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Research Article

σ-Approximately Contractible Banach Algebras

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We investigate σ-approximate contractibility and σ-approximate amenability of Banach algebras, which are extensions of usual notions of contractibility and amenability, respectively, where σ is a dense range or an idempotent bounded endomorphism of the corresponding Banach algebra.

1. Introduction

For a Banach algebra $\mathcal{A}$, an $\mathcal{A}$-bimodule will always refer to a Banach $\mathcal{A}$-bimodule $\mathcal{X}$, that is, a Banach space which is algebraically an $\mathcal{A}$-bimodule, and for which there is a constant $c \geq 0$ such that for $a \in \mathcal{A}, x \in \mathcal{X}$, we have

$$\|a \cdot x\| \leq c\|a\|\|x\|, \qquad \|x \cdot a\| \leq c\|a\|\|x\|. \tag{1.1}$$

A derivation $D : \mathcal{A} \to \mathcal{X}$ is a linear map, always taken to be continuous, satisfying

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathcal{A}). \tag{1.2}$$

A Banach algebra $\mathcal{A}$ is amenable if for any $\mathcal{A}$-bimodule $\mathcal{X}$, any derivation $D : \mathcal{A} \to \mathcal{X}^*$ is inner, that is, there exists $x^* \in \mathcal{X}^*$, with

$$D(a) = a \cdot x^* - x^* \cdot a = \delta_{x^*}(a) \quad (a \in \mathcal{A}). \tag{1.3}$$

Let $\mathcal{A}$ be a Banach algebra and σ a bounded endomorphism of $\mathcal{A}$, that is, a bounded Banach algebra homomorphism from $\mathcal{A}$ into $\mathcal{A}$. A σ-derivation from $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is a bounded linear map $D : \mathcal{A} \to \mathcal{X}$ satisfying

$$D(ab) = \sigma(a) \cdot D(b) + D(a) \cdot \sigma(b) \quad (a, b \in \mathcal{A}). \tag{1.4}$$

For each $x \in \mathcal{X}$, the mapping

$$\delta_x^\sigma : \mathcal{A} \longrightarrow \mathcal{X} \tag{1.5}$$

defined by $\delta_x^\sigma(a) = \sigma(a) \cdot x - x \cdot \sigma(a)$, for all $a \in \mathcal{A}$, is a σ-derivation called an inner σ-derivation.

Remark 1.1. Throughout this paper, we will assume that $\mathcal{A}$ is a Banach algebra, and σ is a bounded endomorphism of $\mathcal{A}$ unless otherwise specified. Also, we write (σ-a.i) for σ-approximately inner, (σ-a.a) for σ-approximately amenable, and (σ-a.c) for σ-approximately contractible.

The basic definition for the present paper is as follows.

Definition 1.2. A σ-derivation $D : \mathcal{A} \to \mathcal{X}$ is σ-a.i, if there exists a net $(x_\alpha) \subseteq \mathcal{X}$ such that for every $a \in \mathcal{A}$, $D(a) = \lim_\alpha \sigma(a) \cdot x_\alpha - x_\alpha \cdot \sigma(a)$, the limit being in norm and we write $D = \lim \delta_{x_\alpha}^\sigma$. Note that we do not suppose (x_α) to be bounded.

Definition 1.3. A Banach algebra $\mathcal{A}$ is called σ-a.c if for any $\mathcal{A}$-bimodule $\mathcal{X}$, every σ-derivation $D : \mathcal{A} \to \mathcal{X}$ is σ-a.i.

Definition 1.4. A Banach algebra $\mathcal{A}$ is called σ-a.a if for any $\mathcal{A}$-bimodule $\mathcal{X}$, every σ-derivation $D : \mathcal{A} \to \mathcal{X}^*$ is σ-a.i.

Definition 1.5. Let $\mathcal{A}$ be a Banach algebra, and let $\mathcal{X}$ and $\mathcal{Y}$ be Banach $\mathcal{A}$-bimodules. The linear map $T : \mathcal{X} \to \mathcal{Y}$ is called a σ-$\mathcal{A}$-bimodule homomorphism if

$$T(a \cdot x) = \sigma(a) \cdot T(x), \qquad T(x \cdot a) = T(x) \cdot \sigma(a) \quad (a \in \mathcal{A}, x \in \mathcal{X}). \tag{1.6}$$

2. Basic Properties

Proposition 2.1. *Let $\mathcal{A}$ be a σ-a.c Banach algebra. Then $\sigma(\mathcal{A})$ has a left and right approximate identity.*

Proof. Consider $\mathcal{X} = \mathcal{A}$ as a Banach $\mathcal{A}$-bimodule with the trivial right action, that is,

$$a \cdot x = ax, \qquad x \cdot a = 0 \quad (a \in \mathcal{A}, x \in \mathcal{X}). \tag{2.1}$$

Then $D : \mathcal{A} \longrightarrow \mathcal{X}$ defined by $D(a) = \sigma(a)$ is a σ-derivation, and so there is a net $\{u_\alpha\} \subseteq \mathcal{X}(= \mathcal{A})$ such that $D = \lim_\alpha \delta^\sigma_{u_\alpha}$. Hence for each $a \in A$,

$$\sigma(a) = D(a) = \lim_\alpha \delta^\sigma_{u_\alpha}(a) = \lim_\alpha \sigma(a) \cdot u_\alpha - u_\alpha \cdot \sigma(a) = \lim_\alpha \sigma(a) u_\alpha, \tag{2.2}$$

which shows that $\{u_\alpha\}$ is a right approximate identity for $\sigma(\mathcal{A})$. Similarly, one can find a left approximate identity for $\sigma(\mathcal{A})$. □

Corollary 2.2. *Let $\mathcal{A}$ be a σ-a.c Banach algebra and σ a continuous epimorphism of $\mathcal{A}$. Then $\mathcal{A}$ has a left and right approximate identity.*

Proposition 2.3. *Let φ be a bounded endomorphism of Banach algebra $\mathcal{A}$. If $\mathcal{A}$ is σ-a.c, then $\mathcal{A}$ is $(\varphi o \sigma)$-a.c.*

Proof. Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and let $D : \mathcal{A} \longrightarrow \mathcal{X}$ be a $(\varphi o \sigma)$-derivation. Then $(\mathcal{X}, *)$ is an $\mathcal{A}$-bimodule with the following module actions:

$$a * x = \varphi(a) \cdot x, \qquad x * a = x \cdot \varphi(a) \quad (a \in \mathcal{A}, x \in \mathcal{X}). \tag{2.3}$$

For each $a, b \in \mathcal{A}$, we have

$$D(ab) = (\varphi o \sigma(a)) \cdot D(b) + D(a) \cdot (\varphi o \sigma(b)) = \sigma(a) * D(b) + D(a) * \sigma(b). \tag{2.4}$$

Thus $D : \mathcal{A} \longrightarrow (\mathcal{X}, *)$ is a continuous σ-derivation. Since $\mathcal{A}$ is σ-a.c, there exists a net $\{x_\alpha\} \subseteq \mathcal{X}$ such that $D = \lim \delta^\sigma_{x_\alpha}$. In fact,

$$\begin{aligned} D(a) &= \lim_\alpha (\sigma(a) * x_\alpha - x_\alpha * \sigma(a)) \\ &= \lim_\alpha (\varphi o \sigma(a) \cdot x_\alpha - x_\alpha \cdot \varphi o \sigma(a)) \\ &= \lim_\alpha \delta^{\varphi o \sigma}_{x_\alpha}(a) \quad (a \in \mathcal{A}). \end{aligned} \tag{2.5}$$

Therefore, D is a $(\varphi o \sigma)$-a.i and so $\mathcal{A}$ is $(\varphi o \sigma)$-a.c. □

Corollary 2.4. *Let $\mathcal{A}$ be an a.c Banach algebra. Then $\mathcal{A}$ is σ-a.c for each bounded endomorphism σ of $\mathcal{A}$.*

Proposition 2.5. *Let $\mathcal{A}$ be a σ-a.c Banach algebra, where σ is a bounded epimorphism of $\mathcal{A}$. Then $\mathcal{A}$ is a.c.*

Proof. Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and let $d : \mathcal{A} \longrightarrow \mathcal{X}$ be a continuous derivation. It is easy to see that $do\sigma$ is a σ-derivation. Since $\mathcal{A}$ is σ-a.c, there exists a net $\{x_\alpha\} \subseteq X$ such that

$do\sigma(a) = \lim_\alpha \sigma(a)x_\alpha - x_\alpha\sigma(a)$. Now for $b \in \mathcal{A}$ there exists $a \in \mathcal{A}$ such that $b = \sigma(a)$, and, therefore,

$$\begin{aligned} d(b) = d(\sigma(a)) &= \lim_\alpha x_\alpha \sigma(a) - \sigma(a)x_\alpha \\ &= \lim_\alpha x_\alpha b - bx_\alpha, \end{aligned} \tag{2.6}$$

which shows that d is approximately inner and so $\mathcal{A}$ is a.c. □

Corollary 2.6. *Let φ be a bounded endomorphism of Banach algebra $\mathcal{A}$. If $\mathcal{A}$ is σ-a.a then it is $(\varphi o \sigma)$-a.a too.*

Corollary 2.7. *Let $\mathcal{A}$ be an a.a Banach algebra. For each bounded endomorphism σ, $\mathcal{A}$ is σ-a.a.*

Corollary 2.8. *Let $\mathcal{A}$ be a σ-a.a Banach algebra, where σ is a bounded epimorphism of $\mathcal{A}$. Then $\mathcal{A}$ is a.a.*

Proposition 2.9. *Suppose that $\mathcal{B}$ is a Banach algebra and $\varphi : \mathcal{A} \to \mathcal{B}$ is a continuous epimorphism. If $\mathcal{A}$ is a.c, then $\mathcal{B}$ is σ-a.c for each bounded endomorphism σ of $\mathcal{B}$.*

Proof. Let $\sigma : \mathcal{B} \to \mathcal{B}$ be a bounded endomorphism of $\mathcal{B}$ and $\mathcal{X}$ a Banach $\mathcal{B}$-bimodule, then $(\mathcal{X}, *)$ is an $\mathcal{A}$-bimodule with the following module actions:

$$a * x = \sigma\big(\varphi(a)\big) \cdot x, \qquad x * a = x \cdot \sigma\big(\varphi(a)\big) \quad (a \in \mathcal{A}, x \in \mathcal{X}). \tag{2.7}$$

Now let $D : \mathcal{B} \to \mathcal{X}$ be a continuous σ-derivation. It is easy to check that $Do\varphi : \mathcal{A} \to (\mathcal{X}, *)$ is a derivation. Since $\mathcal{A}$ is approximately contractible, there exists a net $\{x_\alpha\} \subseteq \mathcal{X}$ such that $Do\varphi(a) = \lim_\alpha \delta_{x_\alpha}(a)$. We have

$$\begin{aligned} D\big(\varphi(a)\big) = Do\varphi(a) = \lim_\alpha \delta_{x_\alpha}(a) &= \lim_\alpha (a * x_\alpha - x_\alpha * a) \\ &= \lim_\alpha \sigma\big(\varphi(a)\big)x_\alpha - x_\alpha\sigma\big(\varphi(a)\big) \quad (a \in \mathcal{A}). \end{aligned} \tag{2.8}$$

Since φ is an epimorphism, so for each $b \in \mathcal{B}$ there exists $a \in A$ such that $b = \varphi(a)$, and we have

$$D(b) = \lim_\alpha \sigma(b)x_\alpha - x_\alpha\sigma(b), \tag{2.9}$$

which shows that D is σ-a.i and so B is σ-a.c. □

Proposition 2.10. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras, and let σ and τ be bounded endomorphism of $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a bounded epimorphism such that $\varphi o \sigma = \tau o \varphi$. If $\mathcal{A}$ is σ-a.c, then $\mathcal{B}$ is τ-a.c.*

Proof. Let $\mathcal{X}$ be a Banach $\mathcal{B}$-bimodule and $D : \mathcal{B} \to \mathcal{X}$ a continuous τ-derivation. Then $(\mathcal{X}, *)$ is an $\mathcal{A}$-bimodule with the following actions:

$$a * x = \varphi(a) \cdot x, \qquad x * a = x \cdot \varphi(a) \quad (a \in \mathcal{A}, x \in \mathcal{X}). \tag{2.10}$$

It is easy to check that $Do\varphi : \mathcal{A} \to (\mathcal{X}, *)$ is a σ-derivation. Since $\mathcal{A}$ is σ-a.c, there exists a net $\{x_\alpha\} \subseteq \mathcal{X}$ such that $Do\varphi(a) = \lim_\alpha \delta^{\sigma}_{x_\alpha}(a)$, so we have

$$\begin{aligned} D(\varphi(a)) &= \lim_\alpha \sigma(a) * x_\alpha - x_\alpha * \sigma(a) \\ &= \lim_\alpha \varphi(\sigma(a)) \cdot x_\alpha - x_\alpha \cdot \varphi(\sigma(a)) \\ &= \lim_\alpha \tau(\varphi(a)) \cdot x_\alpha - x_\alpha \cdot \tau(\varphi(a)) \quad (a \in \mathcal{A}). \end{aligned} \tag{2.11}$$

Since φ is epimorphism, so $D(b) = \lim_\alpha \tau(b) x_\alpha - x_\alpha \tau(b)$ for all $b \in \mathcal{B}$, and hence $\mathcal{B}$ is τ-a.c. □

3. σ-Approximate Contractibility for Unital Banach Algebras

In this section we state some properties of σ-approximate contractibility when $\mathcal{A}$ has an identity. First we express the following proposition that one can see its proof in [1, Proposition 3.3], and bring some corollaries when $\sigma(\mathcal{A})$ is dense in $\mathcal{A}$.

Proposition 3.1. *Let $\mathcal{A}$ be a unital Banach algebra with unit e, $\sigma(\mathcal{A})$ dense in $\mathcal{A}$, $\mathcal{X}$ a Banach $\mathcal{A}$-bimodule, and $D : \mathcal{A} \to \mathcal{X}$ a σ-derivation. Then, there is a σ-derivation $D_1 : \mathcal{A} \to e \cdot \mathcal{X} \cdot e$ and $\eta \in \mathcal{X}$, such that $D = D_1 + \delta_\eta$.*

The following definition extends the definition of the unital Banach $\mathcal{A}$-module in the classical sense.

Definition 3.2. Let $\mathcal{A}$ be a unital Banach algebra with identity e. Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is called σ-unital if $\mathcal{X} = \sigma(e) \cdot \mathcal{X} \cdot \sigma(e)$.

Corollary 3.3. *Let $\mathcal{A}$ be a unital Banach algebra and $\sigma(\mathcal{A})$ dense in $\mathcal{A}$. Then, $\mathcal{A}$ is σ-a.c (resp.,σ-a.a) if and only if for all σ-unital Banach $\mathcal{A}$-bimodule $\mathcal{X}$, every σ-derivation $D : \mathcal{A} \to \mathcal{X}$ (resp., $D : \mathcal{A} \to \mathcal{X}^*$) is σ-a.i.*

Proof. Since $\sigma(e)$ is a unit for $\sigma(\mathcal{A})$, and $\sigma(A)$ is dense in $\mathcal{A}$, we see that $\sigma(e) = e$, so that $e \cdot \mathcal{X} \cdot e$ is a σ-unital Banach $\mathcal{A}$-bimodule. Now by Proposition 3.1, the proof is complete. □

Corollary 3.4. *Suppose that $\mathcal{A}$ is a unital Banach algebra and $\sigma(A)$ is dense in $\mathcal{A}$. Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and $D : \mathcal{A} \to \mathcal{X}^*$ a σ-derivation. If $\mathcal{A}$ is σ-a.a, then there exists a net $(\eta_\alpha) \subseteq e \cdot \mathcal{X}^* \cdot e$, and $\eta \in \mathcal{X}^*$, such that $D = \lim_\alpha \delta^{\sigma}_{\eta_\alpha} + \delta_\eta$.*

Proof. By Proposition 3.1, $D = D_1 + \delta_\eta$ such that $\eta \in \mathcal{X}^*$ and $D_1 : \mathcal{A} \to e \cdot \mathcal{X}^* \cdot e$ is a σ-derivation. Since $e \cdot \mathcal{X}^* \cdot e \cong (e \cdot \mathcal{X} \cdot e)^*$ and $\mathcal{A}$ is σ-a.a, $D_1 : \mathcal{A} \to (e \cdot \mathcal{X} \cdot e)^*$ is σ-a.i. Hence $D_1 = \lim_\alpha \delta^{\sigma}_{\eta_\alpha}$ for some net $(\eta_\alpha) \subseteq e \cdot \mathcal{X}^* \cdot e$. □

In the following proposition we consider σ-approximate contractibility when σ is an idempotent endomorphism of $\mathcal{A}$. We can see the proof of the following proposition in [1, Proposition 4.1].

Proposition 3.5. *Assume that $\mathcal{A}$ has an element e which is a unit for $\sigma(\mathcal{A})$ and $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule. If σ is a bounded idempotent endomorphism of $\mathcal{A}$, then for each σ-derivation $D : \mathcal{A} \to \mathcal{X}$ there exists a σ-derivation $D_1 : \mathcal{A} \to \sigma(e) \cdot \mathcal{X} \cdot \sigma(e)$ and $\eta \in \mathcal{X}$, such that $D = D_1 + \delta_\eta$.*

Corollary 3.6. *Assume that $\mathcal{A}$ has an element e which is a unit for $\sigma(\mathcal{A})$ and σ is a bounded idempotent endomorphism of $\mathcal{A}$, then $\mathcal{A}$ is σ-a.c (resp., σ-a.a) if and only if for all σ-unital Banach $\mathcal{A}$-bimodule, $\mathcal{X}$, every σ-derivation $D : \mathcal{A} \to \mathcal{X}$ (resp., $D : \mathcal{A} \to \mathcal{X}^*$) is σ-a.i.*

Lemma 3.7. *Assume that $\mathcal{A}$ is a unital Banach algebra with the identity e, and $(\mathcal{X}, *)$ is a σ-unital Banach $\mathcal{A}$-bimodule with the following module actions:*

$$a * x = \sigma(a)x, \quad x * a = x\sigma(a) \quad (a \in \mathcal{A}, x \in \mathcal{X}). \tag{3.1}$$

If $D : \mathcal{A} \to \mathcal{X}^*$ is a σ-derivation, then $D(e) = 0$.

Proof. We have $D(e) = D(ee) = \sigma(e)D(e) + D(e)\sigma(e)$ and

$$\begin{aligned} \langle e * x, D(e)\sigma(e)\rangle &= \langle x, D(e)\sigma(e) * e\rangle = \langle x, D(e)\sigma(e)\sigma(e)\rangle \\ &= \langle x, D(e)\sigma(e)\rangle = \langle e * x, D(e)\rangle \quad (x \in \mathcal{X}). \end{aligned} \tag{3.2}$$

Hence $D(e)\sigma(e) = D(e)$ and so $\sigma(e)D(e) = 0$. Hence $D(e) = 0$. □

Proposition 3.8. *Let σ be a bounded idempotent endomorphism of Banach algebra $\mathcal{A}$. If $\mathcal{A}$ is σ-a.a, then $\mathcal{A}^\#$ is $\widehat{\sigma}$-a.a, where $\widehat{\sigma}$ is the endomorphism of $\mathcal{A}^\#$ induced by σ, that is, $\widehat{\sigma}(a + \alpha) = \sigma(a) + \alpha$.*

Proof. Let $\mathcal{X}$ be a Banach $\mathcal{A}^\#$-bimodule and $D : \mathcal{A}^\# \to \mathcal{X}^*$ a continuous $\widehat{\sigma}$-derivation. By Proposition 3.5, there exits $\eta \in \mathcal{X}^*$ and $D_1 : \mathcal{A}^\# \to \widehat{\sigma}(e) \cdot \mathcal{X}^* \cdot \widehat{\sigma}(e)$ such that $D = D_1 + \delta_\eta$. Set $d : D_1|_{\mathcal{A}} : \mathcal{A} \to \widehat{\sigma}(e) \cdot \mathcal{X}^* \cdot \widehat{\sigma}(e)$. It is easy to check that d is a σ-derivation. Since $\mathcal{A}$ is σ-a.a, there exists a net $(x_\gamma^*) \subseteq \mathcal{X}^*$ such that $d = \lim_\gamma \delta^\sigma_{x_\gamma^*}$. Hence $D_1(a) = \lim_\gamma \sigma(a)x_\gamma^* - x_\gamma^*\sigma(a), (a \in \mathcal{A})$. Since $\widehat{\sigma}(e) \cdot \mathcal{X}^* \cdot \widehat{\sigma}(e)$ is $\widehat{\sigma}$-unital, by Lemma 3.7, $D_1(e) = 0$ and for each $a + \alpha \in \mathcal{A}^\#$ we have

$$\begin{aligned} D_1(a + \alpha) &= D_1(a) + \alpha D_1(e) = D_1(a) = \lim_\gamma \sigma(a)x_\gamma^* - x_\gamma^*\sigma(a) \\ &= \lim_\gamma (\widehat{\sigma}(a + \alpha) - \alpha)x_\gamma^* - x_\gamma^*(\widehat{\sigma}(a + \alpha) - \alpha) \\ &= \lim_\gamma \varphi(a + \alpha)x_\gamma^* - x_\gamma^*\varphi(a + \alpha). \end{aligned} \tag{3.3}$$

This shows that D_1 is $\widehat{\sigma}$-a.i, and so $\mathcal{A}^\#$ is $\widehat{\sigma}$-a.a. □

Proposition 3.9. *Let σ be a bounded endomorphism of Banach algebra $\mathcal{A}$. If $A^\#$ is $\widehat{\sigma}$-a.a, then $\mathcal{A}$ is σ-a.a.*

Proof. Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and $D : \mathcal{A} \to \mathcal{X}^*$ a continuous σ-derivation. $\mathcal{X}$ is a Banach $A^{\#}$-bimodule with the following module actions:

$$(a+\alpha)\cdot x = a\cdot x + \alpha x, \qquad x\cdot(a+\alpha) = x\cdot a + \alpha x, \tag{3.4}$$

for all $a \in \mathcal{A}, x \in \mathcal{X}, \alpha \in \mathbb{C}$. Define $D^{\#} : \mathcal{A}^{\#} \to \mathcal{X}^*$ with $D^{\#}(a+\alpha) = D(a)$. Clearly, $D^{\#}$ is a continuous $\widehat{\sigma}$-derivation. Hence, there is a net $(x^*_\gamma) \subseteq \mathcal{X}^*$ such that $D^{\#} = \lim_\gamma^{\widehat{\sigma}} \delta_{x^*_\gamma}$. Hence, for each $a \in \mathcal{A}$ we have

$$D(a) = D^{\#}(a+\alpha) = \lim_\gamma \widehat{\sigma}(a+\alpha)x^*_\gamma - x^*_\gamma\widehat{\sigma}(a+\alpha) = \lim_\gamma \sigma(a)x^*_\gamma - x^*_\gamma\sigma(a) \tag{3.5}$$

which shows that D is σ-a.i and so $\mathcal{A}$ is σ-a.a. □

4. σ-Approximate Amenability When $\mathcal{A}$ Has a Bounded Approximate Identity

Lemma 4.1. *Let $\mathcal{A}$ be a Banach algebra with bounded approximate identity and $\mathcal{X}$ a Banach $\mathcal{A}$-bimodule with trivial left or right action, then every σ-derivation $D : \mathcal{A} \to \mathcal{X}^*$ is σ-inner.*

Proof. Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule with trivial left action. Hence, $\mathcal{X}^*$ is a Banach $\mathcal{A}$-bimodule with trivial right action, that is,

$$x^*\cdot a = 0, \qquad a\cdot x^* = ax^* \quad (x^* \in \mathcal{X}^*, a \in \mathcal{A}). \tag{4.1}$$

Let $D : \mathcal{A} \to \mathcal{X}^*$ be a continuous σ-derivation and (e_α) a bounded approximate identity of $\mathcal{A}$. By Banach Alaoglu's Theorem, $(D(e_\alpha))$ has a subnet $(D(e_\beta))$ such that $D(e_\beta) \xrightarrow{w^*} x^*_0$, for some $x^*_0 \in \mathcal{X}^*$. Since $a\cdot e_\beta \xrightarrow{\|\cdot\|} a$ and D is continuous, $D(a\cdot e_\beta) \xrightarrow{\|\cdot\|} D(a)$. Hence, $D(a\cdot e_\beta) \xrightarrow{w^*} D(a)$.

On the other hand, $D(a\cdot e_\beta) = \sigma(a)D(e_\beta) \xrightarrow{w^*} \sigma(a)x^*_0$ and so $D(a) = \sigma(a)x^*_0$. Hence, $D(a) = \sigma(a)x^*_0 - x^*_0\sigma(a)$ and D is σ-inner. □

The following definitions extends the definition of the neo-unital and essential Banach $\mathcal{A}$-bimodule in the classical sense.

Definition 4.2. Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. Then $\mathcal{X}$ is called σ-neo-unital (σ-pseudo-unital), if $\mathcal{X} = \sigma(\mathcal{A})\cdot\mathcal{X}\cdot\sigma(\mathcal{A})$. Similarly, one defines σ-neo-unital left and right Banach modules.

Definition 4.3. Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. Then $\mathcal{X}$ is called σ-essential if $\mathcal{X} = \sigma(\mathcal{A})\mathcal{X}\sigma(\mathcal{A}) = \overline{\operatorname{span}}\,\sigma(\mathcal{A})\cdot\mathcal{X}\cdot\sigma(\mathcal{A})$. Similarly, one defines σ-essential left and right Banach modules.

We recall that a bounded approximate identity in Banach algebra $\mathcal{A}$ for Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is a bounded net (e_α) in $\mathcal{A}$ such that for each $x \in \mathcal{X}$, $e_\alpha x \to x$ and $xe_\alpha \to x$.

Proposition 4.4. *Assume that $\mathcal{A}$ has a left bounded approximate identity, σ is a bounded idempotent endomorphism of $\mathcal{A}$, and $\mathcal{X}$ is a left Banach $\mathcal{A}$-module. Then $\mathcal{X}$ is σ-neo-unital if and only if $\mathcal{X}$ is σ-essential.*

Proof. Let $\mathcal{X}$ be a σ-essential Banach $\mathcal{A}$-bimodule. Since σ is idempotent, $\sigma(\mathcal{A})$ is Banach subalgebra of $\mathcal{A}$. Let $(e_\alpha) \subseteq \mathcal{A}$ be left approximate identity with bound m. First suppose that $z \in \operatorname{span}\sigma(\mathcal{A}) \cdot \mathcal{X}$, so there exist $a_1, \ldots, a_n \in \mathcal{A}$, $x_1, \ldots, x_n \in \mathcal{X}$ such that $z = \sum_{i=1}^{n} \sigma(a_i)x_i$. For $1 \le i \le n$, $e_\alpha a_i \to a_i$ and, therefore, $\sigma(e_\alpha)z \to z$.

Now suppose that $z \in \sigma(\mathcal{A})\mathcal{X}$. There exists $\{z_n\} \subseteq \operatorname{span}\sigma(\mathcal{A}) \cdot \mathcal{X}$ such that $z_n \to z$. Thus,

$$\exists n_0 \in \mathbb{N} \text{ s.t.} \quad \forall n\left(n \ge n_0; \|z_n - z\| < \frac{\varepsilon}{2(\|\sigma\|m+1)}\right) \tag{4.2}$$

On the other hand, for each $n \in \mathbb{N}$ we have $\sigma(e_\alpha)z_n \xrightarrow{\alpha} z_n$ and so $\sigma(e_\alpha)z_{n_0} \xrightarrow{\alpha} z_{n_0}$. Therefore,

$$\exists \alpha_0; \quad \forall \alpha\left(\alpha \ge \alpha_0; \|\sigma(e_\alpha)z_{n_0} - z_{n_0}\| < \frac{\varepsilon}{2}\right). \tag{4.3}$$

Now we have

$$\begin{aligned}
\|\sigma(e_\alpha)z - z\| &\le \|\sigma(e_\alpha)z - \sigma(e_\alpha)z_{n_0} + \sigma(e_\alpha)z_{n_0} - z_{n_0} + z_{n_0} - z\| \\
&\le \|\sigma\|\|e_\alpha\|\|z - z_{n_0}\| + \|\sigma(e_\alpha)z_{n_0} - z_{n_0}\| + \|z_{n_0} - z\| \\
&< (\|\sigma\|\|e_\alpha\| + 1)\|z - z_{n_0}\| + \frac{\varepsilon}{2} \\
&< (\|\sigma\|m + 1)\frac{\varepsilon}{(\|\sigma\|m+1)2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned} \tag{4.4}$$

which shows that $(\sigma(\mathrm{e}_\alpha)) \subseteq \sigma(\mathcal{A})$ is a left bounded approximate identity for $\mathcal{X}$. Now by Cohen factorization Theorem, $\mathcal{X} = \sigma(\mathcal{A}) \cdot \mathcal{X}$. So $\mathcal{X}$ is σ-neo-unital. The other side is trivial. □

Corollary 4.5. *Every σ-neo-unital left Banach $\mathcal{A}$-module is essential.*

Proof. Let $\mathcal{X}$ be a σ-neo-unital left Banach $\mathcal{A}$-module. We have $\mathcal{X} = \sigma(\mathcal{A}) \cdot \mathcal{X} \subseteq \mathcal{A} \cdot \mathcal{X} \subseteq \mathcal{A}\mathcal{X} \subseteq \mathcal{X}$ so $\mathcal{X} = \mathcal{A}\mathcal{X}$. □

Proposition 4.6. *Let $\mathcal{A}$ be a Banach algebra with a left bounded approximate identity, σ be a bounded idempotent endomorphism of $\mathcal{A}$, and $\mathcal{X}$ a left Banach $\mathcal{A}$-module. Then $\sigma(\mathcal{A}) \cdot \mathcal{X}$ is closed weakly complemented submodule of $\mathcal{X}$.*

Proof. Set $\mathcal{Y} = \sigma(\mathcal{A})\mathcal{X}$, since $\mathcal{A}$ has a left bounded approximate identity, by Cohen factorization Theorem $\mathcal{A}^2 = \mathcal{A}$, and we have $\sigma(\mathcal{A})\mathcal{Y} = \sigma(\mathcal{A})\sigma(\mathcal{A})\mathcal{X} = \sigma(\mathcal{A}^2)\mathcal{X} = \sigma(\mathcal{A})\mathcal{X} = \mathcal{Y}$, which shows that $\mathcal{Y}$ is σ-essential by Proposition 4.4, $\mathcal{Y}$ is σ-neo unital that is, $\mathcal{Y} = \sigma(\mathcal{A}) \cdot \mathcal{Y}$. Hence, $\sigma(\mathcal{A})\mathcal{X} = \mathcal{Y} = \sigma(\mathcal{A}) \cdot \mathcal{Y} \subseteq \sigma(\mathcal{A}) \cdot \mathcal{X}$ and so $\sigma(\mathcal{A})\mathcal{X} = \sigma(\mathcal{A}) \cdot \mathcal{X}$. Thus $\sigma(\mathcal{A}) \cdot \mathcal{X}$ is closed submodule of $\mathcal{X}$.

Now we prove that $\sigma(\mathcal{A}) \cdot \mathcal{X}$ is weakly complemented in $\mathcal{X}$. Let (e_α) be a left approximate identity in $\mathcal{A}$ with bound m, and define a net (T_α) in $\mathcal{B}(\mathcal{X}^*)$ by setting $T_\alpha(x^*) = x^* \cdot \sigma(e_\alpha)$ $(x^* \in \mathcal{X}^*)$. We have $\|T_\alpha\| \le \|\sigma\|m$. Thus (T_α) is a bounded net in $\mathcal{B}(\mathcal{X}^*)$ since $\mathcal{B}(\mathcal{X}^*) = (\mathcal{X}^* \otimes \mathcal{X})^*$ and ball $\mathcal{B}(\mathcal{X}^*)$ is w^*-compact, so there exists $T \in \mathcal{B}(\mathcal{X}^*)$ such that we may suppose that $w^* - \lim_\alpha T_\alpha = T$ and $\|T\| \leq \|\sigma\| m$. For each $a \in \mathcal{A}$, $x \in \mathcal{X}$, and $x^* \in \mathcal{X}^*$, we have

$$\begin{aligned}\langle \sigma(a) \cdot x, T(x^*) \rangle &= \lim_\alpha \langle \sigma(a) \cdot x, x^* \cdot \sigma(e_\alpha) \rangle \\ &= \lim_\alpha \langle \sigma(e_\alpha)\sigma(a) \cdot x, x^* \rangle \\ &= \langle \sigma(a) \cdot x, x^* \rangle,\end{aligned} \tag{4.5}$$

and so $x^* - Tx^* \in (\sigma(\mathcal{A}) \cdot \mathcal{X})^\perp$. On other hand, for each $x^* \in \mathcal{X}^*$,

$$T^2 x^* = T(Tx^*) = \lim_\alpha T(x^*)\sigma(e_\alpha) = \lim_\alpha x^* \sigma(e_\alpha) = T(x^*). \tag{4.6}$$

Thus T is projection, and $I_{\mathcal{X}^*} - T : \mathcal{X}^* \to (\sigma(\mathcal{A}) \cdot \mathcal{X})^\perp$ is projection. So $\sigma(\mathcal{A}) \cdot \mathcal{X}$ is weakly complemented in $\mathcal{X}$ and, we have $\mathcal{X}^* = (\sigma(\mathcal{A}) \cdot \mathcal{X})^\perp \oplus (\sigma(\mathcal{A}) \cdot \mathcal{X})^*$. □

Corollary 4.7. *Let $\mathcal{A}$ have a bounded approximate identity, and let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and σ a bounded idempotent endomorphism of $\mathcal{A}$. Then*

(i) *$\sigma(\mathcal{A}) \cdot \mathcal{X} \cdot \sigma(\mathcal{A})$ is a closed weakly complemented submodule of $\mathcal{X}$,*

(ii) *$\mathcal{A}$ is σ-a.a if and only if for every σ-neo-unital Banach $\mathcal{A}$-bimodule $\mathcal{X}$, every σ-derivation $D : \mathcal{A} \to \mathcal{X}^*$ is σ-approximately inner.*

Proof. Set $\mathcal{Y} = \sigma(\mathcal{A}) \cdot \mathcal{X}$. By Proposition 4.6, $\mathcal{Y}$ is a closed and weakly complemented submodule of $\mathcal{X}$, and $T : \mathcal{X}^* \to \mathcal{Y}^*$ and $I - T : \mathcal{X}^* \to \mathcal{Y}^\perp$ are projection maps. Let $D : \mathcal{A} \to \mathcal{X}^*$ be a σ-derivation, so ToD and $(I - T)oD$ are σ-derivations and $D = (ToD) + (I - T)oD$. Since $A \cdot (X/Y) = \{0\}$ by Lemma 4.1, $(I - T)oD$ is σ-inner. So there exists $J_0 \in \mathcal{Y}^\perp$ such that $(I - T)oD = \delta^\sigma_{J_0}$. Thus $D = ToD + \delta^\sigma_{J_0}$ and so D is σ-a.i if and only if $ToD : \mathcal{A} \to \mathcal{Y}^*$ is σ-a.i.

Now let $\mathcal{Z} = \mathcal{Y} \cdot \sigma(\mathcal{A})$. By Proposition 4.6, $\mathcal{Z}$ is a closed weakly complemented in $\mathcal{Y}$, and $T' : \mathcal{Y}^* \to \mathcal{Z}^*$ and $I - T' : \mathcal{Y}^* \to \mathcal{Z}^\perp$ are projection maps. Assume that $D_1 : \mathcal{A} \to \mathcal{Y}^*$ is a σ-derivation, thus $T'oD$ and $(I - T')oD$ are σ-derivations, and we have $D_1 = T'oD_1 + (I - T') \cdot D_1$. Since $(\mathcal{Y}/\mathcal{Z}) \cdot \mathcal{A} = \{0\}$, by Lemma 4.1, $(I - T') \cdot D_1$ is σ-inner and so there exists $z_0 \in Z^\perp$ such that $(I - T')oD_1 = \delta^\sigma_{Z_0}$. Therefore, $D_1 = T'oD_1 + \delta^\sigma_{Z_0}$. Thus, D_1 is σ.a.i if and only if $T'oD_1$ is σ.a.i. Set $DoT = D_1$. Thus, $D = T'oD_1 + \delta^\sigma_{Z_0} + \delta^\sigma_{J_0}$. Therefore, D is σ-a.i, if and only if $T'oD_1 : \mathcal{A} \to \mathcal{Z}^* = (\sigma(\mathcal{A}) \cdot \mathcal{X} \cdot \sigma(\mathcal{A}))^*$ is σ.a.i. Recall that $\mathcal{Z}$ is σ-neo-unital. Thus, $\mathcal{A}$ is σ-a.a if and only if for every σ-neo-unital Banach $\mathcal{A}$-bimodul, $\mathcal{X}$, every σ-derivation $D : A \to X^*$ is σ-a.i. □

Corollary 4.8. *Let $\mathcal{A}$ have a bounded approximate identity, and let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and σ a bounded idempotent endomorphism of $\mathcal{A}$. Then $\mathcal{A}$ is σ-a.a if and only if for every σ-essential Banach $\mathcal{A}$-bimodule $\mathcal{X}$, every σ-derivation $D : \mathcal{A} \to \mathcal{X}^*$ is σ-approximately inner.*

Proposition 4.9. *Suppose that σ is a bounded idempotent endomorphism of $\mathcal{A}$ and define $\widehat{\sigma} : \mathcal{A}^\# \to \mathcal{A}^\#$ with $\widehat{\sigma}(a + \alpha) = \sigma(a) + \alpha$. The following statements are equivalent.*

(1) *$\mathcal{A}$ is σ-a.a.*

(2) *There is a net $(\mu_\alpha) \subseteq (A^\# \widehat{\otimes} \mathcal{A}^\#)^{**}$ such that for each $a \in \mathcal{A}^\#$, $\widehat{\sigma}(a) \cdot \mu_\alpha - \mu_\alpha \cdot \widehat{\sigma}(a) \to 0$ and $\pi^{**}(\mu_\alpha) \to \widehat{e}$.*

(3) *There is a net* $(\mu'_\alpha) \subseteq (A^{\#}\widehat{\otimes}\mathcal{A}^{\#})^{**}$ *such that for each* $a \in \mathcal{A}^{\#}$, $\widehat{\sigma}(a)\cdot\mu_\alpha - \mu_\alpha\cdot\widehat{\sigma}(a) \to 0$ *and for every* α, $\pi^{**}(\mu'_\alpha) = \widehat{e}$.

Proof. (1⇒3) Suppose that $\mathcal{A}$ is σ-a.a, by Proposition 3.8, $\mathcal{A}^{\#}$ is $\widehat{\sigma}$-a.a. Let $u = e\otimes e \in \mathcal{A}^{\#}\widehat{\otimes}\mathcal{A}^{\#}$. $\mathcal{A}^{\#}\widehat{\otimes}\mathcal{A}^{\#}$ is a Banach $\mathcal{A}^{\#}$-bimodule with the following module actions:

$$a\cdot(b\otimes c) = \widehat{\sigma}(a)(b\otimes c),\quad (b\otimes c)\cdot a = (b\otimes c)\widehat{\sigma}(a)\quad \left(a,b,c\in\mathcal{A}^{\#}\right). \tag{4.7}$$

Set $\delta_{\widehat{u}} : \mathcal{A}^{\#} \to \ker\pi^{**}$ with definition $\delta_{\widehat{u}}(a) = \widehat{\sigma}(a)\cdot\widehat{u} - \widehat{u}\cdot\widehat{\sigma}(a)$ $(a\in\mathcal{A}^{\#})$. $\delta_{\widehat{u}}$ is $\widehat{\sigma}$-derivation. Recall that $\ker\pi^{**} = (\ker\pi)^{**}$. Since $\mathcal{A}^{\#}$ is $\widehat{\sigma}$-a.a, thus there exists $(e_\alpha)\subseteq\ker\pi^{**}$ such that

$$\delta_{\widehat{u}}(a) = \lim_\alpha \widehat{\sigma}(a)e_\alpha - e_\alpha\widehat{\sigma}(a)\quad\left(a\in\mathcal{A}^{\#}\right). \tag{4.8}$$

Set $\mu'_\alpha = \widehat{u} - e_\alpha \in (\mathcal{A}^{\#}\widehat{\otimes}\mathcal{A}^{\#})^{**}$. We have

$$\widehat{\sigma}(a)\mu'_\alpha - \mu'_\alpha\widehat{\sigma}(a) = \widehat{\sigma}(a)\widehat{u} - \widehat{u}\widehat{\sigma}(a) - (\widehat{\sigma}(a)e_\alpha - e_\alpha\widehat{\sigma}(a)) \longrightarrow 0, \tag{4.9}$$

and for each α,

$$\pi^{**}(\mu'_\alpha) = \pi^{**}(\widehat{u} - e_\alpha) = \pi^{**}(\widehat{u}) - \pi^{**}(e_\alpha) = \pi(u) = e. \tag{4.10}$$

(3⇒2) is clear.

(2⇒1) By Proposition 3.9, it is sufficient to show that $A^{\#}$ is $\widehat{\sigma}$-a.a.

Let $D : A^{\#} \to \mathcal{X}^*$ be a derivation. By Corollary 4.7, we may take $\mathcal{X}$ to be σ-neo-unital. We run the standard argument, so for each $\alpha\in I$, set $f_\alpha(x) = \mu_\alpha(\psi_x)$, where for $a,b\in\mathcal{A}^{\#}$, $x\in\mathcal{X}$, we have $\psi_x(a\otimes b) = \langle x, \widehat{\sigma}(a)D(b)\rangle$. Then, $(m^\gamma_\alpha)\subset A^{\#}\widehat{\otimes}\mathcal{A}^{\#}$ converging ω^* to μ_α $(\alpha\in I)$ and noting that for $m\in\mathcal{A}^{\#}\widehat{\otimes}\mathcal{A}^{\#}, a\in\mathcal{A}^{\#}, x\in\mathcal{X}$, then

$$\psi_{\widehat{\sigma}(a)x - x\widehat{\sigma}(a)}(m) = \left(\widehat{\sigma}(a)\psi_x - \psi_x\widehat{\sigma}(a)\right)(m) - \langle x, \widehat{\sigma}(\pi(m))D(a)\rangle. \tag{4.11}$$

Since $\mathcal{X}$ is $\widehat{\sigma}$-neo-unital, so $\mathcal{X} = \mathcal{X}\widehat{\sigma}(\mathcal{A}^{\#})$. So for each $a\in A$ and $x\in\mathcal{X}$, we have

$$\begin{aligned}\left\langle\widehat{\sigma}(a)x - x\widehat{\sigma}(a), f_\alpha\right\rangle &= \left\langle\psi_{\widehat{\sigma}(a)x - x\widehat{\sigma}(a)}, \mu_\alpha\right\rangle\\ &= \lim_\gamma\left\langle m^\gamma_\alpha, \psi_{\widehat{\sigma}(a)x - x\widehat{\sigma}(a)}\right\rangle\\ &= \left\langle\widehat{\sigma}(a)\psi_x - \psi_x\widehat{\sigma}(a), \mu_\alpha\right\rangle - \lim_\gamma\left\langle x, \widehat{\sigma}\left(\pi\left(m^\gamma_\alpha\right)D(a)\right)\right\rangle\\ &= \left\langle\psi_x, \mu_\alpha\widehat{\sigma}(a) - \widehat{\sigma}(a)\mu_\alpha\right\rangle - \left\langle x, \pi^{**}(\mu_\alpha)D(a)\right\rangle.\end{aligned} \tag{4.12}$$

Thus,

$$\begin{aligned}
&\left\|\left\langle x, \widehat{\sigma}(a) f_\alpha - f_\alpha \widehat{\sigma}(a)\right\rangle - \langle x, D(a)\rangle\right\| \\
&\quad \leq \left\|\left\langle \varphi_x, \widehat{\sigma}(a)\mu_\alpha - \mu_\alpha \widehat{\sigma}(a)\right\rangle\right\| + \|x\|\left\|\pi^{**}(\mu_\alpha) - \widehat{e}\right\|\|D(a)\| \\
&\quad \leq \|D\| \cdot \|x\|\left\|\widehat{\sigma}(a)\mu_\alpha - \mu_\alpha \widehat{\sigma}(a)\right\| + \|x\|\left\|\pi^{*}(\mu_\alpha) - \widehat{e}\right\|\|D(a)\|,
\end{aligned} \tag{4.13}$$

and, therefore, $D = \lim_\alpha \delta^{\widehat{\sigma}}_{f_\alpha}$. It follows that $\mathcal{A}^{\#}$ is $\widehat{\sigma}$-a.a and so $\mathcal{A}$ is σ-a.a. □

Proposition 4.10. *Suppose that $\mathcal{A}$ is σ-a.a, and let*

$$\Sigma : 0 \longrightarrow \mathcal{X}^* \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z} \longrightarrow 0, \tag{4.14}$$

be an admissible short exact sequence of left $\mathcal{A}$-module and left σ-$\mathcal{A}$-module homomorphism. Then Σ, σ-approximately split, that is, there is a net $G_\alpha : \mathcal{Z} \to \mathcal{Y}$ of right inverse maps to g such that $\lim_\alpha(\sigma(a)G_\alpha - G_\alpha\sigma(a)) = 0$ for $a \in \mathcal{A}$, and a net $F_\alpha : \mathcal{Y} \to \mathcal{X}^*$ of left inverse maps to f such that $\lim_\alpha(\sigma(a)f_\alpha - f_\alpha\sigma(a)) = 0$ for $a \in \mathcal{A}$.

Proof. Following the proof of [2, Theorem 2.3], for a right inverse G for g, σ-approximate amenability gives a net $(\varphi_\alpha) \subseteq \mathcal{B}(\mathcal{Z}, \mathcal{X}^*)$ such that

$$\sigma(a) \cdot G - G \cdot \sigma(a) = \lim_\alpha\left(\sigma(a) \cdot fG_\alpha - fG_\alpha \cdot \sigma(a)\right) \quad (a \in \mathcal{A}). \tag{4.15}$$

Setting $G_\alpha = G - f\varphi_\alpha$ gives the required net. Applying the same argument as [2, Proposition 1.1] provides (F_α). □

We recall that if $\mathcal{A}$ is a Banach algebra with a weak left (right) approximate identity, then $\mathcal{A}$ has a left (right) approximate identity [1, Lemma 2.2].

Corollary 4.11. *Suppose that Banach algebra $\mathcal{A}$ is σ-a.a, then $\sigma(\mathcal{A})$ has left and right approximate identities.*

Corollary 4.12. *Suppose that Banach algebra $\mathcal{A}$ is σ-a.a and σ is a bounded epimorphism of $\mathcal{A}$, then $\mathcal{A}$ has left and right approximate identities.*

Lemma 4.13. *Let σ be a bounded idempotent endomorphism of Banach algebra $\mathcal{A}$ and $\mathcal{X}$ a σ-neo-unital Banach $\mathcal{A}$-module. If $(e_\alpha)_\alpha$ is a bounded approximate identity in $\mathcal{A}$, then $(\sigma(e_\alpha))_\alpha$ is a bounded approximate identity for $\mathcal{X}$.*

Proof. For every $a \in \mathcal{A}$ we have $e_\alpha\sigma(a) \to \sigma(a)$. Since σ is idempotent, $\sigma(e_\alpha)\sigma(a) \to \sigma(a)$. For each $x \in \mathcal{X}$, there exists $a \in \mathcal{A}$ and $y \in \mathcal{X}$ such that $x = \sigma(a) \cdot y$. Therefore,

$$\sigma(e_\alpha) \cdot x = \sigma(e_\alpha)\sigma(a) \cdot y \longrightarrow \sigma(a) \cdot y = x, \tag{4.16}$$

which shows that $(\sigma(e_\alpha))$ is a bounded approximate identity for $\mathcal{X}$. □

It is often convenient to extend a derivation to a large algebra. If a Banach algebra I is contained as a closed ideal in another Banach algebra $\mathcal{A}$, then the strict topology on $\mathcal{A}$ with respect to I is defined through the family of seminorms $(P_i)_{i\in I}$, where

$$P_i(a) := \|ai\| + \|ia\| \quad (a \in \mathcal{A}). \tag{4.17}$$

Note that the strict topology is Hausdorff only if $\{a \in \mathcal{A} : a \cdot I = I \cdot a = \{0\}\} = \{0\}$ [3].

Proposition 4.14. *Let $\mathcal{A}$ be a Banach algebra and I a closed ideal in $\mathcal{A}$. let σ be a bounded idempotent endomorphism of $\mathcal{A}$ and I has a bounded approximate identity. Let $\mathcal{X}$ be a σ-neo-unital Banach I-module and $D : I \to \mathcal{X}^*$ a σ-derivation. Then, $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule in a canonical fashion, and there is a unique σ-derivation $\widetilde{D} : \mathcal{A} \to \mathcal{X}^*$ such that*

(i) $\widetilde{D}|_I = D$,

(ii) *$\widetilde{D}$ is continuous with respect to the strict topology on $\mathcal{A}$ and the ω^*-topology on $\mathcal{X}^*$.*

Proof. Since $\mathcal{X}$ is a σ-neo-unital Banach I-module, so for each $x \in \mathcal{X}$, there exists $i \in I$ and $y \in \mathcal{X}$ such that $x = \sigma(i) \cdot y$. Define $a \cdot x = \sigma(ai) \cdot y (a \in \mathcal{A})$.

We claim that $a \cdot x$ is well defined, that is, independent of the choices of i and y. Let $i' \in I$ and $y' \in \mathcal{X}$ be such that $x = \sigma(i') \cdot y'$, and let $(e_\alpha)_\alpha$ be a bounded approximate identity for I. For each $a \in \mathcal{A}$ and $x \in \mathcal{X}$ we have

$$\begin{aligned} a \cdot x = \sigma(ai) \cdot y &= \lim_\alpha \sigma(ae_\alpha i) \cdot y \\ &= \lim_\alpha \sigma(ae_\alpha)\sigma(i) \cdot y = \lim_\alpha \sigma(ae_\alpha)x \\ &= \lim_\alpha \sigma(ae_\alpha)\sigma(i') \cdot y' = \lim_\alpha \sigma(ae_\alpha i') \cdot y' \\ &= \sigma(ai') \cdot y'. \end{aligned} \tag{4.18}$$

It is obvious that this operation of $\mathcal{A}$ on $\mathcal{X}$ turns $\mathcal{X}$ into a left Banach $\mathcal{A}$-module. Similarly, one defines a right Banach $\mathcal{A}$-module structure on $\mathcal{X}$. So that, eventually, $\mathcal{X}$ becomes a Banach $\mathcal{A}$-bimodule. To extend D, let

$$\widetilde{D} : \mathcal{A} \longrightarrow \mathcal{X}^*, \qquad a \longrightarrow \omega^* - \lim_\alpha (D(ae_\alpha) - \sigma(a) \cdot D(e_\alpha)). \tag{4.19}$$

We claim that $\widetilde{D}$ is well-defined, that is, the limit in (4.19) does exist. Let $x \in \mathcal{X}$, and let $i \in I$ and $y \in \mathcal{X}$ such that $x = y \cdot \sigma(i)$. By Lemma 4.13, $\sigma(e_\alpha)$ is bounded approximate identity for $\mathcal{X}$, and we have

$$\begin{aligned} \langle x, D(ae_\alpha) - \sigma(a) \cdot D(e_\alpha)\rangle &= \langle y \cdot \sigma(i), D(ae_\alpha) - \sigma(a) \cdot D(e_\alpha)\rangle \\ &= \langle y, \sigma(i)D(ae_\alpha) - \sigma(ia) \cdot D(e_\alpha)\rangle \end{aligned}$$

$$\begin{aligned}
&= \left\langle y, D(iae_\alpha) - D(i)\sigma(ae_\alpha) - D(iae_\alpha) + D(ia)\sigma(e_\alpha)\right\rangle \\
&= \left\langle \sigma(e_\alpha)\cdot y, D(ia)\right\rangle - \left\langle \sigma(ae_\alpha)\cdot y, D(i)\right\rangle \\
&\xrightarrow{\alpha} \left\langle y, D(ia)\right\rangle - \left\langle \sigma(a)\cdot y, D(i)\right\rangle \quad (a\in\mathcal{A}).
\end{aligned} \tag{4.20}$$

So the limit in (4.19) exists. Furthermore, for $i \in I$,

$$\begin{aligned}
\widetilde{D}(i) &= \omega^* - \lim_\alpha (D(ie_\alpha) - \sigma(i)\cdot D(e_\alpha)) \\
&= \omega^* - \lim_\alpha D(ie_\alpha) - D(ie_\alpha) + D(i)\sigma(e_\alpha) = D(i),
\end{aligned} \tag{4.21}$$

so $\widetilde{D}$ is an extension of D. Also for $a \in \mathcal{A}$ and $i \in I$ we have

$$\begin{aligned}
\left(\widetilde{D}a\right)\cdot\sigma(i) &= \omega^* - \lim_\alpha (D(ae_\alpha)\cdot\sigma(i) - \sigma(a)\cdot D(e_\alpha)\cdot\sigma(i)) \\
&= \omega^* - \lim_\alpha (D(ae_\alpha i) - \sigma(ae_\alpha)\cdot D(i) - \sigma(a)\cdot D(e_\alpha i) + \sigma(a)\sigma(e_\alpha)\cdot D(i)) \\
&= \omega^* - \lim_\alpha (D(ae_\alpha i) - \sigma(a)\cdot D(e_\alpha i)) = D(ai) - \sigma(a)\cdot D(i).
\end{aligned} \tag{4.22}$$

We claim that $\widetilde{D}$ is continuous with respect to the strict topology on $\mathcal{A}$ and the ω^*-topology an $\mathcal{X}^*$.

Let $a_n \overset{\text{strict}}{\to} a$ in $\mathcal{A}$.

$$\forall i \in I, \quad \|a_n i\| + \|ia_n\| \longrightarrow \|ai\| + \|ia\|. \tag{4.23}$$

For each $x \in \mathcal{X}$,

$$\begin{aligned}
&\left|\left\langle x, \widetilde{D}(a_n)\right\rangle - \left\langle x, \widetilde{D}(a)\right\rangle\right| \\
&\quad = \lim_\alpha \left|\langle x, D(a_n e_\alpha) - \sigma(a_n)\cdot D(e_\alpha)\rangle - \langle x, D(ae_\alpha) - \sigma(a)\cdot D(e_\alpha)\rangle\right| \\
&\quad = \lim_\alpha \left|\langle x, D(a_n e_\alpha) - D(ae_\alpha) - \sigma(a_n)D(e_\alpha) + \sigma(a)D(e_\alpha)\rangle\right| \\
&\quad \le \lim_\alpha \|x\| \|(D(a_n e_\alpha) - D(ae_\alpha)) - (\sigma(a_0)\cdot D(e_\alpha) - \sigma(a_n)D(e_\alpha))\| \\
&\quad \le \lim_\alpha \|x\| (\|D\| \|a_n e_\alpha - ae_\alpha\| + \|\sigma(a_n) - \sigma(a)\| \|D(e_\alpha)\|) \\
&\quad \le \lim_\alpha \|x\| (\|D\| \|a_n - a\| \|e_\alpha\| + \|\sigma\| \|a_n - a\| \|D(e_\alpha)\|) \longrightarrow 0,
\end{aligned} \tag{4.24}$$

so $\widetilde{D}$ is continuous.

It remains to show that $\widetilde{D}$ is a σ-derivation. From the definition of the strict topology, we have $ae_\alpha \to a$ in the strict topology for all $a \in \mathcal{A}$ because $\|ae_\alpha i\|+\|iae_\alpha\| \xrightarrow{\alpha} \|ai\|+\|ia\|$ $(i \in I)$ and so $\widetilde{D}(ae_\alpha) \xrightarrow{w^*} \widetilde{D}(a)$. Therefore,

$$\begin{aligned}\widetilde{D}(ab) &= \omega^* - \lim_\alpha \lim_\beta \widetilde{D}((ae_\alpha)(be_\beta)) \\ &= \omega^* - \lim_\alpha \lim_\beta D((ae_\alpha)(be_\beta)) \\ &= \omega^* - \lim_\alpha \lim_\beta (\sigma(ae_\alpha)D(be_\beta) + D(ae_\alpha)\cdot\sigma(be_\beta)) \\ &= \omega^* - \lim_\alpha \lim_\beta \Big(\sigma(ae_\alpha)\widetilde{D}(be_\beta) + \widetilde{D}(ae_\alpha)\cdot\sigma(be_\beta)\Big) \\ &= \sigma(a)\widetilde{D}(b) + \widetilde{D}(a)\sigma(b),\end{aligned} \tag{4.25}$$

that is, $\widetilde{D}$ is σ-derivation. □

Corollary 4.15. *Suppose that $\mathcal{A}$ is σ-a.a, where σ is bounded idempotent endomorphism of $\mathcal{A}$, I is a closed ideal in $\mathcal{A}$. If I has a bounded approximate identity, then I is σ-a.a.*

Proof. Suppose that I has a bounded approximate identity, $\mathcal{X}$ is a σ-neo-unital Banach I-bimodule, and $D : I \to \mathcal{X}^*$ is a σ-derivation. By Proposition 4.14, $\mathcal{X}$ becomes to a Banach $\mathcal{A}$-bimodule and D has a unique extension $\widetilde{D} : \mathcal{A} \to \mathcal{X}^*$ which is a σ-derivation. Since $\mathcal{A}$ is σ-a.a,

$$\exists\{x_\alpha^*\} \subseteq \mathcal{X}^* \text{ s.t. } \quad \widetilde{D}(a) = \lim_\alpha \sigma(a)\cdot x_\alpha^* - x_\alpha^* \cdot \sigma(a) \quad (a \in \mathcal{A}). \tag{4.26}$$

So we have $D(i) = \widetilde{D}(i) = \lim_\alpha \sigma(i)\cdot x_\alpha^* - x_\alpha^* \cdot \sigma(i)$, which shows that $D = \lim_\alpha \delta^\sigma_{x_\alpha^*}$ is σ-a.i, and I is σ-a.a. □

Corollary 4.16. *Let $\mathcal{A}$ be an a.a Banach algebra and I a closed ideal of $\mathcal{A}$. Then $\mathcal{A}/I$ is σ-a.a for each bounded endomorphism σ of $\mathcal{A}/I$.*

Proposition 4.17. *Let I be a closed ideal of $\mathcal{A}$ such that $\sigma(I) \subseteq I$. If $\mathcal{A}$ is σ-a.a, then $\mathcal{A}/I$ is $\widehat{\sigma}$-a.c, where $\widehat{\sigma}$ is an endomorphism of $\mathcal{A}/I$ induced by σ (i.e., $\widehat{\sigma}(a+I) = \sigma(a) + I$ for $a \in \mathcal{A}$).*

Proof. Let $\mathcal{X}$ be a Banach $\mathcal{A}/I$-bimodule and $D : \mathcal{A}/I \to \mathcal{X}$ a $\widehat{\sigma}$-derivation. Then $\mathcal{X}$ becomes an $\mathcal{A}$-bimodule with the following module actions:

$$a \cdot x = \pi(a)\cdot x, \qquad x \cdot a = x \cdot \pi(a) \quad (a \in \mathcal{A}, x \in \mathcal{X}), \tag{4.27}$$

where π is the canonical homomorphism $\pi : \mathcal{A} \to \mathcal{A}/I$. It is easy to see that $Do\pi : A \to X$ becomes a σ-derivation. Since $\mathcal{A}$ is σ-a.c, there exists a net $\{x_\alpha\} \subseteq \mathcal{X}$ such that $Do\pi(a) = \lim_\alpha \sigma(a) \cdot x_\alpha - x_\alpha \cdot \sigma(a)$ $(a \in \mathcal{A})$. Therefore, for each $(a \in A)$,

$$\begin{aligned} D(a+I) = Do\pi(a) &= \lim_\alpha \sigma(a) \cdot x_\alpha - x_\alpha \cdot \sigma(a) \\ &= \lim_\alpha \pi(\sigma(a)) \cdot x_\alpha - x_\alpha \cdot \pi(\sigma(a)) \\ &= \lim_\alpha (\sigma(a) + I) \cdot x_\alpha - x_\alpha \cdot (\sigma(a) + I) \\ &= \lim_\alpha \widehat{\sigma}(a+I) x_\alpha - x_\alpha \widehat{\sigma}(a+I). \end{aligned} \tag{4.28}$$

Thus, $\mathcal{A}/I$ is $\widehat{\sigma}$-a.c. □

Proposition 4.18. *Suppose that I is a closed ideal in $\mathcal{A}$. If I is σ-amenable and A/I is a.a, then $\mathcal{A}$ is σ-a.a.*

Proof. Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and $D : \mathcal{A} \to \mathcal{X}^*$ a σ-derivation. $\mathcal{X}$ is a Banach I-bimodule too.

Clearly, $d = D|_I : I \to \mathcal{X}^*$ is a σ-derivation, and by σ-amenability of I there exists $x_0^* \in \mathcal{X}^*$ such that $D = \delta^\sigma_{x_0^*}$, and, therefore, for each $i \in I$ we have $d(i) = \sigma(i) \cdot x_0^* - x_0^* \cdot \sigma(i)$. Set $D_1 = D - \delta^\sigma_{x_0^*}$. Clearly, D_1 is σ-derivation and $D_1|_I = 0$. Now let $\mathcal{X}_0 = \overline{\text{span}}(\mathcal{X} \cdot \sigma(I) \cup \sigma(I) \cdot \mathcal{X}) \cdot$ $(\mathcal{X}/\mathcal{X}_0)$ is a Banach $\mathcal{A}/I$-bimodule via the following module actions:

$$(a+I)(x+\mathcal{X}_0) = \sigma(a)x + \mathcal{X}_0, \qquad (x+\mathcal{X}_0)(a+I) = x\sigma(a) + \mathcal{X}_0 \quad (x \in \mathcal{X}, a \in \mathcal{A}). \tag{4.29}$$

Now we define

$$\widetilde{D} : \frac{\mathcal{A}}{I} \longrightarrow \left(\frac{\mathcal{X}}{\mathcal{X}_0}\right)^*; \qquad \left\langle x + \mathcal{X}_0, \widetilde{D}(a+I) \right\rangle = \langle x, D_1(a) \rangle \quad (a \in \mathcal{A}, x \in \mathcal{X}). \tag{4.30}$$

Let $a + I = a' + I$ and $x + \mathcal{X}_0 = x' + \mathcal{X}_0$ for some $a, a' \in \mathcal{A}$ and $x, x' \in \mathcal{X}$. So $a - a' \in I$, and we have $D_1(a - a') = 0$. Thus, $D_1(a) = D(a')$. Now we have

$$\left\langle x + \mathrm{X}_0, \widetilde{D}(a+I) \right\rangle = \left\langle x' + \mathrm{X}_0, \widetilde{D}(a'+I) \right\rangle. \tag{4.31}$$

Thus, $\langle x, D_1(a) \rangle = \langle x', D_1(a') \rangle = \langle x', D_1(a) \rangle$, and, therefore,

$$\left\langle x - x', D_1(a) \right\rangle = 0. \tag{4.32}$$

It is enough to show that $D_1(a)$ is zero on $\mathcal{X}_0$. Suppose that $\sigma(i)x \in \mathcal{X}_0$, we have

$$\begin{aligned} \langle \sigma(i)x, D_1(a) \rangle &= \langle x, D_1(a)\sigma(i) \rangle = \langle x, D_1(ai) - \sigma(a)D_1(i) \rangle = 0, \\ \langle x\sigma(i), D_1(a) \rangle &= \langle x, \sigma(i)D_1(a) \rangle = \langle x, D_1(ia) - D_1(i)\sigma(a) \rangle = 0. \end{aligned} \tag{4.33}$$

So for all $a \in \mathcal{A}$, $D_1(a) = 0$ on $\sigma(I) \cdot \mathcal{X} \cup \mathcal{X} \cdot \sigma(I)$ and so for all $a \in \mathcal{A}$, $D_1(a) = 0$ on $\mathcal{X}_0$. Since $x - x' \in \mathcal{X}_0$, therefore $\langle x - x', D_1(a) \rangle = 0$ which shows that D_1 is well defined. We claim that $\widetilde{D}$ is a derivation;

$$\begin{aligned}
\left\langle x + \mathcal{X}_0, \widetilde{D}((a+I)(b+I)) \right\rangle &= \langle x, D_1(ab) \rangle \\
&= \langle x, \sigma(a)D_1(b) + D_1(a)\sigma(b) \rangle \\
&= \langle x\sigma(a), D_1(b) \rangle + \langle \sigma(b)x, D_1(a) \rangle \\
&= \left\langle x\sigma(a) + \mathcal{X}_0, \widetilde{D}(b+I) \right\rangle \\
&\quad + \left\langle \sigma(b)x + \mathcal{X}_0, \widetilde{D}(a+\mathrm{I}) \right\rangle \\
&= \left\langle (x + \mathcal{X}_0)(a+I), \widetilde{D}(b+I) \right\rangle \\
&\quad + \left\langle (b+I)(x + \mathcal{X}_0), \widetilde{D}(a+I) \right\rangle.
\end{aligned} \tag{4.34}$$

So there exists a net $(\varphi_\alpha) \subseteq (\mathcal{X}/\mathcal{X}_0)^*$ such that $\widetilde{D} = \lim_\alpha \delta_{\varphi_\alpha}$. Let $q : \mathcal{X} \to \mathcal{X}/\mathcal{X}_0$ be the quotient map. For every α, $(\varphi_\alpha o q) \in \mathcal{X}^*$. Set $(x_\alpha^*) = (\varphi_\alpha o q) \subseteq \mathcal{X}^*$. We have

$$\begin{aligned}
\langle x, D_1(a) \rangle &= \left\langle x + \mathrm{X}_0, \widetilde{D}(a+I) \right\rangle \\
&= \left\langle x + \mathrm{X}_0, \lim_\alpha (a+I)\varphi_\alpha - \varphi_\alpha(a+I) \right\rangle \\
&= \lim_\alpha \langle x\sigma(a) + \mathrm{X}_0, \varphi_\alpha \rangle - \langle \sigma(a)x + \mathrm{X}_0, \varphi_\alpha \rangle \\
&= \lim_\alpha \langle q(x\sigma(a)), \varphi_\alpha \rangle - \langle q(\sigma(a)x), \varphi_\alpha \rangle \\
&= \lim_\alpha \varphi_\alpha o q(x\sigma(a) - \sigma(a)x) = \langle x\sigma(a) - \sigma(a)x, x_\alpha^* \rangle \\
&= \lim_\alpha \langle x, \sigma(a)x_\alpha^* - x_\alpha^*\sigma(a) \rangle \\
&= \left\langle x, \lim_\alpha \delta_{x_\alpha^*}^\sigma(a) \right\rangle.
\end{aligned} \tag{4.35}$$

So $D_1 = D - \delta_{x^*}^\sigma = \lim_\alpha \delta_{x_\alpha^*}^\sigma$, and, therefore, $D = \lim_\alpha \delta_{(x_\alpha^* - x_0^*)}^\sigma$. Which shows that D is σ-a.i and so $\mathcal{A}$ is σ-a.a. □

Example 4.19. Let $\mathcal{A}$ be a Banach algebra and let $0 \neq \varphi \in \mathrm{Ball}(\mathcal{A}^*)$. Then $\mathcal{A}$ with the product $a \cdot a' = \varphi(a)a'$ becomes a Banach algebra. We denote this algebra with $\mathcal{A}_\varphi$. It is easy to see that $\mathcal{A}_\varphi$ has a left identity e, while it has not right approximate identity, so $\mathcal{A}_\varphi$ is not contractible and is not approximately contractible. Also $\mathcal{A}_\varphi$ is biprojective. Now suppose that $\sigma : \mathcal{A}_\varphi \to \mathcal{A}_\varphi$ be defined by $\sigma(a) = \varphi(a)e$. We have

$$\sigma^2(a) = \sigma(\varphi(a)e) = \varphi(a)\sigma(e) = \varphi(a)\varphi(e)e = \varphi(a)e = \sigma(a). \tag{4.36}$$

Thus σ is idempotent. It is easy to see that e is identity for $\sigma(\mathcal{A}_\varphi)$, and since $\mathcal{A}$ is biprojective by [1, Corollary 5.3], $\mathcal{A}_\varphi$ is σ-biprojective. Thus by [1, Theorem 4.3], $\mathcal{A}_\varphi$ is σ-contractible and so $\mathcal{A}_\varphi$ is σ-a.c.

It is easy to see that $\ker\varphi$ and all subspaces of $\ker\varphi$ are all ideals of $\mathcal{A}_\varphi$ and $\sigma(\ker\varphi) \subseteq \ker\varphi$ so $\sigma(I) \subseteq I$ for each ideal of $\mathcal{A}$. Therefore, by Proposition 4.17, $\mathcal{A}_\varphi/I$ is $\widehat{\sigma}$-a.c for each ideal I of $\mathcal{A}_\varphi$, where $\widehat{\sigma}(a+I) = \sigma(a) + I = \varphi(a)e + I$.

Corollary 4.20. *Suppose that σ is a bounded idempotent endomorphism of Banach algebra $\mathcal{A}$. Then $\mathcal{A}$ is σ-a.a if and only if there are nets (μ''_α) in $(\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ and $(F_\alpha), (G_\alpha) \subseteq \mathcal{A}^{**}$, such that for each $a \in \mathcal{A}$,*

(1) $\sigma(a) \cdot \mu''_\alpha - \mu''_\alpha \cdot \sigma(a) + F_\alpha \otimes \sigma(a) - \sigma(a) \otimes G_\alpha \to 0$,

(2) $\sigma(a) \cdot F_\alpha \to \sigma(a), G_\alpha \cdot \sigma(a) \to \sigma(a)$,

(3) $\pi^{**}(\mu''_\alpha) \cdot \sigma(a) - F_\alpha \cdot \sigma(a) - G_\alpha \cdot \sigma(a) \to 0$.

Proof. Suppose that $\mathcal{A}$ is σ-a.a, take the net (μ_α) given in Proposition 4.9 and write

$$\mu_\alpha = \mu''_\alpha - F_\alpha \otimes \widehat{e} - \widehat{e} \otimes G_\alpha + c_\alpha \widehat{e} \otimes \widehat{e}, \tag{4.37}$$

where $(\mu''_\alpha) \subseteq (A\widehat{\otimes}A)^{**}$, $(F_\alpha), (G_\alpha) \subseteq A^{**}$, and $(c_\alpha) \subseteq \mathbb{C}$. Applying π^{**}, $\pi^{**}(\mu''_\alpha) - F_\alpha - G_\alpha + c_\alpha\widehat{e} \to \widehat{e}$, hence $c_\alpha \to 1$, then

$$\pi^{**}(\mu''_\alpha) \cdot \sigma(a) - F_\alpha \cdot \sigma(a) - G_\alpha \cdot \sigma(a) + \widehat{e} \cdot \sigma(a) \longrightarrow \widehat{e} \cdot \sigma(a) \quad (a \in \mathcal{A}). \tag{4.38}$$

So we have (iii) further, by Proposition 4.9, for $a \in \mathcal{A}^{\#}$,

$$\begin{aligned} &\widehat{\sigma}(a) \cdot \mu''_\alpha - \widehat{\sigma}(a) \cdot F_\alpha \otimes \widehat{e} - \widehat{\sigma}(a) \otimes G_\alpha + \widehat{\sigma}(a) \otimes \widehat{e} \\ &\quad + \mu''_\alpha \cdot \widehat{\sigma}(a) + F_\alpha \otimes \widehat{\sigma}(a) + \widehat{e} \otimes G_\alpha \cdot \widehat{\sigma}(a) - \widehat{e} \otimes \widehat{\sigma}(a) \longrightarrow 0. \end{aligned} \tag{4.39}$$

Thus $\widehat{\sigma}(a) \cdot \mu''_\alpha - \mu''_\alpha \cdot \widehat{\sigma}(a) + F_\alpha \otimes \widehat{\sigma}(a) - \widehat{\sigma}(a) \otimes G_\alpha \to 0$, and $\widehat{\sigma}(a) \cdot F_\alpha \to \widehat{\sigma}(a), G_\alpha \cdot \widehat{\sigma}(a) \to \widehat{\sigma}(a)$. So for $a \in \mathcal{A}$,

$$\begin{gathered} \sigma(a) \cdot \mu''_\alpha - \mu''_\alpha \cdot \sigma(a) + F_\alpha \otimes \sigma(a) - \sigma(a) \otimes G_\alpha \longrightarrow 0, \\ \sigma(a) \cdot F_\alpha \longrightarrow \sigma(a), \qquad G_\alpha \cdot \sigma(a) \longrightarrow \sigma(a). \end{gathered} \tag{4.40}$$

Conversely, set $c_\alpha = 1$ and $\mu_\alpha = \mu''_\alpha - F_\alpha \otimes \widehat{e} - \widehat{e} \otimes G_\alpha + \widehat{e} \otimes \widehat{e}$. We have

$$\begin{aligned}
\widehat{\sigma}(a+\alpha) \cdot \mu_\alpha - \mu_\alpha \cdot \widehat{\sigma}(a+\alpha) &= (\sigma(a)+\alpha) \cdot \mu_\alpha - \mu_\alpha \cdot (\sigma(a)+\alpha) \\
&= \sigma(a) \cdot \mu_\alpha - \mu_\alpha \cdot \sigma(a) + a\mu_\alpha - \alpha\mu_\alpha \\
&= \sigma(a) \cdot \mu_\alpha - \mu_\alpha \cdot \sigma(a) \\
&= \sigma(a) \cdot \mu''_\alpha - \sigma(a) F_\alpha \otimes e - \sigma(a) \otimes G_\alpha \\
&\quad + \sigma(a) \otimes e\big(-\mu''_\alpha \cdot \sigma(a) \\
&\qquad + F_\alpha \otimes \sigma(a) + e \otimes G_\alpha \sigma(a) - e \otimes \sigma(a)\big) \\
&= \sigma(a) \cdot \mu''_\alpha - \mu''_\alpha \cdot \sigma(a) \\
&\quad + F_\alpha \otimes \sigma(a) - \sigma(a) \otimes G_\alpha \to 0 \quad (a \in \mathcal{A}).
\end{aligned} \tag{4.41}$$

So $\widehat{\sigma}(a) \cdot \mu_\alpha - \mu_\alpha \cdot \widehat{\sigma}(a) \to 0$ $(a \in \mathcal{A}^\#)$. Also

$$\begin{aligned}
\pi^{**}(\mu_\alpha) \cdot \sigma(a) &= \pi^{**}\big(\mu''_\alpha - F_\alpha \otimes \widehat{e} - \widehat{e} \otimes G_\alpha + \widehat{e} \otimes \widehat{e}\big)\sigma(a) \\
&= \pi^{**}(\mu''_\alpha)\sigma(a) - F_\alpha \cdot \sigma(a) \\
&\quad - G_\alpha \cdot \sigma(a) + \sigma(a) \longrightarrow \sigma(a) \quad (a \in \mathcal{A}),
\end{aligned} \tag{4.42}$$

and so $\pi^{**}(\mu_\alpha) \to \widehat{e}$. Now, by Proposition 4.9, $\mathcal{A}$ is σ-a.a. □

For σ-approximate contractibility we have the following parallel result.

Proposition 4.21. *$\mathcal{A}$ is σ-a.c if and only if any of the following equivalent conditions hold:*

(1) *there is a net $(\mu_\alpha) \subset \mathcal{A}^\# \widehat{\otimes} \mathcal{A}^\#$ such that for each $a \in \mathcal{A}^\#$, $\sigma(a) \cdot \mu_\alpha - \mu_\alpha \cdot \sigma(a) \to 0$ and $\pi(\mu_\alpha) \to e$;*

(2) *there is a net $(\mu'_\alpha) \subset \mathcal{A}^\# \widehat{\otimes} \mathcal{A}^\#$ such that for each $a \in \mathcal{A}^\#$, $\sigma(a) \cdot \mu'_\alpha - \mu'_\alpha \cdot \sigma(a) \to 0$ and $\pi(\mu'_\alpha) = e$;*

(3) *there are nets $(\mu''_\alpha) \subset \mathcal{A} \widehat{\otimes} \mathcal{A}$, $(F_\alpha), (G_\alpha) \subset \mathcal{A}$, such that for each $a \in \mathcal{A}$,*

   (i) *$\sigma(a) \cdot \mu''_\alpha - \mu''_\alpha \cdot \sigma(a) + F_\alpha \otimes \sigma(a) - \sigma(a) \otimes G_\alpha \to 0$;*

   (ii) *$\sigma(i) \cdot F_\alpha \to \sigma(a), G_\alpha \cdot \sigma(a) \to \sigma(a)$;*

   (iii) *$\pi(\mu''_\alpha) \cdot \sigma(a) - F_\alpha \cdot \sigma(a) - G_\alpha \cdot \sigma(a) \to 0$.*

We know Banach algebra $\mathcal{A}$ is amenable if and only if $\mathcal{A}$ has bounded approximate diagonal [3].

Proposition 4.22. *Banach algebra $\mathcal{A}$ is σ-amenable if and only if $\mathcal{A}$ has bounded approximate σ-diagonal, that is, there is a bounded net $(\mu_\alpha) \subseteq \mathcal{A} \widehat{\otimes} \mathcal{A}$ such that for each $a \in \mathcal{A}$, $\sigma(a) \cdot \mu_\alpha - \mu_\alpha \cdot \sigma(a) \to 0$ and $\pi(\mu_\alpha) \cdot \sigma(a) \to \sigma(a)$.*

Proposition 4.23. *If Banach algebra $\mathcal{A}$ is σ-amenable, then $\mathcal{A}$ is σ-a.c.*

Proof. Suppose that $\mathcal{A}$ is σ-amenable. Then there exists a bounded net (μ_α) in $\mathcal{A}\otimes\mathcal{A}$ such that for each $a \in \mathcal{A}$,

$$\sigma(a)\cdot\mu_\alpha - \mu_\alpha\cdot\sigma(a) \longrightarrow 0, \qquad \pi(\mu_\alpha)\cdot\sigma(a) \longrightarrow \sigma(a). \tag{4.43}$$

Set $f_\alpha = \pi(\mu_\alpha)$. It is easy to see that (f_α) is a bounded approximate identity. Then $\mu''_\alpha = \mu_\alpha + f_\alpha \otimes f_\alpha$ and $F_\alpha = G_\alpha = f_\alpha$ satisfy (i)–(iii) of Proposition 4.21, because

(i) $\sigma(a)\cdot\mu''_\alpha - \mu''_\alpha\cdot\sigma(a) + f_\alpha\otimes\sigma(a) - \sigma(a)\otimes f_\alpha = \sigma(a)\cdot\mu_\alpha - \mu_\alpha\cdot\sigma(a) + \sigma(a)f_\alpha\otimes f_\alpha - f_\alpha\otimes\sigma(a) + f_\alpha\otimes\sigma(a) - \sigma(a)\otimes f_\alpha \to 0\ (a\in\mathcal{A})$,

(ii) $\sigma(a)\cdot f_\alpha = \sigma(a)\cdot\pi(\mu_\alpha) \to \sigma(a)$, $f_\alpha\cdot\sigma(a) = \pi(\mu_\alpha)\cdot\sigma(a) \to \sigma(a)$,

(iii) $\pi(\mu''_\alpha)\cdot\sigma(a) = \pi(\mu_\alpha + f_\alpha\otimes f_\alpha)\cdot\sigma(a) = f_\alpha\cdot\sigma(a) + f_\alpha^2\cdot\sigma(a)$.

So

$$\pi(\mu''_\alpha)\cdot\sigma(a) - F_\alpha\cdot\sigma(a) - G_\alpha\cdot\sigma(a) = f_\alpha\cdot\sigma(a) + f_\alpha^2\cdot\sigma(a) - f_\alpha\cdot\sigma(a) - f_\alpha\cdot\sigma(a) \longrightarrow 0. \tag{4.44}$$

Note that f_α^2 is a bounded approximate identity too, thus, by Proposition 4.21, $\mathcal{A}$ is σ-a.c. □

Corollary 4.24. *Suppose that $\mathcal{A}$ is a σ-a.a Banach algebra where σ is an idempotent endomorphism of $\mathcal{A}$ and I is a closed two-sided ideal of $\mathcal{A}$ which $\sigma(I)$ has a bounded approximate identity and $\sigma(I)\subseteq I$. Then, I is σ-a.a.*

Proof. Let $\{e_\alpha\}$ be a bounded approximate identity in $\sigma(I)$, so $\{\widehat{e}_\alpha\}$ is bounded net in $\sigma(I)^{**}$, and so by Banach-Alaoglu theorem there exists a subnet $\{\widehat{e}_\beta\}\subseteq\{\widehat{e}_\alpha\}$ and $E\in\sigma(I)^{**}$ such that $\widehat{e}_\beta \xrightarrow{w^*} E$. E is a right identity in $\sigma(I)^{**}$ because for each $F\in\sigma(I)^{**}$ and $f\in\sigma(I)^*$,

$$\langle f, F\square E\rangle = \langle f\cdot F, E\rangle = \lim_\beta\langle e_\beta, fF\rangle = \lim_\beta\langle e_\beta f, F\rangle = \langle f, F\rangle. \tag{4.45}$$

Also E acts as an identity on $\sigma(I)$ itself. Let $(\mu_\alpha), (F_\alpha), (G_\alpha)$ be the nets given by Corollary 4.20 for $\mathcal{A}$. Define $\mu'_\alpha = E\cdot\mu_\alpha\cdot E \in (I\widehat{\otimes}I)^{**}$, $F'_\alpha = E\cdot F_\alpha \in I^{**}$, and $G'_\alpha = G_\alpha\cdot E\in I^{**}$. Then, for $i\in I$,

(i) we consider

$$\begin{aligned}
\sigma(i)\cdot\mu'_\alpha - \mu'_\alpha\cdot\sigma(i) + F'_\alpha\otimes\sigma(i) - \sigma(i)\otimes G'_\alpha \\
&= \sigma(i)\cdot E\cdot\mu_\alpha\cdot E - E\cdot\mu_\alpha\cdot E\cdot\sigma(i) + E\cdot F_\alpha\otimes\sigma(i) - \sigma(i)\otimes G_\alpha\cdot E \\
&= \sigma(i)\cdot\mu_\alpha\cdot E - E\cdot\mu_\alpha\sigma(i) + E\cdot F_\alpha\otimes\sigma(i) - \sigma(i)\otimes G_\alpha\cdot E \\
&= E\cdot\sigma(i)\cdot\mu_\alpha\cdot E - E\cdot\mu_\alpha\cdot\sigma(i)\cdot E \\
&\quad + E\cdot F_\alpha\otimes\sigma(i)\cdot E - E\cdot\sigma(i)\otimes G_\alpha\cdot E \\
&= E\big(\sigma(i)\cdot\mu_\alpha - \mu_\alpha\cdot\sigma(i) + F_\alpha\otimes\sigma(i) - \sigma(i)\otimes G_\alpha\big)\cdot E \longrightarrow 0,
\end{aligned} \tag{4.46}$$

(ii) we consider

$$\sigma(i) \cdot F'_\alpha = \sigma(i) \cdot E \cdot F_\alpha = \sigma(i) \cdot F_\alpha \longrightarrow \sigma(i),$$
$$G'_\alpha \cdot \sigma(i) = G_\alpha \cdot E \cdot \sigma(i) = G_\alpha \cdot \sigma(i) \longrightarrow \sigma(i) \tag{4.47}$$

(iii) we consider

$$\begin{aligned}
\pi^{**}(\mu'_\alpha) \cdot \widehat{\sigma(a)} - F'_\alpha \cdot \widehat{\sigma(a)} - G'_\alpha - \widehat{\sigma(a)} \\
&= \pi^{**}(E \cdot \mu_\alpha \cdot E) \cdot \sigma(a) - E \cdot F_\alpha \cdot \sigma(a) - G_\alpha \cdot E \cdot \sigma(a) \\
&= E \cdot \pi^{**}(\mu_\alpha) \cdot E \cdot \sigma(a) - E \cdot F_\alpha \cdot \sigma(a) - G_\alpha \cdot \sigma(a) \\
&= E \cdot \pi^{**}(\mu_\alpha) \cdot \sigma(a) - E \cdot F_\alpha \cdot \sigma(a) - G_\alpha \cdot \sigma(a) - E \cdot G_\alpha \sigma(a) + E \cdot G_\alpha \sigma(a) \\
&= E \cdot \left(\pi^{**}(\mu_\alpha) \cdot \sigma(a) - F_\alpha . \sigma(a) - G_\alpha \sigma(a)\right) + (E - \widehat{e}) G_\alpha \sigma(a) \longrightarrow 0.
\end{aligned} \tag{4.48}$$

An alternative proof would be to follow the standard argument stated in Corollary 4.15. □

References

[1] P. C. Curtis Jr. and R. J. Loy, "The structure of amenable Banach algebras," *Journal of the London Mathematical Society*, vol. 40, no. 1, pp. 89–104, 1989.

[2] M. Eshaghi Gordji, "Point derivations on second duals and unitization of Banach algebras," *Nonlinear Functional Analysis and Applications*, vol. 13, no. 2, pp. 271–275, 2008.

[3] M. Eshaghi Gordji, "Homomorphisms, amenability and weak amenability of Banach algebras," *Vietnam Journal of Mathematics*, vol. 36, no. 3, pp. 253–260, 2008.