [28–32]; (iii) the state of the agent and its neighbors are affected by heterogeneous delays. Note that (i) and (ii) are special cases of (iii). Compared to the cases of (i) and (ii), consensus for the case of (iii) receives less attention. Under the restricted assumptions that the graph is undirected and the communication delays are constants, Münz et al. [32, 33] investigated the robustness of the third kind of consensus protocols in both identical and nonidentical agent dynamics.

We are here concerned with the third type of consensus protocols, where the heterogeneous delays are time varying, and the involved graph is directed. It seems to us that the frequency-domain analysis method in [32, 33] becomes invalid in this case, and it is difficult to employ the Lyapunov method directly. In this paper, we will apply the linear matrix inequality method to this problem. In [21–25], the linear matrix inequality method has been used successfully in the case of (i). When applying this method to the case of (iii), we have to overcome some difficulties caused by heterogeneous time-varying delays. For this sake, we first skillfully rewrite the multiagent system by virtue of a partially-reduced-order system and an integral system. Then, by defining a particular Lyapunov function based on the above partially-reduced-order system and the integral system, sufficient conditions to consensus are derived in both cases of fixed topology and switching topologies by using the linear matrix inequality method.

We also consider the H_∞ consensus problem for the single integrator multiagent system with heterogeneous time-varying delays. So far, there are few H_∞ consensus results for continuous time multiagent systems. Lin et al. [34] studied distributed robust H_∞ consensus problem in directed networks of agents with identical delays. Ugrinovskii [35] considered a problem of design of distributed robust filters with H_∞ consensus by using the recent vector dissipativity theory. Sun and Wang [36] investigated the H_∞ consensus for the double integrator multiagent system with asymmetric delays.

To the best of our knowledge, little has been known about the H_∞ consensus problem for the single integrator multiagent systems with heterogeneous time-varying delays and directed network topologies. Therefore, another purpose of this paper is to establish H_∞ consensus criteria in the cases of heterogeneous delays and directed fixed topology (or switching topologies) by using the linear matrix inequality technique.

This paper is structured as follows. The problem statement and the transformation of the multiagent system are summarized in Section 2. In Section 3, sufficient conditions in terms of linear matrix inequalities are established for consensus and H_∞ consensus of the single integrator multiagent system with heterogeneous delays. These results are also extended to the case of switching topologies. Numerical examples and simulation results are given in Section 4. The paper is concluded in Section 5.

The notation used throughout this paper is fairly standard. A^T means the transpose of the matrix A. I_m is an $m \times m$-dimensional identity matrix. $\mathbf{a}_m = (a, a, \ldots, a)^T$ is an m-dimensional column vector with $a \in \mathbb{R}$. We say $X > Y$ if $X - Y$ is positive definite, where X and Y are symmetric matrices of the same dimensions. We use an asterisk $*$ to represent a term that is induced by symmetry and diag$\{\cdots\}$ stands for a block-diagonal matrix. $L_2[0,\infty)$ denotes the space of square-integrable vector functions over $[0,\infty)$.

2. Preliminaries

Throughout this paper, we denote a weighted digraph by $\mathcal{G} = (V, E, A)$, where $V = \{v_1, v_2, \ldots, v_n\}$ is the set of nodes with $n \geq 2$ and node v_i represents the ith agent; $E \subseteq V \times V$ is the set of edges, an edge of $\mathcal{G}$ is denoted by an order pair (i, j), and $(i, j) \in E$ if and only if $a_{ji} > 0$; $A = [a_{ij}]$ is an $n \times n$-dimensional weighted adjacency matrix with $a_{ii} = 0$. If (i, j) is an edge of $\mathcal{G}$, node i is called the parent of node j. A directed tree is a directed graph, where every node, except one special node without any parent, which is called the root, has exactly one parent, and the root can be connected to any other nodes through paths. The Laplacian matrix $L = [l_{ij}]$ of digraph $\mathcal{G}$ is defined by $l_{ii} = -\sum_{j=1}^{n} a_{ij}$ and $l_{ij} = a_{ij}$ for $i \neq j$, $i, j \in \mathcal{N} = \{1, 2, \ldots, n\}$.

Consider the following multiagent system with heterogeneous delays:

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij} \left(x_j \left(t - d_1(t) \right) - x_i \left(t - d_2(t) \right) \right), \quad i \in \mathcal{N}, \tag{1}$$

where $t \geq 0$, $a_{ij} \geq 0$, $i, j \in \mathcal{N}$, are entries of the weighted adjacency matrix A, $d_1(t)$ and $d_2(t)$ are different piecewise continuous communication delays satisfying $0 \leq d_1(t) \leq h_1$ and $0 \leq d_2(t) \leq h_2$ for $t \geq 0$, and h_1 and h_2 are positive constants.

Denote $x = (x_1, x_2, \ldots \text{sd}, x_n)^T$ and $y = (y_1, y_2, \ldots, y_{n-1})^T$. If we set $y_i = x_{i+1} - x_1$ for $i = 1, 2, \ldots, n-1$, by the straightforward computation, we get

$$y = Ex, \qquad x = x_1 \mathbf{1}_n + Fy, \tag{2}$$

where the $(n-1) \times n$-dimensional matrix E and the $n \times (n-1)$-dimensional matrix F are defined as follows:

$$E = \left(-\mathbf{1}_{n-1}, I_{n-1} \right), \qquad F = \begin{pmatrix} \mathbf{0}_{n-1}^T \\ I_{n-1} \end{pmatrix}. \tag{3}$$

Next, we make use of transformation (2) to derive two descriptor systems of the system (1). Rewrite (1) as follows:

$$\begin{aligned} \dot{x}_i(t) = {} & \sum_{j=1}^{n} a_{ij} \left(x_j(t) - x_i(t) \right) + \sum_{j=1}^{n} a_{ij} \left(x_j \left(t - d_1(t) \right) - x_j(t) \right) \\ & + \sum_{j=1}^{n} a_{ij} \left(x_i(t) - x_i \left(t - d_2(t) \right) \right), \quad i \in \mathcal{N}. \end{aligned} \tag{4}$$

Therefore, by the Newton-Leibniz formula, we get a matrix form of (4):

$$\dot{x}(t) = Lx(t) + A\int_{t}^{t-d_1(t)} \dot{x}(s)\,ds + \Delta\int_{t-d_2(t)}^{t} \dot{x}(s)\,ds, \quad (5)$$

where $\Delta = A - L$.

Denote $z(t) = \dot{x}(t)$. By transformation (2) and the property of the Laplacian matrix L that $L\mathbf{1}_n = \mathbf{0}_n$, we get from (5) the following two descriptor systems:

$$\begin{aligned}\dot{y}(t) &= ELFy(t) + EA\int_{t}^{t-d_1(t)} z(s)\,ds + E\Delta\int_{t-d_2(t)}^{t} z(s)\,ds,\\ z(t) &= LFy(t) + A\int_{t}^{t-d_1(t)} z(s)\,ds + \Delta\int_{t-d_2(t)}^{t} z(s)\,ds.\end{aligned} \quad (6)$$

When the system involves disturbance input, we consider the following multiagent system of the form

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}\left(x_j(t - d_1(t)) - x_i(t - d_2(t)) + w_{ij}(t)\right), \quad i \in \mathcal{N}, \quad (7)$$

where a_{ij}, $d_1(t)$, and $d_2(t)$ are defined as above and $w_{ij}(t)$ is the disturbance input satisfying $w_{ij}(t) \in L_2[0,\infty)$.

Denote the ith row of the matrix A by α_i, $\Sigma = \mathrm{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, $w_i(t) = (w_{i1}(t), w_{i2}(t), \ldots, w_{in}(t))^T$, and $w(t) = (w_1^T(t), w_2^T(t), \ldots, w_n^T(t))^T$. Similar to the above procedure, we can get the following matrix form of (7):

$$\begin{aligned}\dot{y}(t) &= ELFy(t) + EA\int_{t}^{t-d_1(t)} z(s)\,ds\\ &\quad + E\Delta\int_{t-d_2(t)}^{t} z(s)\,ds + E\Sigma w(t),\\ z(t) &= LFy(t) + A\int_{t}^{t-d_1(t)} z(s)\,ds\\ &\quad + \Delta\int_{t-d_2(t)}^{t} z(s)\,ds + \Sigma w(t).\end{aligned} \quad (8)$$

In this paper, say that system (1) achieves consensus asymptotically if

$$\lim_{t\to\infty}\left[x_i(t) - x_j(t)\right] = 0, \quad \forall i \neq j,\ i, j \in \mathcal{N}. \quad (9)$$

Say that system (7) solves H_∞ consensus if system (1) achieves consensus asymptotically, and there exists a constant $\gamma > 0$ such that

$$\int_0^\infty y^T(t)\,y(t)\,dt \le \gamma\int_0^\infty w^T(t)\,w(t)\,dt \quad (10)$$

for all nonzero $w \in L_2[0,\infty)$ under zero initial condition.

3. Main Results

The following two lemmas can be concluded from [21, 22].

Lemma 1. *The digraph $\mathcal{G} = \{V, E, A\}$ has a spanning tree if and only if the matrix ELF is Hurwitz.*

Lemma 2. *For any continuous vector $u(t) \in \mathbb{R}^m$ on $[t - h_k, t]$ for $k = 1, 2$ and $t \ge 0$ and any $m \times m$-dimensional matrix $W > 0$, the following inequality holds:*

$$\begin{aligned}&\left(\int_{t}^{t-d_1(t)} u(\alpha)\,d\alpha\right)^T W\int_{t}^{t-d_1(t)} u(\alpha)\,d\alpha\\ &\quad \le h_1\int_{t-h_1}^{t} u^T(\alpha)\,Wu(\alpha)\,d\alpha,\\ &\left(\int_{t-d_2(t)}^{t} u(\alpha)\,d\alpha\right)^T W\int_{t-d_2(t)}^{t} u(\alpha)\,d\alpha\\ &\quad \le h_2\int_{t-h_2}^{t} u^T(\alpha)\,Wu(\alpha)\,d\alpha.\end{aligned} \quad (11)$$

Based on Lemmas 1 and 2 and the preliminaries given in Section 2, we have the following consensus result for system (1).

Theorem 3. *If the digraph $\mathcal{G}$ has a spanning tree, system* (1) *achieves consensus asymptotically for appropriate constants $h_1 > 0$ and $h_2 > 0$ which satisfy the following matrix inequality:*

$$\begin{pmatrix} P\widetilde{L} + \widetilde{L}^T P & P\widetilde{A} & P\widetilde{\Delta} & h_1\overline{L}^T Q_1 & h_2\overline{L}^T Q_2\\ * & -Q_1 & 0 & h_1 A^T Q_1 & h_2 A^T Q_2\\ * & * & -Q_2 & h_1\Delta^T Q_1 & h_2\Delta^T Q_2\\ * & * & * & -Q_1 & 0\\ * & * & * & * & -Q_2 \end{pmatrix} < 0, \quad (12)$$

where $\widetilde{L} = ELF$, $\widetilde{A} = EA$, $\widetilde{\Delta} = E\Delta$, $\overline{L} = LF$, and P, Q_1, and Q_2 are positive-definite matrices of appropriate dimensions.

Proof. First, we see that (12) is equivalent to

$$\Omega + h_1^2 \Phi^T Q_1 \Phi + h_2^2 \Phi^T Q_2 \Phi < 0, \tag{13}$$

where

$$\Phi = \left(\overline{L}, A, \Delta\right), \qquad \Omega = \begin{pmatrix} P\widetilde{L} + \widetilde{L}^T P & P\widetilde{A} & P\widetilde{\Delta} \\ * & -Q_1 & 0 \\ * & * & -Q_2 \end{pmatrix}. \tag{14}$$

On the other hand, we show that (13) is feasible if the digraph $\mathscr{G}$ has a spanning tree. By Lemma 1, we see that there exists a positive-definite matrix P such that $P\widetilde{L} + \widetilde{L}^T P < 0$. Once the positive-definite matrix P is determined, it is not difficult to see that $\Omega < 0$ for certain positive-definite matrices Q_1 and Q_2. Once the matrices $P > 0$, $Q_1 > 0$, and $Q_2 > 0$ are determined, (13) is valid for sufficiently small $h_1 > 0$ and $h_2 > 0$. Therefore, (13) always holds for appropriate matrices $P > 0$, $Q_1 > 0$, and $Q_2 > 0$ and constants $h_1 > 0$, $h_2 > 0$ if $\mathscr{G}$ has a spanning tree.

Next, by the definition of vector y, it is sufficient to show that $\lim_{t\to\infty} y(t) = 0$ if (13) holds. We construct the Lyapunov function as follows:

$$\begin{aligned} V(t) &= y^T(t) P y(t) \\ &\quad + h_1 \int_{t-h_1}^{t} (s - t + h_1) z^T(s) Q_1 z(s)\, ds \\ &\quad + h_2 \int_{t-h_2}^{t} (s - t + h_2) z^T(s) Q_2 z(s)\, ds, \end{aligned} \tag{15}$$

where positive constants h_1 and h_2 and positive-definite matrices P, Q_1, and Q_2 satisfy (13). We now consider the derivative of $V(t)$ along the trajectory of system (6). For the sake of convenience, set

$$\eta_1(t) = \int_{t}^{t-d_1(t)} z(s)\, ds, \qquad \eta_2(t) = \int_{t-d_2(t)}^{t} z(s)\, ds. \tag{16}$$

A straightforward computation yields that

$$\begin{aligned} \dot V(t) &= 2y^T(t) P \left[\widetilde{L} y(t) + \widetilde{A}\eta_1(t) + \widetilde{\Delta}\eta_2(t)\right] \\ &\quad + h_1^2 z^T(t) Q_1 z(t) - h_1 \int_{t-h_1}^{t} z^T(s) Q_1 z(s)\, ds \\ &\quad + h_2^2 z^T(t) Q_2 z(t) - h_2 \int_{t-h_2}^{t} z^T(s) Q_2 z(s)\, ds. \end{aligned} \tag{17}$$

By Lemma 1, (6), and (17), we have that

$$\begin{aligned} \dot V(t) &\le 2y^T(t) P \left[\widetilde{L} y(t) + \widetilde{A}\eta_1(t) + \widetilde{\Delta}\eta_2(t)\right] \\ &\quad + h_1^2 z^T(t) Q_1 z(t) - \eta_1^T(t) Q_1 \eta_1(t) \\ &\quad + h_2^2 z^T(t) Q_2 z(t) - \eta_2^T(t) Q_2 \eta_2(t) \\ &= \xi^T(t) \left(\Omega + h_1^2 \Phi^T Q_1 \Phi + h_2^2 \Phi^T Q_2 \Phi\right) \xi(t), \end{aligned} \tag{18}$$

where $\xi(t) = (y^T(t), \eta_1^T(t), \eta_2^T(t))^T$ and Φ and Ω are defined by (14). From (13) and (18), we have that $\dot V(t) < 0$ for $t \ge 0$. Therefore, we eventually conclude that $\lim_{t\to\infty} y(t) = 0$. This completes the proof of Theorem 3. □

The method used in Theorem 3 can also be applied to the H_∞ consensus problem of system (7).

Theorem 4. *For given constants $h_1 > 0$ and $h_2 > 0$, if there exist positive-definite matrices P, Q_1, and Q_2 of appropriate dimensions and positive constants $\alpha > 0$ and $\beta > 0$ such that the following linear matrix inequality holds:*

$$\begin{pmatrix} P\widetilde{L} + \widetilde{L}^T P + \alpha I_{n-1} & P\widetilde{A} & P\widetilde{\Delta} & P\widetilde{\Sigma} & h_1 \overline{L}^T Q_1 & h_2 \overline{L}^T Q_2 \\ * & -Q_1 & 0 & 0 & h_1 A^T Q_1 & h_2 A^T Q_2 \\ * & * & -Q_2 & 0 & h_1 \Delta^T Q_1 & h_2 \Delta^T Q_2 \\ * & * & * & -\beta I_{n-1} & h_1 \Sigma^T Q_1 & h_2 \Sigma^T Q_2 \\ * & * & * & * & -Q_1 & 0 \\ * & * & * & * & * & -Q_2 \end{pmatrix} < 0, \tag{19}$$

where $\widetilde{\Sigma} = E\Sigma$, $\widetilde{A}$, $\widetilde{\Delta}$, $\widetilde{L}$, and $\overline{L}$ are defined as above, then system (7) *solves H_∞ consensus with $\gamma = \beta/\alpha$.*

Proof. Choose the Lyapunov function defined by (15). First, (19) implies that (12) holds. By Theorem 3, system (1) achieves consensus asymptotically. Next, we show that (10) holds with $\gamma = \beta/\alpha$ for all nonzero $w \in L_2[0,\infty)$ under zero initial condition. In fact, similar to the computation in Theorem 3, we have

$$\begin{aligned} \dot V(t) &\le 2y^T(t) P \left[\widetilde{L} y(t) + \widetilde{A}\eta_1(t) + \widetilde{\Delta}\eta_2(t) + \widetilde{\Sigma} w(t)\right] \\ &\quad + h_1^2 z^T(t) Q_1 z(t) - \eta_1^T(t) Q_1 \eta_1(t) \end{aligned}$$

$$+h_2^2 z^T(t) Q_2 z(t) - \eta_2^T(t) Q_2 \eta_2(t), \tag{20}$$

which implies that

$$\begin{aligned}\dot{V}(t) &+ \alpha y^T(t) y(t) - \beta w^T(t) w(t) \\ &\leq \tilde{\xi}^T \left(\tilde{\Omega} + h_1^2 \tilde{\Phi}^T Q_1 \tilde{\Phi} + h_2^2 \tilde{\Phi}^T Q_2 \tilde{\Phi}\right) \tilde{\xi},\end{aligned} \tag{21}$$

where $\tilde{\xi} = (y^T(t), \eta_1^T(t), \eta_2^T(t), w^T(t))^T$,

$$\begin{gathered}\tilde{\Phi} = \left(\overline{L}, A, \Delta, \Sigma\right), \\ \Omega = \begin{pmatrix} P\tilde{L} + \tilde{L}^T P + \alpha I_{n-1} & P\tilde{A} & P\tilde{\Delta} & P\tilde{\Sigma} \\ * & -Q_1 & 0 & 0 \\ * & * & -Q_2 & 0 \\ * & * & * & -\beta I_{n-1} \end{pmatrix}.\end{gathered} \tag{22}$$

By (19), (21), and (22), we have that

$$\dot{V}(t) + \alpha y^T(t) y(t) - \beta w^T(t) w(t) < 0, \quad t \geq 0. \tag{23}$$

Integrating (23) from 0 to ∞ under zero initial condition, we get

$$\int_0^\infty y^T(t) y(t)\, dt \leq \frac{\beta}{\alpha} \int_0^\infty w^T(t) w(t)\, dt. \tag{24}$$

This completes the proof of Theorem 4. □

Remark 5. Similar to the analysis in the proof of Theorem 3, we see that a necessary condition to Theorem 3 is that the digraph $\mathcal{G}$ has a spanning tree. That is, the linear matrix inequality (19) implies that $\mathcal{G}$ has a spanning tree.

Remark 6. Unlike most of consensus analysis for multiagent systems, it does not require that $a_{ij} \geq 0$ for all $i \neq j$ in the proofs of Theorems 3 and 4. Therefore, even when $a_{ij} < 0$ for some $i \neq j$, system (7) can also solve H_∞ consensus under appropriate conditions.

The method used in this paper can also be extended to the case of switching topology. Consider the following multiagent system with switched topologies and heterogeneous delays:

$$\dot{x}_i(t) = \sum_{j=1}^n a_{ij}^\sigma \left(x_j(t - d_1(t)) - x_i(t - d_2(t))\right), \quad i \in \mathcal{N}, \tag{25}$$

where $t \geq 0$, $\sigma(t) : [0, \infty) \to \Gamma = \{1, 2, \ldots, p\}$ is a switching signal that determines which subsystem is active at time t, and $a_{ij}^k \geq 0$, $i, j \in \mathcal{N}$, $k \in \Gamma$, are entries of the weighted adjacency matrix A_k. When $\sigma(t) = k \in \Gamma$, we denote the involved digraph by $\mathcal{G}_k = (V, A_k, E_k)$.

Under the reduced-order transformation (2), system (25) can be described by the following two systems:

$$\begin{aligned}\dot{y}(t) &= EL_\sigma F y(t) + EA_\sigma \int_t^{t-d_1(t)} z(s)\, ds \\ &\quad + E\Delta_\sigma \int_{t-d_2(t)}^t z(s)\, ds, \\ z(t) &= L_\sigma F y(t) + A_\sigma \int_t^{t-d_1(t)} z(s)\, ds \\ &\quad + \Delta_\sigma \int_{t-d_2(t)}^t z(s)\, ds,\end{aligned} \tag{26}$$

where y and z are defined as above and L_k, A_k, and Δ_k ($k \in \Gamma$) relative to the digraph $\mathcal{G}_k$ are defined as L, A, and Δ, respectively. Let the Lyapunov function defined by (15) be the common Lyapunov function for the switched system (26). Then, similar to the proof of Theorem 3, we have the following consensus result in the case of switching topology.

Theorem 7. *For given constants $h_1 > 0$ and $h_2 > 0$, system (25) solves consensus under arbitrary switching, if there exist positive-definite matrices P, Q_1, and Q_2 of appropriate dimensions such that the following linear matrix inequality*

$$\begin{pmatrix} P\tilde{L}_k + \tilde{L}_k^T P & P\tilde{A}_k & P\tilde{\Delta}_k & h_1 \overline{L}_k^T Q_1 & h_2 \overline{L}_k^T Q_2 \\ * & -Q_1 & 0 & h_1 A_k^T Q_1 & h_2 A_k^T Q_2 \\ * & * & -Q_2 & h_1 \Delta_k^T Q_1 & h_2 \Delta_k^T Q_2 \\ * & * & * & -Q_1 & 0 \\ * & * & * & * & -Q_2 \end{pmatrix} < 0 \tag{27}$$

holds for $k \in \Gamma$, where $\tilde{L}_k = EL_k F$, $\tilde{A}_k = EA_k$, $\tilde{\Delta}_k = E\Delta_k$, and $\overline{L}_k = L_k F$.

Similarly, for the following switched multiagent system with disturbance input and heterogeneous delays

$$\dot{x}_i(t) = \sum_{j=1}^n a_{ij}^\sigma \left(x_j(t - d_1(t)) - x_i(t - d_2(t)) + w_{ij}(t)\right), \quad i \in \mathcal{N}, \tag{28}$$

we have the following H_∞ consensus result.

Theorem 8. *For given constants $h_1 > 0$ and $h_2 > 0$, if there exist positive-definite matrices P, Q_1, and Q_2 of appropriate dimensions and positive constants $\alpha > 0$ and $\beta > 0$ such that the following linear matrix inequality*

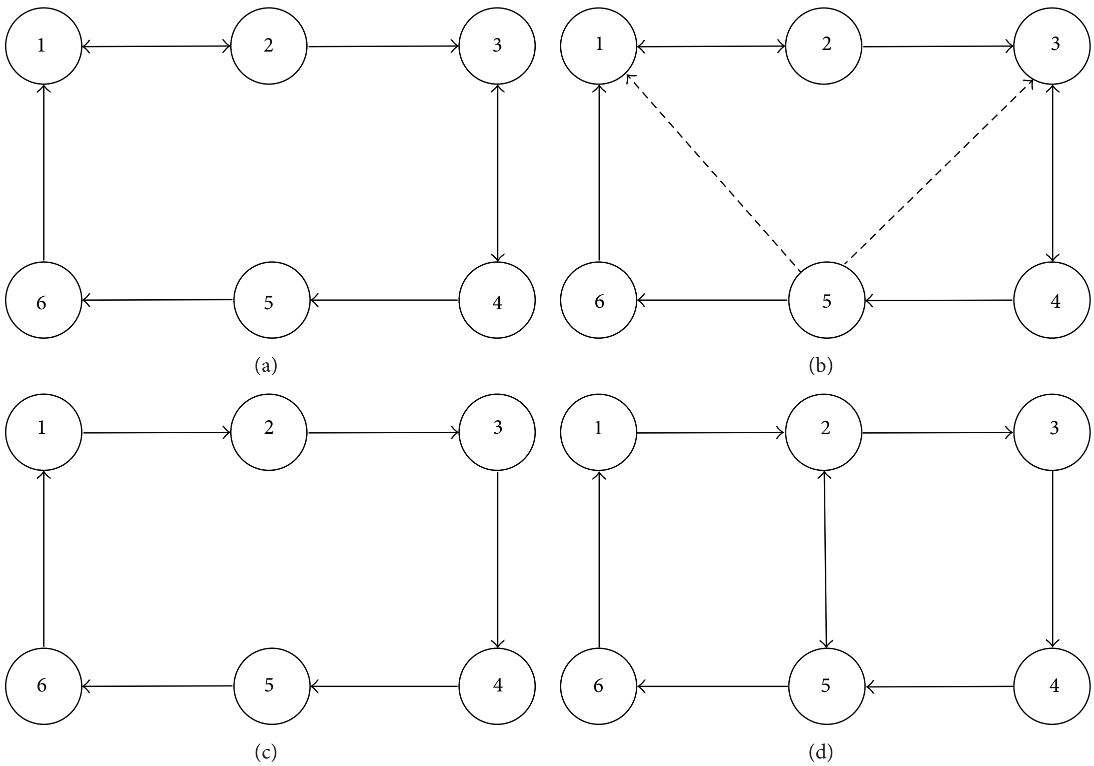

FIGURE 1: Four digraphs: (a) $\mathcal{G}_a$, (b) $\mathcal{G}_b$, (c) $\mathcal{G}_c$, and (d) $\mathcal{G}_d$.

$$\begin{pmatrix} P\widetilde{L}_k + \widetilde{L}_k^T P + \alpha I_{n-1} & P\widetilde{A}_k & P\widetilde{\Delta}_k & P\widetilde{\Sigma}_k & h_1\overline{L}_k^T Q_1 & h_2\overline{L}_k^T Q_2 \\ * & -Q_1 & 0 & 0 & h_1 A_k^T Q_1 & h_2 A_k^T Q_2 \\ * & * & -Q_2 & 0 & h_1 \Delta_k^T Q_1 & h_2 \Delta_k^T Q_2 \\ * & * & * & -\beta I_{n-1} & h_1 \Sigma_k^T Q_1 & h_2 \Sigma_k^T Q_2 \\ * & * & * & * & -Q_1 & 0 \\ * & * & * & * & * & -Q_2 \end{pmatrix} < 0 \tag{29}$$

holds for $k \in \Gamma$, *where* $\widetilde{L}_k$, $\widetilde{A}_k$, $\widetilde{\Delta}_k$, $\widetilde{\Sigma}_k$, Σ_k, *and* $\overline{L}_k$ *relative to the digraph* $\mathcal{G}_k$ *are defined as* $\widetilde{L}$, $\widetilde{A}$, $\widetilde{\Delta}$, $\widetilde{\Sigma}$, Σ, *and* $\overline{L}$ *in Theorem 4, then system* (28) *solves* H_∞ *consensus with* $\gamma = \beta/\alpha$ *under arbitrary switching.*

4. Numerical Examples

Consider the following four digraphs with six nodes shown in Figure 1, where the weights relative to the edges shown by solid lines and dashed lines are 1 and −0.5, respectively.

When $\mathcal{G} = \mathcal{G}_a$, we have that (12) is feasible for given $h_1 \le 0.38$ and $h_2 \le 0.24$. By Theorem 3, system (1) achieves consensus asymptotically. The state trajectory under the stochastic initial condition is shown in Figure 2. In Remark 6, we show that Theorem 3 can also be applied to the extreme case when part of weights a_{ij} is negative. For example, if $\mathcal{G} = \mathcal{G}_b$, we have that (12) is still feasible for given $h_1 \le 0.26$ and $h_2 \le 0.22$. Therefore, system (1) also solves consensus asymptotically even when there exist negative weights $a_{51} = a_{53} = -0.5$. The state trajectory under the stochastic initial condition is shown in Figure 3.

For the case of disturbance input, assume that $\mathcal{G} = \mathcal{G}_c$ and $w_{ij}(t) = w_j(t)$ for $i, j \in \mathcal{N}$. Therefore, $\Sigma = A$. For given

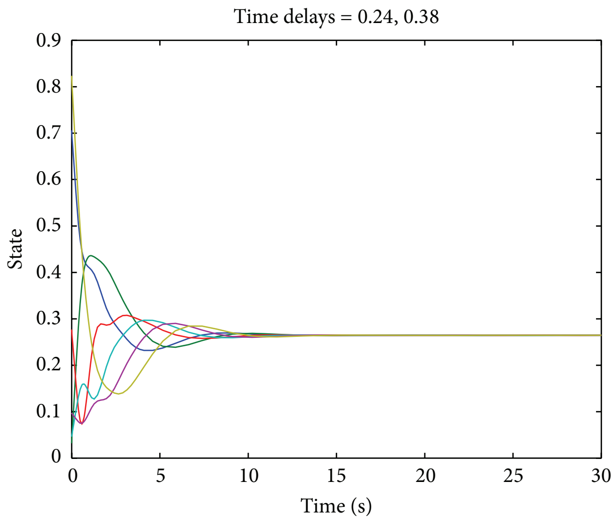

FIGURE 2: State trajectory of system (1) with $\mathscr{G} = \mathscr{G}_a$.

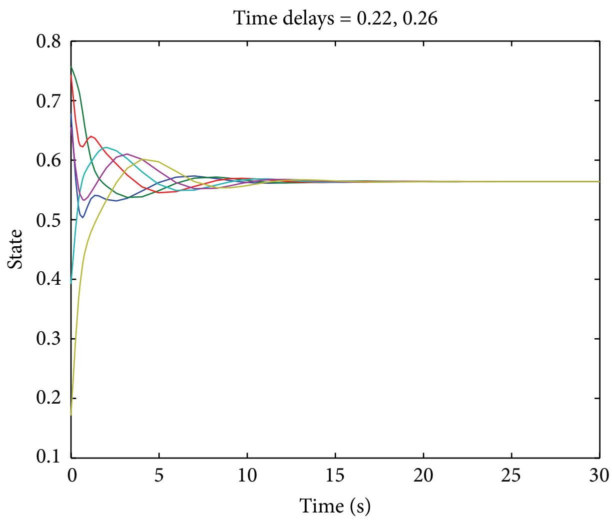

FIGURE 3: State trajectory of system (1) with $\mathscr{G} = \mathscr{G}_b$.

$h_1 = 0.2$ and $h_2 = 0.1$, we get from (19) that $\gamma = 43.0731$. By Theorem 4, we have that system (7) solves H_∞ consensus with $\gamma = 43.0731$. If we let

$$w(t) = \begin{cases} (0.1 \sin t)\,\mathbf{1}, & 0 \leq t \leq 30, \\ 0.0, & \text{otherwise}, \end{cases} \tag{30}$$

the trajectory of system (7) under a stochastic initial condition is shown in Figure 4.

For the case of switching topologies $\{\mathscr{G}_c, \mathscr{G}_d\}$, we have that (27) is feasible for given $h_1 \leq 0.24$ and $h_2 \leq 0.18$. By Theorem 7, system (25) solves consensus under arbitrary switching.

5. Conclusions

In this paper, we first apply the linear matrix inequality method to consensus of the single integrator multiagent system with heterogeneous time-varying delays in directed networks. Unlike the case of identical delays, the multiagent system with heterogeneous delays usually cannot be transformed to a reduced-order system. To overcome such difficulty, we introduce a partially-reduced-order system and an integral system. As a result, the linear matrix inequality method becomes useful in the analysis of consensus and H_∞ consensus by constructing a particular Lyapunov function. The main results are also extended to the case of switching topologies. Finally, numerical examples and simulation results are given to illustrate the theoretical results.

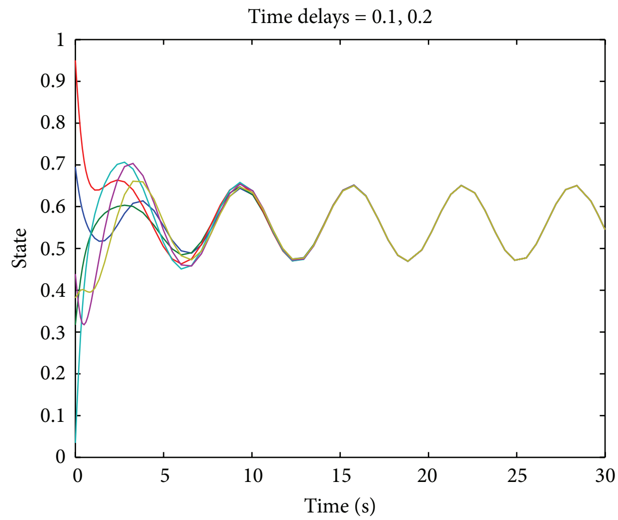

FIGURE 4: State trajectory of system (7) with $\mathscr{G} = \mathscr{G}_c$ and given disturbance.

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