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Research Article

Three Solutions for Inequalities Dirichlet Problem Driven by $p(x)$-Laplacian-Like

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A class of nonlinear elliptic problems driven by $p(x)$-Laplacian-like with a nonsmooth locally Lipschitz potential was considered. Applying the version of a nonsmooth three-critical-point theorem, existence of three solutions of the problem is proved.

1. Introduction

Since many free boundary problems and obstacle problems may be reduced to partial differential equations with discontinuous nonlinearities, the existence of multiple solutions of the problems with discontinuous nonlinearities has been widely investigated in recent years. In 1981, Chang [1] extended the variational methods to a class of nondifferentiable functionals and directly applied the variational methods for nondifferentiable functionals to prove some existence theorems for PDE with discontinuous nonlinearities. Soon thereafter, Kourogenis and Papageorgiou [2] extend the nonsmooth critical point theory of Chang [1], by replacing the compactness and the boundary conditions. In [3], by using the Ekeland variational principle and a deformation theorem, Kandilakis et al. obtained the local linking theorem for locally Lipschitz functions. In the celebrated work [4], Ricceri elaborated a Ricceri-type variational principle for Gateaux differentiable functionals. Later, Marano and Motreanu [5] extended Ricceri's result to a large class of nondifferentiable functionals and gave an application to a Neumann-type problem involving the p-Laplacian with discontinuous nonlinearities.

In this paper, we consider a nonlinear elliptic problem driven by $p(x)$-Laplacian-like with a nonsmooth locally Lipschitz potential (hemivariational inequality):

$$-\operatorname{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)\in\lambda\partial F(x,u), \quad \text{a.e. in } \Omega,$$
$$u=0, \quad \text{on } \partial\Omega, \tag{P}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with C^1-boundary $\partial\Omega$. $p\in C(\Omega)$, $2\le N<p^-:=\inf_{x\in\Omega}p(x)\le p^+:=\sup_{x\in\Omega}p(x)<+\infty$, $F\in C(\overline{\Omega}\times\mathbb{R})$, and $F:\Omega\times\mathbb{R}\to\mathbb{R}$ is a locally Lipschitz with respect to the second variable. By $\partial F(x,u)$, we denote the generalized subdifferential of the locally Lipschitz function $u\to F(x,u)$. Our goal is to establish the same results under different assumptions.

The study of differential equations and variational problems with variable exponent has been a new and interesting topic. It arises from nonlinear elasticity theory, electrorheological fluids, and so forth (see [6, 7]). The study on variable exponent problems attracts more and more interest in recent years. Many results have been obtained on this kind of problems, for example, [8–14]. Neumann-type problems involving the $p(x)$-Laplacian have been studied, for instance, in [15–18].

Recently, Rodrigues [19] has considered the existence of nontrivial solution for the Dirichlet problem involving the $p(x)$-Laplacian-like of the type

$$-\operatorname{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)=\lambda f(x,u), \quad \text{a.e. in } \Omega,$$
$$u=0, \quad \text{on } \partial\Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $p \in C(\overline{\Omega})$ with $p(x) > 2$, for all $x \in \Omega$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory condition. We emphasize that, in our approach, no continuity hypothesis will be required for the function f with respect to the second argument. So, (P) need not have a solution. To avoid this situation, we consider such function $f(x,\cdot)$ which is locally essentially bounded and fill the discontinuity gap of $f(x,\cdot)$, replacing f by the interval $[f_1, f_2]$, where

$$f_1(x,t) := \lim_{s\to 0^+} \operatorname{ess}\inf_{|s-t|<\delta} f(x,s),$$
$$f_2(x,t) := \lim_{s\to 0^+} \operatorname{ess}\sup_{|s-t|<\delta} f(x,s). \tag{2}$$

On the other hand, it is well known that if $F(x,u) = \int_0^u f(x,t)dt$, then F become locally Lipschitz and $\partial F(x,u) = [f_1(x,u), f_2(x,u)]$ (see [1, 20]).

The aim of the present paper is to establish a three-solution theorem for the nonlinear elliptic problem driven by $p(x)$-Laplacian-like with nonsmooth potential (see Theorem 6) by using a consequence (see Theorem 4) of the three-critical-point theorem established firstly by Marano and Motreanu in [20], which is a non-smooth version of Ricceri's three-critical-point theorem (see [21]). The paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable exponent Sobolev spaces and the generalized gradient of the locally Lipschitz function. In Section 3, we give the main result of this paper and use the non-smooth three-critical-point theorem to prove it.

2. Preliminary

In order to discuss problem (P), we need some theories on $W_0^{1,p(x)}(\Omega)$ and the generalized gradient of the locally Lipschitz function. Firstly we state some basic properties of space $W_0^{1,p(x)}(\Omega)$ which will be used later (for details, see [10–12]). Denote by $S(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $S(\Omega)$ are considered as the same element of $S(\Omega)$ when they are equal almost everywhere.

Put $C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : p(x) > 1,\ \forall x \in \overline{\Omega}\}$.

If $p \in C(\overline{\Omega})$, then write

$$L^{p(x)}(\Omega) = \left\{u \in S(\Omega) : \int_\Omega |u(x)|^{p(x)}dx < +\infty\right\}, \tag{3}$$

with the norm $|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |u(x)/\lambda|^{p(x)}dx \le 1\}$, and

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\}, \tag{4}$$

with the norm $\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}$. Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

We remember that the variable exponent Lebesgue spaces are separable and reflexive Banach spaces. Denote by $L^{q(x)}(\Omega)$ the conjugate Lebesgue space of $L^{p(x)}(\Omega)$ with $1/p(x) + 1/q(x) = 1$; then the Hölder-type inequality

$$\int_\Omega |uv|\,dx \le \left(\frac{1}{p^-}+\frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)},$$
$$u \in L^{p(x)}(\Omega),\ v \in L^{q(x)}(\Omega) \tag{5}$$

holds. Furthermore, if we define the mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ by

$$\rho(u) = \int_\Omega |u(x)|^{p(x)}dx, \tag{6}$$

then the following relations hold:

$$|u|_{p(x)} > 1 \Longrightarrow |u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+},$$
$$|u|_{p(x)} < 1 \Longrightarrow |u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}. \tag{7}$$

Proposition 1 (see [12]). *In $W_0^{1,p(x)}(\Omega)$ Poincare's inequality holds; that is, there exists a positive constant C_0 such that*

$$|u|_{p(x)} \le C_0|\nabla u|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega). \tag{8}$$

So $|\nabla u|_{p(x)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$.

We will use the equivalent norm in the following discussion and write $\|u\| = |\nabla u|_{p(x)}$ for simplicity.

Proposition 2 (see [10]). *If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous.*

Consider the following function:

$$J(u) = \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}}\right)dx,$$
$$u \in W_0^{1,p(x)}(\Omega). \tag{9}$$

We know that (see [1]).

If one denotes $A = J' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$, then

$$\langle A(u), v\rangle = \int_\Omega \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)(\nabla u, \nabla v)_{\mathbb{R}^N}dx, \tag{10}$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$.

Proposition 3 (see [19]). *Set $X = W_0^{1,p(x)}(\Omega)$; A is as shown, then*

(1) *$A : X \to X^*$ is a convex, bounded previously; and strictly monotone operator;*

(2) *$A : X \to X^*$ is a mapping of type $(S)_+$; that is, $u_n \xrightarrow{w} u$ in X and $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \le 0$ implies $u_n \to u$ in X;*

(3) *$A : X \to X^*$ is a homeomorphism.*

Let $(X, \|\cdot\|)$ be a real Banach space, and let X^* be its topological dual. A function $f : X \to \mathbb{R}$ is called locally Lipschitz if each point $u \in X$ possesses a neighborhood Ω_u such that $|f(u_1) - f(u_2)| \le L\|u_1 - u_2\|$ for all $u_1, u_2 \in \Omega_u$, for a positive constant L depending on Ω_u. The generalized directional derivative of f at the point $u \in X$ in the direction $h \in X$ is

$$f^0(u;h) = \limsup_{v\to u;\, t\downarrow 0} \frac{f(v+th) - f(v)}{t}. \tag{11}$$

The generalized gradient of f at $u \in X$ is defined by

$$\partial f(u) = \left\{u^* \in X^* : \langle u^*, h\rangle \le f^0(u;h)\ \ \forall h \in X\right\}, \tag{12}$$

which is a nonempty, convex, and w^*-compact subset of X, where $\langle\cdot,\cdot\rangle$ is the duality pairing between X^* and X. One says that $u \in X$ is a critical point of f if $0 \in \partial f(u)$.

For further details, we refer the reader to the work of Chang [1].

Finally, for proving our results in the next section, we introduce the following theorem.

Theorem 4 (see [22, 23]). *Let X be a separable and reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz functions. Assume that there exists $u_0 \in X$ such that $\Phi(u_0) = \Psi(u_0) = 0$ and $\Phi(u) \ge 0$ for every $u \in X$ and that there exist $u_1 \in X$ and $r > 0$ such that*

(1) *$r < \Phi(u_1)$;*

(2) *$\sup_{\Phi(u)<r}\Psi(u) < r(\Psi(u_1)/\Phi(u_1))$, and further, one assumes that function $\Phi - \lambda\Psi$ is sequentially lower semicontinuous and satisfies the (PS)-condition;*

(3) *$\lim_{\|u\|\to\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty$ for every $\lambda \in [0, \overline{a}]$, where*

$$\overline{a} = \frac{hr}{r(\Psi(u_1)/\Phi(u_1)) - \sup_{\Phi(u)<r}\Psi(u)}, \quad \text{with } h > 1. \tag{13}$$

Then, there exist an open interval $\Lambda_1 \subseteq [0, \overline{a}]$ and a positive real number σ such that, for every $\lambda \in \Lambda_1$, the function $\Phi(u) - \lambda\Psi(u)$ admits at least three critical points whose norms are less than σ.

3. Existence Results

In this part, we will prove that there exist three solutions for problem (P) under certain conditions.

Definition 5. We say that I satisfies $(\mathrm{PS})_c$-condition if any sequence $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$, such that $I(u_n) \to c$ and $m(u_n) \to 0$, as $n \to +\infty$, has a strongly convergent subsequence, where $m(u_n) = \inf\{\|u^*\|_{X^*} : u^* \in \partial I(u_n)\}$.

By a solution of (P), we mean a function $u \in W_0^{1,p(x)}(\Omega)$ to which there corresponds a mapping $\Omega \ni x \to w(x)$ with $w(x) \in \partial F(x,u)$ for almost every $x \in \Omega$ having the property that, for every $v \in W_0^{1,p(x)}(\Omega)$, the function $x \to w(x)v(x) \in L^1(\Omega)$ and

$$\begin{aligned}\int_\Omega \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)(\nabla u, \nabla v)_{\mathbb{R}^N}dx \\ = \lambda\int_\Omega w(x)\,v(x)\,dx.\end{aligned} \tag{14}$$

We know that $W_0^{1,p(x)}(\Omega)$ is compactly embedded into $C(\overline{\Omega})$ (by $N < p^- < p^*(x)$). So there is a constant $c_0 > 0$ such that $|u|_\infty \le c_0\|u\|$, for all $u \in W_0^{1,p(x)}(\Omega)$.

Set $\Phi(u) = \int_\Omega (1/p(x))(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}})dx$, $\Psi(u) = \int_\Omega F(x,u)dx$, $u \in W_0^{1,p(x)}(\Omega)$ and $\varphi(u) = \Phi(u) - \lambda\Psi(u)$, for all $u \in W_0^{1,p(x)}(\Omega)$.

We know that the critical points of φ are just the weak solutions of (P).

We consider a non-smooth potential function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ such that $F(x,0) = 0$ a.e. on Ω satisfying the following conditions:

$\mathbf{H(j)}$:

($\mathbf{h_1}$) $F(\cdot,t)$ is measurable for all $t \in \mathbb{R}$;

($\mathbf{h_2}$) $F(x,\cdot)$ is locally Lipschitz for a.e. $x \in \Omega$;

($\mathbf{h_3}$) there exist $a \in L^\infty(\Omega)_+$, $c > 0$ such that

$$|w| \le a(x) + c|t|^{\alpha(x)-1}, \quad \text{a.e. } x \in \Omega,\ \forall t \in \mathbb{R}, \tag{15}$$

where $w \in \partial F(x,t)$ and $1 < \alpha^- \le \alpha^+ < p^-$;

($\mathbf{h_4}$) there exists $q \in C(\overline{\Omega})$ with $p^+ < q^- \le q(x) < p^*(x)$, such that $\lim_{|t|\to 0}(F(x,t)/|t|^{q(x)}) = 0$ uniformly a.e. $x \in \Omega$;

($\mathbf{h_5}$) $\sup_{t\in\mathbb{R}}F(x,t) > 0$, for all $x \in \overline{\Omega}$.

Theorem 6. *Let $(\mathbf{h_1})$–$(\mathbf{h_5})$ hold. Then, there are an open interval $\Lambda \subseteq [0, +\infty)$ and a number σ such that, for every λ belonging to Λ, problem (P) possesses at least three solutions in $W_0^{1,p(x)}(\Omega)$ whose norms are less than σ.*

Proof. We observe that $\Psi(u)$ is Lipschitz on $L^{\alpha(x)}(\Omega)$ and, taking into account that $\alpha(x) < p^*(x)$, Ψ is also locally Lipschitz on $W_0^{1,p(x)}(\Omega)$ (see Proposition 2.2 of [15]). Moreover it results in $\partial\Psi(u) \subseteq \int_\Omega \partial F(x,u)dx$ (see [24]). The interpretation of $\partial\Psi(u) \subseteq \int_\Omega \partial F(x,u)dx$ is as follows: to every $w \in \partial\Psi(u)$ there corresponds a mapping $w(x) \in \partial F(x,u)$ for almost all $x \in \Omega$ having the property that for every $v \in W_0^{1,p(x)}(\Omega)$ the function $w(x)v(x) \in L^1(\Omega)$ and $\langle w, v\rangle = \int_\Omega w(x)v(x)dx$ (see [24]). The proof is divided into the following five steps.

Step 1. We show that φ is coercive.

By ($\mathbf{h_2}$), for almost all $x \in \Omega$, $t \mapsto F(x,t)$ is differentiable almost everywhere on $\mathbb{R}$ and we have

$$\frac{d}{dt}F(x,t) \in \partial F(x,t). \tag{16}$$

From ($\mathbf{h_3}$), there exist positive constants a_1, a_2 such that

$$\begin{aligned} F(x,t) &= F(x,0) + \int_0^t \frac{d}{ds}F(x,s)\,ds \\ &\le a(x)t + \frac{c}{\alpha(x)}|t|^{\alpha(x)} \le a_1 + a_2|t|^{\alpha(x)} \end{aligned} \tag{17}$$

for a.e. $x \in \Omega$ and $t \in \mathbb{R}$.

Note that $1 < \alpha(x) \le \alpha^+ < p^- < p^*(x)$; then by Proposition 2, we have $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$ (compact embedding). Furthermore, there exists c_1 such that $|u|_{\alpha(x)} \le c_1\|u\|$.

So, for $|u|_{\alpha(x)} > 1$ and $\|u\| > 1$, we have $\int_\Omega |u|^{\alpha(x)}dx \le |u|_{\alpha(x)}^{\alpha^+} \le c_1^{\alpha^+}\|u\|^{\alpha^+}$.

Hence,

$$\begin{aligned} &\varphi(u) \\ &\quad = \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}}\right)dx - \lambda \\ &\qquad \times \int_\Omega F(x,u)\,dx \\ &\quad \ge \frac{2}{p^+}\int_\Omega |\nabla u|^{p(x)}dx - \lambda\int_\Omega F(x,u)\,dx \\ &\quad \ge \frac{2}{p^+}\|u\|^{p^-} - \lambda a_1 \operatorname{meas}(\Omega) - \lambda a_2 c_1^{\alpha^+}\|u\|^{\alpha^+} \longrightarrow +\infty, \end{aligned} \tag{18}$$

as $\|u\| \to +\infty$.

Step 2. We show that φ is weakly lower semicontinuous.

Let $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$, and by Proposition 2, we obtain the following results:

$$\begin{gathered} W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega); \quad u_n \longrightarrow u \text{ in } L^{p(x)}(\Omega); \\ u_n \longrightarrow u \quad \text{for a.a. } x \in \Omega; \\ F(x,u_n(x)) \longrightarrow F(x,u(x)) \quad \text{for a.a. } x \in \Omega. \end{gathered} \tag{19}$$

By Fatou's lemma, we have

$$\limsup_{n\to\infty}\int_\Omega F(x,u_n(x))\,dx \le \int_\Omega F(x,u(x))\,dx. \tag{20}$$

Thus,

$$\begin{aligned} &\liminf_{n\to\infty}\varphi(u_n) \\ &\quad = \int_\Omega \frac{1}{p(x)}\left(|\nabla u_n|^{p(x)} + \sqrt{1+|\nabla u_n|^{2p(x)}}\right)dx \\ &\qquad - \lambda\limsup_{n\to\infty}\int_\Omega F(x,u_n)\,dx \\ &\quad \ge \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}}\right)dx \\ &\qquad - \lambda\int_\Omega F(x,u)\,dx = \varphi(u). \end{aligned} \tag{21}$$

Step 3. We show that (PS)-condition holds.

Suppose $\{u_n\}_{n\ge1} \subseteq W_0^{1,p(x)}(\Omega)$ such that $|\varphi(u_n)| \le c$ and $m(u_n) \to 0$ as $n \to +\infty$. If $u_n^* \in \partial\varphi(u_n)$ is such that $m(u_n) = \|u_n^*\|_{(W_0^{1,p(x)})^*}$, $n \ge 1$, then we know that

$$u_n^* = \Phi'(u_n) - \lambda w_n, \tag{22}$$

where the nonlinear operator $\Phi' : W_0^{1,p(x)} \to (W_0^{1,p(x)})^*$ is defined as

$$\begin{aligned} &\left\langle \Phi'(u), v\right\rangle \\ &\quad = \int_\Omega \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)(\nabla u, \nabla v)_{\mathbb{R}^N}dx, \end{aligned} \tag{23}$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$. From the work of Chang [1], we know that if $w_n \in \partial\Psi(u_n)$, then $w_n \in L^{\alpha'(x)}(\Omega)$, where $1/\alpha'(x) + 1/\alpha(x) = 1$.

Since φ is coercive, $\{u_n\}_{n\ge1}$ is bounded in $W_0^{1,p(x)}(\Omega)$ and there exists $u \in W_0^{1,p(x)}(\Omega)$ such that a subsequence of $\{u_n\}_{n\ge1}$, which is still denoted as $\{u_n\}_{n\ge1}$, satisfies $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$. Next we will prove that $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$.

By $W_0^{1,p(x)}(\Omega) \to L^{\alpha(x)}(\Omega)$, we have $u_n \to u$ in $L^{\alpha(x)}(\Omega)$. Moreover, since $\|u_n^*\|_* \to 0$, we get $|\langle u_n^*, u_n\rangle| \le \varepsilon_n$.

Since $u_n^* = \Phi'(u_n) - \lambda w_n$, we obtain

$$\left\langle \Phi'(u_n), u_n - u\right\rangle - \lambda\int_\Omega w_n(u_n - u)\,dx \le \varepsilon_n, \quad \forall n \ge 1. \tag{24}$$

Moreover, since $u_n \to u$ in $L^{\alpha(x)}(\Omega)$ and $\{w_n\}_{n\ge1}$ are bounded in $L^{\alpha'(x)}(\Omega)$, where $1/\alpha(x) + 1/\alpha'(x) = 1$, one has $\int_\Omega w_n(u_n - u)dx \to 0$. Therefore,

$$\limsup_{n\to\infty}\left\langle \Phi'(u_n), u_n - u\right\rangle \le 0. \tag{25}$$

But we know that Φ' is a mapping of type (S_+) (by Proposition 3). Thus we obtain

$$u_n \longrightarrow u \quad \text{in } W_0^{1,p(x)}(\Omega). \tag{26}$$

Step 4. There exists a $u_1 \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ such that $\Psi(u_1) > 0$.

By $(\mathbf{h_5})$, for each $x \in \overline{\Omega}$, there is $t_x \in \mathbb{R}$ such that $F(x, t_x) > 0$.

For $x \in \mathbb{R}^N$, denote by N_x a neighborhood of x which is the product of N compact intervals. From $(\mathbf{h_5})$ and $F(x,t) \in C(\overline{\Omega} \times \mathbb{R})$, for any $x_0 \in \overline{\Omega}$, there are $N_{x_0} \subset \mathbb{R}^N$, $t_{x_0} \in \mathbb{R}$ and $\delta_0 > 0$, such that $F(x, t_{x_0}) > \delta_0 > 0$ for all $x \in N_{x_0} \bigcap \overline{\Omega}$.

Since $\Omega \subseteq \mathbb{R}^N$ is bounded, $\overline{\Omega}$ is compact. Then we can find $N_{x_1}, N_{x_2}, \ldots, N_{x_n}$ such that $\Omega \subset \bigcup_{i=1}^n N_{x_i}$ and $N_{x_i} \bigcap N_{x_j} = \partial N_{x_i} \bigcap \partial N_{x_j}$, $(i \neq j)$ and, also, we can find $t_{x_1}, t_{x_2}, \ldots, t_{x_n} \in \mathbb{R}$, and n positive numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that

$$F\left(x, t_{x_i}\right) > \delta_i > 0 \text{ uniformly for } x \in N_{x_i} \bigcap \overline{\Omega}, \quad i = 1, 2, \ldots, n. \tag{27}$$

Now, set $\delta_0 = \min\{\delta_1, \delta_2, \ldots, \delta_n\}$, and $t_0 = \max\{t_{x_1}, t_{x_2}, \ldots, t_{x_n}\}$, and

$$\sup_{|t| < |t_0|;\, x \in \overline{\Omega}} |F(x,t)| = M. \tag{28}$$

Then, we can find a closed set $\Omega_{x_i} \subset \operatorname{int}(N_{x_i} \bigcap \Omega)$ such that

$$\operatorname{meas}\left(\Omega_{x_i}\right) > \frac{M \operatorname{meas}\left(N_{x_i} \bigcap \overline{\Omega}\right)}{\delta_0 + M}, \tag{29}$$

where meas(A) denote the Lebesgue measure of set A. We consider a function $u_1 \in W_0^{1,p(x)}(\Omega)$ such that $|u_1(x)| \in [0, t_0]$ and $u_1(x) \equiv t_{x_i}$ for all $x \in \Omega_{x_i}$. For instance, we can set $u_1(x) = \sum_{i=1}^n u_1^i(x)$, where $u_1^i \in C_0^\infty(N_{x_i} \bigcap \Omega)$ and

$$u_1^i(x) = \begin{cases} t_{x_i}, & x \in \Omega_{x_i}, \\ 0 \le u_1^i(x) < t_{x_i}, & x \in \left(N_{x_i} \cap \Omega\right) \setminus \Omega_{x_i}. \end{cases} \tag{30}$$

Then, from (27)–(29), we have

$$\begin{aligned} \Psi(u_1) &= \int_\Omega F(x, u_1)\, dx = \int_{\bigcup_{i=1}^n N_{x_i} \cap \Omega} F(x, u_1)\, dx \\ &= \int_{\bigcup_{i=1}^n \Omega_{x_i}} F(x, u_1)\, dx \\ &\quad + \int_{(\bigcup_{i=1}^n N_{x_i} \cap \Omega) \setminus \bigcup_{i=1}^n \Omega_{x_i}} F(x, u_1)\, dx \\ &\ge \sum_{i=1}^n \delta_i \operatorname{meas}\left(\Omega_{x_i}\right) \\ &\quad - \sum_{i=1}^n M\left[\operatorname{meas}\left(N_{x_i} \bigcap \Omega\right) - \operatorname{meas}\left(\Omega_{x_i}\right)\right] \\ &> \sum_{i=1}^n \left[(\delta_0 + M) \operatorname{meas}\left(\Omega_{x_i}\right) - M \operatorname{meas}\left(N_{x_i} \bigcap \overline{\Omega}\right)\right] \\ &> 0. \end{aligned} \tag{31}$$

Step 5. We show that Φ, Ψ satisfy conditions (1) and (2) of Theorem 4.

Let $u_0 = 0$; then we can easily find $\Phi(u_0) = \Psi(u_0) = 0$.

From (7) and Proposition 1, we have the following:

if $\|u\| \ge 1$, then

$$\frac{2}{p^+} \|u\|^{p^-} \le \Phi(u) \le \frac{2 + |\Omega|}{p^-} \|u\|^{p^+}; \tag{32}$$

if $\|u\| < 1$, then

$$\frac{2}{p^+} \|u\|^{p^+} \le \Phi(u) \le \frac{2 + |\Omega|}{p^-}. \tag{33}$$

From $(\mathbf{h_4})$, there exist $\eta \in]0, 1[$ and $C_3 > 0$ such that

$$F(x,t) \le C_3 |t|^{q(x)} \le C_3 |t|^{q^-}, \quad \forall t \in [-\eta, \eta], \ x \in \Omega. \tag{34}$$

In view of $(\mathbf{h_3})$, if we put

$$C_4 = \max\left\{C_3, \sup_{\eta \le |t| < 1} \frac{a_1 + a_2 |t|^{\alpha^-}}{|t|^{q^-}}, \sup_{|t| \ge 1} \frac{a_1 + a_2 |t|^{\alpha^+}}{|t|^{q^-}}\right\}, \tag{35}$$

then we have

$$F(x,t) \le C_4 |t|^{q^-}, \quad \forall t \in \mathbb{R}, \ x \in \Omega. \tag{36}$$

Fix r such that $0 < r < 1$. And when $(2/p^+)\max\{\|u\|^{p^-}, \|u\|^{p^+}\} < r < 1$, by Sobolev Embedding Theorem ($W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q^-}(\Omega)$), we have (for suitable positive constants C_5, C_6)

$$\begin{aligned} \Psi(u) &= \int_\Omega F(x,u)\, dx \le C_4 \int_\Omega |u|^{q^-} dx \le C_5 \|u\|^{q^-} \\ &< C_6 r^{q^-/p^-} \left(\text{or} \quad C_6 r^{q^-/p^+}\right). \end{aligned} \tag{37}$$

Since $q^- > p^+ \ge p^-$, we have

$$\lim_{r \to 0^+} \frac{\sup_{(2/p^+)\max\{\|u\|^{p^-}, \|u\|^{p^+}\} < r} \Psi(u)}{r} = 0. \tag{38}$$

And so, taking into account (32) and (33),

$$\lim_{r \to 0^+} \frac{\sup_{\Phi(u) < r} \Psi(u)}{r} = 0. \tag{39}$$

From Step 4, there exists $u_1 \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ such that $\Psi(u_1) > 0$. Thanks to (32) and (33), we have

$$0 < \frac{2}{p^+} \max\left\{\|u_1\|^{p^-}, \|u_1\|^{p^+}\right\} \le \Phi(u_1), \tag{40}$$

and so

$$\frac{\Psi(u_1)}{\Phi(u_1)} > 0. \tag{41}$$

By (32), (33), and (39), there exists $r_0 < (2/p^+)\max\{\|u_1\|^{p^-}, \|u_1\|^{p^+}\} \le \Phi(u_1)$ such that, for each $r \in]0, r_0[$,

$$\sup_{\Phi(u) < r} \Psi(u) < r \frac{\Psi(u_1)}{\Phi(u_1)}. \tag{42}$$

By choosing $r \in]0, r_0[$, conditions (1) and (2) requested in Theorem 4 are verified and so the proof is complete. □

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