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Research Article

Implicit Ishikawa Approximation Methods for Nonexpansive Semigroups in CAT(0) Spaces

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This paper is devoted to the convergence of the implicit Ishikawa iteration processes for approximating a common fixed point of nonexpansive semigroup in CAT(0) spaces. We obtain the Δ-convergence results of the implicit Ishikawa iteration sequences for a family of nonexpansive mappings in CAT(0) spaces. Under certain and different conditions, we also get the strong convergence theorems of implicit Ishikawa iteration sequences for nonexpansive semigroups in the CAT(0) spaces. The results presented in this paper extend and generalize some previous results.

1. Introduction

Let (X, d) be a metric space and K be a subset of X. A mapping $T : K \to X$ is said to be nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$. We denote the set of all nonnegative elements in $\mathbb{R}$ by $\mathbb{R}^+$ and denote the set of all fixed points of T by $F(T)$, that is,

$$F(T) = \{x \in K : Tx = x\}. \tag{1}$$

For each $n \in \mathbb{N}$, let $T_n : K \to K$ be nonexpansive mappings and denote the common fixed points set of the family $\{T_n\}$ by $\bigcap_{n=1}^{\infty} F(T_n)$. A family of mappings $\{T_n\}$ is said to be uniformly asymptotically regular if, for any bounded subset B of K,

$$\lim_{n\to\infty} \sup_{z\in B} d(T_n z, T_i(T_n z)) = 0, \tag{2}$$

for all $i \in \mathbb{N}$.

A nonexpansive semigroup is a family,

$$\Gamma := \{T(t) : t \geq 0\}, \tag{3}$$

of mappings $T(t)$ on K such that

(1) $T(s + t)x = T(s)(T(t)x)$ for all $x \in K$ and $s, t \geq 0$;
(2) $T(t) : K \to K$ is nonexpansive for each $t \geq 0$;
(3) for each $x \in K$, the mapping $T(\cdot)x$ from $\mathbb{R}^+$ to K is continuous.

We denote by $F(\Gamma)$ the common fixed points set of nonexpansive semigroup Γ, that is,

$$F(\Gamma) = \bigcap_{t\in\mathbb{R}^+} F(T(t)) = \{x \in X : T(t)x = x \text{ for each } t \geq 0\}. \tag{4}$$

Note that, if K is compact, then $F(\Gamma)$ is nonempty (see [1, 2, 28]).

A geodesic from x to y in X is a mapping Ψ from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $\Psi(0) = x$, $\Psi(l) = y$, and $d(\Psi(t), \Psi(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, Ψ is an isometry and $d(x, y) = l$. The image Θ of Ψ is called a geodesic (or metric) segment joining x and y. The space (X, d) is said to be a geodesic space if any two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for any $x, y \in X$, which is denoted by $[x, y]$ and is called the segment joining x and y. A subset K of a geodesic space X is said to be convex if for any $x, y \in K$, $[x, y] \subset K$.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) = \Delta(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for all $i, j \in \{1, 2, 3\}$. It is known that such a triangle always exists (see [3]). A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom (CA).

Let Δ be a geodesic triangle in (X, d), and let $\overline{\Delta} \subset \mathbb{R}^2$ be a comparison triangle for Δ. Then, Δ is said to satisfy the CAT(0) inequality if, for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,

$$d(x, y) \leq d(\overline{x}, \overline{y}). \tag{5}$$

The complete CAT(0) spaces are often called, Hadamard spaces (see [4]). For any $x, y \in X$, we denote by $\alpha x \oplus (1-\alpha) y$ the unique point $z \in [x, y]$ which satisfies

$$\begin{aligned} d(x, \alpha x \oplus (1-\alpha) y) &= (1-\alpha) d(x, y), \\ d(y, \alpha x \oplus (1-\alpha) y) &= \alpha d(x, y). \end{aligned} \tag{6}$$

It is known that if (X, d) is a CAT(0) space and $x, y \in X$, then for any $\beta \in [0, 1]$, there exists a unique point $\beta x \oplus (1-\beta) y \in [x, y]$. For any $z \in X$, the following inequality holds:

$$d(z, \beta x \oplus (1-\beta) y) \leq \beta d(z, x) + (1-\beta) d(z, y), \tag{7}$$

where $\beta x \oplus (1-\beta) y \in [x, y]$ (for metric spaces of hyperbolic type, see [5]).

Recently, Cho et al. [6] proved the strong convergence of an explicit iterative sequence $\{z_n\}$ in a CAT(0) space, where $\{z_n\}$ is generated by the following iterative scheme for a nonexpansive semigroup Γ:

$$z_1 \in K, \quad z_{n+1} = \alpha z_n \oplus (1-\alpha) T(t_n) z_n, \quad \forall n \geq 1, \tag{8}$$

where $\alpha \in (0, 1)$ and $\{t_n\} \subset \mathbb{R}^+$. The existence of fixed points, an invariant approximation, and convergence theorems for several mappings in CAT(0) spaces have been studied by many authors (see [7–18]).

On the other hand, Thong [19] considered an implicit Mann iteration process for a nonexpansive semigroup $\Gamma = \{T(t) : t \in \mathbb{R}^+\}$ on a closed convex subset C of a Banach space as follows:

$$x_1 \in C, \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n) T(t_n) x_{n+1}, \quad n \geq 1. \tag{9}$$

Under different conditions, Thong [19] proved the weak convergence and strong convergence results of the implicit Mann iteration scheme (9) for nonexpansive semigroups in certain Banach spaces. Many authors have studied the convergence of implicit iteration sequences for nonexpansive mappings, nonexpansive semigroups and pseudocontractive semigroups in the Banach spaces (see [20–23]). Readers may consult [24, 25] for the convergence of the Ishikawa iteration sequences for nonexpansive mappings and nonexpansive semigroups in certain Banach spaces. However, to our best knowledge, there is no paper to study the convergence of the implicit Ishikawa type iteration processes for nonexpansive semigroups in CAT(0) spaces. Therefore, it is of interest to investigate the convergence of implicit Ishikawa type iteration processes for nonexpansive semigroups in CAT(0) spaces under some suitable conditions.

Motivated and inspired by the work mentioned previously, we consider the following implicit Ishikawa iteration scheme for a family of nonexpansive mappings in a CAT(0) space:

$$\begin{aligned} &x_1 \in K, \\ &x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)((1-\theta_n) T_n x_n \oplus \theta_n T_n x_{n+1}), \\ &\qquad \forall n \geq 1, \end{aligned} \tag{10}$$

where $\{\alpha_n\} \subset (0, 1]$ and $\{\theta_n\} \subset [0, 1]$ are given sequences. We prove that $\{x_n\}$ generated by (10) is Δ-convergent to some point in $\bigcap_{n=1}^{\infty} F(T_n)$ under appropriate conditions. We also consider the following implicit Ishikawa iteration process for a nonexpansive semigroup $\Gamma = \{T(t) : t \in \mathbb{R}^+\}$ in a CAT(0) space:

$$\begin{aligned} &x_1 \in K, \\ &x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)((1-\theta_n) T(t_n) x_n \oplus \theta_n T(t_n) x_{n+1}), \\ &\qquad \forall n \geq 1, \end{aligned} \tag{11}$$

where $\{\alpha_n\} \subset (0, 1]$ and $\{\theta_n\} \subset [0, 1]$ are given sequences. Under various and appropriate conditions, we obtain that $\{x_n\}$ generated by (11) converges strongly to a common fixed point of Γ. The results presented in this paper extend and generalize some previous results in [6, 19].

2. Definitions and Lemmas

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space (X, d). For any $x \in X$, denote

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n). \tag{12}$$

Consider the following:

(i) $r(\{x_n\}) = \inf\{r(x, x_n) : x \in X\}$ is called the asymptotic radius of $\{x_n\}$;

(ii) $r_K(\{x_n\}) = \inf\{r(x, x_n) : x \in K\}$ is called the asymptotic radius of $\{x_n\}$ with respect to K;

(iii) the set $A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$ is called the asymptotic center of $\{x_n\}$;

(iv) the set $A_K(\{x_n\}) = \{x \in K : r(x, \{x_n\}) = r_K(\{x_n\})\}$ is called the asymptotic center of $\{x_n\}$ with respect to K.

Definition 1 (see [12, 26]). A sequence $\{x_n\}$ in a CAT(0) space X is said to be Δ-convergent to a point x in X, if x is the unique asymptotic center of $\{x_{n_j}\}$ for all subsequences $\{x_{n_j}\} \subseteq \{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$, and x is called the Δ-limit of $\{x_n\}$.

For the sake of convenience, we restate the following lemmas that shall be used.

Lemma 2 (see [10]). *Let (X, d) be a CAT(0) space. Then,*

$$d((1-t)x \oplus ty, z) \le (1-t)d(x,z) + td(y,z), \quad (13)$$

for all $x, y, z \in X$ and $t \in [0,1]$.

Lemma 3 (see [10]). *Let (X, d) be a CAT(0) space. Then,*

$$\begin{aligned}&[d((1-t)x \oplus ty, z)]^2 \\ &\quad \le (1-t)[d(x,z)]^2 + t[d(y,z)]^2 - t(1-t)[d(x,y)]^2,\end{aligned} \quad (14)$$

for all $x, y, z \in X$ and $t \in [0,1]$.

Lemma 4 (see [10]). *Let K be a closed convex subset of a complete CAT(0) space and $T : K \to K$ be a nonexpansive mapping. Suppose that $\{x_n\}$ is a bounded sequence in K such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\{d(x_n, p)\}$ converges for all $p \in F(T)$. Then, $\omega_w(x_n) = \bigcup A(\{x_{n_j}\}) \subset F(T)$, where the union is taken over all subsequences $\{x_{n_j}\}$ of $\{x_n\}$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

Lemma 5 (see [6]). *Let $\{z_n\}$ and $\{w_n\}$ be bounded sequences in a CAT(0) space X. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ such that $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}\alpha_n < 1$. Define $z_{n+1} = \alpha_n z_n \oplus (1-\alpha_n)w_n$ for all $n \in \mathbb{N}$ and suppose that*

$$\limsup_{n\to\infty}\left[d(w_{n+1}, w_n) - d(z_{n+1}, z_n)\right] \le 0. \quad (15)$$

Then, $\lim_{n\to\infty} d(w_n, z_n) = 0$.

3. Main Results

It is necessary for us to show that the implicit Ishikawa iteration sequences generated by schemes (10) and (11) are well defined, before providing the main results of this present paper.

Lemma 6. *Let K be a nonempty, closed, and convex subset of a complete CAT(0) space X and $T_n : K \to K$ be nonexpansive mappings. Suppose that $\{\alpha_n\} \subset (0,1]$ and $\{\theta_n\} \subset [0,1]$ are given parameter sequences. Then, the sequence $\{x_n\}$ generated by the implicit Ishikawa iteration process (10) is well defined.*

Proof. For each $n \in \mathbb{N}$ and any given $u \in K$, define a mapping $S_{n+1} : K \to K$ by

$$S_{n+1}x := \alpha_n u \oplus (1-\alpha_n)((1-\theta_n)T_n u \oplus \theta_n T_n x), \quad \forall n \ge 1. \quad (16)$$

It can be verified that for any fixed $n \in \mathbb{N}$, S_{n+1} is a contractive mapping. Indeed, if setting $p_n = (1-\theta_n)T_n u \oplus \theta_n T_n x$ and $q_n = (1-\theta_n)T_n u \oplus \theta_n T_n y$, then we have $S_{n+1}x = \alpha_n u \oplus (1-\alpha_n)p_n$ and $S_{n+1}y = \alpha_n u \oplus (1-\alpha_n)q_n$. It follows from Lemmas 3 and 2 that

$$\begin{aligned}&[d(p_n, q_n)]^2 \\ &\quad = [d(p_n, (1-\theta_n)T_n u \oplus \theta_n T_n y)]^2 \\ &\quad \le \theta_n[d(p_n, T_n y)]^2 + (1-\theta_n)[d(p_n, T_n u)]^2 \\ &\qquad - \theta_n(1-\theta_n)[d(T_n u, T_n y)]^2 \\ &\quad \le \theta_n\left\{(1-\theta_n)[d(T_n u, T_n y)]^2 + \theta_n[d(T_n x, T_n y)]^2\right. \\ &\qquad\qquad \left.-\theta_n(1-\theta_n)[d(T_n u, T_n x)]^2\right\} \\ &\qquad + (1-\theta_n)\theta_n^2[d(T_n u, T_n x)]^2 \\ &\qquad - \theta_n(1-\theta_n)[d(T_n u, T_n y)]^2 \\ &\quad = \theta_n^2[d(T_n x, T_n y)]^2.\end{aligned} \quad (17)$$

Consequently, $d(p_n, q_n) \le \theta_n d(T_n x, T_n y) \le \theta_n d(x,y)$, and thus,

$$\begin{aligned}d(S_{n+1}x, S_{n+1}y) &\le (1-\alpha_n)d(p_n, q_n) \\ &\le (1-\alpha_n)\theta_n d(x,y),\end{aligned} \quad (18)$$

which shows that for each $n \in \mathbb{N}$, S_{n+1} is a contractive mapping. By induction, Banach's fixed theorem yields that the sequence $\{x_n\}$ generated by (10) is well defined. This completes the proof. □

We need the following lemma for our main results. The analogs of [6, Lemma 3.1] and [27, Lemma 2.2] are given in what follows. We sketch the proof here for the convenience of the reader.

Lemma 7. *Let K be a nonempty, closed, and convex subset of a complete CAT(0) space X and $T_n : K \to K$ be nonexpansive mappings. Let $\{\alpha_n\} \subset (0,1]$ and $\{\theta_n\} \subset [0,1]$ be given sequences such that $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}\alpha_n < 1$. Suppose that $\{x_n\}$ generated by (10) is bounded and either*

$$\begin{aligned}&\lim_{n\to\infty} d(T_{n+1}x_{n+1}, T_n x_{n+1}) = 0 \\ &\text{or} \quad \lim_{n\to\infty} d(T_{n+1}x_n, T_n x_n) = 0\end{aligned} \quad (19)$$

holds. If $\lim_{n\to\infty}\theta_n = 0$, then $\lim_{n\to\infty} d(T_n x_n, x_n) = 0$.

Proof. First, we show that the boundedness of $\{x_n\}$ implies the boundedness of $\{T_n x_n\}$. If $\{x_n\}$ is bounded, then set

$$M = \sup\{d(x_n, x) : n \in \mathbb{N}\} < +\infty, \quad (20)$$

for some given point $x \in X$ and $\alpha = \liminf_{n\to\infty}\alpha_n > 0$, $\beta = \limsup_{n\to\infty}\alpha_n < 1$. With $0 < a < \alpha \le \beta < b < 1$, there exists $n_1 \in \mathbb{N}$ such that for all $n \ge n_1$,

$$0 < a < \alpha_n < b < 1. \quad (21)$$

It follows from Lemma 2 that

$$\begin{aligned}
d(T_nx_n, x) &\le d(T_nx_n, x_{n+1}) + d(x_{n+1}, x) \\
&= d(T_nx_n, \alpha_nx_n \oplus (1-\alpha_n)((1-\theta_n)T_nx_n \oplus \theta_nT_nx_{n+1})) \\
&\quad + d(x_{n+1}, x) \\
&\le \alpha_n d(T_nx_n, x_n) \\
&\quad + (1-\alpha_n) d(T_nx_n, (1-\theta_n)\ T_nx_n \oplus \theta_nT_nx_{n+1}) \\
&\quad + d(x_{n+1}, x) \\
&\le \alpha_n d(T_nx_n, x_n) + (1-\alpha_n)\theta_n d(T_nx_n, T_nx_{n+1}) \\
&\quad + d(x_{n+1}, x) \\
&\le \alpha_n d(T_nx_n, x) + \alpha_n d(x_n, x) + (1-\alpha_n)\theta_n d(x_n, x_{n+1}) \\
&\quad + d(x_{n+1}, x) \\
&\le \alpha_n d(T_nx_n, x) + [\alpha_n + (1-\alpha_n)\theta_n] d(x_n, x) \\
&\quad + [1 + (1-\alpha_n)\theta_n] d(x_{n+1}, x).
\end{aligned} \tag{22}$$

Hence, for all $n \ge n_1$, from (21), we have

$$\begin{aligned}
d(T_nx_n, x) &\le \frac{1}{1-\alpha_n}[2d(x_n, x) + 2d(x_{n+1}, x)] \\
&\le \frac{4}{1-b}M,
\end{aligned} \tag{23}$$

which means that $\{T_nx_n\}$ is bounded.

Next, we prove the conclusion of Lemma 7. If $\lim_{n\to\infty} d(T_{n+1}x_{n+1}, T_nx_{n+1}) = 0$, then we have

$$\begin{aligned}
&\limsup_{n\to\infty} [d((1-\theta_{n+1})T_{n+1}x_{n+1} \oplus \theta_{n+1}T_{n+1}x_{n+2}, \\
&\qquad (1-\theta_n)T_nx_n \oplus \theta_nT_nx_{n+1}) - d(x_{n+1}, x_n)] \\
&\le \limsup_{n\to\infty} [d((1-\theta_{n+1}) \\
&\qquad \times T_{n+1}x_{n+1} \oplus \theta_{n+1}T_{n+1}x_{n+2}, T_{n+1}x_{n+1}) \\
&\qquad + d(T_{n+1}x_{n+1}, (1-\theta_n)T_nx_n \oplus \theta_nT_nx_{n+1}) \\
&\qquad - d(x_{n+1}, x_n)] \\
&\le \limsup_{n\to\infty} [\theta_{n+1} d(T_{n+1}x_{n+2}, T_{n+1}x_{n+1}) \\
&\qquad + d(T_{n+1}x_{n+1}, T_nx_{n+1}) \\
&\qquad + d(T_nx_{n+1}, (1-\theta_n)T_nx_n \oplus \theta_nT_nx_{n+1}) \\
&\qquad - d(x_{n+1}, x_n)] \\
&\le \limsup_{n\to\infty} [\theta_{n+1} d(x_{n+2}, x_{n+1}) + (1-\theta_n) d(x_{n+1}, x_n) \\
&\qquad - d(x_{n+1}, x_n)] \\
&\le \limsup_{n\to\infty} (\theta_{n+1} + \theta_n)[d(x_{n+2}, x) + 2d(x_{n+1}, x) \\
&\qquad + d(x_n, x)] \\
&\le \limsup_{n\to\infty} 4M(\theta_{n+1} + \theta_n) = 0.
\end{aligned} \tag{24}$$

Similarly, if $\lim_{n\to\infty} d(T_{n+1}x_n, T_nx_n) = 0$, then we have

$$\begin{aligned}
&\limsup_{n\to\infty} [d((1-\theta_{n+1})T_{n+1}x_{n+1} \oplus \theta_{n+1}T_{n+1}x_{n+2}, \\
&\qquad (1-\theta_n)T_nx_n \oplus \theta_nT_nx_{n+1}) - d(x_{n+1}, x_n)] \\
&\le \limsup_{n\to\infty} [d((1-\theta_{n+1}) \\
&\qquad \times T_{n+1}x_{n+1} \oplus \theta_{n+1}T_{n+1}x_{n+2}, T_{n+1}x_{n+1}) \\
&\qquad + d(T_{n+1}x_{n+1}, T_{n+1}x_n) + d(T_{n+1}x_n, T_nx_n) \\
&\qquad + d(T_nx_n, (1-\theta_n)T_nx_n \oplus \theta_nT_nx_{n+1}) \\
&\qquad - d(x_{n+1}, x_n)] \\
&\le \limsup_{n\to\infty} [\theta_{n+1} d(T_{n+1}x_{n+2}, T_{n+1}x_{n+1}) \\
&\qquad + \theta_n d(T_nx_n, T_nx_{n+1})] \\
&\le \limsup_{n\to\infty} [\theta_{n+1} d(x_{n+2}, x_{n+1}) + \theta_n d(x_{n+1}, x_n)] \\
&\le \limsup_{n\to\infty} (\theta_{n+1} + \theta_n)[d(x_{n+2}, x) + 2d(x_{n+1}, x) \\
&\qquad + d(x_n, x)] \\
&\le \limsup_{n\to\infty} 4M(\theta_{n+1} + \theta_n) = 0.
\end{aligned} \tag{25}$$

It follows from Lemma 5 that $\lim_{n\to\infty} d((1-\theta_n)T_nx_n \oplus \theta_nT_nx_{n+1}, x_n) = 0$. Since

$$\begin{aligned}
d(x_n, T_nx_n) &\le d(x_n, (1-\theta_n)T_nx_n \oplus \theta_nT_nx_{n+1}) \\
&\quad + d((1-\theta_n)T_nx_n \oplus \theta_nT_nx_{n+1}, T_nx_n) \\
&\le d(x_n, (1-\theta_n)T_nx_n \oplus \theta_nT_nx_{n+1}) + \theta_n d(T_nx_{n+1}, T_nx_n) \\
&\le d(x_n, (1-\theta_n)T_nx_n \oplus \theta_nT_nx_{n+1}) \\
&\quad + \theta_n[d(x_{n+1}, x) + d(x_n, x)] \\
&\le d(x_n, (1-\theta_n)T_nx_n \oplus \theta_nT_nx_{n+1}) + 2M\theta_n \longrightarrow 0,
\end{aligned} \tag{26}$$

we obtain that $\lim_{n\to\infty} d(T_nx_n, x_n) = 0$. This completes the proof. □

As a direct consequence of Lemma 7, the following lemma is immediate.

Lemma 8. *Let K be a nonempty, closed, and convex subset of a complete CAT(0) space X and $T_n : K \to K$ be nonexpansive mappings. Let $\{\alpha_n\} \subset (0, 1]$ and $\{\theta_n\} \subset [0, 1]$ be given sequences such that $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}\alpha_n < 1$. Suppose that $\{x_n\}$ generated by (10) is bounded and*

$$\lim_{n\to\infty} \sup_{x\in K} d(T_{n+1}x, T_nx) = 0 \tag{27}$$

holds. If $\lim_{n\to\infty}\theta_n = 0$, then $\lim_{n\to\infty} d(T_nx_n, x_n) = 0$.

We now present our main results in this paper. The following theorem discusses the Δ-convergence of the implicit Ishikawa iteration sequence (10) for a family of nonexpansive mappings in CAT(0) spaces.

Theorem 9. *Let K be a nonempty, closed, and convex subset of a complete CAT(0) space X and $T_n : K \to K$ be uniformly asymptotically regular and nonexpansive mappings such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\alpha_n\} \subset (0,1]$ and $\{\theta_n\} \subset [0,1]$ be given sequences such that $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}\alpha_n < 1$. Then, the sequence $\{x_n\}$ generated by (10) is well defined. Suppose that either*

$$\begin{aligned} &\lim_{n\to\infty} d\left(T_{n+1}x_n, T_n x_n\right) = 0 \\ \text{or} \quad &\lim_{n\to\infty} d\left(T_{n+1}x_{n+1}, T_n x_{n+1}\right) = 0 \end{aligned} \tag{28}$$

holds. If $\lim_{n\to\infty}\theta_n = 0$, then $\{x_n\}$ Δ-converges to some point in $\bigcap_{n=1}^{\infty} F(T_n)$.

Proof. By Lemma 6, we know that the sequence $\{x_n\}$ generated by (10) is well defined. For any $p \in \bigcap_{n=1}^{\infty} F(T_n)$, from (10) and Lemma 2, we have

$$\begin{aligned} d\left(x_{n+1}, p\right) &= d\left(\alpha_n x_n \oplus (1-\alpha_n)\left((1-\theta_n) T_n x_n \oplus \theta_n T_n x_{n+1}\right), p\right) \\ &\le \alpha_n d\left(x_n, p\right) + (1-\alpha_n) \\ &\quad \times \left((1-\theta_n) d\left(T_n x_n, p\right) + \theta_n d\left(T_n x_{n+1}, p\right)\right) \\ &\le \alpha_n d\left(x_n, p\right) + (1-\alpha_n)(1-\theta_n) d\left(x_n, p\right) \\ &\quad + (1-\alpha_n)\theta_n d\left(x_{n+1}, p\right). \end{aligned} \tag{29}$$

Since $\alpha_n + (1-\alpha_n)(1-\theta_n) = 1 - (1-\alpha_n)\theta_n$, it follows that

$$d\left(x_{n+1}, p\right) \le d\left(x_n, p\right). \tag{30}$$

Consequently, $\{d(x_n, p)\}$ converges, and $\{x_n\}$ is thus bounded.

It follows from (10) and (21) that for sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned} d\left(x_{n+1}, x_n\right) &= d\left(\alpha_n x_n \oplus (1-\alpha_n)\left((1-\theta_n) T_n x_n \oplus \theta_n T_n x_{n+1}\right), x_n\right) \\ &\le (1-\alpha_n)(1-\theta_n) d\left(x_n, T_n x_n\right) \\ &\quad + (1-\alpha_n)\theta_n d\left(T_n x_{n+1}, x_n\right) \\ &\le (1-\alpha_n)\theta_n d\left(T_n x_{n+1}, T_n x_n\right) \\ &\quad + (1-\alpha_n)\theta_n d\left(T_n x_n, x_n\right) + d\left(x_n, T_n x_n\right) \\ &\le (1-a) d\left(x_{n+1}, x_n\right) + 2d\left(x_n, T_n x_n\right). \end{aligned} \tag{31}$$

Applying Lemma 7, we have $\lim_{n\to\infty} d(x_n, T_n x_n) = 0$. Hence,

$$d\left(x_{n+1}, x_n\right) \le \frac{2}{a} d\left(x_n, T_n x_n\right) \longrightarrow 0, \tag{32}$$

and thus,

$$d\left(x_{n+1}, T_n x_n\right) \le d\left(x_{n+1}, x_n\right) + d\left(x_n, T_n x_n\right) \longrightarrow 0. \tag{33}$$

We prove that for each $i \in \mathbb{N}$, $\lim_{n\to\infty} d(x_{n+1}, T_i x_{n+1}) = 0$. Since

$$d\left(T_i x_{n+1}, T_i\left(T_n x_n\right)\right) \le d\left(x_{n+1}, T_n x_n\right), \tag{34}$$

we know that $\lim_{n\to\infty} d(T_i x_{n+1}, T_i(T_n x_n)) = 0$. Because the family of nonexpansive mappings $\{T_i\}$ is uniformly asymptotically regular, we have

$$\begin{aligned} d&\left(x_{n+1}, T_i x_{n+1}\right) \\ &\le d\left(x_{n+1}, T_n x_n\right) + d\left(T_n x_n, T_i\left(T_n x_n\right)\right) \\ &\quad + d\left(T_i\left(T_n x_n\right), T_i x_{n+1}\right) \\ &\le 2d\left(x_{n+1}, T_n x_n\right) + \sup_{z\in\{x_n\}} d\left(T_n z, T_i\left(T_n z\right)\right) \longrightarrow 0. \end{aligned} \tag{35}$$

Since $\{d(x_n, p)\}$ converges for any $p \in \bigcap_{n=1}^{\infty} F(T_n)$, an application of Lemma 4 yields that $\omega_w(x_n)$ consists of exactly one point and is contained in $F(T_i)$, for all $i \in \mathbb{N}$. This shows that $\{x_n\}$ Δ-converges to some point in $\bigcap_{n=1}^{\infty} F(T_n)$. This completes the proof. □

In the special case where $\theta_n \equiv 0$, from Theorem 9, we have the following corollary.

Corollary 10 (see [6, Theorem 3.4]). *Let K be a nonempty, closed, and convex subset of a complete CAT(0) space X and $T_n : K \to K$ be uniformly asymptotically regular and nonexpansive mappings such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\alpha_n\} \subset (0,1]$ be a given sequence of real numbers such that $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}\alpha_n < 1$. Let $\{x_n\}$ be a sequence defined by*

$$x_1 \in K, \quad x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) T_n x_n, \quad \forall n \ge 1. \tag{36}$$

Suppose that either

$$\begin{aligned} &\lim_{n\to\infty} d\left(T_{n+1}x_n, T_n x_n\right) = 0 \\ \text{or} \quad &\lim_{n\to\infty} d\left(T_{n+1}x_{n+1}, T_n x_{n+1}\right) = 0 \end{aligned} \tag{37}$$

holds. Then, $\{x_n\}$ Δ-converges to some point in $\bigcap_{n=1}^{\infty} F(T_n)$.

Remark 11. Theorem 9 extends and improves [6, Theorem 3.4] from the explicit Mann iteration schemes to the implicit Ishikawa iteration schemes.

By Lemma 8 and Theorem 9, the following theorem holds trivially.

Theorem 12. *Let K be a nonempty, closed, and convex subset of a complete CAT(0) space X and $T_n : K \to K$ be uniformly asymptotically regular and nonexpansive mappings such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\alpha_n\} \subset (0,1]$ and $\{\theta_n\} \subset [0,1]$ be given sequences such that $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}\alpha_n < 1$. Then, the sequence $\{x_n\}$ generated by (10) is well defined. Suppose that*

$$\lim_{n\to\infty} \sup_{x\in K} d\left(T_{n+1}x, T_n x\right) = 0 \tag{38}$$

holds. If $\lim_{n\to\infty}\theta_n = 0$, then $\{x_n\}$ Δ-converges to some point in $\bigcap_{n=1}^{\infty} F(T_n)$.

Finally, we study the strong convergence of the implicit Ishikawa iteration sequence (11) for nonexpansive semigroups in CAT(0) spaces, under various and appropriate conditions.

Theorem 13. *Let C be a compact convex subset of a complete CAT(0) space and $\Gamma = \{T(t) : t \in \mathbb{R}^+\}$ be a nonexpansive semigroup on C. Let $\{\alpha_n\} \subset (0,1]$ and $\{\theta_n\} \subset [0,1]$ be given sequences of real numbers such that $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}\alpha_n < 1$. Then, the sequence $\{x_n\}$ generated by the implicit Ishikawa iteration process* (11) *is well defined. Suppose that $\{t_n\}$ is a sequence in $\mathbb{R}^+$ such that*

$$\liminf_{n\to\infty} t_n < \limsup_{n\to\infty} t_n, \qquad \lim_{n\to\infty} (t_{n+1} - t_n) = 0. \tag{39}$$

If $\lim_{n\to\infty}\theta_n = 0$, then $\{x_n\}$ converges strongly to some point in $F(\Gamma)$.

Proof. It is known that $F(\Gamma)$ is nonempty (see [1, 2, 28]). From Lemma 6, we know that the sequence $\{x_n\}$ generated by (11) is well defined. Then, we show that

$$\lim_{n\to\infty}\sup_{x\in C} d\left(T\left(t_{n+1}\right)x, T\left(t_n\right)x\right) = 0. \tag{40}$$

Assume for the contrary that (40) does not hold. There exist a subsequence $\{t_{n_k}\} \subset \{t_n\}$, a sequence $\{y_k\} \subset C$, and an $\eta > 0$ such that for all $k \in \mathbb{N}$,

$$d\left(T\left(t_{n_k+1}\right)y_k, T\left(t_{n_k}\right)y_k\right) \ge \eta. \tag{41}$$

Since C is compact, there exists a convergent subsequence contained in $\{y_k\}$. Without loss of generality, we assume that $\lim_{k\to\infty}y_k = y$ with $y \in C$. Consequently,

$$\begin{aligned} 0 &< \eta \\ &\le \limsup_{k\to\infty} d\left(T\left(t_{n_k+1}\right)y_k, T\left(t_{n_k}\right)y_k\right) \\ &\le \limsup_{k\to\infty} d\left(T\left(\left|t_{n_k+1} - t_{n_k}\right|\right)y_k, T(0)\,y_k\right) \\ &\le \limsup_{k\to\infty} \Big[d\left(T\left(\left|t_{n_k+1} - t_{n_k}\right|\right)y_k, T\left(\left|t_{n_k+1} - t_{n_k}\right|\right)y\right) \\ &\qquad + d\left(T\left(\left|t_{n_k+1} - t_{n_k}\right|\right)y, T(0)\,y\right) \\ &\qquad + d\left(T(0)\,y, T(0)\,y_k\right)\Big] \\ &\le \limsup_{k\to\infty} \left[2d\left(y_k, y\right) + d\left(T\left(\left|t_{n_k+1} - t_{n_k}\right|\right)y, T(0)\,y\right)\right] = 0, \end{aligned} \tag{42}$$

which is a contradiction. Formula (40) follows readily. Now, Lemma 8 yields that

$$\lim_{n\to\infty} d\left(T\left(t_n\right)x_n, x_n\right) = 0. \tag{43}$$

Similar to the proof of [6, Theorem 3.5], it is easy to see that there exists a subsequence $\{x_{n_j}\}$ which converges to x^*, where x^* is a common fixed point in $F(\Gamma)$. Since x^* is a cluster of $\{x_n\}$, we have $\liminf_{n\to\infty}d(x_n, x^*) = 0$. It follows from (11) and (30) that $\lim_{n\to\infty}d(x_n, x^*)$ exists. Hence, we obtain $\lim_{n\to\infty}d(x_n, x^*) = 0$, which completes the proof. □

Remark 14. The proof of Theorem 13 is an analog of [6, Theorem 3.5]. If $\alpha_n \equiv \lambda \in (0,1)$ and $\theta_n \equiv 0$, then Theorem 13 reduces to [6, Theorem 3.5]. Therefore, Theorem 13 extends and generalizes [6, Theorem 3.5] from the explicit Mann iteration processes to the implicit Ishikawa iteration processes.

We prove another strong convergence theorem which differs from Theorem 13.

Theorem 15. *Let C be a compact convex subset of a complete CAT(0) space and $\Gamma = \{T(t) : t \in \mathbb{R}^+\}$ be a nonexpansive semigroup on C. Let $\{\alpha_n\} \subset (0,1]$ and $\{\theta_n\} \subset [0,1]$ be given sequences. Then, the sequence $\{x_n\}$ generated by the implicit Ishikawa iteration process* (11) *is well defined. Moreover, if*

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n + 1 - \theta_n}{t_n} = 0, \tag{44}$$

then $\{x_n\}$ converges strongly to a common fixed point x^* of Γ.

Proof. It is known that $F(\Gamma)$ is nonempty (see [1, 2, 28]). From Lemma 6, we know that $\{x_n\}$ generated by (11) is well defined.

Claim 1. If $\{r_n\}$ is a sequence of nonnegative real numbers such that $\lim_{n\to\infty}r_n = 0$, then

$$\lim_{n\to\infty}\sup_{x\in C} d\left(T\left(r_n\right)x, T(0)\,x\right) = 0. \tag{45}$$

Assume for the contrary that (45) does not hold. There exist a subsequence $\{r_{n_k}\} \subset \{r_n\}$, a sequence $\{y_k\} \subset C$, and an $\eta > 0$ such that for all $k \in \mathbb{N}$,

$$d\left(T\left(r_{n_k}\right)y_k, T(0)\,y_k\right) \ge \eta. \tag{46}$$

Since C is compact, there exists a convergent subsequence of $\{y_k\}$. Without loss of generality, we assume that $\lim_{n\to\infty}y_k = y$ with $y \in C$. Consequently,

$$\begin{aligned} 0 &< \eta \\ &\le \limsup_{k\to\infty} d\left(T\left(r_{n_k}\right)y_k, T(0)\,y_k\right) \\ &\le \limsup_{k\to\infty} \Big[d\left(T\left(r_{n_k}\right)y_k, T\left(r_{n_k}\right)y\right) \\ &\qquad + d\left(T\left(r_{n_k}\right)y, T(0)\,y\right) + d\left(T(0)\,y, T(0)\,y_k\right)\Big] \\ &\le \limsup_{k\to\infty} \left[2d\left(y_k, y\right) + d\left(T\left(r_{n_k}\right)y, T(0)\,y\right)\right] = 0, \end{aligned} \tag{47}$$

which is a contradiction. Formula (45) follows readily.

Claim 2. Consider that $\lim_{n\to\infty}d(x_{n+1}, T(t)x_{n+1}) = 0$. Since C is a compact convex subset of X, there exists a subsequence

$\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to x^*$ as $j \to \infty$. It follows from (11) and Lemma 2 that

$$\begin{aligned} & d(x_{n+1}, T(t_n) x_{n+1}) \\ & \quad = d(\alpha_n x_n \oplus (1-\alpha_n)((1-\theta_n) T(t_n) x_n \\ & \qquad\qquad \oplus \theta_n T(t_n) x_{n+1}), T(t_n) x_{n+1}) \\ & \quad \le \alpha_n d(x_n, T(t_n) x_{n+1}) \\ & \qquad + (1-\alpha_n)(1-\theta_n) d(T(t_n) x_n, T(t_n) x_{n+1}) \\ & \quad \le \alpha_n d(x_{n+1}, T(t_n) x_{n+1}) \\ & \qquad + \alpha_n d(x_n, x_{n+1}) + (1-\theta_n) d(x_n, x_{n+1}). \end{aligned} \tag{48}$$

Hence, we have

$$d(x_{n+1}, T(t_n) x_{n+1}) \le \frac{\alpha_n + 1 - \theta_n}{1 - \alpha_n} d(x_n, x_{n+1}). \tag{49}$$

For any given $x \in X$, let $M = \sup\{d(x_n, x) : n \in \mathbb{N}\}$. Since C is compact and $\lim_{n\to\infty} \alpha_n = 0$, we know that $M < +\infty$ and $\alpha_n \le 2/3$ for sufficiently large $n \in \mathbb{N}$. Consequently,

$$\begin{aligned} & d(x_{n+1}, T(t_n) x_{n+1}) \\ & \quad \le 3(\alpha_n + 1 - \theta_n)[d(x_n, x) + d(x_{n+1}, x)] \\ & \quad \le 6M(\alpha_n + 1 - \theta_n) \longrightarrow 0. \end{aligned} \tag{50}$$

Thus, from (45) we get

$$\begin{aligned} & \limsup_{n\to\infty} d(T(0) x_{n+1}, x_{n+1}) \\ & \quad \le \limsup_{n\to\infty} [d(T(0) x_{n+1}, T(t_n) x_{n+1}) \\ & \qquad\qquad + d(T(t_n) x_{n+1}, x_{n+1})] \\ & \quad \le \lim_{n\to\infty} \sup_{x\in C} d(T(0) x, T(t_n) x) \\ & \qquad + \limsup_{n\to\infty} d(x_{n+1}, T(t_n) x_{n+1}) = 0. \end{aligned} \tag{51}$$

For any given $t > 0$, it follows from (50) that

$$\begin{aligned} & d(T(0) x_{n+1}, T(t) x_{n+1}) \\ & \quad \le \sum_{k=0}^{[t/t_n]-1} d(T(kt_n) x_{n+1}, T((k+1) t_n) x_{n+1}) \\ & \qquad + d\left(T\left(\left[\frac{t}{t_n}\right] t_n\right) x_{n+1}, T(t) x_{n+1}\right) \\ & \quad \le \left[\frac{t}{t_n}\right] d(x_{n+1}, T(t_n) x_{n+1}) \\ & \qquad + d\left(T\left(t - \left[\frac{t}{t_n}\right] t_n\right) x_{n+1}, x_{n+1}\right) \\ & \quad \le 6Mt \frac{\alpha_n + 1 - \theta_n}{t_n} \\ & \qquad + \max\{d(T(s) x_{n+1}, x_{n+1}) : 0 \le s \le t_n\} \\ & \quad = 6Mt \frac{\alpha_n + 1 - \theta_n}{t_n} \\ & \qquad + d(T(s_n) x_{n+1}, x_{n+1}) \quad (\text{say } s_n \in [0, t_n]) \\ & \quad \le 6Mt \frac{\alpha_n + 1 - \theta_n}{t_n} + d(T(s_n) x_{n+1}, T(0) x_{n+1}) \\ & \qquad + d(T(0) x_{n+1}, x_{n+1}) \\ & \quad \le 6Mt \frac{\alpha_n + 1 - \theta_n}{t_n} + d(T(0) x_{n+1}, x_{n+1}) \\ & \qquad + \sup_{x\in C} d(T(s_n) x, T(0) x), \end{aligned} \tag{52}$$

where $[t/t_n]$ is the integer part of t/t_n. Since $\lim_{n\to\infty}((\alpha_n + 1 - \theta_n)/t_n) = 0$, it follows from (45) and (51) that

$$\lim_{n\to\infty} d(x_{n+1}, T(t) x_{n+1}) = 0. \tag{53}$$

Therefore, from the previous formula, we know that $x^* \in F(\Gamma)$. Iteration process (11) and inequality (30) imply that $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Remark 16. If X is a Banach space, the notation "$\beta x \oplus (1-\beta) y$" with $\beta \in [0,1]$ is replaced by "$\beta x + (1-\beta) y$" and $\theta_n \equiv 1$, then Theorem 15 reduces to [19, Theorem 2.3]. Therefore, Theorem 15 extends and generalizes [19, Theorem 2.3] from the implicit Mann iteration processes in the Banach spaces to the implicit Ishikawa iteration processes in CAT(0) spaces.

Remark 17. The results presented in this paper can be immediately applied to any CAT(k) space with $k \le 0$, because any CAT(k) space is a CAT(k') space for any $k' > k$ (see [3, 6]).

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