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Research Article

Fixed Point of Strong Duality Pseudocontractive Mappings and Applications

Baowei Liu

Department of Mathematics, Cangzhou Normal University, Cangzhou 061001, China

Correspondence should be addressed to Baowei Liu, lbw8880148@sina.com

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Let E be a smooth Banach space with the dual E^*, an operator $T : E \to E^*$ is said to be α-strong duality pseudocontractive if $\langle x - y, Tx - Ty\rangle \leq \langle x - y, Jx - Jy\rangle - \alpha\|Jx - Jy - (Tx - Ty)\|^2$, for all $x, y \in E$, where α is a nonnegative constant. An element $x \in E$ is called a duality fixed point of T if $Tx = Jx$. The purpose of this paper is to introduce the definition of α-strong duality pseudocontractive mappings and to study its fixed point problem and applications for operator equation and variational inequality problems.

1. Introduction and Preliminaries

Let E be a real Banach space with the dual E^*: let T be an operator from E into E^*. We consider the first operator equation problem of finding an element $x^* \in E$ such that

$$\langle Tx^*, x^*\rangle = \|Tx^*\|^2 = \|x^*\|^2. \tag{1.1}$$

We also consider the second variational inequality problem of finding an element $x^* \in E$ such that

$$\langle Tx^*, x^* - x\rangle \geq 0, \quad \forall \|x\| \leq \|x^*\|. \tag{1.2}$$

Let E be a real Banach space with the dual E^*. Let p be a given real number with $p > 1$. The generalized duality mapping J_p from E into 2^{E^*} is defined by

$$J_p(x) = \left\{ f \in E^* : \langle x, f\rangle = \|f\|^p, \|f\| = \|x\|^{p-1} \right\}, \quad \forall x \in E, \tag{1.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In particular, $J = J_2$ is called the normalized duality mapping and $J_p(x) = \|x\|^{p-2} J(x)$ for all $x \neq 0$. If E is a Hilbert space, then $J = I$, where I is the identity mapping. The duality mapping J has the following properties:

(i) if E is smooth, then J is single valued;

(ii) if E is strictly convex, then J is one to one;

(iii) if E is reflexive, then J is a mapping of E onto $^*E$;

(iv) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E;

(v) if E^* is uniformly convex, then J is uniformly continuous on each bounded subsets of E and J is single valued and also one to one.

For more details, see [1, 2].

Let E be a smooth Banach space with the dual E^*. Let $T : E \to E^*$ be an operator; an element $x^* \in E$ is called a duality fixed point of T, if $Tx^* = Jx^*$.

We also consider the third variational inequality problem of finding an element $x^* \in E$ such that

$$\langle Tx^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1.4}$$

where C is a closed convex subset of E. The set of solutions of the variational inequality problem (1.4) is denoted by $VI(C, T)$.

We also consider the fourth variational inequality problem of finding an element $x^* \in E$ such that

$$\langle Jx^* - Tx^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1.5}$$

where C is a closed convex subset of E. The set of solutions of the variational inequality problem (1.5) is denoted by $VI(C, J, T)$.

Conclusion 1. If x^* is a duality fixed point of T, then x^* must be a solution of problem (1.1).

Proof. If x^* is a normalized fixed point of T, then $Tx^* = Jx^*$, so that

$$\langle Tx^*, x^* \rangle = \langle Jx^*, x^* \rangle = \|Jx^*\|^2 = \|Tx^*\|^2 = \|x^*\|^2. \tag{1.6}$$

This completes the proof. □

Conclusion 2. If x^* is a duality fixed point of T, then x^* must be a solution of variational inequality problem (1.2).

Proof. Suppose x^* is a duality fixed point of T; then

$$\langle Tx^*, x^* \rangle = \|Tx^*\|^2 = \|x^*\|^2. \tag{1.7}$$

Obverse that

$$\begin{aligned}\langle Tx^*, x^* - x\rangle &= \langle Tx^*, x^*\rangle - \langle Tx^*, x\rangle \\ &\geq \|Tx^*\|^2 - \|Tx^*\|\|x\| \\ &= \|Tx^*\|(\|Tx^*\| - \|x\|) \\ &= \|Tx^*\|(\|x^*\| - \|x\|) \geq 0,\end{aligned} \tag{1.8}$$

for all $\|x\| \leq \|x^*\|$. This completes the proof. □

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be strictly convex if for any $x, y \in U$, $x \neq y$ implies $\|(x + y)/2\| < 1$. It is also said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$, $\|x - y\| \geq \varepsilon$ implies $\|(x + y)/2\| < 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex. And we define a function $\delta : [0, 2] \to [0, 1]$ called the modulus of convexity of E as follows:

$$\delta(\varepsilon) = \left\{1 - \left\|\frac{x + y}{2}\right\| : \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\right\}. \tag{1.9}$$

It is known that E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Let p be a fixed real number with $p \geq 2$. Then E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0, 2]$. For example, see [3, 4] for more details. The constant $1/c$ is said to be uniformly convexity constant of E.

A Banach space E is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.10}$$

exists for all $x, y \in U$. It is also said to be uniformly smooth if the above limit is attained uniformly for $x, y \in U$. One should note that no Banach space is p-uniformly convex for $1 < p < 2$; see [5] for more details. It is well known that the Hilbert and the Lebesgue $L^q(1 < q \leq 2)$ spaces are 2-uniformly convex and uniformly smooth. Let X be a Banach space, and let $L^q(X) = \{\Omega, \Sigma, \mu; X\}, 1 < q \leq \infty$ be the Lebesgue-Bochner space on an arbitrary measure space (Ω, Σ, μ). Let $2 \leq p < \infty$, and let $1 < q \leq p$. Then $L^q(X)$ is p-uniformly convex if and only if X is p-uniformly convex; see [4].

In this paper, we first propose the definition of generalized α-strongly pseudocontractive mappings from a smooth Banach E into its dual E^* as follows. We also discuss the problem of fixed point for generalized α-strongly pseudocontractive mappings and its applications.

Let E be a smooth Banach space and E^* denote the dual of E. An operator $A : E \to E^*$ is said to be

(1) α-inverse-strongly monotone if there exists nonnegative real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in E. \tag{1.11}$$

(2) α-strong duality pseudocontractive mapping, if there exists a nonnegative real number α such that

$$\langle x-y, Ax-Ay\rangle \leq \langle x-y, Jx-Jy\rangle - \alpha\|Jx-Jy-(Ax-Ay)\|^2 \tag{1.12}$$

for all $x, y \in E$.

It is easy to show that A is α-strong duality pseudocontractive if and only if $(J-A)$ is α-inverse-strongly monotone.

Let E be a smooth Banach space and E^* denote the dual of E. Let $A : E \to E^*$ be an operator. The set of zero points of A is defined by $A^{-1}0 = \{x \in E : Ax = 0\}$. The set of duality fixed points of A is defined by $F(A) = \{x \in E : Ax = Jx\}$. It is also easy to show that, an element $u \in E$ is a zero point of an α-inverse-strongly monotone operator A if and only if u is a duality fixed point of the α-strong duality pseudocontractive mapping $(J-A)$.

2. Main Results and Applications

Recently, Zegeye and Shahzad [6] proved the following result.

Theorem 2.1 (see, [6]). *Let E be a uniformly smooth and 2-uniformly convex real Banach space with the dual E^*: let $A : E \to E^*$ be a γ-inverse-strongly monotone mapping and $T : E \to E$ a relatively weak nonexpansive mapping with $A^{-1}0 \cap F(T) \neq \emptyset$. Assume that $0 < \alpha \leq \lambda_n \leq \gamma c^2/2$, where $1/c$ is the uniformly convexity constant. Define a sequence $\{x_n\}$ in E by the following algorithm:*

$$\begin{gathered}
x_0 \in E \text{ chosen arbitrarily}, \\
z_n = J^{-1}(Jx_n - \lambda_n Ax_n), \\
y_n = Tz_n, \\
C_n = \left\{z \in C_{n-1} \bigcap Q_{n-1} : \phi(z, y_n) \leq \phi(z, z_n) \leq \phi(z, x_n)\right\}, \\
C_0 = \{z \in E : \phi(z, y_0) \leq \phi(z, z_0) \leq \phi(z, x_0)\}, \\
Q_n = \left\{z \in C_{n-1} \bigcap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\right\}, \\
Q_0 = E, \\
x_{n+1} = \Pi_{C_n \cap Q_n}(x_0),
\end{gathered} \tag{2.1}$$

where J is the duality mapping on E. Then $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0\cap F(T)}x_0$, where $\Pi_{A^{-1}0\cap F(T)}$ is the generalized projection from E onto $A^{-1}0 \cap F(T)$.

IF taking $T = I$, then Theorem 2.1 reduces to the following result.

Theorem 2.2. *Let E be a uniformly smooth and 2-uniformly convex real Banach space with the dual E^*, let $A : E \to E^*$ be a γ-inverse strongly monotone mapping with $A^{-1}0 \neq \emptyset$. Assume that*

$0 < \alpha \leq \lambda_n \leq \gamma c^2/2$, *where* $1/c$ *is the uniformly convexity constant. Define a sequence* $\{x_n\}$ *in* E *by the following algorithm:*

$$\begin{gathered}
x_0 \in E \text{ chosen arbitrarily}, \\
y_n = J^{-1}(Jx_n - \lambda_n A x_n), \\
C_n = \left\{z \in C_{n-1} \bigcap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\right\}, \\
C_0 = \{z \in E : \phi(z, y_0) \leq \phi(z, x_0)\}, \\
Q_n = \left\{z \in C_{n-1} \bigcap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\right\}, \\
Q_0 = E, \\
x_{n+1} = \Pi_{C_n \cap Q_n}(x_0),
\end{gathered} \tag{2.2}$$

where J *is the duality mapping on* E. *Then* $\{x_n\}$ *converges strongly to* $\Pi_{A^{-1}0}x_0$, *where* $\Pi_{A^{-1}0}$ *is the generalized projection from* E *onto* $A^{-1}0$.

Theorem 2.3. *Let* E *be a uniformly smooth and 2-uniformly convex real Banach space; let* $A : E \to E^*$ *be an* α*-strong duality pseudocontractive mapping with nonempty set of duality fixed points* $F(A)$. *Let* $T : E \to E$ *be a relatively weak nonexpansive mapping and* $F(A) \cap F(T) = \emptyset$. *Assume* $0 < a \leq \lambda_n \leq \alpha c^2/2L$. *Define a sequence* $\{x_n\}$ *in* E *by the following algorithm:*

$$\begin{gathered}
x_0 \in E \quad \text{chosen arbitrarily}, \\
z_n = J^{-1}((1 - \lambda_n)Jx_n + \lambda_n A x_n), \\
y_n = Tz_n, \\
C_n = \left\{z \in C_{n-1} \bigcap Q_{n-1} : \phi(z, y_n) \leq \phi(z, z_n) \leq \phi(z, x_n)\right\}, \\
C_0 = \{z \in E : \phi(z, y_0) \leq \phi(z, z_0) \leq \phi(z, x_0)\}, \\
Q_n = \left\{z \in C_{n-1} \bigcap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\right\}, \\
Q_0 = E, \\
x_{n+1} = \Pi_{C_n \cap Q_n}(x_0),
\end{gathered} \tag{2.3}$$

where J *is the duality mapping on* E. *Then* $\{x_n\}$ *converges strongly to a common element* $x^* \in F(A) \cap F(T)$. *This element is also a common solution of operator equation* (1.1) *and variational inequality* (1.2).

Proof. Let $B = J - A$, then $B : E \to E^*$ is α/L-inverse-strongly monotone and α-strongly monotone, so that $B^{-1}0 = F(A)$ has only one element. On the other hand, we have

$$z_n = J^{-1}((1 - \lambda_n)Jx_n + \lambda_n A x_n) = J^{-1}(Jx_n - \lambda_n B x_n). \tag{2.4}$$

By using Theorem 2.1 and Conclusions 1 and 2, we obtain the conclusion of Theorem 2.3.

Taking $T = I$ in Theorem 2.3, we get the following result. □

Theorem 2.4. *Let E be a uniformly smooth and 2-uniformly convex real Banach space; let $A : E \to E^*$ be a L-Lipschitz and α-strongly duality pseudocontractive mapping with nonempty set of duality fixed points $F(A)$. Assume $0 < a \leq \lambda_n \leq \alpha c^2/2L$. Define a sequence $\{x_n\}$ in E by the following algorithm:*

$$
\begin{gathered}
x_0 \in E \quad \text{chosen arbitrarily}, \\
y_n = J^{-1}((1-\lambda_n)Jx_n + \lambda_n A x_n), \\
C_n = \left\{z \in C_{n-1} \bigcap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\right\}, \\
C_0 = \{z \in E : \phi(z, y_0) \leq \phi(z, x_0)\}, \\
Q_n = \left\{z \in C_{n-1} \bigcap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\right\}, \\
Q_0 = E, \\
x_{n+1} = \Pi_{C_n \cap Q_n}(x_0),
\end{gathered}
\tag{2.5}
$$

where J is the duality mapping on E. Then $\{x_n\}$ converges strongly to a duality fixed point $x^* \in F(A)$. This element x^* is also a common solution of operator equation (1.1) *and variational inequality* (1.2).

Iiduka and Takahashi [7] introduce an iterative scheme for finding a solution of the variational inequality problem for an operator A that satisfies the following conditions (i)–(iii) in a 2-uniformly convex and uniformly smooth Banach space E:

(i) A is α-inverse-strongly monotone;

(ii) $VI(C, A) \neq \emptyset$;

(iii) $\|Ay\| \leq \|Ay - Au\|$ for all $y \in E$ and $u \in VI(C, A)$.

They proved the following convergence theorem.

Theorem 2.5 (see, [7]). *Let E be a 2-uniformly convex and uniformly smooth Banach space, whose duality mapping J is weakly sequentially continuous, and C a nonempty, closed convex subset of E. Assume that A is an operator of C into E^*, that satisfies the conditions (i)–(iii). Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$
x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n),
\tag{2.6}
$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence of positive numbers. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, then the sequence $\{x_n\}$ converges weakly to some element $z \in VI(C, A)$, where $1/c$ is the 2-uniformly convexity constant of E. Further $z = \lim_{n\to\infty} \Pi_{VI(C,A)} x_n$.

In this paper, we introduce an iterative scheme for finding a solution of the variational inequality problem for an operator T that satisfies the following conditions (iv)–(vi) in a 2-uniformly convex and uniformly smooth Banach space E:

(iv) T is α-strong duality pseudocontractive,

(v) $VI(C, J, T) \neq \emptyset$,

(vi) $\|Jy - Ty\| \leq \|(J - T)y - (J - T)u\|$ for all $y \in E$ and $u \in VI(C, J, T)$.

By using Theorem 2.5, we prove the following convergence theorem.

Theorem 2.6. *Let E be a 2-uniformly convex and uniformly smooth Banach space, whose duality mapping J is weakly sequentially continuous, and C a nonempty, closed convex subset of E. Assume that T is an operator of C into E^*. that satisfies the conditions (iv)–(vi). Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \Pi_C J^{-1}((1-\lambda_n)Jx_n + \lambda_n T x_n), \tag{2.7}$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence of positive numbers. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, then the sequence $\{x_n\}$ converges weakly to some element $z \in VI(C, J, T)$, where $1/c$ is the 2-uniformly convexity constant of E. Further $z = \lim_{n\to\infty} \Pi_{VI(C,J,T)} x_n$.

Proof. Let $A = J - T$, then $B : E \to E^*$ is α-inverse-strongly monotone, so that $B^{-1}0 = F(A)$. On the other hand, we have

$$x_{n+1} = \Pi_C J^{-1}((1-\lambda_n)Jx_n + \lambda_n T x_n) = J^{-1}(Jx_n - \lambda_n A x_n). \tag{2.8}$$

By using Theorem 2.5, we obtain the conclusion of Theorem 2.6. □

In fact, from condition (vi), we have $F(T) = VI(C, J, T)$, so that under the conditions of Theorem 2.6, the $\{x_n\}$ converges strongly to a duality fixed point $z \in F(T)$. This element z is also a common solution of operator equation (1.1) and variational inequality (1.2). where $\{x_n\}$ is defined by Algorithm (2.7).

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