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Research Article

A Connection between Basic Univalence Criteria

Horiana Tudor

Department of Mathematics, Faculty of Mathematics and Computer Science, "Transilvania" University of Braşov, 500091 Braşov, Romania

Correspondence should be addressed to Horiana Tudor; horianatudor@yahoo.com

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We obtain a sufficient condition for the analyticity and the univalence of a class of functions defined by an integral operator. The well-known univalence criteria of Alexander, Noshiro-Warschawski, Nehari, Goluzin, Ozaki-Nunokawa, Becker, and Lewandowski would follow upon specializing the functions and the parameters involved in the main result. The results obtained not only reduce to those earlier works, but they also extend the previous results.

1. Introduction

Let $\mathcal{U}_r = \{z \in \mathbb{C} : |z| < r\}$, $0 < r \le 1$, be the disk of radius r centered at 0, let $\mathcal{U} = \mathcal{U}_1$ be the unit disk, and let $I = [0, \infty)$.

Denote by $\mathcal{A}$ the class of analytic functions in $\mathcal{U}$ which satisfy the usual normalization $f(0) = f'(0) - 1 = 0$.

The first results concerning univalence criteria are related to the univalence of an analytic function in the unit disk. Among the most important sufficient conditions for univalence we mention those obtained by Alexander [1], Noshiro [2] and Warschawski [3], Nehari [4], Goluzin [5], Ozaki and Nunokawa [6], Becker [7], and Lewandowski [8].

Furthermore, the first extension of univalence criteria was obtained by Pascu in [9]. In his paper, starting from an analytic function f in the unit disk he established not only the univalence of f but also the analyticity and the univalence of a whole class of functions defined by an integral operator.

Other extensions of the univalence criteria, for an integral operator, were obtained in the papers [10–14]. From the main result of this paper, we found all the univalence criteria mentioned earlier and at the same time other new ones.

2. Loewner Chains

Before proving our main result we need a brief summary of theory of Loewner chains.

A function $L(z, t) : \mathcal{U} \times I \to \mathbb{C}$ is said to be a *Loewner chain* or a *subordination chain* if

(i) $L(z, t)$ is analytic and univalent in $\mathcal{U}$ for all $t \in I$;

(ii) $L(z, t) \prec L(z, s)$ for all $0 \le t \le s < \infty$, where the symbol "$\prec$" stands for subordination.

The following result due to Pommerenke is often used to obtain univalence criteria.

Theorem 1 (see [15, 16]). *Let $L(z, t) = a_1(t)z + a_2(t)z^2 + \cdots$, $a_1(t) \neq 0$, be an analytic function in $\mathcal{U}_r$ for all $t \in I$, locally absolutely continuous in I, locally uniform with respect to $\mathcal{U}_r$. For almost all $t \in I$, suppose that*

$$z \frac{\partial L(z,t)}{\partial z} = p(z,t) \frac{\partial L(z,t)}{\partial t}, \quad z \in \mathcal{U}_r, \tag{1}$$

where $p(z, t)$ is analytic in $\mathcal{U}$ and satisfying $\Re p(z, t) > 0$ for all $z \in \mathcal{U}$, $t \in I$. If $\lim_{t\to\infty} |a_1(t)| = \infty$ and $\{L(z, t)/a_1(t)\}_{t \ge 0}$ forms a normal family in $\mathcal{U}_r$, then for each $t \in I$, the function $L(z, t)$ has an analytic and univalent extension to the whole disk $\mathcal{U}$.

3. Main Result

Making use of Theorem 1, the essence of which is the construction of suitable Loewner chain, we can prove our main result.

Theorem 2. *Let α, β, and c be complex numbers such that $\Re\alpha > 0$, $|\beta| < \Re(\alpha+\beta)$, $\Re c > -1/2$, and*

$$\left|\frac{\alpha c-\beta}{\alpha(1+c)}\right| \le 1. \tag{2}$$

For $f \in \mathcal{A}$, if there exist two analytic functions in $\mathcal{U}$, $g(z) = 1 + b_1 z + \cdots$, $h(z) = c_0 + c_1 z + \cdots$ such that the inequalities

$$\left|\frac{f'(z)}{(1+c)\,g(z)} - 1\right| < 1, \tag{3}$$

$$\begin{aligned}\Bigg|&\left(\frac{f'(z)}{(1+c)\,g(z)}-1\right)|z|^{2(\alpha+\beta)}\\&+\frac{1-|z|^{2(\alpha+\beta)}}{\alpha+\beta}\left(2\frac{\alpha+\beta}{1+c}\frac{zf'(z)\,h(z)}{g(z)}+\frac{zg'(z)}{g(z)}-\beta\right)\\&+\frac{\left(1-|z|^{2(\alpha+\beta)}\right)^2}{(\alpha+\beta)\,|z|^{2(\alpha+\beta)}}z^2\\&\times\left(\frac{\alpha+\beta}{1+c}\frac{f'(z)\,h^2(z)}{g(z)}+\frac{g'(z)\,h(z)}{g(z)}\right.\\&\left.+(\alpha-1)\frac{h(z)}{z}-h'(z)\right)\Bigg| \le 1\end{aligned} \tag{4}$$

are true for all $z \in \mathcal{U}\setminus\{0\}$, then the function F_α,

$$F_\alpha(z) = \left(\alpha\int_0^z u^{\alpha-1}f'(u)\,du\right)^{1/\alpha}, \tag{5}$$

is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Proof. We consider the function $\rho_1(z,t)$ defined by

$$\rho_1(z,t) = 1 + \left(e^{2(\alpha+\beta)t}-1\right)\cdot e^{-t}zh\left(e^{-t}z\right). \tag{6}$$

For all $t \ge 0$ and $z \in \mathcal{U}$ we have $e^{-t}z \in \mathcal{U}$, and from the analyticity of h in $\mathcal{U}$ it follows that $\rho_1(z,t)$ is also analytic in $\mathcal{U}$. Since $\rho_1(0,t) = 1$, there exists a disk $\mathcal{U}_{r_1}$, $0 < r_1 \le 1$, in which $\rho_1(z,t) \ne 0$ for all $t \ge 0$. Since $f \in \mathcal{A}$, it is easy to see that the function

$$\rho_2(z,t) = (\alpha+\beta)\int_0^{e^{-t}z} u^{\alpha-1}f'(u)\,du \tag{7}$$

can be written as $\rho_2(z,t) = z^\alpha\cdot\rho_3(z,t)$, where $\rho_3(z,t)$ is analytic in $\mathcal{U}_{r_1}$, for all $t \ge 0$, and $\rho_3(0,t) = ((\alpha+\beta)/\alpha)e^{-\alpha t}$. It follows that the function

$$\rho_4(z,t) = \rho_3(z,t) + (1+c)\left(e^{(\alpha+2\beta)t}-e^{-\alpha t}\right)\frac{g\left(e^{-t}z\right)}{\rho_1(z,t)} \tag{8}$$

is also analytic in a disk $\mathcal{U}_{r_2}$, $0 < r_2 \le r_1$, and

$$\rho_4(0,t) = e^{(\alpha+2\beta)t}\left[(1+c)+\left(\frac{\beta}{\alpha}-c\right)e^{-2(\alpha+\beta)t}\right]. \tag{9}$$

Let us prove that $\rho_4(0,t) \ne 0$ for all $t \ge 0$. We have $\rho_4(0,0) = 1 + \beta/\alpha$. From $|\beta| < \Re(\alpha+\beta)$ and since $\Re(\alpha+\beta) \le |\alpha+\beta|$, we see that $|\beta| < |\alpha+\beta|$ which is equivalent to $\mathrm{Re}(\beta/\alpha) > -1/2$. It follows that $\rho_4(0,0) \ne 0$. Assume now that there exists $t_0 > 0$ such that $\rho_4(0,t_0) = 0$. Then $e^{2(\alpha+\beta)t_0} = (\alpha c-\beta)/\alpha(1+c)$. From $\Re(\alpha+\beta) > 0$, $t_0 > 0$, it results that $e^{2\Re(\alpha+\beta)t_0} > 1$, and from inequality (2), we conclude that $\rho_4(0,t) \ne 0$ for all $t \ge 0$. Therefore, there is a disk $\mathcal{U}_{r_3}$, $0 < r_3 \le r_2$, in which $\rho_4(z,t) \ne 0$, for all $t \ge 0$, and we can choose an analytic branch of $[\rho_4(z,t)]^{1/\alpha}$, denoted by $\rho(z,t)$. We fix a determination of $(1+\beta/\alpha)^{1/\alpha}$, denoted by δ. For $\delta(t)$ we fix, for $t = 0$, the determination equal to δ, where

$$\delta(t) = e^{(1+2(\beta/\alpha))t}\left[(1+c)+\left(\frac{\beta}{\alpha}-c\right)e^{-2(\alpha+\beta)t}\right]^{1/\alpha}. \tag{10}$$

From these considerations it follows that the function $L(z,t) = z\cdot\rho(z,t)$ is analytic in $\mathcal{U}_{r_3}$, for all $t \ge 0$, and can be written as follows:

$$\begin{aligned}L(z,t) = \Bigg[&(\alpha+\beta)\int_0^{e^{-t}z}u^{\alpha-1}f'(u)\,du + (1+c)\\&\times\frac{\left(e^{(\alpha+2\beta)t}-e^{-\alpha t}\right)z^\alpha g\left(e^{-t}z\right)}{1+\left(e^{2(\alpha+\beta)t}-1\right)\cdot e^{-t}zh\left(e^{-t}z\right)}\Bigg]^{1/\alpha}.\end{aligned} \tag{11}$$

If $L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$ is the Taylor expansion of $L(z,t)$ in $\mathcal{U}_{r_3}$, we have $a_1(t) = \delta(t)$. Since $\Re(\alpha+\beta) > 0$, $\Re(\beta/\alpha) > -1/2$, we have $\lim_{t\to\infty}|a_1(t)| = \infty$. We saw also that $a_1(t) \ne 0$ for all $t \in I$.

From the analyticity of $L(z,t)$ in $\mathcal{U}_{r_3}$, it follows that there exists a number r_4, $0 < r_4 \le r_3$, and a constant $K = K(r_4)$ such that

$$\left|\frac{L(z,t)}{a_1(t)}\right| < K, \quad \forall z \in \mathcal{U}_{r_4},\ t \in I, \tag{12}$$

and thus $\{L(z,t)/a_1(t)\}$ is a normal family in $\mathcal{U}_{r_4}$. From the analyticity of $\partial L(z,t)/\partial t$, for all fixed numbers $T > 0$ and r_5, $0 < r_5 \le r_4$, there exists a constant $K_1 > 0$ (that depends on T and r_5) such that

$$\left|\frac{\partial L(z,t)}{\partial t}\right| < K_1, \quad \forall z \in \mathcal{U}_{r_5},\ t \in [0,T]. \tag{13}$$

It follows that the function $L(z,t)$ is locally absolutely continuous in $[0,\infty)$, locally uniform with respect to $z \in \mathcal{U}_{r_5}$. The function $p(z,t)$ defined by (1) is analytic in a disk $\mathcal{U}_r$, $0 < r \le r_5$, for all $t \ge 0$. In order to prove that the function $p(z,t)$ is analytic and has positive real part in $\mathcal{U}$, we will show that the function $w(z,t) = (p(z,t)-1)/(p(z,t)+1)$, $z \in \mathcal{U}_r$, $t \in I$, is analytic in $\mathcal{U}$, and

$$|w(z,t)| < 1 \quad \forall z \in \mathcal{U},\ t \in I. \tag{14}$$

Elementary calculation gives

$$w(z,t) = \left(\frac{f'(e^{-t}z)}{(1+c)\,g(e^{-t}z)} - 1\right)e^{-2(\alpha+\beta)t} + \frac{1-e^{-2(\alpha+\beta)t}}{\alpha+\beta}\left[2\frac{\alpha+\beta}{1+c}\frac{e^{-t}zf'(e^{-t}z)\,h(e^{-t}z)}{g(e^{-t}z)} + \frac{e^{-t}zg'(e^{-t}z)}{g(e^{-t}z)} - \beta\right] + \frac{\left(1-e^{-2(\alpha+\beta)t}\right)^2}{(\alpha+\beta)\,e^{-2(\alpha+\beta)t}}e^{-2t}z^2 \times \left[\frac{\alpha+\beta}{1+c}\frac{f'(e^{-t}z)\,h^2(e^{-t}z)}{g(e^{-t}z)} + \frac{g'(e^{-t}z)\,h(e^{-t}z)}{g(e^{-t}z)} + (\alpha-1)\frac{h(e^{-t}z)}{e^{-t}z} - h'\left(e^{-t}z\right)\right]. \quad (15)$$

From (3) and (4) we deduce that $g(z) \neq 0$, for all $z \in \mathcal{U}$, and then the function $w(z,t)$ is analytic in the unit disk $\mathcal{U}$. For $t = 0$, in view of (3), we have

$$|w(z,0)| = \left|\frac{f'(z)}{(1+c)\,g(z)} - 1\right| < 1. \quad (16)$$

In order to evaluate $|w(0,t)|$, we will use the following inequality (see [17]):

$$\left|\frac{1-|c|^{2\alpha}}{\alpha}\right| \leq \frac{1-|c|^{2\Re\alpha}}{\Re\alpha}, \quad c \in \mathcal{U},\ c \neq 0,\ \Re\alpha > 0. \quad (17)$$

For $z = 0$ and $t > 0$, from (15), we have

$$|w(0,t)| = \left|\frac{-c}{1+c}e^{-2(\alpha+\beta)t} + \frac{1-e^{-2(\alpha+\beta)t}}{\alpha+\beta}(-\beta)\right| \leq \left|\frac{c}{1+c}\right|e^{-2\Re(\alpha+\beta)t} + \frac{1-e^{-2\Re(\alpha+\beta)t}}{\Re(\alpha+\beta)}|\beta|. \quad (18)$$

From $\Re c > -1/2$ which is equivalent to $|c| < |c+1|$ and since $|\beta| < \Re(\alpha+\beta)$, we have $|w(0,t)| < 1$.

Let t be a fixed number, $t > 0$, and let $z \in \mathcal{U}, z \neq 0$. Since $|e^{-t}z| \leq e^{-t} < 1$ for all $z \in \overline{\mathcal{U}} = \{z \in \mathbb{C} : |z| \leq 1\}$, the function $w(z,t)$ is analytic in $\overline{\mathcal{U}}$. Using the maximum modulus principle it follows that for each $t > 0$, arbitrary fixed, there exists $\theta = \theta(t) \in \mathbb{R}$ such that

$$|w(z,t)| < \max_{|\xi|=1}|w(\xi,t)| = \left|w\left(e^{i\theta},t\right)\right|. \quad (19)$$

Denote that $u = e^{-t}\cdot e^{i\theta}$. Then $|u| = e^{-t} < 1$, and from (15) we obtain

$$w\left(e^{i\theta},t\right) = \left(\frac{f'(u)}{(1+c)\,g(u)} - 1\right)|u|^{2(\alpha+\beta)} + \frac{1-|u|^{2(\alpha+\beta)}}{\alpha+\beta} \times \left(2\frac{\alpha+\beta}{1+c}\frac{uf'(u)\,h(u)}{g(u)} + \frac{ug'(u)}{g(u)} - \beta\right) + \frac{\left(1-|u|^{2(\alpha+\beta)}\right)^2}{(\alpha+\beta)\,|u|^{2(\alpha+\beta)}}u^2 \times \left[\frac{\alpha+\beta}{1+c}\frac{f'(u)\,h^2(u)}{g(u)} + \frac{g'(u)\,h(u)}{g(u)} + (\alpha-1)\frac{h(u)}{u} - h'(u)\right]. \quad (20)$$

Since $u \in \mathcal{U}$, inequality (4) implies that $|w(e^{i\theta},t)| \leq 1$, and from (16), (18), and (19) we conclude that inequality (14) holds true for all $z \in \mathcal{U}$ and $t \geq 0$. Since all the conditions of Theorem 1 are satisfied, it follows that $L(z,t)$ is a Loewner chain, for each $t \geq 0$. For $t = 0$ it results that the function

$$L(z,0) = \left[(\alpha+\beta)\int_0^z u^{\alpha-1}f'(u)\,du\right]^{1/\alpha} \quad (21)$$

is analytic and univalent in $\mathcal{U}$, and then the function F_α defined by (5) is analytic and univalent in $\mathcal{U}$. □

4. Specific Cases and Examples

Suitable choices of the functions g and h and special values of the parameter c yield various types of univalence criteria. So, if in Theorem 2 we take $c = 0$ and $h(z) \equiv 0$, we get the following result.

Theorem 3. *Let α and β be complex numbers such that $\Re\alpha > 0$, $|\beta| < \Re(\alpha+\beta)$, and $|\beta| \leq |\alpha|$. For $f \in \mathcal{A}$, if there exists an analytic function in $\mathcal{U}$, $g(z) = 1 + b_1z + \cdots$, such that the inequalities*

$$\left|\frac{f'(z)}{g(z)} - 1\right| < 1,$$
$$\left|\left(\frac{f'(z)}{g(z)} - 1\right)|z|^{2(\alpha+\beta)} + \frac{1-|z|^{2(\alpha+\beta)}}{\alpha+\beta}\left(\frac{zg'(z)}{g(z)} - \beta\right)\right| \leq 1 \quad (22)$$

are true for all $z \in \mathcal{U}\setminus\{0\}$, then the function F_α defined by (5) *is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.*

Theorem 3 gives us a "continuous" passage from Becker's criterion to Lewandowski's criterion. Indeed, for $g(z) \equiv f'(z)$, we have the following.

Corollary 4. *Let α and β be complex numbers, $\Re\alpha > 0$, $|\beta| < \Re(\alpha+\beta)$, $|\beta| \le |\alpha|$, and $f \in \mathcal{A}$. If for all $z \in \mathcal{U} \setminus \{0\}$*

$$\left|\frac{1-|z|^{2(\alpha+\beta)}}{\alpha+\beta}\left(\frac{zf''(z)}{f'(z)}-\beta\right)\right| \le 1, \tag{23}$$

then the function F_α defined by (5) *is analytic and univalent in $\mathcal{U}$.*

Remark 5. Corollary 4 generalizes the well-known univalence criterion due to Becker. For $\beta = 0$ we found the result from [9]. In the case when $\beta = 0$ and $\alpha = 1$, the previous corollary reduces to Becker's criterion [7].

For $g(z) \equiv f'(z) \cdot (p(z)+1)/2$, where p is analytic in $\mathcal{U}$, $p(0) = 1$, from Theorem 3 we have the following.

Corollary 6. *Let α and β be complex numbers, $\Re\alpha > 0$, $|\beta| < \Re(\alpha+\beta)$, $|\beta| \le |\alpha|$, and $f \in \mathcal{A}$. If there exists an analytic function p with positive real part in $\mathcal{U}$, $p(0) = 1$, such that the inequality*

$$\begin{aligned}&\left|\frac{1-p(z)}{1+p(z)}|z|^{2(\alpha+\beta)}+\frac{1-|z|^{2(\alpha+\beta)}}{\alpha+\beta}\right.\\&\quad\left.\times\left(\frac{zf''(z)}{f'(z)}+\frac{zp'(z)}{p(z)+1}-\beta\right)\right| \le 1\end{aligned} \tag{24}$$

is true for all $z \in \mathcal{U} \setminus \{0\}$, then the function F_α defined by (5) *is analytic and univalent in $\mathcal{U}$.*

Remark 7. Corollary 6 represents a generalization of the univalence criterion due to Lewandowski. For $\beta = 0$ we found the result from [12]. In the case when $\beta = 0$ and $\alpha = 1$, the previous corollary reduces to Lewandowski's criterion [8].

For $c = \beta$ and $h(z) \equiv 0$, from Theorem 2 we can derived some results from paper [18].

Theorem 8. *Let α and β be complex numbers such that $\Re\alpha \ge 1/2$, $|\beta| < \Re(\alpha+\beta)$. For $f \in \mathcal{A}$, if there exists an analytic function in $\mathcal{U}$, $g(z) = 1 + b_1 z + \cdots$, such that the inequalities*

$$\left|\frac{f'(z)}{(1+\beta)g(z)}-1\right| < 1, \tag{25}$$

$$\begin{aligned}&\left|\left(\frac{f'(z)}{(1+\beta)g(z)}-1\right)|z|^{2(\alpha+\beta)}\right.\\&\quad\left.+\frac{1-|z|^{2(\alpha+\beta)}}{\alpha+\beta}\left(\frac{zg'(z)}{g(z)}-\beta\right)\right| \le 1\end{aligned} \tag{26}$$

are true for all $z \in \mathcal{U} \setminus \{0\}$, then the function F_α defined by (5) *is analytic and univalent in $\mathcal{U}$.*

Proof. In view of assumption $\Re\alpha \ge 1/2$ and since $\Re(\alpha+\beta) > 0$, it follows that $\Re\beta > -1/2$. But $\Re\alpha \ge 1/2$ is equivalent to $|\alpha - 1| \le |\alpha|$ and $\Re\beta > -1/2$ with $|\beta| < |\beta+1|$. It results that inequality (2) is true. From (3) and (4) we get immediately inequalities (25) and (26). □

For $\alpha = 1$ and $g(z) \equiv f(z)/z$, from Theorem 8 we obtain the following.

Corollary 9. *Let β be a complex number, $|\beta| < \Re(1+\beta)$. If for all $z \in \mathcal{U}$ the function $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf'(z)}{f(z)}-(\beta+1)\right| < |\beta+1|, \tag{27}$$

then the function f is univalent in $\mathcal{U}$. Moreover, it is a spiral-like function.

Proof. For $\alpha = 1$, we have $F_1 = f$, and in view of (27), inequality (25) of Theorem 8 is verified and inequality (26) is also reduced to (25). It follows that f is univalent in $\mathcal{U}$. The condition (27) of the corollary can be written as $|(1/(\beta+1))(zf'(z)/f(z)) - 1| < 1$. It follows that $\Re((1/(\beta+1))(zf'(z)/f(z))) > 0$. If we put $\beta+1 = |\beta+1|e^{\imath\varphi}$, where from $\Re(1+\beta) > 0$ we have $|\varphi| < \pi/2$, then for all $z \in \mathcal{U}$ we have $\Re(e^{-\imath\varphi}(zf'(z)/f(z))) > 0$, which shows that f is spiral-like in $\mathcal{U}$. □

Taking $g(z) \equiv f(z)/z$, we get the following useful corollary which generalizes the result from [19].

Corollary 10. *Let α and β be complex numbers such that $\Re\alpha \ge 1/2$, $|\beta+1| \le \Re(\alpha+\beta)$, and $f \in \mathcal{A}$. If the inequality*

$$\left|\frac{zf'(z)}{f(z)}-(\beta+1)\right| < |\beta+1| \tag{28}$$

holds true for all $z \in \mathcal{U}$, then the function F_α defined by (5) *is analytic and univalent in $\mathcal{U}$.*

Proof. It is easy to check that inequality (28) implies inequality (26) of Theorem 8. Indeed, for $|\beta+1| \le \Re(\alpha+\beta)$ and making use of (17), we have

$$\begin{aligned}&\left|\frac{1}{\beta+1}\left(\frac{zf'(z)}{f(z)}-(\beta+1)\right)|z|^{2(\alpha+\beta)}\right.\\&\quad\left.+\frac{1-|z|^{2(\alpha+\beta)}}{\alpha+\beta}\left(\frac{zf'(z)}{f(z)}-(\beta+1)\right)\right|\\&\le |z|^{2\Re(\alpha+\beta)}+\frac{|\beta+1|}{\Re(\alpha+\beta)}\left(1-|z|^{2\Re(\alpha+\beta)}\right) \le 1.\end{aligned} \tag{29}$$

□

For the function $g(z) \equiv 1$, from Theorem 8 we get the following.

Corollary 11. *Let α and β be complex numbers such that $\Re\alpha \ge 1/2$, $|\beta| < \Re(\alpha+\beta)$, and $f \in \mathcal{A}$. If the inequality*

$$\left|f'(z)-(\beta+1)\right| < |\beta+1| \tag{30}$$

holds true for all $z \in \mathcal{U}$, then the function F_α defined by (5) *is analytic and univalent in $\mathcal{U}$. In particular, the function f is univalent in $\mathcal{U}$, where $|\beta| < \Re(1+\beta)$.*

Remark 12. From inequality (30), only for β real number, $\beta > -1/2$, we get $\Re f'(z) > 0$. For β complex number, if we put $\beta + 1 = |\beta + 1| e^{\imath\varphi}$, where from $\Re(1+\beta) > 0$ we have $|\varphi| < \pi/2$, then from inequality (28) we obtain $\Re e^{-\imath\varphi} f'(z) > 0$. So, in both cases, we can also conclude that f is univalent in $\mathcal{U}$ from Alexander's theorem [1], and respectively, from Noshiro-Warschawski's theorem [2, 3].

Example 1. Consider the function $f(z) = z + (\beta/4)z^2 + (\beta/6)z^3$, where $\beta \in \mathbb{C}$, $|\beta - 1/3| \le 2/3$. The condition (30) of Corollary 11 is satisfied. Indeed, since $|\beta - 1/3| \le 2/3$ is equivalent with $2|\beta| \le |\beta + 1|$, we get

$$\left|f'(z) - (\beta+1)\right| = \left|\frac{\beta}{2}z + \frac{\beta}{2}z^2 - \beta\right| < 2|\beta| \le |\beta+1|. \quad (31)$$

Then, for all complex numbers α, $\Re\alpha \ge 1/2$, and $|\beta| < \Re(\alpha+\beta)$, by using (5), we obtain that

$$\begin{aligned} F_\alpha(z) &= \left(\alpha \int_0^z u^{\alpha-1} f'(u)\,du\right)^{1/\alpha} \\ &= z\left(1 + \frac{\alpha\beta}{2(\alpha+1)}z + \frac{\alpha\beta}{2(\alpha+2)}z^2\right)^{1/\alpha} \end{aligned} \quad (32)$$

is analytic and univalent in $\mathcal{U}$.

If in Theorem 2 we take $c = 0$ and $g(z) \equiv f'(z)$, then we have the following.

Theorem 13. *Let α and β be complex numbers such that $\Re\alpha > 0$, $|\beta| < \Re(\alpha+\beta)$, and $|\beta| \le |\alpha|$. For $f \in \mathcal{A}$, if there exists an analytic function in $\mathcal{U}$, $h(z) = c_0 + c_1 z + \cdots$, such that the inequality*

$$\begin{aligned} &\left|\frac{1-|z|^{2(\alpha+\beta)}}{\alpha+\beta}\left(2(\alpha+\beta)zh(z) + \frac{zf''(z)}{f'(z)} - \beta\right)\right. \\ &\quad + \frac{\left(1-|z|^{2(\alpha+\beta)}\right)^2}{(\alpha+\beta)|z|^{2(\alpha+\beta)}} \cdot z^2 \\ &\quad \left.\times\left((\alpha+\beta)h^2(z) + \frac{f''(z)h(z)}{f'(z)} + (\alpha-1)\frac{h(z)}{z} - h'(z)\right)\right| \\ &\le 1 \end{aligned} \quad (33)$$

is true for all $z \in \mathcal{U} \setminus \{0\}$, then the function F_α defined by (5) *is analytic and univalent in $\mathcal{U}$.*

For $h(z) \equiv -(1/2(\alpha+\beta))(f''(z)/f'(z))$ the following results.

Corollary 14. *Let α and β be complex numbers, $\Re\alpha > 0$, $|\beta| < \Re(\alpha+\beta)$, $|\beta| \le |\alpha|$, and $f \in \mathcal{A}$. If for all $z \in \mathcal{U} \setminus \{0\}$*

$$\begin{aligned} &\left|\frac{\left(1-|z|^{2(\alpha+\beta)}\right)^2}{2(\alpha+\beta)^2|z|^{2(\alpha+\beta)}}\left(z^2\{f;z\} + (1-\alpha)\frac{zf''(z)}{f'(z)}\right)\right. \\ &\quad \left. - \frac{\beta}{\alpha+\beta}\left(1-|z|^{2(\alpha+\beta)}\right)\right| \le 1, \end{aligned} \quad (34)$$

where

$$\{f;z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2, \quad (35)$$

then the function F_α defined by (5) *is analytic and univalent in $\mathcal{U}$.*

Remark 15. For special values of the parameters α and β, from Corollary 14 we get some known results. For $\beta = 0$, we get the result given in [13]. For $\alpha = 1$, since $F_1(z) = f(z)$, Corollary 14 generalizes the criterion of univalence due to Nehari, and for $\alpha = 1$ and $\beta = 0$ we obtain the univalence criterion due to Nehari [4].

For $h(z) \equiv (1/(\alpha+\beta))(1/z - f'(z)/f(z))$ we have the following.

Corollary 16. *Let α and β be complex numbers, $\Re\alpha > 0$, $|\beta| < \Re(\alpha+\beta)$, $|\beta| \le |\alpha|$, and $f \in \mathcal{A}$. If for all $z \in \mathcal{U} \setminus \{0\}$*

$$\begin{aligned} &\left|\frac{1-|z|^{2(\alpha+\beta)}}{\alpha+\beta}\left[z\frac{d}{dz}\log\left(\frac{z^2 f'(z)}{f^2(z)}\right) - \beta\right]\right. \\ &\quad \left. + \frac{\left(1-|z|^{2(\alpha+\beta)}\right)^2}{(\alpha+\beta)^2|z|^{2(\alpha+\beta)}} \cdot z\frac{d}{dz}\log\left(\frac{z^{1+\alpha}f'(z)}{f^{1+\alpha}(z)}\right)\right| \le 1, \end{aligned} \quad (36)$$

then the function F_α defined by (5) *is analytic and univalent in $\mathcal{U}$.*

Remark 17. Corollary 16 represents a generalization of the univalence criterion due to Goluzin. For $\beta = 0$ we obtain the results from paper [11]. For $\alpha = 1$ and $\beta = 0$ we get Goluzin's criterion [5].

For $c = 0$, $g(z) \equiv (f(z)/z)^2$, and $h(z) \equiv (1/(\alpha+\beta))(1/z - f(z)/z^2)$, from Theorem 2 we get the following.

Corollary 18. *Let α and β be complex numbers, $\Re\alpha > 0$, $|\beta| < \Re(\alpha+\beta)$, $|\beta| \le |\alpha|$, and $f \in \mathcal{A}$. If f satisfies the inequalities*

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1, \tag{37}$$

$$\begin{aligned}&\left|\left(\frac{z^2 f'(z)}{f^2(z)} - 1\right)|z|^{2(\alpha+\beta)} + \frac{1-|z|^{2(\alpha+\beta)}}{\alpha+\beta}\right.\\ &\quad\times\left[2\left(\frac{z^2 f'(z)}{f^2(z)} - 1\right) - \beta\right] + \frac{\left(1-|z|^{2(\alpha+\beta)}\right)^2}{(\alpha+\beta)^2|z|^{2(\alpha+\beta)}}\\ &\quad\left.\times\left[\left(\frac{z^2 f'(z)}{f^2(z)} - 1\right) + (1-\alpha)\left(\frac{f(z)}{z} - 1\right)\right]\right| \le 1,\end{aligned} \tag{38}$$

then the function F_α defined by (5) *is analytic and univalent in $\mathcal{U}$.*

Remark 19. Corollary 18 represents a generalization of the univalence criterion due to Ozaki and Nunokawa. For $\beta = 0$ we found the result from [14]. In the case when $\beta = 0$ and $\alpha = 1$, Corollary 18 reduces to the univalence criterion of Ozaki and Nunokawa [6].

Example 2. Let n be a natural number, $n \ge 3$. We consider the function

$$f(z) = \frac{z}{1 - z^{n+1}/n}. \tag{39}$$

We note that

$$\frac{z^2 f'(z)}{f^2(z)} - 1 = z^{n+1}, \qquad \frac{f(z)}{z} - 1 = \frac{z^{n+1}}{n - z^{n+1}}. \tag{40}$$

The condition (37) of Corollary 18 is verified, and it assures the univalence of the function f. Taking into account (40), for $\alpha = n$ and $\beta = (1-n)/2$, from (38) we have that

$$\begin{aligned}&\left|z^{n+1}|z|^{n+1} + 2\frac{1-|z|^{n+1}}{n+1}\left(2z^{n+1} + \frac{n-1}{2}\right)\right.\\ &\quad\left.+4\frac{\left(1-|z|^{n+1}\right)^2}{(n+1)^2|z|^{n+1}}\left(z^{n+1} + (1-n)\frac{z^{n+1}}{n-z^{n+1}}\right)\right|\\ &\le |z|^{2(n+1)} + \frac{1-|z|^{n+1}}{n+1}\left(4|z|^{n+1} + n - 1\right)\\ &\quad+\frac{8}{(n+1)^2}\left(1-|z|^{n+1}\right)^2\\ &= \frac{1}{(n+1)^2}\left[\left(n^2 - 2n + 5\right)|z|^{2(n+1)}\right.\\ &\qquad\left.-\left(n^2 - 4n + 11\right)|z|^{n+1} + \left(n^2 + 7\right)\right] \le 1,\end{aligned} \tag{41}$$

because the greatest value of the function

$$g(x) = \left(n^2 - 2n + 5\right)x^2 - \left(n^2 - 4n + 11\right)x + \left(n^2 + 7\right), \tag{42}$$

for $x \in [0,1]$, $n \ge 3$, is taken for $x = 1$ and is $g(1) = (n+1)^2$. It follows that all the conditions of Corollary 18 are satisfied, and therefore the function F_n defined by (5) is analytic and univalent in $\mathcal{U}$.

Remark 20. Theorem 2 gives us a connection between Alexander's theorem, Noshiro-Warschawski's theorem, and the univalence criteria of Becker, Lewandowski, Nehari, Goluzin, and Ozaki and Nunokawa as well as their generalizations.

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