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Research Article

Approximation by the q-Szász-Mirakjan Operators

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This paper deals with approximating properties of the q-generalization of the Szász-Mirakjan operators in the case $q > 1$. Quantitative estimates of the convergence in the polynomial-weighted spaces and the Voronovskaja's theorem are given. In particular, it is proved that the rate of approximation by the q-Szász-Mirakjan operators ($q > 1$) is of order q^{-n} versus $1/n$ for the classical Szász-Mirakjan operators.

1. Introduction

The approximation of functions by using linear positive operators introduced via q-Calculus is currently under intensive research. The pioneer work has been made by Lupaş [1] and Phillips [2] who proposed generalizations of Bernstein polynomials based on the q-integers. The q-Bernstein polynomials quickly gained the popularity, see [3–11]. Other important classes of discrete operators have been investigated by using q-Calculus in the case $0 < q < 1$, for example, q-Meyer-König operators [12–14], q-Bleimann, Butzer and Hahn operators [15–17], q-Szász-Mirakjan operators [18–21], and q-Baskakov operators [22, 23].

In the present paper, we introduce a q-generalization of the Szász operators in the case $q > 1$. Notice that different q-generalizations of Szász-Mirakjan operators were introduced and studied by Aral and Gupta [18, 19], by Radu [20], and by Mahmudov [21] in the case $0 < q < 1$. Since we define q-Szász-Mirakjan operators for $q > 1$, the rate of approximation by the q-Szász-Mirakjan operators ($q > 1$) is of order q^{-n}, which is essentially better than $1/n$ (rate of approximation for the classical Szász-Mirakjan operators). Thus our q-Szász-Mirakjan operators have better approximation properties than the classical Szász-Mirakjan operators and the other q-Szász-Mirakjan operators.

The paper is organized as follows. In Section 2, we give standard notations that will be used throughout the paper, introduce q-Szász-Mirakjan operators, and evaluate the moments of $M_{n,q}$. In Section 3 we study convergence properties of the q-Szász-Mirakjan operators in the polynomial-weighted spaces. In Section 4, we give the quantitative Voronovskaja-type asymptotic formula.

2. Construction of $M_{n,q}$ and Estimation of Moments

Throughout the paper we employ the standard notations of q-calculus, see [24, 25].

q-integer and q-factorial are defined by

$$[n]_q := \begin{cases} \dfrac{1-q^n}{1-q}, & \text{if } q \in R^+ \setminus \{1\}, \\ n, & \text{if } q = 1, \end{cases} \quad \text{for } n \in \mathbb{N}, \ [0] = 0,$$
$$[n]_q! := [1]_q[2]_q \dots [n]_q, \quad \text{for } n \in \mathbb{N}, \ [0]! = 1. \tag{2.1}$$

For integers $0 \le k \le n$ q-binomial is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}. \tag{2.2}$$

The q-derivative of a function $f(x)$, denoted by $D_q f$, is defined by

$$(D_q f)(x) := \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0, \qquad (D_q f)(0) := \lim_{x \to 0} (D_q f)(x). \tag{2.3}$$

The formula for the q-derivative of a product and quotient are

$$D_q(u(x)v(x)) = D_q(u(x))v(x) + u(qx)D_q(v(x)). \tag{2.4}$$

Also, it is known that

$$D_q x^n = [n]_q x^{n-1}, \qquad D_q E(ax) = aE(qax). \tag{2.5}$$

If $|q| > 1$, or $0 < |q| < 1$ and $|z| < 1/(1-q)$, the q-exponential function $e_q(x)$ was defined by Jackson

$$e_q(z) := \sum_{k=0}^{\infty} \frac{z^k}{[k]_q!}. \tag{2.6}$$

If $|q| > 1$, $e_q(z)$ is an entire function and

$$e_q(z) = \prod_{j=0}^{\infty}\left(1 + (q-1)\frac{z}{q^{j+1}}\right), \quad |q| > 1. \tag{2.7}$$

There is another q-exponential function which is entire when $0 < |q| < 1$ and which converges when $|z| < 1/|1-q|$ if $|q| > 1$. To obtain it we must invert the base in (2.6), that is, $q \to 1/q$:

$$E_q(z) := e_{1/q}(z) = \sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}z^k}{[k]_q!}. \tag{2.8}$$

We immediately obtain from (2.7) that

$$E_q(z) = \prod_{j=0}^{\infty}\left(1 + (1-q)zq^j\right), \quad 0 < |q| < 1. \tag{2.9}$$

The q-difference equations corresponding to $e_q(z)$ and $E_q(z)$ are

$$\begin{gathered} D_q e_q(az) = a e_q(qz), \qquad D_q E_q(az) = a E_q(qaz), \\ D_{1/q} e_q(z) = D_{1/q} E_{1/q}(z) = E_{1/q}\left(q^{-1}z\right) = e_q\left(q^{-1}z\right), \quad q \neq 0. \end{gathered} \tag{2.10}$$

Let C_p be the set of all real valued functions f, continuous on $[0,\infty)$, such that $w_p f$ is uniformly continuous and bounded on $[0,\infty)$ endowed with the norm

$$\|f\|_p := \sup_{x\in[0,\infty)} w_p(x)|f(x)|. \tag{2.11}$$

Here

$$w_0(x) := 1, \qquad w_p(x) := (1+x^p)^{-1}, \quad \text{if } p \in \mathbb{N}. \tag{2.12}$$

The corresponding Lipschitz classes are given for $0 < \alpha \leq 2$ by

$$\begin{gathered} \Delta_h^2 f(x) := f(x+2h) - 2f(x+h) + f(x), \\ \omega_p^2(f;\delta) := \sup_{0<h\leq\delta}\left\|\Delta_h^2 f\right\|_p, \qquad \mathrm{Lip}_p^2\alpha := \left\{f \in C_p : \omega_p^2(f;\delta) = 0(\delta^\alpha),\ \delta \to 0^+\right\}. \end{gathered} \tag{2.13}$$

Now we introduce the q-parametric Szász-Mirakjan operator.

Definition 2.1. Let $q > 1$ and $n \in \mathbb{N}$. For $f : [0, \infty) \to R$ one defines the Szász-Mirakjan operator based on the q-integers

$$M_{n,q}(f;x) := \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{[n]_q}\right) \frac{1}{q^{k(k-1)/2}} \frac{[n]_q^k x^k}{[k]_q!} e_q\left(-[n]_q q^{-k} x\right). \tag{2.14}$$

Similarly as a classical Szász-Mirakjan operator S_n, the operator $M_{n,q}$ is linear and positive. Furthermore, in the case of $q \to 1^+$ we obtain classical Szász-Mirakjan operators.

Moments $M_{n,q}(t^m;x)$ are of particular importance in the theory of approximation by positive operators. From (2.14) one easily derives the following recurrence formula and explicit formulas for moments $M_{n,q}(t^m;x)$, $m = 0, 1, 2, 3, 4$.

Lemma 2.2. *Let $q > 1$. The following recurrence formula holds*

$$M_{n,q}\left(t^{m+1};x\right) = \sum_{j=0}^{m} \binom{m}{j} \frac{xq^j}{[n]_q^{m-j}} M_{n,q}\left(t^j;q^{-1}x\right). \tag{2.15}$$

Proof. The recurrence formula (2.15) easily follows from the definition of $M_{n,q}$ and $q[k]_q + 1 = [k+1]_q$ as show below:

$$\begin{aligned}
&M_{n,q}\left(t^{m+1};x\right) \\
&= \sum_{k=0}^{\infty} \frac{[k]_q^{m+1}}{[n]_q^{m+1}} \frac{1}{q^{k(k-1)/2}} \frac{[n]_q^k x^k}{[k]_q!} e_q\left(-[n]_q q^{-k} x\right) \\
&= \sum_{k=1}^{\infty} \frac{[k]_q^{m}}{[n]_q^{m}} \frac{1}{q^{k(k-1)/2}} \frac{[n]_q^{k-1} x^k}{[k-1]_q!} e_q\left(-[n]_q q^{-k} x\right) \\
&= \sum_{k=0}^{\infty} \frac{\left(q[k]_q + 1\right)^m}{[n]_q^{m}} \frac{1}{q^{k(k+1)/2}} \frac{[n]_q^{k} x^{k+1}}{[k]_q!} e_q\left(-[n]_q q^{-k} q^{-1} x\right) \\
&= \sum_{k=0}^{\infty} \frac{1}{[n]_q^{m}} \sum_{j=0}^{m} \binom{m}{j} q^j [k]_q^j \frac{1}{q^{k(k+1)/2}} \frac{[n]_q^{k} x^{k+1}}{[k]_q!} e_q\left(-[n]_q q^{-k} q^{-1} x\right) \\
&= \sum_{j=0}^{m} \binom{m}{j} \frac{xq^j}{[n]_q^{m-j}} \sum_{k=0}^{\infty} \frac{[k]_q^j}{[n]_q^j} \frac{1}{q^{k(k-1)/2}} \frac{[n]_q^{k} x^{k}}{[k]_q! q^k} e_q\left(-[n]_q q^{-k} q^{-1} x\right) \\
&= \sum_{j=0}^{m} \binom{m}{j} \frac{xq^j}{[n]_q^{m-j}} M_{n,q}\left(t^j;q^{-1}x\right).
\end{aligned} \tag{2.16}$$

□

Lemma 2.3. *The following identities hold for all $q > 1$, $x \in [0, \infty)$, $n \in \mathbb{N}$, and $k \geq 0$:*

$$x D_q s_{nk}(q; x) = [n]_q \left(\frac{[k]_q}{[n]_q} - x \right) s_{nk}(q; x),$$
$$M_{n,q}\left(t^{m+1}; x\right) = \frac{x}{[n]_q} D_q M_{n,q}(t^m; x) + x M_{n,q}(t^m; x), \tag{2.17}$$

where $s_{nk}(q; x) := (1/q^{k(k-1)/2})([n]_q^k x^k / [k]_q!) e_q(-[n]_q q^{-k} x)$.

Proof. The first identitiy follows from the following simple calculations

$$\begin{aligned} &x D_q s_{nk}(q; x) \\ &= [k]_q \frac{1}{q^{k(k-1)/2}} \frac{[n]_q^k x^k}{[k]_q!} e_q\left(-[n] q^{-k} x\right) - x q^{-k} [n]_q \frac{1}{q^{k(k-1)/2}} \frac{[n]_q^k q^k x^k}{[k]_q!} e_q\left(-[n]_q q^{-k} x\right) \\ &= [k]_q s_{nk}(q; x) - x[n]_q s_{nk}(q; x) = [n]_q \left(\frac{[k]_q}{[n]_q} - x \right) s_{nk}(q; x). \end{aligned} \tag{2.18}$$

The second one follows from the first:

$$\begin{aligned} x D_q M_{n,q}(t^m; x) &= [n]_q \sum_{k=0}^{\infty} \left(\frac{[k]_q}{[n]_q} \right)^m \left(\frac{[k]_q}{[n]_q} - x \right) s_{nk}(q; x) \\ &= [n]_q \sum_{k=0}^{\infty} \left(\frac{[k]_q}{[n]_q} \right)^{m+1} s_{nk}(q; x) - [n]_q x \sum_{k=0}^{\infty} \left(\frac{[k]_q}{[n]_q} \right)^m s_{nk}(q; x) \\ &= [n]_q M_{n,q}\left(t^{m+1}; x\right) - [n]_q x M_{n,q}(t^m; x). \end{aligned} \tag{2.19}$$

□

Lemma 2.4. *Let $q > 1$. One has*

$$\begin{aligned} &M_{n,q}(1; x) = 1, \qquad M_{n,q}(t; x) = x, \qquad M_{n,q}\left(t^2; x\right) = x^2 + \frac{1}{[n]_q} x, \\ &M_{n,q}\left(t^3; x\right) = x^3 + \frac{2+q}{[n]_q} x^2 + \frac{1}{[n]_q^2} x, \\ &M_{n,q}\left(t^4; x\right) = x^4 + \left(3 + 2q + q^2\right) \frac{x^3}{[n]_q} + \left(3 + 3q + q^2\right) \frac{x^2}{[n]_q^2} + \frac{1}{[n]_q^3} x. \end{aligned} \tag{2.20}$$

Proof. For a fixed $x \in R_+$, by the q-Taylor theorem [24], we obtain

$$\varphi_n(t) = \sum_{k=0}^{\infty} \frac{(t-x)_{1/q}^k}{[k]_{1/q}!} D_{1/q}^k \varphi_n(x). \tag{2.21}$$

Choosing $t = 0$ and taking into account

$$(-x)_{1/q}^{k} = (-1)^{k} x^{k} q^{-k(k-1)/2}, \qquad D_{1/q}^{k} e_q\left(-[n]_q x\right) = (-1)^{k} q^{-k(k-1)/2} [n]_q^{k} e_q\left(-[n]_q q^{-k} x\right) \quad (2.22)$$

we get for $\varphi_n(x) = e_q(-[n]_q x)$ that

$$\begin{aligned}
1 = \varphi_n(0) &= \sum_{k=0}^{\infty} \frac{(-1)^{k} x^{k}}{q^{k(k-1)/2} [k]_{1/q}!} D_{1/q}^{k} \varphi_n(x) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^{k} x^{k}}{[k]_q!} (-1)^{k} q^{-k(k-1)/2} [n]_q^{k} e_q\left(-[n]_q q^{-k} x\right) \\
&= \sum_{k=0}^{\infty} \frac{[n]_q^{k} x^{k}}{[k]_q! q^{k(k-1)/2}} e_q\left(-[n]_q q^{-k} z\right).
\end{aligned} \tag{2.23}$$

In other words $M_{n,q}(1; x) = 1$.

Calculation of $M_{n,q}(t^i; x)$, $i = 1, 2, 3, 4$, based on the recurrence formula (2.17) (or (2.15)). We only calculate $M_{n,q}(t^3; x)$ and $M_{n,q}(t^4; x)$:

$$\begin{aligned}
M_{n,q}\left(t^3; x\right) &= \frac{x}{[n]_q} D_q M_{n,q}\left(t^2; x\right) + x M_{n,q}\left(t^2; x\right) \\
&= \frac{x}{[n]_q} \left([2]_q x + \frac{1}{[n]_q}\right) + x\left(x^2 + \frac{1}{[n]_q} x\right) \\
&= \frac{1}{[n]_q^2} x + \frac{2+q}{[n]_q} x^2 + x^3, \\
M_{n,q}\left(t^4; x\right) &= \frac{x}{[n]_q} D_q M_{n,q}\left(t^3; x\right) + x M_{n,q}\left(t^3; x\right) \\
&= \frac{x}{[n]_q} \left(\frac{1}{[n]_q^2} + \frac{2+q}{[n]_q} [2]_q x + [3]_q x^2\right) + x\left(\frac{1}{[n]_q^2} x + \frac{2+q}{[n]_q} x^2 + x^3\right) \\
&= \frac{1}{[n]_q^3} x + \left(3 + 3q + q^2\right) \frac{x^2}{[n]_q^2} + \left(3 + 2q + q^2\right) \frac{x^3}{[n]_q} + x^4.
\end{aligned} \tag{2.24}$$

□

Lemma 2.5. *Assume that $q > 1$. For every $x \in [0, \infty)$ there hold*

$$M_{n,q}\left((t-x)^2; x\right) = \frac{x}{[n]_q}, \tag{2.25}$$

$$M_{n,q}\left((t-x)^3; x\right) = \frac{1}{[n]_q^2} x + (q-1)\frac{x^2}{[n]_q}, \tag{2.26}$$

$$M_{n,q}\left((t-x)^4; x\right) = \frac{1}{[n]_q^3} x + \left(q^2 + 3q - 1\right)\frac{x^2}{[n]_q^2} + (q-1)^2 \frac{x^3}{[n]_q}. \tag{2.27}$$

Proof. First of all we give an explicit formula for $M_{n,q}((t-x)^4; x)$.

$$\begin{aligned}
M_{n,q}\left((t-x)^3; x\right) &= M_{n,q}\left(t^3; x\right) - 3x M_{n,q}\left(t^2; x\right) + 3x^2 M_{n,q}(t; x) - x^3 \\
&= x^3 + \frac{2+q}{[n]_q} x^2 + \frac{1}{[n]_q^2} x - 3x\left(x^2 + \frac{x}{[n]_q}\right) + 3x^3 - x^3 \\
&= \frac{1}{[n]_q^2} x + (q-1)\frac{x^2}{[n]_q}, \\
M_{n,q}\left((t-x)^4; x\right) &= M_{n,q}\left(t^4; x\right) - 4x M_{n,q}\left(t^3; x\right) + 6x^2 M_{n,q}\left(t^2; x\right) - 4x^3 M_{n,q}(t; x) + x^4 \\
&= \frac{1}{[n]_q^3} x + \left(3 + 3q + q^2\right)\frac{x^2}{[n]_q^2} + \left(3 + 2q + q^2\right)\frac{x^3}{[n]_q} + x^4 \\
&\quad - 4x\left(\frac{1}{[n]_q^2} x + \frac{2+q}{[n]_q} x^2 + x^3\right) + 6x^2\left(x^2 + \frac{x}{[n]_q}\right) - 4x^4 + x^4 \\
&= \frac{1}{[n]_q^3} x + \left(-1 + 3q + q^2\right)\frac{x^2}{[n]_q^2} + (q-1)^2 \frac{x^3}{[n]_q}.
\end{aligned} \tag{2.28}$$

□

Now we prove explicit formula for the moments $M_{n,q}(t^m; x)$, which is a q-analogue of a result of Becker, see [26, Lemma 3].

Lemma 2.6. *For $q > 1$, $m \in \mathbb{N}$ there holds*

$$M_{n,q}(t^m; x) = \sum_{j=1}^{m} \mathbb{S}_q(m, j)\frac{x^j}{[n]_q^{m-j}}, \tag{2.29}$$

where

$$\begin{gathered}\mathbb{S}_q(m+1,j) = [j]\mathbb{S}_q(m,j) + \mathbb{S}_q(m,j-1), \quad m \ge 0,\ j \ge 1, \\ \mathbb{S}_q(0,0) = 1, \qquad \mathbb{S}_q(m,0) = 0, \quad m > 0, \qquad \mathbb{S}_q(m,j) = 0, \quad m < j.\end{gathered} \tag{2.30}$$

In particular $M_{n,q}(t^m;x)$ is a polynomial of degree m without a constant term.

Proof. Because of $M_{n,q}(t;x) = x$, $M_{n,q}(t^2;x) = x^2 + x/[n]_q$, the representation (2.29) holds true for $m = 1,2$ with $\mathbb{S}_q(2,1) = 1$, $\mathbb{S}_q(1,1) = 1$.

Now assume (2.29) to be valued for m then by Lemma 2.3 we have

$$\begin{aligned} M_{n,q}\left(t^{m+1};x\right) &= \frac{x}{[n]_q} D_q M_{n,q}(t^m;x) + xM_{n,q}(t^m;x) \\ &= \frac{x}{[n]_q}\sum_{j=1}^{m}[j]_q\mathbb{S}_q(m,j)\frac{x^{j-1}}{[n]_q^{m-j}} + x\sum_{j=1}^{m}\mathbb{S}_q(m,j)\frac{x^j}{[n]_q^{m-j}} \\ &= \sum_{j=1}^{m}[j]_q\mathbb{S}_q(m,j)\frac{x^j}{[n]_q^{m-j+1}} + \sum_{j=1}^{m}\mathbb{S}_q(m,j)\frac{x^{j+1}}{[n]_q^{m-j}} \\ &= \frac{x}{[n]^m}\mathbb{S}_q(m,1) + x^{m+1}\mathbb{S}_q(m,m) \\ &\quad + \sum_{j=2}^{m}\left([j]_q\mathbb{S}_q(m,j) + \mathbb{S}_q(m,j-1)\right)\frac{x^j}{[n]_q^{m-j+1}}. \end{aligned} \tag{2.31}$$

□

Remark 2.7. Notice that $\mathbb{S}_q(m,j)$ are Stirling numbers of the second kind introduced by Goodman et al. in [8]. For $q = 1$ the formulae (2.30) become recurrence formulas satisfied by Stirling numbers of the second type.

3. $M_{n,q}$ in Polynomial-Weighted Spaces

Lemma 3.1. *Let $p \in \mathbb{N} \cup \{0\}$ and $q \in (1,\infty)$ be fixed. Then there exists a positive constant $K_1(q,p)$ such that*

$$\left\|M_{n,q}(1/w_p;x)\right\|_p \le K_1(q,p), \quad n \in \mathbb{N}. \tag{3.1}$$

Moreover for every $f \in C_p$ one has

$$\left\|M_{n,q}(f)\right\|_p \le K_1(q,p)\|f\|_p, \quad n \in \mathbb{N}. \tag{3.2}$$

Thus $M_{n,q}$ is a linear positive operator from C_p into C_p for any $p \in \mathbb{N} \cup \{0\}$.

Proof. The inequality (3.1) is obvious for $p = 0$. Let $p \geq 1$. Then by (2.29) we have

$$w_p(x)M_{n,q}(1/w_p;x) = w_p(x) + w_p(x)\sum_{j=1}^{p}\mathbb{S}_q(p,j)\frac{x^j}{[n]_q^{p-j}} \leq K_1(q,p), \tag{3.3}$$

$K_1(q,p)$ is a positive constant depending on p and q. From this follows (3.1). On the other hand

$$\left\| M_{n,q}(f) \right\|_p \leq \|f\|_p \left\| M_{n,q}\left(\frac{1}{w_p}\right) \right\|_p, \tag{3.4}$$

for every $f \in C_p$. By applying (3.1), we obtain (3.2). □

Lemma 3.2. *Let $p \in \mathbb{N} \cup \{0\}$ and $q \in (1,\infty)$ be fixed. Then there exists a positive constant $K_2(q,p)$ such that*

$$\left\| M_{n,q}\left(\frac{(t-\cdot)^2}{w_p(t)};\cdot\right) \right\|_p \leq \frac{K_2(q,p)}{[n]_q}, \quad n \in \mathbb{N}. \tag{3.5}$$

Proof. The formula (2.25) imply (3.5) for $p = 0$. We have

$$M_{n,q}\left(\frac{(t-x)^2}{w_p(t)};x\right) = M_{n,q}\left((t-x)^2;x\right) + M_{n,q}\left((t-x)^2 t^p;x\right), \tag{3.6}$$

for $p, n \in \mathbb{N}$. If $p = 1$ then we get

$$\begin{aligned} M_{n,q}\left((t-x)^2(1+t);x\right) &= M_{n,q}\left((t-x)^2;x\right) + M_{n,q}\left((t-x)^2 t;x\right) \\ &= M_{n,q}\left((t-x)^3;x\right) + (1+x)M_{n,q}\left((t-x)^2;x\right), \end{aligned} \tag{3.7}$$

which by Lemma 2.5 yields (3.5) for $p = 1$.

Let $p \geq 2$. By applying (2.29), we get

$$\begin{aligned} &w_p(x)M_{n,q}\left((t-x)^2 t^p;x\right) \\ &= w_p(x)\left(M_{n,q}\left(t^{p+2};x\right) - 2xM_{n,q}\left(t^{p+1};x\right) + x^2 M_{n,q}(t^p;x)\right) \\ &= w_p(x)\left(x^{p+2} + \sum_{j=1}^{p+1}\mathbb{S}_q(p+2,j)\frac{x^j}{[n]_q^{p+2-j}} - 2x^{p+2} - 2\sum_{j=1}^{p}\mathbb{S}_q(p+1,j)\frac{x^{j+1}}{[n]_q^{p+1-j}}\right. \\ &\qquad \left. + x^{p+2} + \sum_{j=1}^{p-1}\mathbb{S}_q(p,j)\frac{x^{j+2}}{[n]_q^{p-j}}\right) \end{aligned}$$

$$
\begin{aligned}
&= w_p(x)\Bigg(\sum_{j=2}^{p}(\mathbb{S}_q(p+2,j)-2\mathbb{S}_q(p+1,j)+\mathbb{S}_q(p,j))\frac{x^{j+1}}{[n]_q^{p+1-j}}\\
&\quad+\mathbb{S}_q(p+2,1)\frac{x}{[n]_q^{p+1}}+(\mathbb{S}_q(p+2,2)-2\mathbb{S}_q(p+2,1))\frac{x^2}{[n]_q^{p}}\Bigg)\\
&= w_p(x)\frac{x}{[n]_q}\mathcal{P}_p(q;x),
\end{aligned}
\tag{3.8}
$$

where $\mathcal{P}_p(q;x)$ is a polynomial of degree p. Therefore one has

$$
w_p(x)M_{n,q}\left((t-x)^2t^p;x\right)\le K_2(q,p)\frac{x}{[n]_q}. \tag{3.9}
$$

□

Our first main result in this section is a local approximation property of $M_{n,q}$ stated below.

Theorem 3.3. *There exists an absolute constant $C>0$ such that*

$$
w_p(x)\left|M_{n,q}(g;x)-g(x)\right|\le K_3(q,p)\|g''\|\frac{x}{[n]_q}, \tag{3.10}
$$

where $g\in C_p^2$, $q>1$ and $x\in[0,\infty)$.

Proof. Using the Taylor formula

$$
g(t)=g(x)+g'(x)(t-x)+\int_x^t\int_x^s g''(u)du\,ds,\quad g\in C_p^2, \tag{3.11}
$$

we obtain that

$$
\begin{aligned}
w_p(x)\left|M_{n,q}(g;x)-g(x)\right| &= w_p(x)\left|M_{n,q}\left(\int_x^t\int_x^s g''(u)du\,ds;x\right)\right|\\
&\le w_p(x)M_{n,q}\left(\left|\int_x^t\int_x^s g''(u)du\,ds\right|;x\right)\\
&\le w_p(x)M_{n,q}\left(\|g''\|_p\left|\int_x^t\int_x^s(1+u^m)du\,ds\right|;x\right)\\
&\le w_p(x)\frac{1}{2}\|g''\|_pM_{n,q}\left((t-x)^2(1/w_p(x)+1/w_p(t));x\right)
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{1}{2}\|g''\|_p\Big(M_{n,q}\big((t-x)^2;x\big)+w_p(x)M_{n,q}\big((t-x)^2w_p(t);x\big)\Big)\\
&\leq K_3(q,x)\|g''\|_p\frac{x}{[n]_q}.
\end{aligned} \tag{3.12}$$

□

Now we consider the modified Steklov means

$$f_h(x) := \frac{4}{h^2}\iint_0^{h/2}\left[2f(x+s+t)-f(x+2(s+t))\right]ds\,dt. \tag{3.13}$$

$f_h(x)$ has the following properties:

$$f(x)-f_h(x)=\frac{4}{h^2}\iint_0^{h/2}\Delta_{s+t}^2 f(x)ds\,dt, \qquad f_h''(x)=h^{-2}\Big(8\Delta_{h/2}^2 f(x)-\Delta_h^2 f(x)\Big) \tag{3.14}$$

and therefore

$$\|f-f_h\|_p\leq \omega_p^2(f;h), \qquad \|f_h''\|_p\leq \frac{1}{9h^2}\omega_p^2(f;h). \tag{3.15}$$

We have the following direct approximation theorem.

Theorem 3.4. *For every $p\in\mathbb{N}\cup\{0\}$, $f\in C_p$ and $x\in[0,\infty)$, $q>1$, one has*

$$w_p(x)\left|M_{n,q}(f;x)-f(x)\right|\leq M_p\omega_p^2\left(f;\sqrt{\frac{x}{[n]_q}}\right)=M_p\omega_p^2\left(f;\sqrt{\frac{(q-1)x}{(q^n-1)}}\right). \tag{3.16}$$

Particularly, if $Lip_p^2\alpha$ for some $\alpha\in(0,2]$, then

$$w_p(x)\left|M_{n,q}(f;x)-f(x)\right|\leq M_p\left(\frac{x}{[n]_q}\right)^{\alpha/2}. \tag{3.17}$$

Proof. For $f\in C_p$ and $h>0$

$$\left|M_{n,q}(f;x)-f(x)\right|\leq\left|M_{n,q}((f-f_h);x)-(f-f_h)(x)\right|+\left|M_{n,q}(f_h;x)-f_h(x)\right| \tag{3.18}$$

and therefore

$$\begin{aligned}
w_p(x)\left|M_{n,q}(f;x)-f(x)\right|\leq \|f-f_h\|_p&\left(w_p(x)M_{n,q}\left(\frac{1}{w_p(t)};x\right)+1\right)\\
&+K_3(q,p)\|f_h''\|_p\frac{x}{[n]_q}.
\end{aligned} \tag{3.19}$$

Since $w_p(x)M_{n,q}(1/w_p(t);x) \le K_1(q,p)$, we get that

$$w_p(x)\left|M_{n,q}(f;x) - f(x)\right| \le M(q,p)\omega_p^2(f;h)\left[1 + \frac{x}{[n]_q h^2}\right]. \tag{3.20}$$

Thus, choosing $h = \sqrt{x/[n]_q}$, the proof is completed. □

Corollary 3.5. *If $p \in \mathbb{N} \cup \{0\}$, $f \in C_p$, $q > 1$ and $x \in [0,\infty)$, then*

$$\lim_{n\to\infty} M_{n,q}(f;x) = f(x). \tag{3.21}$$

This converegnce is uniform on every $[a,b]$, $0 \le a < b$.

Remark 3.6. Theorem 3.4 shows the rate of approximation by the q-Szász-Mirakjan operators ($q > 1$) is of order q^{-n} versus $1/n$ for the classical Szász-Mirakjan operators.

4. Convergence of q-Szász-Mirakjan Operators

An interesting problem is to determine the class of all continuous functions f such that $M_{n,q}(f)$ converges to f uniformly on the whole interval $[0,\infty)$ as $n \to \infty$. This problem was investigated by Totik [27, Theorem 1] and de la Cal and Cárcamo [28, Theorem 1]. The following result is a q-analogue of Theorem 1 [28].

Theorem 4.1. *Assume that $f : [0,\infty) \to R$ is bounded or uniformly continuous. Let*

$$f^*(z) = f\left(z^2\right), \quad z \in [0,\infty). \tag{4.1}$$

One has, for all $t > 0$ and $x \ge 0$,

$$\left|M_{n,q}(f;x) - f(x)\right| \le 2\omega\left(f^*;\sqrt{\frac{1}{[n]_q}}\right). \tag{4.2}$$

Therefore, $M_{n,q}(f;x)$ converges to f uniformly on $[0,\infty)$ as $n \to \infty$, whenever f^* is uniformly continuous.

Proof. By the definition of f^* we have

$$M_{n,q}(f;x) = M_{n,q}(f^*(\sqrt{\cdot});x). \tag{4.3}$$

Thus we can write

$$
\begin{aligned}
\left|M_{n,q}(f;x)-f(x)\right| &= \left|M_{n,q}\left(f^{*}(\sqrt{\cdot});x\right)-f^{*}(\sqrt{x})\right| \\
&= \left|\sum_{k=0}^{\infty}\left(f^{*}\left(\sqrt{\frac{[k]_q}{[n]_q}}\right)-f^{*}(\sqrt{x})\right)s_{n,k}(q;x)\right| \\
&\leq \sum_{k=0}^{\infty}\left|\left(f^{*}\left(\sqrt{\frac{[k]_q}{[n]_q}}\right)-f^{*}(\sqrt{x})\right)\right|s_{n,k}(q;x) \\
&\leq \sum_{k=0}^{\infty}\omega\left(f^{*};\left|\sqrt{\frac{[k]_q}{[n]_q}}-\sqrt{x}\right|\right)s_{n,k}(q;x) \\
&\leq \sum_{k=0}^{\infty}\omega\left(f^{*};\frac{\left|\sqrt{[k]_q/[n]_q}-\sqrt{x}\right|}{M_{n,q}\left(\left|\sqrt{\cdot}-\sqrt{x}\right|;x\right)}M_{n,q}\left(\left|\sqrt{\cdot}-\sqrt{x}\right|;x\right)\right)s_{n,k}(q;x).
\end{aligned}
\tag{4.4}
$$

Finally, from the inequality

$$
\omega(f^{*};\alpha\delta) \leq (1+\alpha)\omega(f^{*};\delta), \quad \alpha,\delta \geq 0, \tag{4.5}
$$

we obtain

$$
\begin{aligned}
\left|M_{n,q}(f;x)-f(x)\right| &\leq \omega\left(f^{*};M_{n,q}\left(\left|\sqrt{\cdot}-\sqrt{x}\right|;x\right)\right)\sum_{k=0}^{\infty}\left(1+\frac{\left|\sqrt{[k]_q/[n]_q}-\sqrt{x}\right|}{M_{n,q}\left(\left|\sqrt{\cdot}-\sqrt{x}\right|;x\right)}\right)s_{n,k}(q;x) \\
&= 2\omega\left(f^{*};M_{n,q}\left(\left|\sqrt{\cdot}-\sqrt{x}\right|;x\right)\right).
\end{aligned}
\tag{4.6}
$$

In order to complete the proof we need to show that we have for all $t>0$ and $x>0$,

$$
M_{n,q}\left(\left|\sqrt{\cdot}-\sqrt{x}\right|;x\right) \leq \sqrt{\frac{1}{[n]_q}}. \tag{4.7}
$$

Indeed we obtain from the Cauchy-Schwarz inequality:

$$
\begin{aligned}
M_{n,q}\left(\left|\sqrt{\cdot}-\sqrt{x}\right|;x\right) &= \sum_{k=0}^{\infty}\left|\sqrt{\frac{[k]_q}{[n]_q}}-\sqrt{x}\right|s_{n,k}(q;x) \\
&= \sum_{k=0}^{\infty}\frac{\left|[k]_q/[n]_q-x\right|}{\sqrt{[k]_q/[n]_q}+\sqrt{x}}s_{n,k}(q;x) \leq \frac{1}{\sqrt{x}}\sum_{k=0}^{\infty}\left|\frac{[k]_q}{[n]_q}-x\right|s_{n,k}(q;x) \\
&\leq \frac{1}{\sqrt{x}}\sqrt{\sum_{k=0}^{\infty}\left|\frac{[k]_q}{[n]_q}-x\right|^2 s_{n,k}(q;x)} = \frac{1}{\sqrt{x}}\sqrt{M_{n,q}\left((\cdot-x)^2;x\right)} \\
&= \frac{1}{\sqrt{x}}\sqrt{\frac{1}{[n]_q}x} = \sqrt{\frac{1}{[n]_q}}
\end{aligned}
\tag{4.8}
$$

showing (4.2), and completing the proof. □

Next we prove Voronovskaja type result for q-Szász-Mirakjan operators.

Theorem 4.2. *Assume that $q \in (1,\infty)$. For any $f \in C_p^2$ the following equality holds*

$$
\lim_{n\to\infty}[n]_q\left(M_{n,q}(f;x)-f(x)\right)=\frac{1}{2}f''(x)x,
\tag{4.9}
$$

for every $x \in [0,\infty)$.

Proof. Let $x \in [0,\infty)$ be fixed. By the Taylor formula we may write

$$
f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+r(t;x)(t-x)^2,
\tag{4.10}
$$

where $r(t;x)$ is the Peano form of the remainder, $r(\cdot;x) \in C_p$, and $\lim_{t\to x}r(t;x)=0$. Applying $M_{n,q}$ to (4.10) we obtain

$$
\begin{aligned}
[n]\left(M_{n,q}(f;x)-f(x)\right) &= f'(x)[n]_q M_{n,q}(t-x;x) \\
&\quad + \frac{1}{2}f''(x)[n]_q M_{n,q}\left((t-x)^2;x\right)+[n]_q M_{n,q}\left(r(t;x)(t-x)^2;x\right).
\end{aligned}
\tag{4.11}
$$

By the Cauchy-Schwartz inequality, we have

$$
M_{n,q}\left(r(t;x)(t-x)^2;x\right) \leq \sqrt{M_{n,q}(r^2(t;x);x)}\sqrt{M_{n,q}\left((t-x)^4;x\right)}.
\tag{4.12}
$$

Observe that $r^2(x;x) = 0$. Then it follows from Corollary 3.5 that

$$\lim_{n\to\infty} M_{n,q}\left(r^2(t;x);x\right) = r^2(x;x) = 0. \tag{4.13}$$

Now from (4.12), (4.13), and Lemma 2.5 we get immediately

$$\lim_{n\to\infty} [n]_q M_{n,q}\left(r(t;x)(t-x)^2;x\right) = 0. \tag{4.14}$$

The proof is completed. □

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