THE GEOMETRY OF A NET OF QUADRICS IN FOUR-DIMENSIONAL SPACE.

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Among the reasons why the study of the geometry of a net of quadrics in four-dimensional space should prove interesting and attractive there are two which immediately present themselves even before this study is commenced; they are, first, that the base curve of the net, through which all the quadrics pass, is a canonical curve and, second, that the Jacobian curve of the net — the locus of the vertices of the cones belonging to it — is birationally equivalent to a plane quintic.

In space of any number $n (> 2)$ of dimensions a net of quadrics has a base locus, of order eight, and a Jacobian curve; these must both figure prominently in any account of the geometry of the net of quadrics. The polar primes[1] of any point in regard to the quadrics of the net have in common an $[n-3]$ *except when the point lies on the Jacobian curve, when they have in common an* $[n-2]$; there is thus a singly-infinite family of $[n-2]$'s in $(1, 1)$ correspondence with the points of the Jacobian curve, and it is found that each $[n-2]$ has $\frac{1}{2}n(n-1)$ intersections with the Jacobian curve.[2] This is analogous to the well-known result in [3] that, when a point lies on the twisted sextic which is the locus of vertices of cones belonging to a net of quadric surfaces, the polar planes of the point in regard to the quadrics all pass through a trisecant of the sextic. The

[1] When we are concerned with geometry in a linear space $[n]$ of n dimensions the word *prime* is used to denote a linear space of $n-1$ dimensions; the word *primal* is used to denote any locus, other than a linear space, of $n-1$ dimensions. In [4] we also use the term *solid* to denote a three-dimensional space.

[2] Cf. Edge: *Proc. Edinburgh Math. Soc.* (2), 3 (1933), 259—268.

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$[n-2]$'s may be called *secant spaces* of the Jacobian curve. Any two of the secant spaces intersect in an $[n-4]$ and, when $n>4$ and the net of quadrics is perfectly general, there are no two secant spaces having an $[n-3]$ in common. But, when $n=4$, it is found that, for a general net of quadrics, a finite number of pairs of secant planes of the Jacobian curve ϑ do intersect in lines. It is also found, further, that a line which is common to two secant planes of ϑ also lies in a third secant plane, that such a line is a trisecant of ϑ, that, conversely, any trisecant of ϑ lies in three secant planes and that the trisecants of ϑ are associated in pairs (cf. § 10). These results give an added interest to the net of quadrics in four-dimensional space, and bring the Jacobian curve into greater prominence in the four-dimensional case than in any other.

The Jacobian curve of a net of quadrics in $[n]$ is a particular example of those curves in $[n]$ which are generated by $n+1$ projectively related doubly-infinite systems of primes; these curves are of order $\frac{1}{2}n(n+1)$ and, being birationally equivalent to plane curves of order $n+1$, are of genus $\frac{1}{2}n(n-1)$. Moreover such a curve has, like the Jacobian curve, ∞^1 secant spaces $[n-2]$ each meeting it in $\frac{1}{2}n(n-1)$ points, these secant spaces being in $(1, 1)$ correspondence with the points of the curve and $\frac{1}{2}n(n-1)$ of them passing through each point of the curve.[1] We shall not however in this paper be concerned with those curves which are not Jacobians of nets of quadrics, and we shall obtain the properties of the Jacobian curve ϑ in $[4]$ without further reference to the more general type of curve.

It is to be understood throughout that the net of quadrics is perfectly general. The ways in which it may be specialised are manifold: the base curve or Jacobian curve may have multiple points or may break up into component curves; the net may include one or more cones with line vertices; there may be singular pencils of quadrics belonging to the net — i. e. pencils of quadrics whose members are all cones. Many of these specialisations are of great interest, but they will not be considered here.

The processes employed in obtaining the results below rest almost entirely on two fundamental ideas — the idea of conjugacy of points in regard to quadrics and the idea of the projective generation of loci. There is no need to elaborate

[1] Cf. White: *Proc. Camb. Phil. Soc.* 22 (1924), 1—10.

these ideas here as they are both very well known and have been very widely used; the idea of conjugacy was first used in higher space by Segre in his memoir on quadric loci[1], while that of projective generation, which is fundamental in the geometry of von Staudt and Reye, was extended to space of higher dimensions by Veronese.[2] Nor is there anything novel in combining the two ideas; they were both used freely by Segre in the above-mentioned paper and, to mention only one other instance, by Reye in his *Geometrie der Lage* when, for example, he studies the properties of a net of quadric surfaces.

A brief résumé of some of the results which are obtained may now be given.

After one or two preliminary definitions the Jacobian curve ϑ, of order ten and genus six, is introduced at once, and the cubic complex which is generated by the lines on the quadrics is also mentioned. It is then pointed out that the polar solids of any point O in regard to all the quadrics of the net have in common a line j, which is called the line conjugate to O; there is of course an exception to this statement, since the polar solids of a point on ϑ have in common a plane which is a secant plane of ϑ, meeting it in six points; but the statement is always true so long as O is not a point of ϑ. These lines j form a system J of ∞^4 lines. It is found that those lines which are conjugate to the points of a line λ generate a cubic scroll, the planes of the ∞^2 directrix conics of the scroll being the polar planes of λ in regard to the ∞^2 quadrics of the net; the scroll is a cone if λ belongs to J. If however λ meets ϑ in a point P the scroll consists of a quadric and the secant plane which is conjugate to P, while if λ is a chord of ϑ the scroll is made up of the secant planes conjugate to its two intersections with ϑ and of another plane. If λ should happen to be one of the trisecants of ϑ it is found, in § 10, that the lines conjugate to the points of λ all coincide with a second trisecant of ϑ; these two trisecants are mutually related to each other, and are called a pair of conjugate trisecants.

The ∞^2 lines which are conjugate to the points of a plane π are found to generate a six-nodal cubic primal Π; there is also a second mode of generation of Π, namely by means of the polar lines of π in regard to all the quadrics of the net. The six nodes of Π have as their conjugate lines six lines belonging

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[1] »Studio sulle quadriche in uno spazio lineare ad un numero qualunque di dimensioni.« *Mem. Acc. Torino* (2), 36 (1884), 3.

[2] *Math. Annalen* 19 (1882), 161.

to the system J and lying in π, and these are the only lines of J which can lie in π. It therefore follows that the lines of J which lie in an arbitrary solid generate a congruence of order three and class six. The primal Π will be specialised when π is not of general position; if π meets ϑ then Π acquires an extra node, so that there are primals with seven, eight, nine or ten nodes associated with planes which meet ϑ in one, two, three or four points. When π is a secant plane of ϑ, meeting it in six points, Π becomes a cubic cone whose vertex is that point of ϑ to which the secant plane is conjugate. If π contains a trisecant of ϑ then Π has the conjugate trisecant of ϑ as a double line, having also three nodes not on this line; it has also a fourth isolated node if π meets ϑ in a further point. We thus obtain, corresponding to different positions of π in regard to ϑ, all the different types of cubic primals which can be generated by means of three projectively related nets of solids.

The secant planes of ϑ are studied in detail in §§ 5 *et seq.* They generate a primal of order fifteen on which ϑ is a sextuple curve, and the six secant planes which pass through any point of ϑ are met by an arbitrary solid in six lines forming one half of a double-six. The solid which joins a point P of ϑ to its conjugate secant plane is the common tangent solid at P of all those quadrics of the net which pass through P; there thus arises a singly-infinite family of solids, and it is found that through an arbitrary point there pass twenty-five of them.

The properties of a pair of conjugate trisecants of ϑ, some of which have already been mentioned, are obtained in § 10; there are *ten pairs* of conjugate trisecants, and the solid which contains a pair of conjugate trisecants is such that there are four cones belonging to the net which meet it in plane-pairs.

The locus of the poles of an arbitrary solid S_3 in regard to the ∞^2 quadrics of the net is a determinantal sextic surface, and the trisecants of the surface are the lines which are conjugate to the points of S_3. The line which is conjugate to any point of the surface lies in S_3 and, conversely, any line of the system J which lies in S_3 is conjugate to a point of the surface. We thus see that the lines of the $(3, 6)$ congruence which is generated by the ∞^2 lines of J lying in S_3 can be represented by the points of a determinantal sextic surface in [4]. The surface is particularised in various ways when S_3 occupies special positions; if, for example, S_3 is the solid which joins a point P of ϑ to its conjugate secant plane, the surface has a triple point at P.

In § 16 the loci of lines which are conjugate to the points of a curve or of a surface are referred to, and it is found that of the secant planes of ϑ there are 120 which touch the curve.

In § 17 the locus M_3^{16} generated by the chords of C, the base curve of the net of quadrics, is considered; C is a sextuple curve and ϑ a quadruple curve on the locus, which has a double surface of order sixty. It is found that every chord of ϑ which meets C is a chord of C, and that there are 120 of these common chords of ϑ and C, each of these chords being such that the tangents of C at its two intersections with the chord meet each other.

The quadrics of the net can be represented, in Hesse's manner, by the points of a plane σ; the cones of the net are then represented by the points of a quintic curve ζ, without multiple points.[1] From § 19 until the end of the paper the work centres round the (1, 1) correspondence between ϑ and ζ; several features of this correspondence are of course exactly analogous to those of Hesse's correspondence between the Jacobian curve of a net of quadric surfaces and a plane quartic; for example those quadrics of the net which touch an arbitrary solid are represented in σ by the points of a contact quartic of the quintic ζ, the ten points of contact not lying on a cubic curve; we thus obtain a system of ∞^4 contact quartics of ζ, any two sets of contacts of two curves of the system making up the complete intersection of ζ with a quartic curve. This system of ∞^4 contact quartics is one of 2080 systems all of which have similar properties and there are, beside these 2080 systems, 2015 systems of contact quartics of ζ of a different kind.

There is thus associated with each solid of the [4] in which the net of quadrics lies a contact quartic of ζ; this contact quartic has special forms when the solid has special positions. When, for example, the solid joins a point P of ϑ to its conjugate secant plane the contact quartic breaks up into the tangent of ζ at that point which, in the correspondence between ϑ and ζ, corresponds to P, and a cubic curve which passes through the remaining three intersections of this tangent with ζ and touches ζ in six other points. When we consider the solid containing a pair of conjugate trisecants of ϑ it is found that the associated contact quartic *breaks up into a pair of conics;* the four intersections of the two conics all lie on ζ while each conic is a tritangent conic of ζ, the two sets of three contacts corresponding, in the correspondence between ϑ and ζ,

[1] The existence of the (1, 1) correspondence between the locus of vertices of cones, belonging to a net of quadrics in [4], and a plane quintic was pointed out by Wiman: *Stock. Akad. Bihang* 21 (1895), Afd. 1, No. 3.

to the two sets of three points of ϑ which lie on the two conjugate trisecants. The configuration of points on ζ which is associated with a pair of conjugate trisecants of ϑ is studied in §§ 22 *et seq*, and a form is obtained for the equation of ζ.

A set of ten points of ϑ which correspond to the ten intersections of ζ with a conic is a canonical set on ϑ; it is shown in § 28 that these canonical sets on ϑ are cut out by quadrics passing through any one of the sets, and also that all the quadrics which pass through nine of the points of a canonical set on ϑ also pass through the tenth point of the set. Every canonical set on ϑ is such that there is a quadric touching ϑ at every point of the set; thus there arises a set of ∞^5 contact quadrics of ϑ.

The quadrics of the net which are represented in σ by the points of a conic have as their envelope a quartic primal on which C is a double curve; a few properties of such primals are given in §§ 29—33; they are of class 28, having no bispatial points on C. If the conic touches ζ the associated quartic primal has a node at the corresponding point of ϑ; hence, associated with the 2015 contact conics of ζ, there are 2015 five-nodal quartic primals; of these 2015 primals 992 are such that their five nodes lie in a solid, such a solid meeting the primal in a quartic surface with a double twisted cubic. Since the three intersections of ϑ with any one of its trisecants correspond to three points of ζ which are points of contact of ζ with a tritangent conic there is a quartic primal, with C as a double curve, having nodes at the three intersections of ϑ with any one of its trisecants.

In conclusion a *canonical form* is obtained for the equations of the quadrics of the net.

1. We consider a doubly-infinite linear system, or *net*, N say, of quadrics in [4]. Algebraically, if Q_0, Q_1, Q_2 are three linearly independent homogeneous quadratic functions of five variables, such a net is given by an equation

$$x Q_0 + y Q_1 + z Q_2 = 0,$$

where $x:y:z$ are varying parameters. Through two points of general position there passes one and only one quadric of the system. Through an arbitrary point of [4] there pass ∞^1 quadrics of the net; these quadrics have in common a quartic surface, and such a quartic surface, the base surface of a pencil of quadrics belonging to N, will be called a *cyclide*.[1] A cyclide contains, in general,

[1] The term *cyclide* is already in use for the surface in [3] which is the projection of the quartic surface of intersection of two quadrics in [4].

sixteen lines. All the quadrics of N have in common a curve C, the base curve of N, of order eight and genus five; C is met by any solid in a set of eight associated points.

Among the quadrics of N there are ∞^1 which are cones; the locus of the vertices of these cones is a curve ϑ — the *Jacobian curve of the net of quadrics.* Algebraically ϑ is given by the vanishing of all the three-rowed determinants of a matrix of three rows and five columns, the elements of the matrix being linear in the five homogeneous coordinates of the space. Hence ϑ is of order ten.[1] It will naturally be expected to play a very important part in the geometry of the net of quadrics.

If we regard the parameters $x:y:z$ as the homogeneous coordinates of a point in a plane σ then the quadrics of N are *represented* by the points of σ, and the cones of N must be represented by the points of some plane curve ζ. Since the condition that a quadric should be a cone is that its discriminant should vanish, the left hand side of the equation of ζ is a symmetrical determinant, of five rows and columns, whose elements are homogeneous linear functions of x, y, z; thus ζ is a quintic curve. The two curves ϑ and ζ are in $(1, 1)$ correspondence; any point of ϑ is the vertex of a cone of N which is represented by the corresponding point of ζ; any point of ζ represents a cone of N whose vertex is the corresponding point of ϑ. This correspondence, which is analogous to the $(1, 1)$ correspondence established by Hesse between a plane quartic and a twisted sextic, will be considered in detail later (§§ 19 *et seq*), for the present it will suffice to remark that the two curves ϑ and ζ have the same genus. Since ζ is in fact without double points it is of genus 6; hence ϑ is also of genus 6.

An arbitrary line of [4] does not lie on a quadric of N; for three conditions must be imposed on a quadric in order that it should contain a line and the quadrics of N have only freedom 2. But each quadric of N has ∞^3 lines upon it, so that, of the ∞^6 lines of [4], there are ∞^5 which do lie on quadrics of N; the lines of [4] which lie on quadrics of N therefore form a complex V. V may also be defined as the complex of lines which are cut in involution by the quadrics of N. Moreover, since the two double points of the involution are

[1] Salmon: *Higher Algebra* (Dublin, 1885), Lesson 19. The two statements that ϑ is of order ten and that it is birationally equivalent to a plane quintic also follow easily from the fact that ϑ is the locus of points which are common to corresponding solids of five projectively related doubly-infinite systems.

conjugate points in regard to every quadric of N, we have a third definition of V as the complex of lines which join pairs of points that are conjugate in regard to every quadric of N. The lines of V which pass through any point O of [4] are those lines which lie on quadrics of N and pass through O; but the quadrics of N which pass through O form a pencil whose base surface is a cyclide passing through O, so that, since a line through O lies on a quadric of N if and only if it is a chord of this cyclide, the lines of V which pass through O are the chords of the cyclide which pass through O. Since these chords form a three-dimensional cubic cone *V is a cubic complex.*

It has been remarked that an arbitrary line of [4] does not lie on a quadric of N; but if a line meets C only two conditions need be imposed on a quadric of N in order that it should contain the line, so that the line does lie on a quadric of N. Hence *all the lines which meet C belong to V.* A chord of C lies not merely on one quadric of N but on all the quadrics of N belonging to a pencil, since a quadric of N only has to satisfy one linear condition in order that it should contain the line, which already meets it in two fixed points. Conversely: if a line lies on all the quadrics of N which belong to a pencil it must be one of the sixteen lines on the base cyclide of the pencil, and is a chord of C. In particular the chord may be a tangent of C.

2. The ∞^2 polar solids of O in regard to the quadrics of N have a line in common; since every point of this line is conjugate to O in regard to every quadric of N we shall speak of the line as the line conjugate to O in regard to N. There is thus associated with each point of [4], with certain exceptions to be noted later (§ 5), a conjugate line; these conjugate lines are ∞^4 in aggregate and form a system of lines which will be denoted by J. The ∞^2 polar solids of O all pass through the line j conjugate to O and are related to the quadrics of N in such a way that to each quadric of N there corresponds one and only one solid through j and that to quadrics of N which belonging to the same pencil there correspond solids containing the same plane through j, and conversely; we may therefore say that the ∞^2 solids through j are related projectively to the ∞^2 quadrics of N. Associated with two arbitrary points O and O' are the conjugate lines j and j'; the solids through j and the solids through j' form two doubly-infinite systems *which are projectively related to each other,* solids of the two systems corresponding when they are the polar solids of O and O' respectively in regard to the same quadric of N.

The lines which join a point O to the points of its conjugate line j all belong to V because each of them is the join of a pair of points which are conjugate in regard to every quadric of N. The quadrics of N which pass through O all contain a cyclide passing through O; the plane which joins O to j is common to the tangent solids of all these quadrics at O and is the tangent plane of the cyclide at O.

The line conjugate to a point of C is the line common to the tangent solids of all the quadrics at this point, and so is the tangent of C at the point. Thus all the tangents of C belong to J; we have remarked already that they belong to V. If the point O lies on a tangent of C the polar solids of O in regard to the quadrics of N all pass through the point of contact of the tangent with C, so that the line conjugate to O must pass through this point. Hence the lines which are conjugate to the points of a tangent of C all pass through the point of contact of the tangent with C, forming a cone with vertex at that point. Hence, since all these lines meet C, they belong not only to J but also to V. Whence we can identify at once ∞^2 of the lines common to J and V, namely the lines which are conjugate to the points of the surface formed by the tangents of C.

3. The lines j conjugate to the points of a line λ generate a scroll Δ. Now the polar solid of any point of λ in regard to a quadric contains the polar plane of λ in regard to that quadric; hence the ∞^2 polar planes of λ in regard to the ∞^2 quadrics of N all meet all the generators of Δ. If we take three planes which are the polar planes of λ in regard to three quadrics of N which do not belong to the same pencil, then the three pencils of solids which join these planes to the generators of Δ are projectively related to each other, each pencil being related projectively to the range of points on λ. The generators of Δ may therefore be obtained as the intersections of corresponding solids of three projectively related pencils, so that[1] *the lines which are conjugate to the points of a line λ generate a cubic scroll Δ, the polar planes of λ in regard to the quadrics of N being the planes of the directrix conics of Δ.* The cubic scroll λ has a directrix λ'; the line conjugate to any point O of λ therefore meets λ', say in O'. The two points O and O' are conjugate in regard to every quadric of N, so that the line conjugate to O' passes through O. Whence *if the lines which are conjugate to the points of a line λ generate a cubic scroll whose directrix*

[1] Cf. Veronese: *Math. Annalen* 19 (1882), 229—230.

is λ', then the lines which are conjugate to the points of λ' generate a cubic scroll whose directrix is λ. There is thus established, by means of the net N, an involutory correspondence between the lines λ and λ' of [4]. If λ is a chord of C then λ' coincides with it, for the line which is conjugate to a point O on a chord of C meets this chord in the point O' which is harmonically conjugate to O in regard to the two intersections of the chord with C.

Through a general point L of Δ there pass ∞^1 of its directrix conics, and the planes of these conics are the polar planes of λ in regard to the quadrics of a pencil belonging to N; one of the planes is that which joins the generator through L to the directrix λ'. Conversely: the polar planes of λ in regard to the quadrics of any pencil belonging to N all pass through the same point L of Δ; so that each pencil of quadrics belonging to N includes a quadric such that the polar plane of λ in regard to it passes through λ'. If, however, we take a point on λ', the ∞^1 planes of the directrix conics of Δ passing through the point are the planes which join λ' to the generators of Δ; there is a particular pencil of quadrics belonging to N which is such that the polar planes of λ in regard to the quadrics of the pencil all pass through λ'. Also the polar planes of λ' in regard to the quadrics of the pencil all pass through λ.

The preceding arguments are not valid if λ belongs to J, for then the polar planes of λ in regard to the quadrics of N all pass through that point to which λ is conjugate. If j is the line conjugate to a point O the lines which are conjugate to the points of j are determined as the intersections of corresponding solids of three projectively related pencils, but now the planes which are the bases of the pencils all pass through O. Hence the lines which are conjugate to the points of a line of J generate a two-dimensional cubic cone, the vertex of the cone being the point to which the line of J is itself conjugate. There can be no other lines which pass through the vertex of the cone and belong to J; hence the lines of J which pass through an arbitrary point of [4] form a two-dimensional cubic cone.

The lines conjugate to the points of a line λ_1 generate a cubic scroll Δ_1 and the lines conjugate to the points of a line λ_2 generate a cubic scroll Δ_2; Δ_1 and Δ_2 have nine common points and the line conjugate to any one of these points meets both λ_1 and λ_2. Conversely, if a line of J meets both λ_1 and λ_2 the point to which it is conjugate lies on both Δ_1 and Δ_2. Hence there are nine lines of J meeting two arbitrary lines. This is equivalent to the statement that the lines of J which meet an arbitrary line λ and which lie in a solid S_3 passing through λ form a ruled surface of order nine. The line λ is a triple line on this ruled surface, three generators of the surface passing through each point of it; for the lines of J which pass through any point of λ form a cubic cone, three of whose generators lie in S_3.

4. Take now an arbitrary plane π; we shall show that the ∞^2 lines j which are conjugate to the points of π and the ∞^2 lines k which are the polar lines of π in regard to the quadrics of N generate the same three dimensional cubic locus Π.

Let j be the line conjugate to a point O of π and k the polar line of π in regard to a quadric Q; the solid jk is the polar solid of O in regard to Q. Corresponding to different points O in π we obtain different solids passing through k; to the points O which lie on a line in π there correspond the solids which contain the polar plane of the line in regard to Q. Thus we may say that the solids through k are related projectively to the points of π. Suppose now that we take the polar lines, k, k', k'' of π in regard to three quadrics Q, Q', Q'' which belong to N and which do not all belong to the same pencil; then the lines j which are conjugate to the points of π are determined as the intersections of corresponding solids of three projectively related doubly-infinite systems, so that they generate a cubic primal Π. We know that the lines which are conjugate to the points of a line λ in π generate a cubic scroll Δ; hence we have a system $\mathfrak{L}$ of ∞^2 cubic scrolls on Π. Since any two lines in π have a point of intersection any two scrolls of the system $\mathfrak{L}$ have a common generator.

Let j be the line conjugate to a point O of π, and let O' be any point of j; then the line j' conjugate to O' passes through O and the solid $\pi j'$ is the polar solid of O' in regard to some quadric Q belonging to N; the polar line of π in regard to Q therefore passes through O'. Hence any point of any line which is conjugate to a point of π lies on the polar line of π in regard to some quadric of N, and therefore the ∞^2 polar lines of π in regard to the quadrics of N generate the same locus Π as do the ∞^2 lines which are conjugate to the points of π.

If we take the lines j and j' which are conjugate to two points O and O' of π then the polar solids of O in regard to the quadrics of N all pass through j while those of O' all pass through j'; if solids through j and j' correspond to each other when they are the polar solids of O and O' respectively in regard to the same quadric of N then the two systems of solids are projectively related to each other. Hence, if we take the lines j, j', j'' which are conjugate to three non-collinear points of π, the polar lines of π in regard to the quadrics of N are determined as the intersections of corresponding solids of three projectively related doubly-infinite systems. We thus obtain again the primal Π. It contains two systems of lines; the first system consists of the ∞^2 lines j conjugate to the points of π and the second system consists of the ∞^2 lines k which are the polar lines of π in regard to the quadrics of N. There is a system $\mathfrak{M}$ of ∞^2 cubic scrolls associated with this second generation of Π, just as there was a system $\mathfrak{L}$ of ∞^2 cubic scrolls associated with the first generation; for the polar lines of π in regard to the quadrics of a pencil generate a cubic scroll, and there are ∞^2 pencils of quadrics belonging to the net N. Since any two pencils of quadrics belonging to the same net have a quadric in common, any two scrolls of the system $\mathfrak{M}$ have a common generator.

Since the polar line of a plane in regard to a cone passes through the vertex of the cone the primal Π contains the curve ϑ. Moreover; since a pencil of quadrics contains five cones each scroll of the system $\mathfrak{M}$ meets ϑ in five points.

We have already mentioned the fact that the line which is conjugate to a point of C is the tangent of C at the point. Suppose now that the tangent of C at a point T meets π; then the line conjugate to its point of intersection with π must pass through T, so that T is an intersection of C and Π. Conversely, if Π meets C in T the tangent of C at T must meet π, because the only lines of J which pass through T are those which are conjugate to the points of the tangent of C at T. Hence *the tangents of C at its twenty-four intersections with Π, and only these tangents, meet the plane π.*

Let p be an intersection of π and Π; then through p there passes a line k which is the polar line of π in regard to some quadric Q belonging to N. The point p is therefore the pole, in regard to Q, of some solid passing through π; this solid must then be the tangent solid of Q at p, and so meets Q in a cone vertex p. Hence π meets Q in a line-pair intersecting at p. The cubic curve in which π meets Π is therefore the locus of points in which π is touched by quadrics of N, and is the Jacobian curve of the net of conics in which the quadrics meet π.

The cubic primal Π has six nodes; its properties are obtained in a paper by Castelnuovo.[1] All the cubic scrolls of both systems $\mathfrak{L}$ and $\mathfrak{M}$ pass through

[1] *Atti del R. Istituto veneto* (6) 5 (1887), 1249.

all the nodes, and each system of scrolls includes six cones, the vertices of the cones being the nodes of Π. Since the system $\mathfrak{L}$ contains six cubic cones whose vertices are nodes of Π it follows that the lines which are conjugate to the six nodes of Π lie in π. Hence an arbitrary plane contains *six* lines which belong to J. Those lines of J which lie in an arbitrary solid S_3 generate a congruence of order 3 and class 6. It will be seen in § 6 that all the chords of ϑ belong to J; hence the ten intersections of S_3 with ϑ must be singular points of the congruence, as there are at least nine lines of the congruence of order 3 passing through any one of them.

Any scroll of the system $\mathfrak{L}$ and any scroll of the system $\mathfrak{M}$ form the complete intersection of Π with a quadric cone[1]; hence, since each scroll of the system $\mathfrak{M}$ meets ϑ in five points, each scroll of the system $\mathfrak{L}$ meets ϑ in fifteen points.

The Secant Planes of ϑ.

5. Take an arbitrary point P of ϑ; we shall denote the cone of N whose vertex is at P by (P), and similarly for any other point of ϑ. The quadrics of N which pass through P all have a common tangent solid ϖ at P and form a pencil whose base cyclide has a node at P. The five cones which belong to the pencil consist of three cones (A), (B), (C), whose vertices, A, B, C lie in ϖ, and of the cone (P) counted twice. Any line lying in ϖ and passing through P is a generator of a quadric of the pencil so that, whereas the lines of V which pass through an arbitrary point of [4] form a cubic cone, if the point lies on ϑ the cubic cone consists of the cone of N whose vertex is at the point and of the lines which pass through the point and lie in the common tangent solid of all the quadrics of N which pass through the point. If a line of V passes through P and is not a generator of (P) then it must lie in ϖ.

The solid ϖ meets ϑ in the four points P, A, B, C and in six other points X_1, X_2, X_3, X_4, X_5, X_6. If X is any one of these last six points the line PX belongs to V and is therefore cut in involution by the quadrics of N; the two double points of the involution must be P and X since the line is not a generator either of (P) or of (X). Hence P and X are conjugate in regard to every quadric of N. The six points X are thus all conjugate to P in regard to every quadric of N, so that the polar solids of P in regard to the quadrics of N all pass through the six points X, which must therefore be coplanar. In this way

[1] Castelnuovo: *loc. cit.*, 1261.

we obtain ∞^1 planes each of which has six intersections with ϑ, the planes being in (1, 1) correspondence with the points of ϑ and therefore forming a family of genus 6; we shall call them the *secant planes* of ϑ. *A point P of ϑ is conjugate not to the points of a line j but to all the points of a secant plane of ϑ*, which may therefore be called the *secant plane conjugate to P*. All lines in a secant plane of ϑ belong to J. If the secant plane conjugate to P meets ϑ in a point Q then P and Q are conjugate points in regard to every quadric of N and the secant plane conjugate to Q passes through P; hence through each point of ϑ there pass *six* secant planes.

In general a curve in [4] has only a finite number of planes which meet it in six points; the curve ϑ is thus exceptional in this respect.

If five points of ϑ are the vertices of the five cones which belong to some pencil of quadrics of N the polar solid of any one of the five points, in regard to any quadric of the pencil, is the solid which contains the other four. Hence, if four points of ϑ form, together with the point P of ϑ, a set of five points which are the vertices of the five cones belonging to a pencil of quadrics of N, the solid containing the four points must pass through the secant plane α which is conjugate to P. Conversely: any solid passing through α meets ϑ further in four points not lying in α; these four points are vertices of cones of N which all belong to the same pencil, the fifth cone of the pencil being (P).

The lines of V passing through an arbitrary point o of [4] form a cubic cone, so that thirty of them meet ϑ; five of these are generators of cones of N — they join O to the vertices of the five cones which belong to the pencil of quadrics of N which pass through O. Hence there are twenty-five lines of V which meet ϑ, pass through O and are not generators of cones of N; the solids ϖ associated with the intersections of ϑ with these lines must therefore pass through O, and they are the only solids ϖ which can do so. Whence *there are twenty-five solids ϖ passing through an arbitrary point of* [4].

6. The secant planes of ϑ form a three-dimensional locus on which ϑ is a sextuple curve and whose section by an arbitrary plane is a curve of genus 6. The order of this locus is the number of its intersections with an arbitrary line λ. But if O is a point of intersection of λ with a secant plane of ϑ the point of ϑ which is conjugate to this secant plane must lie on the line j which is conjugate to O; conversely, if the line j which is conjugate to O meets ϑ in a point P the secant plane conjugate to P must pass through O. But we have seen

that the lines which are conjugate to the points of λ generate a cubic scroll $\varDelta$ which meets ϑ in fifteen points; hence *the secant planes of* ϑ *form a locus* R_3^{15} *of order fifteen.*

The order of R_3^{15} can also be obtained as follows. The points of the curve ϑ, of order ten, are in (1, 1) correspondence with the secant planes; the solid joining a point of ϑ to the corresponding secant plane is a solid ϖ, and therefore the solids so obtained, by joining the points of ϑ to the secant planes which correspond to them, form a singly-infinite family of which there are twenty-five members passing through an arbitrary point of [4]. Hence, if n is the order of the locus formed by the secant planes, we have, since no point of ϑ lies in its conjugate secant plane,

$$10 + n = 25;$$

we therefore again find that the locus is of order fifteen.

Yet a further remark may be made concerning the order of R_3^{15}. A curve in [4] is usually such that there is only a finite number of planes which meet it in six points; but there are in general ∞^1 planes which meet the curve in five points. These ∞^1 five-secant planes form a three-dimensional locus M on which the six-secant planes are sextuple planes. The order of M has been obtained by Severi[1]. Now in the particular case when the curve has ∞^1 six-secant planes these form a locus which counts six times as part of the locus M, since each six-secant plane must be regarded as six five-secant planes, each of its six intersections being omitted in turn. Since the secant planes of ϑ form a locus of order fifteen Severi's formula should, when applied to a curve of order ten and genus 6 in [4], give the value ninety, assuming that ϑ has not got a family of five-secant planes distinct from the six-secant planes we have been discussing; and, in fact, the value ninety is actually obtained.

Since any two secant planes of ϑ have a point of intersection there is a double surface on R_3^{15}; the order of this double surface is the number of its intersections with an arbitrary plane π, these intersections being double points of the curve in which π meets R_3^{15}. But this curve, since it is of order 15 and genus 6, has 85 double points, so that the double surface of R_3^{15} is of order 85.

[1] *Memorie Torino*, 51 (1902), 104. The ∞^1 planes which meet a curve of order n and genus p in [4] each in five points generate a locus whose order is

$$\frac{1}{24}(n-2)(n-3)(n-4)^2(n-5) - \frac{1}{3}(n-3)(n-4)(n-5)p + \frac{1}{2}(n-4)p(p-1).$$

In the particular case when $n = p + 4$ this reduces to $\frac{1}{24}p^2(p-1)(p-2)(p-3)$.

If the secant planes which are conjugate to the two points P and Q of ϑ intersect in the point E then, since the polar solids of P in regard to all the quadrics of N pass through E and the polar solids of Q in regard to all the quadrics of N also pass through E, the polar solids of E in regard to all the quadrics of N pass both through P and through Q, and therefore through the line PQ. The line conjugate to E is therefore PQ, and *all the chords of* ϑ *belong to* J. The points of the double surface of R_3^{15} are in (1, 1) correspondence with the chords of ϑ, and therefore also with the chords of a plane quintic.

A prime section of R_3^{15} is a ruled surface whose double curve is met by each of its generators in thirteen points; hence every secant plane of ϑ is met by the other secant planes in the points of a curve of order thirteen. This curve has quintuple points at each of the six points in which the plane meets ϑ; since it is of genus 6 it cannot have any other multiple points.

7. The cubic scroll generated by the lines which are conjugate to the points of a line λ which meets ϑ in a point P contains as part of itself the secant plane α which is conjugate to P. The polar planes of λ in regard to the quadrics of N all meet α in lines; the pencils of solids whose bases are these polar planes are projectively related to each other, but now the solids which join the planes to α all correspond to each other. The locus of lines which are conjugate to points of λ is therefore, apart from the plane α, a regulus of which one line lies in α. But if λ belongs to J the lines in which α is met by the polar planes of λ all pass through the point O of α to which λ is conjugate; instead of a regulus we have a quadric cone with vertex O, one generator of the cone lying in α.

The cubic cone generated by the lines which are conjugate to the points of a chord PQ of ϑ contains the secant planes α and β conjugate to P and Q. If E is the point of intersection of α and β the polar planes of PQ in regard to all the quadrics of N pass through E, each of them meeting α and β in lines through E. The locus of the lines which are conjugate to points of PQ is, apart from the planes α and β, a plane passing through E and meeting both α and β in lines through E.

8. Consider now the cubic primal Π associated with a plane π which meets ϑ in a point P. When Π is regarded as the locus of lines which are conjugate to the points of π it is seen that it contains the secant plane α conjugate to P; when it is regarded as the locus of the polar lines of π in regard

to the quadrics of N it is seen that it contains the polar *plane* ϱ of π in regard to the cone (P). The polar lines of π in regard to the quadrics of N all meet α while the lines conjugate to the points of π all meet ϱ. When Π is generated by means of three projectively related nets of solids through the polar lines k, k', k'' of π in regard to three quadrics of N not belonging to the same pencil the three solids αk, $\alpha k'$, $\alpha k''$ correspond to each other in the projectivity, being the polar solids of P in regard to the three quadrics, and have as their intersection the plane α. Similarly, when Π is generated by means of three projectively related nets of solids through the lines j, j', j'' conjugate to three non-collinear points of π, the three solids ϱj, $\varrho j'$, $\varrho j''$, having in common the plane ϱ, correspond in the projectivity. The point of intersection of α and ϱ is a node of Π, in addition to the six nodes that the primal Π in general possesses.

The primal Π thus possesses seven, eight, nine or ten nodes according as π meets ϑ in one, two, three or four points[1]. Suppose, in particular, that π is a quadrisecant plane of ϑ, meeting it in P_1, P_2, P_3, P_4 (ϑ having ∞^2 quadrisecant planes). Then Π is a Segre cubic primal with ten nodes. The polar lines of π in regard to the quadrics of N, which generate Π, meet the secant planes α_1, α_2, α_3, α_4 conjugate to the points P_1, P_2, P_3, P_4; the lines conjugate to the points of π, which also generate Π, meet the polar planes ϱ_1, ϱ_2, ϱ_3, ϱ_4 of π in regard to the cones (P_1), (P_2), (P_3), (P_4). The eight planes lie on Π, and it is known that Π also contains seven further planes. When arranged in the form

$$\begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \varrho_1 & \varrho_2 & \varrho_3 & \varrho_4 \end{matrix}$$

the eight planes form a *double-four*, each plane meeting in lines the three planes which are written in the other row and not in the same column; α_4 and ϱ_2, for example, have a line in common because they both lie in the polar solid of P_4 in regard to (P_2).

None of the six nodes of the primal Π which is associated with a plane π of general position lies on ϑ; but if π meets a secant plane α in a line then this line is one of the six lines of J which lie in π, and the point of ϑ to which α is conjugate is one of the nodes of Π. In particular: the primal Π associated with the plane of intersection of the two solids ϖ and ϖ' has nodes

[1] Concerning the cubic primals with seven, eight, nine or ten nodes see Segre: *Memorie Torino* (2), 39 (1889), pp. 15 *et seq.*

at the points P and P' on ϑ. Since there is one plane in [4] meeting each of three given planes in lines there is a primal Π with nodes at three arbitrary points of ϑ.

9. Let us now take the secant plane α which is conjugate to a point P of ϑ and which meets ϑ in the six points $X_1, X_2, X_3, X_4, X_5, X_6$. The lines which are conjugate to the points of α all pass through P, as also do the polar lines of α in regard to all the quadrics of N. These two sets of lines are in fact the same. For let j be the line through P which is conjugate to the point O of α; the lines which are conjugate to the points of j other than P generate a quadric cone whose vertex is O. Let j' be the generator of this cone which is conjugate to any point O' of j; the solids through j' are the polar solids of O' in regard to the quadrics of N, so that there is one quadric of N in regard to which the polar solid of O' is the solid containing j' and α. The polar plane of j in regard to this quadric is therefore α — the intersection of the polar solids of O' and P; this is the same as saying that the polar line of α in regard to the quadric is j. Hence the line j which is conjugate to any point O of α is the polar line of α in regard to some quadric of N. The converse is also true.

The lines through P which are conjugate to the points of α and are the polar lines of α in regard to the quadrics of N generate a cubic cone Π_P; this is the cubic primal associated with the secant plane α. It contains ϑ and meets α in a cubic curve passing through the six points X. The chords of ϑ passing through P are all generators of Π_P; they are conjugate to the points of the curve of order thirteen and genus 6 in which α is met by the other secant planes.

Take any three generators of Π_P which are the polar lines of α in regard to three quadrics of N which do not all belong to the same pencil. The polar solids of any point O of α in regard to the three quadrics pass respectively through these three lines and meet in the line j conjugate to O. The three systems of solids passing respectively through the three lines are thus related collinearly to the system of points of the plane α; there are six sets of three corresponding solids which meet in planes, and these planes must be the six secant planes $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ which pass through P and are conjugate to the six points $X_1, X_2, X_3, X_4, X_5, X_6$. Hence *the six secant planes which pass through any point of ϑ are met by any solid not passing through the point in six lines forming one half of a double six,* and so any five of the six secant planes through

P are such that there is a plane passing through P and meeting them all in lines.

The cone Π_P also contains the planes $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6$ which are the polar planes of α in regard to the cones $(X_1), (X_2), (X_3), (X_4), (X_5), (X_6)$ whose vertices lie in α. Since the polar solid of X_2 in regard to (X_1) contains the secant plane α_2 conjugate to X_2 and the polar plane β_1 of α in regard to (X_1) the two planes α_2 and β_1 meet in a line. Thus the twelve planes

$$\begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \end{matrix}$$

are met by any solid not passing through P in the twelve lines of a double-six.

The plane β_1, being the polar plane of α in regard to (X_1), contains X_1; but it cannot contain any point of ϑ other than X_1 and P. For if the point Y of ϑ does not coincide either with X_1 or with P and yet lies in β_1 the polar solid of Y in regard to (X_1) must contain both α and the secant plane conjugate to Y, which therefore meets α in a line. But, if α is a general secant plane of ϑ, there are no other secant planes meeting it in lines. Hence we have the following: *each set of five of the six secant planes through a point P of ϑ is such that there is a plane which meets every plane of the set in a line through P; this plane meets ϑ in one point other than P, and this point is the point of ϑ which is conjugate to the sixth secant plane through P.* We thus have a method of obtaining the point of ϑ which is conjugate to a given secant plane, and there are six ways of passing from a given secant plane to the point to which it is conjugate.

In addition to the planes α and β the cone Π_P contains fifteen further planes γ; the solid $\alpha_r\beta_s$, where r and s are different, meets Π_P in the two planes α_r and β_s and in a third plane γ_{rs}. The plane γ_{rs} must meet ϑ in three points other than P, since a solid meets ϑ in ten points altogether; thus the planes γ_{rs} are fifteen of the quadrisecant planes of ϑ which pass through P. The projection of ϑ from P on to a solid is a curve of order nine and genus 6 lying on a cubic surface. There is a double-six on the cubic surface such that the lines of one half of the double-six each meet the curve in five points while the lines of the other half of the double-six each meet the curve in one point; the remaining fifteen lines of the surface are trisecants of the curve. The trisecants of a curve of order nine and genus 6 in [3] form a ruled surface of order seventy[1]; since fifteen trisecants pass through any point of the curve the curve is of multiplicity fifteen on the ruled surface. The ruled surface meets the cubic surface on which the curve lies in a curve of order 210; this is made up of the curve of order nine itself, counted fifteen times, of the six lines which meet the curve in five points, each counted ten times, and of the fifteen trisecants of the curve which lie on the cubic surface.

The Trisecants of ϑ.

10. We now suppose that there are two points P and Q of ϑ such that the secant planes α and β which are conjugate to them meet in a line. The polar solids of P in regard to the quadrics of N all pass through α; any one of them — in particular $\alpha\beta$ — is the polar solid of P in regard to all the quadrics of a pencil. Similarly $\alpha\beta$ is the polar solid of Q in regard to all the quadrics of a second pencil. Hence, since any two pencils of quadrics which belong to N have a quadric in common, there is a quadric of N in regard to which $\alpha\beta$ is the polar solid both of P and of Q, and therefore of every point of the line PQ. This quadric can only be a cone whose vertex is the intersection R of the line PQ with the solid $\alpha\beta$; *PQR is a trisecant of ϑ*. The point R is conjugate to a secant plane γ. The polar solids of Q and R in regard to (P) are, since Q and R are collinear with the vertex of (P), the same solid; this solid contains the secant planes β and γ which are conjugate to Q and R and passes through P. Similarly the polar solid both of R and P in regard to (Q) contains γ and α and passes through Q. The plane γ is therefore the intersection of the two solids $P\beta$ and $Q\alpha$, and so passes through the line of intersection of α and β. Hence *the supposition that two secant planes of ϑ intersect in a line leads to the conclusion that the points to which the secant planes are conjugate lie on a trisecant of ϑ, and that the secant plane which is conjugate to the third intersection of the trisecant with ϑ passes through the line of intersection of the other two.*

The solid containing any two of these three secant planes contains one of the three points P, Q, R and therefore nine, and only nine, further points of ϑ. It follows that *the line of intersection of the three secant planes must also be a trisecant of ϑ*, meeting in three points U, V, W; each of the secant planes

[1] The order of the scroll of trisecants of a curve of order n and genus p, without multiple points, in [3] is $\frac{1}{3}(n-1)(n-2)(n-3)-(n-2)p$.

α, β, γ meets ϑ in U, V, W and three other points. Since the secant planes conjugate to P, Q, R all pass through U, V and W the secant planes conjugate to U, V, W all pass through P, Q, and R. We have a pair of trisecants, PQR and UVW, of ϑ; the three secant planes conjugate to the three points on either trisecant all pass through the other trisecant.

There is a pencil of quadrics belonging to N which contains the two cones (P) and (Q); we may denote the pencil by (PQ), with similar symbols to denote other pencils. Since $\alpha\beta$ is the polar solid of Q in regard to all the quadrics of (QR), and $\alpha\gamma$ is the polar solid of R in regard to the same quadrics, the plane α is the polar plane of the line QR in regard to all the quadrics of (QR) and therefore contains the vertices of the three cones, other than (Q) and (R), which belong to the pencil. Hence, apart from the three points U, V, W, the plane α meets ϑ in the vertices of the three cones, other than (Q) and (R), which belong to (QR). Similarly β meets ϑ in the vertices of the three cones, other than (R) and (P), which belong to (RP) and γ meets ϑ in the vertices of the three cones, other than (P) and (Q), which belong to (PQ). There are similar statements concerning the intersections of ϑ with the three secant planes δ, ε, η which pass through the trisecant PQR and are conjugate to the points U, V, W.

The line which is conjugate to any point of PQR is the line of intersection of the polar solids of the point in regard to (P), (Q) and (R); hence *the line UVW is conjugate to every point of PQR.* Similarly *the line PQR is conjugate to every point of UVW.* We may call PQR and UVW *conjugate trisecants of ϑ.*

The configuration of two conjugate trisecants of ϑ and of six secant planes, three of which pass through each trisecant, has been constructed from a pair of secant planes with a line of intersection. It could also have been constructed by assuming the existence of a trisecant of ϑ. For, if PQR is a trisecant of ϑ, the polar solids of P and Q in regard to (R) are the same solid; this solid passes through R and contains the secant planes α and β conjugate to P and Q, and so these two secant planes have a line of intersection.

The curve ϑ is known to have twenty trisecants[1]; these are therefore made up of ten pairs of conjugate trisecants. We have thus ten configurations, such

[1] The number of trisecants of a curve of order n and genus p in [4] is

$$\frac{1}{6}(n-2)(n-3)(n-4)-(n-4)p.$$

as the above, of two trisecants and six secant planes conjugate to the intersections of the two trisecants with ϑ.

The existence of these ten configurations may be suspected on other grounds. It has been shown that the secant planes of ϑ generate a locus R_3^{15}; the secant planes are therefore dual to the generators of a ruled surface of order fifteen and genus 6. Such a ruled surface has, in general, sixty double points[1]; it is therefore to be expected that there are sixty pairs of secant planes of ϑ which have a line of intersection. But we have shown that any line which lies in a pair of secant planes of ϑ lies also in a third secant plane, so that there are three pairs of secant planes passing through the line. We therefore expect that there are twenty lines through each of which there pass three secant planes of ϑ, and this is actually what does happen. Again: consider the ruled surface which is a prime section of R_3^{15} and, in particular, the number of its triple points. An arbitrary solid meets R_3^{15} in a ruled surface of order fifteen and genus 6; such a ruled surface has, in general, 220 triple points[2]. But the ruled surface has sextuple points at the ten intersections of the solid with ϑ; reckoning each sextuple point as ${}^6C_3 = 20$ triple points we account in this way for 200 of the triple points of the ruled surface. The remaining twenty triple points, which are not accounted for by the intersections of the solid with ϑ, are the intersections of the solid with the twenty trisecants of ϑ; each of these points is a triple point of the ruled surface because each trisecant of ϑ is common to three secant planes.

11. Let us denote by Σ the solid which contains the two conjugate trisecants PQR and UVW; there are four points X, Y, Z, T of ϑ, other than its six intersections with the two trisecants, which lie in Σ; let λ, μ, ν, ϱ be the four lines which meet both PQR and UVW and pass respectively through X, Y, Z, T. Since each point of PQR is conjugate to each point of UVW in regard to every quadric of N a line which meets both PQR and UVW belongs to V; the quadrics of N cut the line in the pairs of points of an involution whose two double points are the intersections of the line with PQR and UVW. In particular the line λ belongs to V and, since the involution cut out on λ by the

[1] A ruled surface of order n and genus p in [4] has, in general, $\frac{1}{2}(n-2)(n-3)-3p$ double points.

[2] A ruled surface of order n and genus p in [3] has, in general, $\frac{1}{6}(n-2)(n-3)(n-4)-(n-4)p$ triple points.

quadrics of N cannot have more than two double points, the cone (X) must contain λ; (X) is that quadric of N of which λ is a generator. Moreover: the tangent solids, at the intersection of λ and PQR, to all the quadrics of N which pass through the point all contain the plane joining the point to UVW while the tangent solids, at the intersection of λ and UVW, to all the quadrics of N passing through this point all contain the plane joining the point to PQR. But the tangent solids of (X) at the intersections of λ with PQR and UVW are the same solid. Hence the tangent solid of (X) along λ is Σ, *which therefore meets* X *in a pair of planes through* λ. Similarly (Y) is met by Σ in a pair of planes through μ, (Z) in a pair of planes through ν and (T) in a pair of planes through ϱ. The net of quadric surfaces in which the quadrics of N are met by Σ includes four plane-pairs.

Since the tangent solids of (Y) at all the points of μ contain λ, the polar solid of any point of λ in regard to (Y) contains the line μ and so meets Σ in a plane through μ. The line which is conjugate to this point of λ therefore meets μ and, similarly, it also meets ν and ϱ. Hence the regulus which is generated by the lines conjugate to the points of λ (apart from the secant plane conjugate to X) consists of the transversals of the three lines μ, ν, ϱ. Similarly the lines conjugate to the points of μ generate a regulus and meet the lines ν, ϱ, λ; and similarly for the lines conjugate to the points of ν or ϱ.

12. Consider now the cubic primal Π associated with a plane π which contains a trisecant PQR of ϑ. Π must contain the secant planes α, β, γ which are conjugate to P, Q, R; hence an arbitrary solid meets Π in a cubic surface on which there are three concurrent non-coplanar lines, the point of concurrence of the lines being the intersection of the solid with that trisecant UVW of ϑ which is conjugate to PQR; this point must be a node on the cubic surface. Hence UVW *is a double line on* Π. Since the polar plane of PQR in regard to any quadric of N passes through UVW, the polar lines of π in regard to the quadrics of N all meet UVW; these lines generate Π. We can also generate Π by means of three projectively related nets of solids, the bases of the nets being the polar lines k_1, k_2, k_3 of π in regard to any three quadrics of N not belonging to the same pencil; solids belonging to the three nets correspond when they are the polar solids of the same point O of π in regard to the three quadrics, and the line of intersection of three corresponding solids is the line j conjugate to O. Since the polar solids of all points of PQR in regard to all

the quadrics of N contain UVW the relation between the three nets of solids is such that the three planes which join the lines k_1, k_2, k_3 to UVW correspond to each other in the projectivity. Π has three nodes which do not lie on UVW[1]; the three lines conjugate to these nodes are the only lines, other than PQR, which belong to J and lie in π. Π also contains σ_1, σ_2, σ_3, the polar planes of π in regard to (P), (Q), (R) respectively; σ_1 lies in the solid $\beta\gamma$, σ_2 in the solid $\gamma\alpha$ and σ_3 in the solid $\alpha\beta$. The lines conjugate to the points of π all meet the three planes σ_1, σ_2, σ_3.

If π joins the trisecant PQR to a further point S of ϑ then it contains, in addition to the planes enumerated above, the polar plane σ of π in regard to (S) and the secant plane s conjugate to S. The polar lines of π in regard to the quadrics of N all meet UVW and s, and Π has four nodes which do not lie on its double line UVW, these four nodes all lying in the plane s.

13. The pole of a solid S_3 in regard to a quadric is the intersection of the polar solids of any four non-coplanar points of S_3 in regard to that quadric. Take then the four lines which are conjugate, in regard to N, to four non-coplanar points of S_3; if the solids through these respective lines correspond to each other when they are the polar solids of the four points of S_3 in regard to the same quadric of N, the four doubly-infinite systems of solids are related projectively to each other. Hence the surface which is the locus of poles of S_3 in regard to the quadrics of N is generated by the intersections of corresponding solids of four projectively related doubly-infinite systems. It is therefore[2] a sextic surface F_2^6 whose prime sections are of genus 3. This surface must contain the vertices of all the cones belonging to N, so that it contains the curve ϑ. There are ten lines on F_2^6, these being the polar lines of S_3 in regard to the cones of N whose vertices are the ten intersections of S_3 with ϑ. The twisted sextic of genus 3 in which S_3 meets F_2^6 is the locus of points in which S_3 is touched by quadrics of N, and is the Jacobian curve of the net of quadric surfaces in which the quadrics of N meet S_3.

Let j be the line which is conjugate to the point O of S_3. The lines which are conjugate to the points of j form a cubic cone whose vertex is O and of which three generators lie in S_3, the points to which these three generators are conjugate lie on j and are the poles of S_3 in regard to quadrics of N; hence

[1] Cf. Segre: *Memorie Torino* (2), 39 (1889), 31.
[2] Cf. Veronese: *loc. cit.*, 232.

the lines which are conjugate to the points of S_3 *are trisecants of* F_2^6. Conversely, let t be a trisecant of F_2^6. The lines conjugate to its three intersections with F_2^6 are generators of a cubic scroll and all lie in S_3; hence they must be concurrent and the cubic scroll a cubic cone, t being conjugate to the vertex of the cone. The trisecants of F_2^6 all belong to J. Through an arbitrary point of [4] there passes one trisecant of F_2^6 because the line conjugate to the point meets S_3 in one point. Through a point of F_2^6 itself there pass ∞^1 of its trisecants, generating a cubic cone; they are the lines which are conjugate to the points of that line of S_3 which is conjugate to the point of F_2^6.

We have seen that the lines of J which lie in S_3 form a congruence K of order three and class six. If a line of J lies in S_3 the point to which it is conjugate lies on F_2^6, and conversely. Hence we have a representation[1] of the lines of the congruence K by means of the points of the surface F_2^6; the three lines of the congruence which pass through a point O of S_3 are represented by the three points of F_2^6 on the trisecant j which is conjugate to O. A point P which is an intersection of ϑ with S_3 is a singular point of K; for all the chords of ϑ belong to J, so that there are at least nine lines of K passing through P, namely those lines which join P to the other intersections of S_3 with ϑ; hence there must be ∞^1 lines of K passing through P. There are in fact ∞^2 lines of J passing through P, these being the generators of the cubic cone Π_P associated with the secant plane conjugate to P; S_3 meets Π_P in an elliptic cubic cone whose ∞^1 generators all belong to K. Each intersection of S_3 with ϑ is thus the vertex of an elliptic cubic cone of lines belonging to K, and K has therefore ten of these singular points. The points of F_2^6 which represent the lines of K passing through P must all lie in the secant plane conjugate to P; in fact this secant plane meets F_2^6 in a cubic curve, for if j is any line in the secant plane the lines conjugate to the points of j form a cubic cone whose vertex is P, three of whose generators lie in S_3, so that j has three intersections with F_2^6. Hence corresponding to the ten singular points of K there are ten plane cubics on F_2^6.

Suppose P and Q are two intersections of S_3 with ϑ; then the secant plane conjugate to P and the polar line of S_3 in regard to (Q) both lie in the polar solid of P in regard to (Q), and therefore meet each other. Thus we see that the secant planes conjugate to the ten intersections of ϑ with S_3 and the polar

[1] Cf. Fano: *Memorie Torino* (2), 51 (1902), 70; Semple: *Proc. London Math. Soc.* (2), 35 (1933), 319.

lines of S_3 in regard to the cones whose vertices are at these ten intersections form a double-ten of lines and planes in [4]. This is in accordance with the well-known property of the ten lines and the planes of the ten cubic curves on a determinantal sextic surface in [4].

14. The curve of intersection of S_3 and F_2^6, or the locus of points in which S_3 is touched by quadrics of N, will have special forms when S_3 has special positions.

Suppose S_3 is the solid ϖ associated with the point P of ϑ; every quadric of N which passes through P touches ϖ at P. The three cones (A), (B), (C) touch ϖ along the lines PA, PB, PC respectively and meet ϖ in plane pairs intersecting in these lines. The sextic curve in which ϖ meets the surface which is the locus of poles of ϖ in regard to quadrics of N consists of the lines PA, PB, PC and a cubic curve. This cubic curve lies in the secant plane α conjugate to P, passing through the six points in which α meets ϑ and also through the three intersections of α with PA, PB, PC; the curve is in fact the intersection of α with the cubic cone Π_P. Those quadrics of N which touch ϖ at points other than P do so at points of α.

If S_3 is the solid Σ containing two conjugate trisecants, PQR and UVW, of ϑ, the sextic curve which is the locus of points in which S_3 is touched by quadrics of N consists of the six lines PQR, UVW, λ, μ, ν, ϱ.

It is to be expected that the surface F_2^6 associated with a solid ϖ should have a singularity at P; this singularity can be investigated by means of the method of generating F_2^6. We take four non-coplanar points of ϖ, three of which are situated in the secant plane α; the lines conjugate to the last three points all pass through P. Then we find that F_2^6 lies on the cubic cone Π_P and has a *triple point* at P, the tangent cone at P of F_2^6 being the cubic cone which is generated by the polar lines of α in regard to the quadrics of N which pass through P. The generators of this cubic cone meet F_2^6 each in four points at P, and four of them lie entirely on the surface; these four lines are PA, PB, PC — the polar lines of α in regard to the three cones (A), (B), (C) which pass through P — and the polar line of α in regard to (P).

15. Since there are ∞^3 solids which pass through an arbitrary point O of [4] there are ∞^3 sextic surfaces which have the line j conjugate to O as a trisecant. In fact there is just one sextic surface passing through three arbitrary points on any line of J, for the lines which are conjugate to the points of this

line of J generate a cubic cone, and any three generators of this cone determine a solid S_3.

It follows immediately, from the definitions of F_2^6 and Π, that the surface F_2^6 lies on all the ∞^3 cubic primals Π associated with the ∞^3 planes of S_3. Since any two solids have a plane of intersection in which there lie six lines of J, any two surfaces F_2^6 lie on a cubic primal Π, the surfaces having in common the curve ϑ and the six nodes of Π. The cubic primals Π_1 and Π_2 associated with the planes π_1 and π_2 of S_3 intersect in F_2^6 and the cubic scroll Δ associated with the line λ in which π_1 and π_2 meet; the primals associated with the planes which pass through λ and lie in S_3 form a pencil, the primal of the pencil which passes through an arbitrary point O of [4] being associated with the plane which joins λ to the point of intersection of S_3 with the line conjugate to O. Ten of the primals belonging to this pencil are seven-nodal; they are associated with the planes which join λ to the ten intersections of S_3 and ϑ. The primals Π which are associated with all the planes of [4] passing through λ all contain the cubic scroll Δ and form a net; the primal of the net which passes through two arbitrary points O and O' of [4] is associated with the plane which passes through λ and meets the lines conjugate to O and O'. Since there are thirty chords of ϑ which meet λ there are thirty eight-nodal primals Π belonging to the net.

16. The lines which are conjugate to the points of a curve generate a ruled surface whose genus is equal to that of the curve. The number of points in which the ruled surface is met by an arbitrary plane π is equal to the number of points in which the curve is met by the cubic primal Π associated with π; hence the order of the ruled surface is three times the order of the curve. The order of the ruled surface will however be reduced by one for each intersection the curve may have with ϑ. If a line of J should be a chord of the curve then the point to which it is conjugate is a double point of the ruled surface. In particular: the lines which are conjugate to the points of a conic γ generate a rational sextic scroll Γ_2^6. Γ_2^6 lies on the cubic primal Π associated with the plane π in which γ lies; there are two generators of Γ_2^6 belonging to each cubic scroll of the system $\mathfrak{L}$ on Π, and the six nodes of Π are double points of Γ_2^6.

The lines which are conjugate to the points of a surface generate a primal; the number of intersections of the primal with an arbitrary line λ is equal to the number of intersections of the surface with the associated cubic scroll Δ;

hence the order of the primal is three times the order of the surface. In particular: the lines conjugate to the points of a quadric surface ξ generate a sextic primal Ξ. The lines conjugate to the points of a generator of ξ form a cubic scroll on Ξ; hence there are two singly-infinite families of cubic scrolls on Ξ, any two scrolls of different families having a generator in common. There are also ∞^3 sextic scrolls Γ_2^6 on Ξ corresponding to the ∞^3 conics γ on ξ; any two of these sextic scrolls have two common generators, and each sextic scroll has six nodes. The quadric surface ξ lies in a solid S_3, the locus of whose poles in regard to the quadrics of N is a surface F_2^6; the lines on Ξ which are conjugate to the points of ξ are all trisecants of F_2^6. Moreover the line conjugate to any point of F_2^6 lies in S_3 and meets ξ in two points; the lines conjugate to these two points both pass through the point of F_2^6, so that F_2^6 is a double surface on Ξ.

The above reasoning seems to indicate that the lines conjugate to the points of F_2^6, which we know all lie in the solid S_3, generate a primal of order 18. But when it is remembered that F_2^6 contains ϑ it is seen at once that the primal R_3^{15} forms part of this primal of order 18; the residual cubic primal is the solid S_3 counted three times, since through each point of S_3 there pass lines conjugate to three different points of F_2^6. The curve ϑ is a multiple curve on any primal which is the locus of lines which are conjugate to the points of a surface, the order of its multiplicity being, in general, equal to the order of the surface; for any secant plane of ϑ meets the surface in n points, where n is its order, and the lines conjugate to these n points all pass through that point of ϑ to which the secant plane is conjugate. If however a surface of order n contains ϑ the lines conjugate to the points of the surface generate, apart from the locus R_3^{15}, a primal of order $3n-15$ on which ϑ is a curve of multiplicity $n-6$. There are similar statements concerned with surfaces on which ϑ is a multiple curve. Of the ∞^4 chords of a surface ∞^2 will belong to J; thus there is, on the primal generated by the lines conjugate to the points of the surface, a double surface.

The ruled surface in which R_3^{15} is met by an arbitrary solid is of order fifteen and genus 6; hence it has, on its double curve, 50 pinch-points.[1] There is therefore, on the double surface of R_3^{15}, a *pinch-curve* of order 50; this curve is the locus of intersection of pairs of 'consecutive' secant planes, and the line

[1] The number of pinch points on the double curve of a ruled surface of order n and genus p in [3] is, in general, $2(n+2p-2)$.

conjugate to any point of the pinch-curve is a tangent of ϑ. Since the tangents of ϑ form a ruled surface of order 30, and since the lines which are conjugate to the points of a curve of order 50 generate a ruled surface of order $150-i$, where i is the number of intersections (if any) of the curve with ϑ, it follows that the pinch-curve must meet ϑ in 120 points. We should expect the two curves to have a finite number of intersections since they both lie on the double surface of R_3^{15}. If G is an intersection of ϑ and the pinch-curve two of the six secant planes which pass through G coincide; hence the secant plane conjugate to G must meet ϑ in six points two of which coincide, or, in other words, it must contain a tangent of ϑ. *Of the* ∞^1 *secant planes of* ϑ *there are 120 which touch* ϑ.

17. Each cone of N has two singly-infinite systems of generating planes, and every plane which lies on a cone of N must be a quadrisecant plane of C because the quadrics of the net N meet the plane in the conics of a *pencil* whose four base points must be points of C. Conversely, every quadrisecant plane of C lies on a quadric of N; hence this quadric must be a cone, and so every quadrisecant plane of C meets ϑ. The ∞^2 quadrisecant planes of C are exactly the same system of planes as the generating planes of the cones of N.

When the curve C is projected from a point of ϑ on to a solid it becomes a curve, of order eight and genus 5, which lies on a quadric surface and meets every generator of that surface, of either system, in four points. Hence the curve must have four double points[1], so that *through each point of* ϑ *there pass four chords of* C. When C is projected from an arbitrary line on to a plane it becomes a plane octavic of genus 5, and therefore has sixteen nodes; when it is projected from a line which meets it in one point it becomes a plane septimic of genus 5, and therefore has ten nodes. Hence *the chords of* C *form a primal* M_3^{16}, *of order sixteen, on which* C *is a sextuple curve and* ϑ *is a quadruple curve.* Every chord of C is met by five other chords, apart from those which pass through its intersections with C; there is thus a double surface on M_3^{16}. It can be shown[2] that this double surface is of order 60.

[1] If a curve on a quadric surface in [3] meets all the generators of one system in α points and all the generators of the other system in β points, its genus is $(\alpha-1)(\beta-1)-d$, where d is the number of its double points. This is easily seen by projecting the curve from a point of the quadric into a plane curve.

[2] If a point of C and a point of a general plane section C' of M_3^{16} correspond when the line joining them is a chord of C, the correspondence is a $(2, 7)$ correspondence, having 144 branch points on C and 24 on C'. Then Zeuthen's formula shows that C' is of genus 45 and so, being a plane curve of order 16, it must have 60 double points.

Since C is of order eight and the secant planes of ϑ form a locus R_3^{15} there are 120 secant planes of ϑ which meet C. Suppose that the secant plane α, conjugate to the point P of ϑ, meets C in a point H. The line conjugate to H is the tangent of C at H; since it is conjugate to a point of α it passes through P. Hence there are 120 tangents of C which meet ϑ; their points of contact with C are the intersections of C with secant planes of ϑ and their intersections with ϑ are the points to which these secant planes are conjugate. We have seen that there are ∞^1 chords of C which meet ϑ; we should then expect there to be a finite number of tangents of C among them.

Suppose now that the chord T_1T_2 of ϑ meets C; since it is a generator both of (T_1) and (T_2) it lies on the cyclide which is the base surface of the pencil (T_1T_2), and so meets C in *two* points. Hence *every chord of ϑ which meets C is a chord of C.* Let T_1T_2 meet C in I and J. The tangent solid of (T_1) along the chord contains the tangents of C at I and J, while the tangent solid of (T_2) along the chord also contains these two tangents of C; hence the tangents of C at I and J lie in the plane of intersection of the two solids. That the tangents of C at I and J are coplanar also follows from § 7; for these two tangents are the lines conjugate to I and J, and the lines conjugate to the points of T_1T_2 all pass through the point of intersection of the secant planes conjugate to T_1 and T_2. When ϑ is projected from an arbitrary line on to a plane it becomes a plane decimic of genus 6, having thirty double points; hence the chords of ϑ form a locus M_3^{30} of order thirty. This locus must meet C in 240 points, so that *there are 120 chords of ϑ which are also chords of C.* Incidentally we have found 120 bitangent planes of C, and these are in fact the only bitangent planes that C possesses.[1] The 120 chords IJ are on the double surface of M_3^{16}.

The line T_1T_2 lies on all the quadrics of the pencil to which (T_1) and (T_2) belong; the five cones which belong to this pencil consist of the cones (T_1) and (T_2), each counted twice, and of one other cone. Let T be the vertex of this last cone. Then T must lie in the bitangent plane of C, for it lies both in the polar solid of T_1 in regard to (T_2) and in the polar solid of T_2 in regard to (T_1); these two solids are the tangent solids of (T_2) and (T_1) respectively along the line T_1T_2, and we have seen that their plane of intersection is the bitangent plane of C. The cone (T) contains the plane TT_1T_2. The conics in which the quadrics of N meet any bitangent plane of C do not form a net of conics, but

[1] Cf. Richmond: *Proc. Camb. Phil. Soc.* 28 (1932), 175—6.

a pencil of conics having double contact; it therefore follows that there must be a quadric of N which contains the plane and this quadric, since it contains a plane, must be a cone with its vertex in the plane.

18. If a plane π meets C in two points the line joining these points lies on the cubic primal Π associated with π. Suppose now that we take π to be a quadrisecant plane of C; then the cubic primal Π must contain the plane π. Moreover, since π meets ϑ in a point P, Π is seven-nodal. Let O_1, O_2, O_3 be the diagonal points of the quadrangle formed by the four intersections of π with C; then the lines which are conjugate to the points O_1, O_2, O_3 are O_2O_3, O_3O_1, O_1O_2 respectively. Hence, since O_1 is an intersection of lines which are conjugate to two different points of π, it must be a node of Π; thus O_1, O_2, O_3 are three of the seven nodes of Π. There is also a fourth node of Π lying in π, namely the intersection of π with the secant plane conjugate to P. The plane π lies on the cone (P), so that every line in π belongs to V. The three points O_1, O_2, O_3 are on the double surface of M_3^{16}, and the lines which are conjugate to them belong not only to J but also to V.

Of the ∞^4 lines belonging to J there must be ∞^3 which belong also to the complex V. Since the lines of J which pass through an arbitrary point O of [4] form a two-dimensional cubic cone and the lines of V which pass through O form a three-dimensional cubic cone, there are nine lines common to J and V passing through an arbitrary point O. If j is the line conjugate to O then there are nine points of j such that their conjugate lines belong to V. In fact if λ is an arbitrary line of [4], not necessarily belonging to J, there are nine points of λ whose conjugate lines belong to V; for the lines conjugate to the points of λ generate a cubic scroll Δ, and there are nine generators of Δ belonging to the cubic complex V. It follows that *the points of* [4] *whose conjugate lines belong to* V *form a primal* W_3^9 *of order nine.* We have seen that the primal contains the double surface of M_3^{16}; any point which is the intersection of two chords of C lies on W_3^9. We also saw previously, in § 2, that W_3^9 contains the surface (of order twenty-four) formed by the tangents of C; the complete intersection of W_3^9 and M_3^{16} consists of the double surface (of order 60) counted twice and of the surface formed by the tangents of C. If a line meets ϑ then the lines which are conjugate to the points of it generate, apart from a secant plane of ϑ, a regulus; there are six lines of this regulus belonging to the cubic complex V, so that the line has only six intersections with W_3^9 apart from its intersection with ϑ. Hence *ϑ is a triple curve on W_3^9.*

Take a point O_1 and the line which is conjugate to it, and assume that this line belongs to V; then the line is cut in involution by the quadrics of N. Let O_2 and O_3 be the double points of the involution. The line conjugate to O_2 passes both through O_3 and O_1, and is therefore $O_3 O_1$; similarly the line conjugate to O_3 is $O_1 O_2$. We thus have a triangle $O_1 O_2 O_3$ whose sides belong both to J and to V; $O_3 O_1$, for example, belongs to V because O_3 and O_1 are conjugate in regard to every quadric of N. We thus see that there is an involution of triads of points on W_3^9, a general point of W_3^9 belonging to one and only one triad. The quadrics of N meet the plane of a triad of points O_1, O_2, O_3 in the net of conics for which $O_1 O_2 O_3$ is a selfconjugate triangle. The three sides of this triangle make up the intersection of the plane with its associated cubic primal, and the vertices of the triangle are three of the six nodes of the primal.

The Birational Correspondence between ϑ and a Plane Quintic.

19. Just as Hesse[1] represented the members of a net of quadric surfaces in [3] by the points of a plane, so we may represent the quadrics $x Q_0 + y Q_1 + z Q_2 = 0$ of the net N in [4] by the points (x, y, z) of a plane σ. The cones of N are then represented by the points of a plane quintic curve ζ without double points[2], the left-hand side of the equation of ζ being a symmetrical determinant, of five rows and columns, whose elements are homogeneous linear functions of the coordinates of the representative point in σ. There is thus established a $(1, 1)$ correspondence between the two curves ϑ and ζ; any point of ϑ is the vertex of a cone of N which is represented in σ by the corresponding point of ζ, while any point of ζ represents a cone of N whose vertex is at the corresponding point of ϑ.

Those quadrics of N which are inpolar to an arbitrary quadric Q of [4] are represented in σ by the points of a quartic curve, because the condition that a quadric should be inpolar to Q is of the fourth degree in the coefficients of its point equation. Since Q is outpolar to any cone whose vertex lies on it the twenty intersections of Q and ϑ correspond, in the $(1, 1)$ correspondence between ϑ and ζ, to the twenty intersections of ζ with the quartic curve. Conversely:

[1] *Journal für Math.*, 49 (1855), 279—332; *Gesammelte Werke,* 345.

[2] Cf. W. P. Milne: *Journal London Math. Soc.*, 2 (1927), 80.

the points of a quartic curve in σ represent quadrics of N which are all inpolar to the same quadric Q. For suppose we take fourteen points of the quartic curve; since there is a unique quadric outpolar to fourteen quadrics in [4], the tangential equations of these fourteen quadrics being supposed to be linearly independent, these fourteen points represent quadrics of N which are all inpolar to a unique quadric Q; the quartic curve in σ which represents all the quadrics of N which are inpolar to Q is the quartic curve from which we started, since it has fourteen points in common with it, and there is only one quartic curve passing through fourteen points of general position in a plane. The ∞^{14} quadrics of [4] are thus associated with the ∞^{14} quartic curves of σ. In the particular case when Q belongs to N the associated quartic curve in σ is the first polar, in regard to ζ, of the point O of σ which represents Q. For if Q, which is now supposed to belong to N, meets ϑ in a point P, the cone (P) counts twice among the five cones of the pencil to which Q and (P) belong; whence, if the quartic curve associated with Q meets ζ in a point p the line Op touches ζ at p. Hence the twenty intersections of the quartic curve with ζ are the points of contact of ζ with its twenty tangents passing through O, so that the quartic curve is the first polar of O in regard to ζ.

In order that a quadric Q should be inpolar to a pair of solids it is necessary and sufficient that the two solids should be conjugate (i. e. that each solid should contain the pole of the other) in regard to Q. Hence those quadrics of N in regard to which two given solids are conjugate are represented in σ by the points of a quartic curve, the twenty intersections of the quartic curve with ζ corresponding to the intersections of the two solids with ϑ. When the two solids coincide we have the following fundamental result: *those quadrics of N which touch a solid S_3 are represented in σ by the points of a quartic curve δ which touches ζ at each of the ten points corresponding to the ten intersections of ϑ with S_3.* There is thus associated with each solid of [4] a *contact quartic* of ζ, and the ∞^4 solids of [4] give rise to a system $ of ∞^4 contact quartics of ζ. The system $ is such that the two sets of contacts of any two of its members with ζ make up the complete intersection of ζ with a quartic curve; if δ is the contact quartic associated with S_3 and δ' that associated with S'_3 the ten contacts of δ and the ten contacts of δ' all lie on that quartic curve whose points represent the quadrics of N in regard to which S_3 and S'_3 are conjugate. Conversely: let δ be any contact quartic of the system $; then, if any quartic curve is taken which passes through the ten contacts of δ with ζ its ten remaining intersections with ζ are the points of contact of another contact quartic δ', also belonging to $\$$. For the points of the quartic curve through the ten contacts of δ and ζ represent quadrics of N which are all inpolar to some quadric Q, and ten of the intersections of Q and ϑ are the intersections of ϑ with S_3, the solid associated with δ. Hence, since ten of the intersections of Q and ϑ lie in a solid S_3, Q must be a pair of solids, and its ten remaining intersections with ϑ must lie in a solid S_3', the intersections of S_3 and ϑ not lying, in general, on a quadric surface. This solid S_3' is associated with the contact quartic δ'.

The points of contact of S_3 with those quadrics of N which touch it lie on the sextic curve in which S_3 meets F_2^6, the surface which is the locus of the poles of S_3 in regard to the quadrics of N; this sextic curve is of genus 3 if S_3 is of general position. The quadrics of N meet S_3 in a net of quadric surfaces whose base points are the eight intersections of S_3 with C; the vertices of the cones belonging to this net of quadric surfaces are the points of the sextic curve in which S_3 meets F_2^6. The points of this sextic curve are thus in (1, 1) correspondence, *exactly in Hesse's manner*, with the points of the contact quartic δ.

If S_3 meets a quadric of N in a plane-pair the point of σ which represents this quadric must be a double point of δ because a plane-pair counts for two among the four cones of any pencil of quadric surfaces to which it belongs. Moreover, since a quadric of [4] cannot contain a plane unless it be a cone, this double point of δ must be on ζ; instead of having an ordinary contact with ζ at the point the quartic curve δ has a node there.

If a plane quartic is made to pass through twelve arbitrary points of a plane cubic it must contain the cubic completely; hence those quadrics of N which are represented in σ by the points of a cubic curve are such that any quadric of [4] which is outpolar to twelve of them is outpolar to them all. The fifteen points of ϑ which correspond to the fifteen intersections of ζ with a cubic curve therefore lie on ∞^2 quadrics, any quadric which contains twelve of them containing them all. But if, of a set of fifteen points on ϑ, ten are the intersections of ϑ with a solid, the fifteen points cannot lie on ∞^2 quadrics; whence *the ten points of contact of ζ with a contact quartic of the system $\$$ cannot lie on a cubic curve.*

20. Since the curve ζ is of odd order any contact curve of ζ, i. e., any curve which has two intersections with ζ wherever it meets it, must be of even

order. It is known that ζ possesses 2015 contact conics.[1] The contact quartics of ζ consist of various systems of curves; each system consists of ∞^4 quartics and there is, in general, one and only one curve of each system touching ζ at four arbitrary points; moreover the two sets of ten contacts with ζ of any two curves belonging to the same system make up the complete intersection of ζ with a quartic curve and, conversely, if a quartic curve passes through the ten points of contact of a contact quartic with ζ its ten remaining intersections with ζ are the points of contact of a second contact quartic belonging to the same system as the former. The sets of contacts of ζ with the quartics belonging to any one system are cut out by the quartic curves which pass through any one of the sets; they form a $g_{10}{}^4$ on ζ.

Suppose now that a contact quartic is such that its ten contacts lie on a cubic curve γ^3; denote for the moment by T that system of contact quartics to which this particular quartic belongs. Then any quartic curve through the ten contacts meets ζ further in ten points which are also the contacts of a quartic belonging to T; we may, in particular, suppose that the quartic curve through the ten contacts consists of γ^3 and any line of the plane. Thus the five remaining intersections of γ^3 with ζ, together with any five collinear points of ζ, make up a set of ten contacts of a contact quartic belonging to T; but a contact quartic five of whose contacts with ζ are collinear must contain the line, repeated, on which the five points lie, so that we conclude that the five remaining intersections of γ^3 with ζ are the five contacts of ζ with one of its 2015 contact conics. Conversely: take any cubic curve γ^3 through the five contacts of ζ with any one of its contact conics $c = 0$. If we take any line $l = 0$ then $cl^2 = 0$ is a contact quartic whose ten contacts with ζ are the five contacts of ζ with $c = 0$ and the five intersections of ζ with $l = 0$; the cubic γ^3 and the line $l = 0$ form a quartic curve which passes through these ten points and whose ten other intersections with ζ are the ten intersections, other than the five contacts of $c = 0$, of ζ with γ^3. These last ten points are therefore the ten points of contact of ζ with a quartic curve which belongs to the same system as the contact quartic $cl^2 = 0$. Since there are ∞^4 cubic curves passing through the five

[1] The number of odd theta-characteristics associated with a curve of genus 6 is 2016, but when the curve is a plane quintic one of these is special; the corresponding odd theta-function not only vanishes for zero values of the argument, but all its first and second derivatives do so too. This particular odd characteristic is associated with the ∞^2 degenerate contact conics which consist of the lines of the plane, counted twice; the remaining 2015 odd characteristics are associated with the proper contact conics.

contacts of ζ with $c=0$ we obtain the ∞^4 contact quartics of the system in this way.

It is now clear that there are two kinds of systems of contact quartics of ζ. If a system of contact quartics contains one curve whose ten contacts lie on a cubic then all the curves of the system have this property, and the cubic through the ten contacts of any one of them meets ζ again in the five points of contact of one of its contact conics, the same contact conic being obtained whichever contact quartic of the system is taken. A system of contact quartics is either such that *no* curve of the system has its ten contacts on a cubic or else such that *every* curve of the system has its ten contacts on a cubic. A system which is such that no curve belonging to it has its ten contacts on a cubic we call a *system of the first kind*; the system $\mathfrak{S}$ that we have already met with is of the first kind. A system which is such that every curve belonging to it has its ten contacts on a cubic we call a *system of the second kind.* The number of different systems of the second kind is 2015, the same as the number of contact conics. The number of different systems of the first kind is[1], in fact, 2080.

The systems of the first kind all have similar properties, and those of the second kind also have similar properties; but the properties possessed by the systems of the first kind are different from those possessed by the systems of the second kind. The very method of distinguishing between the two kinds of systems gives an example of this difference. Another example is given by the number of contact quartics of a system which break up into two contact conics; whereas, of the ∞^4 contact quartics which belong to a system of the first kind, there are 496 quartics which break up into two contact conics, of the ∞^4 contact quartics which belong to a system of the second kind there are only 495 which break up into two contact conics. Every pair of contact conics of ζ forms a contact quartic, this contact quartic belonging of course only to one system. These statements are in accordance with the arithmetical relation

$$\frac{1}{2}(2015 \times 2014) = 2080 \times 496 + 2015 \times 495 .$$

If any contact conic C is given then, since each system of the first kind includes $2 \times 496 = 992$ contact conics, the number of contact quartics, belonging

[1] The number of even theta-characteristics of genus 6 is 2080; one of these is associated with each of the systems of contact quartics of the first kind.

to systems of the first kind, of which C forms a part is $(992 \times 2080)/2015 = 1024$. Similarly the number of contact quartics which belong to systems of the second kind and of which C forms a part is $(990 \times 2015)/2015 = 990$. Hence, given any contact conic C, the remaining 2014 contact conics are such that 1024 of them, when taken with C, make up contact quartics belonging to systems of the first kind while the other 990, when taken with C, make up contact quartics belonging to systems of the second kind.

When a net of quadrics in [4] is given, its Jacobian curve ϑ can immediately be put into (1, 1) correspondence with a plane quintic; it is also true, conversely, that a general plane quintic ζ, without nodes or cusps, can be put into (1, 1) correspondence with the Jacobian curve of some net of quadrics in [4]. In fact, given the curve ζ, such a correspondence can be set up in 2080 different ways, each way of setting up the correspondence being associated with a particular one of the 2080 systems of contact quartics of the first kind.

21. We have seen that, associated with any solid S_3 of [4], there is a contact quartic of ζ belonging to the system \$; the points of the contact quartic represent those quadrics of N which touch S_3. Let us now consider the contact quartic associated in this way with a solid ϖ; ϖ is the common tangent solid, at the point P of ϑ, of all the quadrics of N which pass through P. The five cones of N which pass through P consist of the cone (P), counted twice, and of three other cones (A), (B), (C); we have correspondingly on ζ a point p and the three remaining intersections a, b, c of ζ with its tangent at p. Moreover, since the points A, B, C all lie in ϖ, the four points a, b, c, p are four of the points of contact of ζ with the contact quartic associated with ϖ. But any contact quartic of ζ which has a, b, c, p for four of its points of contact must contain the line $abcp$ as part of itself; it therefore consists of this line and a cubic curve which passes through a, b, c and which touches ζ at six points. The six remaining intersections, other than P, A, B, C of ϑ with ϖ are the six points X_1, X_2, X_3, X_4, X_5, X_6 in which ϑ is met by the secant plane α which is conjugate to P; the six corresponding points x_1, x_2, x_3, x_4, x_5, x_6 on ζ are therefore the six points of contact of the cubic curve. The contact quartic, being made up of this cubic and the line $pabc$, has nodes at a, b, c; this is in accordance with the fact that ϖ meets each of the cones (A), (B), (C) in a plane-pair. The points of the line $pabc$ represent quadrics of N which touch ϖ at P. We have seen that there is a quadric of N touching ϖ at any point

common to α and the cubic cone Π_P; the cubic curve in which α meets Π_P is in $(1, 1)$ correspondence with the cubic curve which touches ζ at the six points x, the points of this last cubic representing quadrics of N which touch ϖ at points of α.

There is, in general, one and only one contact quartic of the system \$ which touches ζ at four arbitrary points; it is associated with the solid which contains the four corresponding points of ϑ. If the four arbitrary points of ζ are taken to be any point p and the three remaining intersections of ζ with its tangent at p, then the corresponding solid in [4] is the solid ϖ associated with the point P of ϑ which corresponds to p; the contact quartic of \$ which touches ζ at the four points breaks up into the tangent of ζ at p and a cubic curve. Of the ∞^6 cubics which pass through the three intersections of ζ with any one of its tangents there are 2080 which touch ζ in six points; these cubics, when taken with the tangent of ζ, make up the 2080 contact quartics, belonging to the different systems of the first kind, four of whose contacts with ζ are the point of contact and the three remaining intersections of the tangent with ζ.

The statement that there is one and only one contact quartic of the system \$ touching ζ at four arbitrary points is true in general; but it is no longer true when the four corresponding points of ϑ are coplanar. In this case there are ∞^1 solids which contain the four points on ϑ, so that there are ∞^1 contact quartics belonging to \$ and touching ζ at the four points. In particular, since the six points $x_1, x_2, x_3, x_4, x_5, x_6$ of ζ correspond to six points of ϑ lying in a secant plane α, there are ∞^1 contact quartics of ζ belonging to the system \$ and touching ζ at the six points x. We have remarked that if any solid is taken which passes through α, its four remaining intersections with ϑ are the vertices of four cones belonging to a pencil, the fifth cone of the pencil being (P). Hence, if we take any one of the ∞^1 contact quartics which belong to \$ and touch ζ at the six points x, the four remaining contacts of the quartic with ζ are collinear, and the line on which they lie meets ζ again in p. Conversely: if any line through p is taken its four remaining intersections with ζ form, when taken with the six points x, a set of ten points which are the points of contact with ζ of a contact quartic belonging to the system \$.

22. Consider now the contact quartic of ζ which is associated with the solid Σ containing two conjugate trisecants PQR and UVW of ϑ. This solid meets ϑ in four further points X, Y, Z, T and meets each of the four cones

(X), (Y), (Z), (T) in a plane-pair. The associated contact quartic therefore has double points at the four corresponding points x, y, z, t of ζ, and so breaks up into two conics intersecting in these four points. One of these conics touches ζ at the three points p, q, r which correspond to the three points P, Q, R of ϑ; the points of this conic represent those quadrics of N which touch Σ at the points of the line PQR. The other conic touches ζ at the three points u, v, w which correspond to the three points U, V, W of ϑ; the points of this conic represent those quadrics of N which touch Σ at the points of the line UVW. Hence *three points of ζ which correspond to three collinear points of ϑ are points of contact of ζ with a tritangent conic; the remaining four intersections of this conic with ζ are such that there is another tritangent conic passing through them, the points of contact of this second tritangent conic corresponding to the three intersections of ϑ with the trisecant conjugate to the former.* The system $ of contact quartics therefore contains ten curves which break up in this way into pairs of tritangent conics. Any other system of contact quartics of the first kind will also contain ten such curves. Since the ten contacts of a quartic belonging to a system of the second kind lie on a cubic curve a system of the second kind cannot contain any contact quartics which break up in this way. Hence *there are in all 20800 contact quartics of ζ which break up into pairs of tritangent conics.*

23. We now consider the configuration on ζ associated with a pair of conjugate trisecants, say PQR and UVW, of ϑ. We suppose that

α, the secant plane conjugate to P, meets ϑ in the six points
$$U,\ V,\ W,\ X_1,\ Y_1,\ Z_1;$$
β, the secant plane conjugate to Q, meets ϑ in the six points
$$U,\ V,\ W,\ X_2,\ Y_2,\ Z_2;$$
γ, the secant plane conjugate to R, meets ϑ in the six points
$$U,\ V,\ W,\ X_3,\ Y_3,\ Z_3.$$

When a point of ϑ is denoted by a certain capital letter we shall always denote the corresponding point of ζ by the corresponding small letter. It has already been stated, in § 10, that the five cones

$$(X_1),\ (Y_1),\ (Z_1),\ (Q),\ (R),$$

belong to the same pencil; hence the five points

$$x_1,\ y_1,\ z_1,\ q,\ r,$$

of ζ are collinear. The vertices of the triangle pqr lie on ζ; the side qr meets ζ again in x_1, y_1, z_1, the side rp meets ζ again in x_2, y_2, z_2 and the side pq meets ζ again in x_3, y_3, z_3. There is a conic touching ζ at each of the points pqr and another touching ζ at each of the points uvw.

Each of the two sets of points

$$u\,v\,w\,x_1\,y_1\,z_1\,x_2\,y_2\,z_2\,r, \qquad u\,v\,w\,x_3\,y_3\,z_3\,x_1\,y_1\,z_1\,q$$

is a set of ten points of ζ corresponding to ten intersections of ϑ with a solid, the first set corresponding to the intersections of ϑ with the solid $\alpha\beta$ and the second to the intersections of ϑ with the solid $\gamma\alpha$; hence each of the sets is a set of contacts of ζ with a contact quartic of the system \$. Wherefore (with an obvious notation) the set of twenty points

$$2\,(u\,v\,w\,x_1\,y_1\,z_1)\,x_2\,y_2\,z_2\,x_3\,y_3\,z_3\,q\,r$$

forms the complete intersection of ζ with a quartic curve. The points of this quartic curve represent those quadrics of N in regard to which the two solids $\alpha\beta$ and $\gamma\alpha$ are conjugate. But this set includes five collinear points x_1, y_1, z_1, q, r; the quartic must therefore consist of the line qr and a cubic, the set of fifteen intersections of ζ with this cubic being

$$2\,(u\,v\,w)\,x_1\,y_1\,z_1\,x_2\,y_2\,z_2\,x_3\,y_3\,z_3$$

Whence we have the following.[1] *Suppose we take any contact quartic of ζ which breaks up into two tritangent conics. Then the points of contact of either of the tritangent conics form a triangle each of whose sides has three other intersections with ζ; the nine intersections so arising are nine associated points, and one of the cubic curves passing through them touches the other tritangent conic at each of its three contacts with ζ.*

24. We suppose now that the equations of the lines qr, rp, pq are $\xi_1 = 0$, $\xi_2 = 0$, $\xi_3 = 0$ respectively, that the equation of the cubic curve obtained above

[1] This result follows easily also by considering residual and coresidual sets of points on ζ. The two sets $2\,(pqr)$ and $2\,(uvw)$ are co-residual, since each set is residual to the four intersections of the two tritangent conics. Hence the set $x_1\,y_1\,z_1\,x_2\,y_2\,z_2\,x_3\,y_3\,z_3$, being residual to $2\,(pqr)$, is also residual to $2\,(uvw)$.

is $\Gamma_3 = 0$ and that the equation of the conic which touches ζ at u, v, w, is $\Gamma_2 = 0$. Then $\Gamma_2 = 0$ is also a tritangent conic of $\Gamma_3 = 0$. The points of $\Gamma_3 = 0$ represent quadrics of N in regard to which each of the three pairs of solids $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ is a pair of conjugate solids.

Since α is the secant plane conjugate to P there is a cubic curve touching ζ at the six points u, v, w, x_1, y_1, z_1 and meeting ζ again in its three intersections with its tangent at p; let these three intersections be called p_1, p_2, p_3, with similar notation for the intersections of ζ with its tangents at q, r, u, v, w. Let this cubic curve have the equation $C_p = 0$. Then the two cubics $C_p = 0$ and $\Gamma_3 = 0$ intersect in nine points, the nine points consisting of three points x_1, y_1, z_1 on the line $\xi_1 = 0$ and of the set uvw counted twice, the conic $\Gamma_2 = 0$ touching both cubics at these three points. Hence we may write

$$C_p \equiv \Gamma_3 + \xi_1 \Gamma_2$$

and, similarly,

$$C_q \equiv \Gamma_3 + \xi_2 \Gamma_2,$$

$$C_r \equiv \Gamma_3 + \xi_3 \Gamma_2.$$

Now the quartic $\xi_1 \Gamma_3 = 0$ meets ζ in the set of twenty points

$$2\,(u v w x_1 y_1 z_1)\, q r\, x_2 y_2 z_2 x_3 y_3 z_3$$

while the quartic $\xi_2 C_p = 0$ meets ζ in the set of twenty points

$$2\,(u v w\, x_1 y_1 z_1)\, r p\, x_2 y_2 z_2 p_1 p_2 p_3.$$

Hence the two quartics belong to a pencil whose sixteen base points are all on ζ, being in fact the set $2\,(uvw\,x_1 y_1 z_1)\, r x_2 y_2 z_2$. The quartic $\xi_1 \Gamma_3 = 0$ meets ζ again in the four collinear points q, x_3, y_3, z_3 and the quartic $\xi_2 C_p = 0$ meets ζ again in the four collinear points p, p_1, p_2, p_3; the lines on which these two sets of four collinear points lie both meet ζ again in p, and the g_4^1 cut out on ζ by the pencil of quartics is the same as that cut out by the pencil of lines through p. One set of this g_4^1 consists of the points r, x_2, y_2, z_2; hence one quartic curve of the pencil meets ζ in the set of twenty points.

$$2\,(u v w\, x_1 y_1 z_1 x_2 y_2 z_2 r),$$

and so is a contact quartic. It is in fact the contact quartic associated with the solid $\alpha\beta$. But, by precisely similar reasoning, this contact quartic also belongs to the pencil determined by the two quartic curves $\xi_2 \Gamma_3 = 0$ and $\xi_1 C_q = 0$, this pencil of quartics cutting out on ζ the g_4^1 given by the lines through q. That there is a curve belonging to both the pencils of quartics is in accordance with the identity

$$\xi_1 \Gamma_3 + \xi_2 C_p \equiv \xi_2 \Gamma_3 + \xi_1 C_q \equiv (\xi_1 + \xi_2) \Gamma_3 + \xi_1 \xi_2 \Gamma_2 .$$

Incidentally we have obtained the equation of the contact quartic associated with the solid $\alpha\beta$. The equations of the contact quartics associated with the solids $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ are therefore

$$\begin{aligned} (\xi_2 + \xi_3) \Gamma_3 + \xi_2 \xi_3 \Gamma_2 &= 0, \\ (\xi_3 + \xi_1) \Gamma_3 + \xi_3 \xi_1 \Gamma_2 &= 0, \\ (\xi_1 + \xi_2) \Gamma_3 + \xi_1 \xi_2 \Gamma_2 &= 0, \end{aligned}$$

respectively. We have already found that the quartic curve which meets ζ in the points of contact of the second and third of these contact quartics is $\xi_1 \Gamma_3 = 0$; the other two pairs give the quartics $\xi_2 \Gamma_3 = 0$ and $\xi_3 \Gamma_3 = 0$. We have the three identities

$$\begin{aligned} &\{(\xi_3 + \xi_1) \Gamma_3 + \xi_3 \xi_1 \Gamma_2\} \{(\xi_1 + \xi_2) \Gamma_3 + \xi_1 \xi_2 \Gamma_2\} - \xi_1^2 \Gamma_3^2 \\ &\qquad \equiv \{(\xi_2 \xi_3 + \xi_3 \xi_1 + \xi_1 \xi_2) \Gamma_3 + \xi_1 \xi_2 \xi_3 \Gamma_2\} (\Gamma_3 + \xi_1 \Gamma_2) \\ &\{(\xi_1 + \xi_2) \Gamma_3 + \xi_1 \xi_2 \Gamma_2\} \{(\xi_2 + \xi_3) \Gamma_3 + \xi_2 \xi_3 \Gamma_2\} - \xi_2^2 \Gamma_3^2 \\ &\qquad \equiv \{(\xi_2 \xi_3 + \xi_3 \xi_1 + \xi_1 \xi_2) \Gamma_3 + \xi_1 \xi_2 \xi_3 \Gamma_2\} (\Gamma_3 + \xi_2 \Gamma_2) \\ &\{(\xi_2 + \xi_3) \Gamma_3 + \xi_2 \xi_3 \Gamma_2\} \{(\xi_3 + \xi_1) \Gamma_3 + \xi_3 \xi_1 \Gamma_2\} - \xi_3^2 \Gamma_3^2 \\ &\qquad \equiv \{(\xi_2 \xi_3 + \xi_3 \xi_1 + \xi_1 \xi_2) \Gamma_3 + \xi_1 \xi_2 \xi_3 \Gamma_2\} (\Gamma_3 + \xi_3 \Gamma_2). \end{aligned}$$

The equation of the quintic curve ζ *is*

$$(\xi_2 \xi_3 + \xi_3 \xi_1 + \xi_1 \xi_2) \Gamma_3 + \xi_1 \xi_2 \xi_3 \Gamma_2 = 0.$$

It follows from this equation, since neither the cubic $\Gamma_3 = 0$ nor the conic $\Gamma_2 = 0$ passes through any of the points p, q, r, that the tangents of ζ at these points are respectively

$$\xi_2 + \xi_3 = 0, \qquad \xi_3 + \xi_1 = 0, \qquad \xi_1 + \xi_2 = 0.$$

Hence *the equation of the conic which is tritangent to* ζ *at* p, q, r *is*

$$\xi_2 \xi_3 + \xi_3 \xi_1 + \xi_1 \xi_2 = 0.$$

Hence the curve

$$(\xi_2 \xi_3 + \xi_3 \xi_1 + \xi_1 \xi_2) \Gamma_2 = 0$$

is a pair tritangent conics of ζ, making up a contact quartic belonging to the system $\$$.

25. If the equations of any two quartics belonging to the system $\$$ are $F_1 = 0$ and $F_2 = 0$, and if $F_{12} = 0$ is the equation of the quartic through these points of contact, then there is always an identity of the form

$$F_1 F_2 + k F_{12}^2 \equiv C \{(\xi_2 \xi_3 + \xi_3 \xi_1 + \xi_1 \xi_2) \Gamma_3 + \xi_1 \xi_2 \xi_3 \Gamma_2\},$$

where k is a constant and $C = 0$ is a cubic curve; the form of the identity shows that $C = 0$ is a contact cubic both of $F_1 = 0$ and $F_2 = 0$. Moreover: if $S_3^{(1)}$ is the solid in [4] which is touched by the quadrics of N which are represented by the points of $F_1 = 0$ and $S_3^{(2)}$ that which is touched by the quadrics of N which are represented by the points of $F_2 = 0$, then the points of $F_{12} = 0$ represent the quadrics of N in regard to which $S_3^{(1)}$ and $S_3^{(2)}$ are conjugate, and the points of $C = 0$ represent the quadrics of N which touch the plane of intersection of $S_3^{(1)}$ and $S_3^{(2)}$.

For example: taking the contact quartic associated with the solid $\beta\gamma$ and the contact quartic associated with the solid Σ we find the identity

$$\begin{aligned}\{(\xi_2 + \xi_3) \Gamma_3 + \xi_2 \xi_3 \Gamma_2\} (\xi_2 \xi_3 + \xi_3 \xi_1 + \xi_1 \xi_2) \Gamma_2 - \xi_2^2 \xi_3^2 \Gamma_2^2 \\ \equiv (\xi_2 + \xi_3) \Gamma_2 \{(\xi_2 \xi_3 + \xi_3 \xi_1 + \xi_1 \xi_2) \Gamma_3 + \xi_1 \xi_2 \xi_3 \Gamma_2\}.\end{aligned}$$

Here the cubic $C = 0$ breaks up into the tangent of ζ at p and the conic $\Gamma_2 = 0$; hence those quadrics of N which touch the plane of intersection of the two solids $\beta\gamma$ and Σ, i. e. the plane $PUVW$, consist of the quadrics which pass through P and of the quadrics which touch Σ at the points of the line UVW. Also the quartic $F_{12} = 0$ here breaks up into the two lines $\xi_2 = 0$, $\xi_2 = 0$ and the conic $\Gamma_2 = 0$; hence those quadrics of N in regard to which $\beta\gamma$ and Σ are conjugate solids are the quadrics of the pencil (RP), the quadrics of the pencil (PQ) and the quadrics which touch Σ at the points of the line UVW. These statements are all easily verified: take, for example, the statement that $\beta\gamma$ and Σ are conjugate solids in regard to all quadrics of the pencil (RP). The simplex $RPX_2Y_2Z_2$ is a common self-conjugate simplex for all the quadrics of (RP); hence, since Σ contains R and P, the poles of Σ in regard to all the quadrics of the pencil lie in the plane $X_2 Y_2 Z_2$; but this is the plane β, and so lies in the solid $\beta\gamma$.

If we now take the contact quartics associated with the solids $P\alpha$ and Σ we obtain

$$(\xi_2+\xi_3)(\Gamma_3+\xi_1\Gamma_2)(\xi_2\xi_3+\xi_3\xi_1+\xi_1\xi_2)\Gamma_2-(\xi_2+\xi_3)^2\xi_1^2\Gamma_2^2$$
$$\equiv(\xi_2+\xi_3)\Gamma_2\{(\xi_2\xi_3+\xi_3\xi_1+\xi_1\xi_2)\Gamma_3+\xi_1\xi_2\xi_3\Gamma_2\}.$$

This gives the same result as before as regards quadrics which touch the plane $PUVW$; it shows in addition that those quadrics of N in regard to which the solids $P\alpha$ and Σ are conjugate are the quadrics passing through P, the quadrics of the pencil (QR) and the quadrics which touch Σ at the points of UVW.

From the relations

$$\begin{aligned}(\xi_3+\xi_1)(\xi_1+\xi_2)(\Gamma_3+\xi_2\Gamma_2)(\Gamma_3+\xi_3\Gamma_2)-(\xi_1\Gamma_3-\xi_2\xi_3\Gamma_2)^2&\\ \equiv(\xi_1+\xi_2)(\xi_2+\xi_3)(\Gamma_3+\xi_3\Gamma_2)(\Gamma_3+\xi_1\Gamma_2)-(\xi_2\Gamma_3-\xi_3\xi_1\Gamma_2)^2&\\ \equiv(\xi_2+\xi_3)(\xi_3+\xi_1)(\Gamma_3+\xi_1\Gamma_2)(\Gamma_3+\xi_2\Gamma_2)-(\xi_3\Gamma_3-\xi_1\xi_2\Gamma_2)^2&\\ \equiv\{\Gamma_3+(\xi_1+\xi_2+\xi_3)\Gamma_2\}\{(\xi_2\xi_3+\xi_3\xi_1+\xi_1\xi_2)\Gamma_3+\xi_1\xi_2\xi_3\Gamma_2\},&\end{aligned}$$

it follows that the points of the cubic

$$\Gamma_3+(\xi_1+\xi_2+\xi_3)\Gamma_2=0$$

represent quadrics of N which touch the three planes of intersection of pairs of the three solids $P\alpha$, $Q\beta$, $R\gamma$. The points in which this cubic curve meets ζ are seen at once from the identity

$$(\xi_2\xi_3+\xi_3\xi_1+\xi_1\xi_2)\Gamma_3+\xi_1\xi_2\xi_3\Gamma_2\equiv(\xi_2\xi_3+\xi_3\xi_1+\xi_1\xi_2)\{\Gamma_3+(\xi_1+\xi_2+\xi_3)\Gamma_2\}$$
$$-(\xi_2+\xi_3)(\xi_3+\xi_1)(\xi_1+\xi_2)\Gamma_2,$$

and we have the following result: the tangents of ζ at its three contacts with a tritangent conic which forms part of a contact quartic each meet ζ in three further points; the nine points so arising are nine associated points, and one of the cubic curves passing through them touches ζ at each of its three contacts with the other tritangent conic which makes up the contact quartic. This result also follows at once from the consideration of residual sets of points on ζ.

26. If a conic circumscribes a triangle, the tangents of the conic at the vertices of the triangle meet the opposite sides in three collinear points; hence

the tangents of ζ at p, q, r meet qr, rp, pq respectively in three collinear points. The point of intersection of qr with the tangent of ζ at p represents a quadric of N which, since the point is on qr, belongs to the pencil (QR) and which, since the point is on the tangent of ζ at p, passes through P; it is therefore that quadric of (QR) which passes through P. Hence the quadric of (QR) which passes through P, the quadric of (RP) which passes through Q and the quadric of (PQ) which passes through R are three quadrics belonging to the same pencil. The line which represents the quadrics of this pencil has the equation $\xi_1 + \xi_2 + \xi_3 = 0$. Since the quadrics of (QR) meet PQR in the pairs of points of an involution whose double points are Q and R, that quadric of (QR) which passes through P passes also through P', the harmonic conjugate of P in regard to Q and R; it may therefore be defined as the quadric of N which passes through P and P'. The fact that the three quadrics belong to the same pencil follows immediately from the known fact that the three pairs of points P, P'; Q, Q'; R, R' (where Q' and R' are defined similarly to P') belong to an involution. The double points of this involution are the Hessian points of PQR; hence we may say that *the points of the line $\xi_1 + \xi_2 + \xi_3 = 0$ represent those quadrics of N in regard to which the Hessian points of PQR are a pair of conjugate points.*

There is a pencil of quadrics belonging to N and in regard to which the points P and P' are conjugate; we call this pencil H_1. The quadric of N which passes through Q and R is clearly a member of H_1, as also is the cone (P); the quadrics through Q being represented by the points of the line $\xi_3 + \xi_1 = 0$, and those through R being represented by the points of the line $\xi_1 + \xi_2 = 0$, that through both Q and R is represented by the point of intersection of these two lines. The pencil H_1 is therefore represented by the line joining P to the intersection of $\xi_3 + \xi_1 = 0$ and $\xi_1 + \xi_2 = 0$, i. e. by the line $\xi_2 = \xi_3$. Similarly we have a pencil H_2 of quadrics belonging to N and represented by the line $\xi_3 = \xi_1$; these are the quadrics of N in regard to which Q and Q' are conjugate; and similarly for the pencil H_3. There is a quadric common to the three pencils H_1, H_2, H_3; it is represented by the point $\xi_1 = \xi_2 = \xi_3$ and may be defined as the quadric of N which contains the Hessian points of PQR.

The contact quartic whose points represent quadrics of N which touch the solid $\alpha\beta$ is $(\xi_1 + \xi_2)\Gamma_3 + \xi_1\xi_2\Gamma_2 = 0$, and that whose points represent quadrics of N which touch the solid $\alpha\gamma$ is $(\xi_1 + \xi_3)\Gamma_3 + \xi_1\xi_3\Gamma_2 = 0$. The identity

$$(\xi_1 + \xi_2) \Gamma_3 + \xi_1 \xi_2 \Gamma_2 - (\xi_1 + \xi_3) \Gamma_3 - \xi_1 \xi_3 \Gamma_2 \equiv (\xi_2 - \xi_3)(\Gamma_3 + \xi_1 \Gamma_2)$$

proves that the four intersections of these two contact quartics — apart from their six contacts at u, v, w, x_1, y_1, z_1, — lie on the line $\xi_2 = \xi_3$. Hence, apart from the six cones whose vertices lie in α, there are four quadrics belonging to N which touch both the solids $\alpha\beta$ and $\alpha\gamma$, and these four quadrics belong to the pencil H_1.

Again the identity

$$\Gamma_3 + \xi_2 \Gamma_2 - \Gamma_3 - \xi_3 \Gamma_2 \equiv (\xi_2 - \xi_3) \Gamma_2$$

shows that, apart from the cones (U), (V), (W), there are three quadrics belonging to N which touch both the planes β and γ, and these three quadrics belong to the pencil H_1.

Similarly we obtain corresponding quadrics belonging to the pencils H_2 and H_3.

27. Up to this we have made use of only one half of the configuration; precisely similar results can be obtained by using the triangle uvw in place of the triangle pqr. We suppose that

δ, the secant plane conjugate to U, meets ϑ in the six points P, Q, R, L_1, M_1, N_1;

ε, the secant plane conjugate to V, meets ϑ in the six points P, Q, R, L_2, M_2, N_2;

η, the secant plane conjugate to W, meets ϑ in the six points P, Q, R, L_3, M_3, N_3.

Then we have three sets of five collinear points on ϑ namely

$$v, w, l_1, m_1, n_1; \quad w, u, l_2, m_2, n_2; \quad u, v, l_3, m_3, n_3.$$

We take $\eta_1 = 0$, $\eta_2 = 0$, $\eta_3 = 0$ to be the equations of vw, wu, uv respectively; $\Delta_2 = 0$ to be the equation of the conic touching ζ at p, q, r, whose equation we have already found to be $\xi_2 \xi_3 + \xi_3 \xi_1 + \xi_1 \xi_2 = 0$. Then there is a cubic whose fifteen intersections with ζ are the set

$$2(pqr)\ l_1 m_1 n_1\ l_2 m_2 n_2\ l_3 m_3 n_3$$

and whose equation we suppose to be $\Delta_3 = 0$. Then the equation of ζ is

$$(\eta_2 \eta_3 + \eta_3 \eta_1 + \eta_1 \eta_2) \Delta_3 + \eta_1 \eta_2 \eta_3 \Delta_2 = 0$$

while the equation of the conic touching ζ at u, v, w is

$$\eta_2 \eta_3 + \eta_3 \eta_1 + \eta_1 \eta_2 = 0.$$

There is also a cubic curve, whose equation is

$$\Delta_3 + (\eta_1 + \eta_2 + \eta_3) \Delta_2 = 0,$$

meeting ζ in the set of fifteen points $2(pqr) u_1 u_2 u_3 v_1 v_2 v_3 w_1 w_2 w_3$. The points of this curve represent quadrics of N which touch the three planes of intersection of pairs of the three solids $U\delta$, $V\varepsilon$, $W\eta$.

We have now obtained two forms of the equation of ζ; the first form may be written

$$(\xi_2 \xi_3 + \xi_3 \xi_1 + \xi_1 \xi_2) \Gamma_3 + \xi_1 \xi_2 \xi_3 (\eta_2 \eta_3 + \eta_3 \eta_1 + \eta_1 \eta_2) = 0,$$

and the second

$$(\eta_2 \eta_3 + \eta_3 \eta_1 + \eta_1 \eta_2) \Delta_3 + \eta_1 \eta_2 \eta_3 (\xi_2 \xi_3 + \xi_3 \xi_1 + \xi_1 \xi_2) = 0.$$

These must be the same equation. Now since $\Gamma_3 = 0$ is tritangent to the conic $\Gamma_2 \equiv \eta_2 \eta_3 + \eta_3 \eta_1 + \eta_1 \eta_2 = 0$ at the vertices of the triangle uvw we may write

$$\Gamma_3 \equiv \eta_1 \eta_2 \eta_3 + L (\eta_2 \eta_3 + \eta_3 \eta_1 + \eta_1 \eta_2),$$

where L is linear in η_1, η_2, η_3; and, similarly,

$$\Delta_3 \equiv \xi_1 \xi_2 \xi_3 + M (\xi_2 \xi_3 + \xi_3 \xi_1 + \xi_1 \xi_2),$$

where M is linear in ξ_1, ξ_2, ξ_3. Then it is clear that L and M must be identical, so that we have the equation of ζ in the form

$$(\xi_2 \xi_3 + \xi_3 \xi_1 + \xi_1 \xi_2) \eta_1 \eta_2 \eta_3 + (\eta_2 \eta_3 + \eta_3 \eta_1 + \eta_1 \eta_2) \xi_1 \xi_2 \xi_3 \\ + L (\xi_2 \xi_3 + \xi_3 \xi_1 + \xi_1 \xi_2)(\eta_2 \eta_3 + \eta_3 \eta_1 + \eta_1 \eta_2) = 0.$$

This may be written as

$$\xi_1^{-1} + \xi_2^{-1} + \xi_3^{-1} + \eta_1^{-1} + \eta_2^{-1} + \eta_3^{-1} + L (\xi_1^{-1} + \xi_2^{-1} + \xi_3^{-1})(\eta_1^{-1} + \eta_2^{-1} + \eta_3^{-1}) = 0.$$

28. We have seen that the points of a quartic curve in σ represent quadrics of N which are all inpolar to the same quadric Q (not necessarily be-

longing to N). When we have a linear system of quartic curves in σ the associated quadrics Q belong to a linear system[1] of quadric primals in [4]. A quartic curve in σ may, in particular, consist of a pair of conics, or of a single conic counted twice. Since each conic of the plane may be paired with ∞^5 other conics to make up a quartic curve, the set of ten points of ϑ which correspond to the ten intersections of ζ with a conic is such that there are ∞^5 quadrics passing through the ten points of the set; hence the set only imposes nine conditions on a quadric primal, and therefore every quadric which passes through nine points of the set passes also through the tenth. This may also be seen otherwise. For the points of a conic in σ represent the quadrics of N given by an equation of the form

$$(a_0\,\Theta^2 + 2\,b_0\,\Theta + c_0)\,Q_0 + (a_1\,\Theta^2 + 2\,b_1\,\Theta + c_1)\,Q_1 + (a_2\,\Theta^2 + 2\,b_2\,\Theta + c_2)\,Q_2 = 0,$$

where Θ is a parameter; these quadrics therefore form a singly-infinite system Σ_2 of index 2, two quadrics of the system passing through an arbitrary point of [4]. The tangential equation of the system, being of degree four in the coefficients of the point equation, is of the form

$$\Theta^8\,T_0 + \Theta^7\,T_1 + \Theta^6\,T_2 + \Theta^5\,T_3 + \Theta^4\,T_4 + \Theta^3\,T_5 + \Theta^2\,T_6 + \Theta\,T_7 + T_8 = 0,$$

and is linearly dependent from the tangential equations of nine quadrics. There are therefore six linearly independent quadrics outpolar to all the quadrics of Σ_2. Each of these outpolar quadrics must contain the vertices of all the cones belonging to Σ_2; these vertices are the ten points of ϑ which correspond to the ten intersections of ζ with the conic, and which we now see to lie on ∞^5 quadrics.

The set of points common to ζ and a conic is a *canonical set* on ζ, the corresponding set of points on ϑ is therefore a canonical set on ϑ. Whence *any two canonical sets of ϑ form the complete intersection of ϑ with a quadric, and all the quadrics through nine points of a canonical set pass also through the tenth.*

When a conic of σ, counted twice, is regarded as a quartic curve, the associated quadric Q must touch ϑ at each of the ten points corresponding to the ten intersections of ϑ with the conic. Hence *there is a quadric touching ϑ at each of the ten points of any canonical set.* There is therefore *a system of* ∞^5

[1] Cf. for the analogous statement in three dimensions Hesse: *Journal für Math.* 49 (1855), 288—289.

contact quadrics of ϑ; the twenty points of contact of ϑ with any two quadrics of the system form the complete intersection of ϑ with a quadric, and *the canonical series is cut out on* ϑ *by those quadrics which pass through any one of its sets.*

Any two sets of five collinear points of ζ make up a canonical set and, in particular, a set counted twice does so; hence any two sets of five points of ϑ, each of which is the set of vertices of the five cones which belong to a pencil of quadrics of N, make up a canonical set and, in particular, any such set counted twice does so. If five points of ϑ are vertices of the five cones belonging to a pencil of quadrics of N *there must be a quadric having four-point contact with* ϑ *at each of these five points.* Any four such sets of five points of ϑ lie on a quadric, while any three such sets make up a set of fifteen points lying on ∞^2 quadrics[1], and therefore on a curve of order eight and genus 5. If, in particular, we consider the pencil of quadrics which belong to N and pass through a point P of ϑ, the five cones of the pencil consist of (P), counted twice, and of three other cones (A), (B), (C); there is a quadric having eight-point contact with ϑ at P and four-point contact at each of the points A, B, C. Again: the five cones which belong to a pencil of quadrics corresponding to one of the 120 bitangents of ζ consist of a cone (T) and of two cones (T_1) and (T_2) both counted twice: there is a quadric having eight-point contact with ϑ at T_1 and at T_2, while it has four-point contact with ϑ at T. Further: the five cones which belong to a pencil of quadrics corresponding to one of the 45 inflectional tangents of ζ consist of a cone (I), counted three times, and of two cones (J) and (K); there is a quadric having twelve-point contact with ϑ at I and four-point contact at each of the points J, K.

There are 2015 contact conics of ζ, and the five contacts of any one of these make up, when counted twice, a canonical set on ζ; hence there are 2015 quadrics each of which has five four-point contacts with ϑ, apart from those quadrics for which the five four-point contacts are the vertices of five cones of N belonging to a pencil. The system $\mathfrak{S}$ of contact quartics of ζ includes 496 curves which break up into pairs of contact conics; hence, of the 2015 sets of five points just found on ϑ there are 496 pairs of sets such that each pair makes up a set of ten intersections of ϑ with a solid. The remaining 1023 sets are sets of five points which do not lie in a solid.

[1] This is of course a particular case of the statement, made in § 19, that the set of fifteen points of ϑ which correspond to the fifteen intersections of ζ and a cubic curve lie on ∞^2 quadrics.

The Quartic Primals having C for a Double Curve.

29. We shall use the symbol $\varDelta$ to denote a quartic primal having C for a double curve. It is clear, in the first place, that if a chord of C meets $\varDelta$ in a point other than its two intersections with C then this chord, as having at least five intersections with $\varDelta$, lies entirely on $\varDelta$. Now the chords of C generate a locus M_3^{16} on which C is a sextuple curve, as was pointed out in § 17; hence those chords of C which lie on $\varDelta$ generate a ruled surface R_2^{64} on which C is a curve of multiplicity 12. The tangent cone of $\varDelta$ at any point G of C is a quadric line-cone whose vertex is the tangent of C at G; this cone meets C in 12 points other than G, and the lines joining these points to G are the 12 generators of R_2^{64} which pass through G. Since ϑ is a quadruple curve on M_3^{16} the forty points in which $\varDelta$ meets ϑ are quadruple points of R_2^{64}. The surface generated by the tangents of C is of order 24; its intersection with $\varDelta$ therefore consists of the curve C, counted four times (since C is double on $\varDelta$ and cuspidal on the other surface), and of 64 tangents of C. This then is the number of tangents of C which lie on $\varDelta$, or the number of generators of R_2^{64} which touch C.

The cyclide which is the base surface of any pencil of quadrics belonging to N contains C; thus the curve C, counted twice, must make up the whole of the intersection of the cyclide with $\varDelta$, unless the cyclide lies entirely on $\varDelta$. There is a cyclide passing through any arbitrary point of [4], and *the cyclide which passes through any point of $\varDelta$ which is not on C lies entirely on $\varDelta$.* We can thus generate $\varDelta$ by means of a singly-infinite set of cyclides, and we can foresee that $\varDelta$ may be defined as the envelope of a singly-infinite family of quadrics belonging to N. Any cyclide which lies on $\varDelta$ contains sixteen generators of R_2^{64}. The primal $\varDelta$ contains, in particular, the cyclide which passes through P, where P is a point common to $\varDelta$ and ϑ; this cyclide has a node at P, the tangent cone being[1] the intersection of (P) with the solid ϖ; since each generator of this tangent cone must also have three intersections with $\varDelta$ at P we see that *the tangent solid of $\varDelta$ at any one of its intersections with ϑ is the solid ϖ associated with that point*, and that the inflectional tangents of $\varDelta$ are those generators of (P) which lie in ϖ. The four chords of C which pass through P lie on $\varDelta$; they must be generators of (P) and lie in ϖ.

[1] Cf. Segre: *Math. Annalen* 24 (1884), 353.

The surface of intersection of Δ and a quadric belonging to N consists of two cyclides, which may coincide. Two quartic primals Δ and Δ', on both of which C is a double curve, intersect in four cyclides.

30. The class of Δ is the number of points, other than those of C, which are common to Δ and the first polars of three arbitrary points O, O', O''; these first polars are cubic primals containing C; they meet in C and in a residual curve K of order 19, and the class of Δ is the number of its intersections with K which do not lie on C. It follows from a known formula[1] that K and C have 24 common points. Now K must pass through any point of C at which the tangent of C meets the plane $OO'\,O''$, and there are precisely 24 such points on C; thus all the intersections of K and C are accounted for, and *Δ has no bispatial points on its double curve.* Each intersection of K and C counts twice among the intersections of K and Δ, so that the number of intersections of K and Δ which are not on C is $4 \times 19 - 2 \times 24 = 28$. Hence *$\Delta$ is of class 28.* Other characters of Δ are obtained at once from those of an arbitrary prime section, such a section being an octadic surface whose nodes are the eight intersections of the prime with C.

The 24 intersections of C with K are, as we saw in § 4, on the cubic primal Π associated with the plane $OO'\,O''$. The primal Π however meets K not only in these 24 points but also in the 28 further points common to K and Δ. For, if A is any one of these latter points, the solid $AOO'\,O''$ is the tangent solid of Δ at A and hence meets Δ in an octadic surface with a ninth node at A. Hence A lies on the Jacobian curve of the net of quadric surfaces[2] in the solid $AOO'\,O''$, and hence on the surface F_2^6 which is the locus of poles of the solid in regard to the quadrics of N. Hence, since F_2^6 lies on Π, A must lie on Π.

We have thus shown that the cubic primal associated with the plane $OO'\,O''$ meets K at its 24 intersections with C and at 28 other points, but we can in fact show that the cubic primal contains the curve K completely. For let π be any plane; the first polars of the points of π in regard to Δ form a net of cu-

[1] If a curve C is the complete intersection of three primals in [4] of orders λ, μ, ν, then three primals of orders l, m, n which pass through C have in common also a curve of order $lmn - \lambda\mu\nu$ having $\lambda\mu\nu\,(l+m+n-\lambda-\mu-\nu)$ points in common with C. See Salmon: *Higher Algebra* (Dublin 1885), ch. 18.

[2] Edge: »Octadic Surfaces and Plane Quartic Curves». *Proc. London Math. Soc.* (2), 34 (1932), 492—525 (502).

bic primals, the base curve of the net consisting of C and a curve K of order 19; we may call K the *polar curve* of π with respect to Δ. If P is any point of [4] there are two quadrics of N which pass through P and belong to that system of quadrics of which Δ is the envelope, and there is one quadric of N, say Q, which contains the two cyclides along which these two quadrics touch Δ. It is easily shown that the polar prime of P in regard to Q is also the polar prime (i. e. the third polar) of P in regard to Δ. Thus associated with any point P of [4] there is a quadric of N such that P has the same polar prime in regard to Δ and this quadric; the same quadric is in fact obtained for all points on the cyclide through P, and the correspondence thus set up by Δ between the quadrics and cyclides of N is the same as the conjugacy, in regard to a conic, of the points and lines of a plane. Now suppose that P is on the polar curve of π with respect to Δ; then, since the first polars of all the points of π contain P, the polar prime of P in regard to Δ contains the plane π. Hence, since this prime is also the polar prime of P in regard to a quadric of N, the line conjugate to P meets π; in other words, the point P lies on the cubic primal Π associated with π. *If Δ is any quartic primal having C as a double curve and π is any plane, the polar curve of π with respect to Δ lies on the primal Π associated with π.* As there are in all ∞^5 quartic primals which have C as a double curve the primal Π which is associated with any given plane π contains ∞^5 of these polar curves, all passing through the 24 intersections of Π and C.

31. Several properties of the primals Δ are analogous to those of octadic surfaces in [3], and are obtained in the same way; as a paper has recently been published in which the properties of these surfaces are obtained[1] the following brief account of certain properties of the primals Δ will suffice.

If the quadrics, $x\,Q_0 + y\,Q_1 + z\,Q_2 = 0$, which belong to N are represented by the points of a plane σ, then those points in σ which lie on the conic

$$a x^2 + b y^2 + c z^2 + 2 f y z + 2 g z x + 2 h x y = 0$$

represent quadrics of N which belong to a singly-infinite family Σ_2 of index 2; the quadrics of Σ_2 have as their envelope the primal Δ whose equation is

$$A Q_0^2 + B Q_1^2 + C Q_2^2 + 2 F Q_1 Q_2 + 2 G Q_2 Q_0 + 2 H Q_0 Q_1 = 0,$$

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[1] Edge: *loc. cit.*

where $A = bc - f^2$, etc. Thus we have, associated with the conics of σ, a linear system of ∞^5 primals Δ. The forty intersections of Δ and ϑ correspond, in the (1, 1) correspondence between ϑ and ζ, to the points of contact of ζ with the forty common tangents of the conic and ζ.

Those primals Δ which pass through an arbitrary point T of [4] all contain the cyclide which passes through T, so that their tangent solids at T all contain the tangent plane of the cyclide; hence, in order that Δ should have a node at T, it is only necessary to impose three linear conditions. It would therefore be expected that there are ∞^2 primals Δ with a node at T, but these all break up into pairs of quadrics belonging to N; in order that Δ should have a node at some point not on C, and not break up into a pair of quadrics, this point must be on ϑ. The primals Δ which pass through a point P of ϑ all touch the solid ϖ there; hence, in order that Δ should have a node at P only two linear conditions need be imposed. The primals Δ which have a node at P are in fact associated with the conics which touch ζ at the point p corresponding to P; *if a conic touches ζ the associated primal Δ has a node at the corresponding point of ϑ.*

Since the quadrics of N which are represented by the points of a conic which touches ζ at p have as their envelope a primal Δ with a node at the corresponding point P of ϑ, it is possible to have primals Δ with one, two, three, four or five nodes on ϑ just as it is possible to have conics touching ζ in one, two, three, four or five points.

32. There are 2015 conics which touch ζ in five points; hence, of the ∞^5 primals Δ, 2015 are five-nodal. Now, of the 2015 contact conics of ζ, there are 496 pairs, each pair making up a contact quartic belonging to the system $\$$. Consider then such a pair of contact conics, γ and γ'; the quadrics of N represented by the points of γ have as their envelope a five-nodal primal Δ, the nodes of Δ corresponding to the contacts of γ and ζ; similarly there is a five-nodal primal Δ' associated with γ'. Since the conics γ and γ' make up a contact quartic belonging to $\$$ the five nodes of Δ and the five nodes of Δ' make up the ten intersections of ϑ with a solid S. Since γ is part of the contact quartic associated with S the quadrics of N represented by the points of γ all touch S, as also do the quadrics of N represented by the points of γ'; the locus of points in which S is touched by quadrics of N therefore consists of two curves δ and δ'; these curves together constitute the Jacobian of the net of quadric surfaces

in S. Now the quadric of N which is represented by any one of the four intersections of γ and γ' touches S not only at a point of δ but also at a point of δ'; but, since an intersection of γ and γ' cannot lie on ζ, this quadric is not a cone and so can only touch S in one point; hence δ and δ' have four common points, and these are nodes of the Jacobian curve of the net of quadric surfaces. Thus the quadric surfaces have fixed tangent lines at each of these four points, and *S is a quadritangent solid of C;* δ and δ' are twisted cubics. Since the quadrics of N which have Δ as their envelope all touch S at points of δ, S is the tangent solid of Δ at each point of δ, and so meets Δ in a quartic surface on which δ is a double curve; it also meets Δ' in a quartic surface on which δ' is a double curve. It is known that C possesses 496 quadritangent solids; hence each quadritangent solid of C meets ϑ in ten points which divide into two groups of five; the five points of either group lie on a twisted cubic passing through the four points of contact of the solid with C[1]. In addition to the 992 contact conics of ζ which make up contact quartics belonging to $\$$, there are 1023 others; hence of the 2015 five-nodal primals Δ 1023 are such that their five nodes do not lie in solids[2]. The solid which contains any four of the five nodes of such a primal meets it in a triply-octadic surface.

Among the contact conics of ζ there are sets of three such that the three conics of any set have a common tangent[3]; each conic belongs to 495 of these sets. Hence among the five-nodal primals Δ there are sets of three such that the three primals of any one set have a cyclide in common; each five-nodal primal Δ belongs to 495 of these sets.

33. It has been shown above that if UVW is any one of the trisecants of ϑ, there is a conic touching ζ at each of the corresponding points u, v, w; hence *there is a primal Δ having nodes at the three intersections of ϑ with any one of its trisecants.* The trisecant UVW lies on Δ. The points of the tritangent conic uvw represent quadrics of N which touch the solid Σ, joining UVW to its conjugate trisecant, at points of UVW; hence Σ is the tangent solid of Δ at each point of UVW, and meets Δ in a quartic surface on which UVW is a double line. This quartic surface is in fact *a complex-surface of Plücker.*

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[1] Cf. W. P. Milne: *Journal London Math. Soc.* 2 (1927), 79—84.
[2] Cf. W. P. Milne: *Proc. London Math. Soc.* (2), 28 (1928), 485.
[3] White; *Proc. London Math. Soc.* (2), 30 (1930), 347—358.

A Canonical Form for the Net of Quadrics.

34. We return now to the pair of conjugate trisecants UVW and PQR. The secant planes α, β, γ which are conjugate respectively to P, Q, R all pass through UVW; we suppose that the three solids $\beta\gamma, \gamma\alpha, \alpha\beta$ have the respective equations $x_1 = 0$, $x_2 = 0$, $x_3 = 0$. Also we suppose that, if $\delta, \varepsilon, \eta$ are the secant planes conjugate to U, V, W respectively, the equations of the three solids $\varepsilon\eta$, $\eta\delta$, $\delta\varepsilon$ are $y_1 = 0$, $y_2 = 0$, $y_3 = 0$; these three solids have PQR as their line of intersection. The six expressions $x_1, x_2, x_3, y_1, y_2, y_3$ are homogneous linear functions of five coordinates; there must therefore be one identical relation between them, and we can suppose this to be

$$x_1 + x_2 + x_3 \equiv y_1 + y_2 + y_3.$$

If $\Phi(x_1, x_2, x_3, y_1, y_2, y_3)$ is any homogenous quadratic polynomial then it is easily shown that the polar prime of any point $(h_1, h_2, h_3, k_1, k_2, k_3)$ in regard to the quadric $\Phi = 0$ is

$$h_1 \frac{\partial \Phi}{\partial x_1} + h_2 \frac{\partial \Phi}{\partial x_2} + h_3 \frac{\partial \Phi}{\partial x_3} + k_1 \frac{\partial \Phi}{\partial y_1} + k_2 \frac{\partial \Phi}{\partial y_2} + k_3 \frac{\partial \Phi}{\partial y_3} = 0,$$

where the partial differentiations are performed as though the six coordinates were independent. Now if $\Phi = 0$ is a quadric belonging to N the polar prime of any point on PQR, whose equations are $y_1 = y_2 = y_3 = 0$, must contain UVW, whose equations are $x_1 = x_2 = x_3 = 0$; it follows that Φ cannot in this case contain any product terms xy and so is of the form

$$a x_1^2 + b x_2^2 + c x_3^2 + 2 f x_2 x_3 + 2 g x_3 x_1 + 2 h x_1 x_2 +$$
$$+ a' y_1^2 + b' y_2^2 + c' y_3^2 + 2 f' y_2 y_3 + 2 g' y_3 y_1 + 2 h' y_1 y_2.$$

The polar prime of P in regard to this quadric has to contain the plane α; hence, since P, being the intersection of the line PQR and the solid $\beta\gamma$, has coordinates $(0, 1, -1, 0, 0, 0)$, the solid

$$h x_1 + b x_2 + f x_3 = g x_1 + f x_2 + c x_3$$

must contain the plane $x_2 = x_3 = 0$. Hence $h = g$. Similarly we obtain $g = f$ as the condition that the polar prime of R should contain γ; so that we must have $f = g = h$. Also, by considering the polar primes of U, V, W we find $f' = g' = h'$. The equation of the quadric may therefore be written in the form

$$a x_1^2 + b x_2^2 + c x_3^2 + 2 d (x_2 x_3 + x_3 x_1 + x_1 x_2) + \\ + a' y_1^2 + b' y_2^2 + c' y_3^2 + 2 d' (y_2 y_3 + y_3 y_1 + y_1 y_2) = 0.$$

This is the same as

$$(a - d) x_1^2 + (b - d) x_2^2 + (c - d) x_3^2 + d (x_1 + x_2 + x_3)^2 + \\ + (a' - d') y_1^2 + (b' - d') y_2^2 + (c' - d') y_3^2 + d' (y_1 + y_2 + y_3)^2 = 0.$$

Any quadric which belongs to N must therefore have an equation of this form; whence the following fundamental result may be enunciated:

The equations of three quadrics in [4] can be reduced simultaneously to the form

$$\xi_1 X_1^2 + \xi_2 X_2^2 + \xi_3 X_3^2 + \tau T^2 + \eta_1 Y_1^2 + \eta_2 Y_2^2 + \eta_3 Y_3^2 = 0,$$

where

$$X_1 + X_2 + X_3 \equiv T \equiv Y_1 + Y_2 + Y_3.$$

A given net of quadrics in [4] can be reduced to this canonical form in ten different ways.

The coefficients which occur are of course different for different quadrics of the net, but the seven linear forms are fixed for all the quadrics. It is always supposed that the net of quadrics is not specialised in any way.

The seven linear forms occuring are the seven solids $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, Σ, $\varepsilon\eta$, $\eta\delta$, $\delta\varepsilon$ where Σ denotes, as previously, the solid which joins the two conjugate trisecants.

35. We can use this canonical form of the net of quadrics to obtain a canonical form for the plane quintic; we have merely to obtain the discriminant of the quadric. Let as then, for the moment, regard X_2, X_3, T, Y_2, Y_3 as the five independent coordinates in [4]; the equation to the quadric may be written

$$\xi_1 (X_2 + X_3 - T)^2 + \xi_2 X_2^2 + \xi_3 X_3^2 + \tau T^2 + \eta_1 (Y_2 + Y_3 - T)^2 + \eta_2 Y_2^2 + \eta_3 Y_3^2 = 0.$$

The discriminant of this is

$$\varDelta \equiv \begin{vmatrix} \xi_1+\xi_2 & \xi_1 & -\xi_1 & 0 & 0 \\ \xi_1 & \xi_1+\xi_3 & -\xi_1 & 0 & 0 \\ -\xi_1 & -\xi_1 & \xi_1+\tau+\eta_1 & -\eta_1 & -\eta_1 \\ 0 & 0 & -\eta_1 & \eta_1+\eta_2 & \eta_1 \\ 0 & 0 & -\eta_1 & \eta_1 & \eta_1+\eta_3 \end{vmatrix}$$

Now add the second and fourth rows of $\varDelta$ to its middle row and, in the modified determinant, add the second and fourth columns to the middle column; we obtain

$$\varDelta \equiv \begin{vmatrix} \xi_1+\xi_2 & \xi_1 & 0 & 0 & 0 \\ \xi_1 & \xi_1+\xi_3 & \xi_3 & 0 & 0 \\ 0 & \xi_3 & \xi_3+\tau+\eta_2 & \eta_2 & 0 \\ 0 & 0 & \eta_2 & \eta_1+\eta_2 & \eta_1 \\ 0 & 0 & 0 & \eta_1 & \eta_1+\eta_3 \end{vmatrix}$$

It follows that *this is a canonical form for the ternary quintic,* where now $\xi_1, \xi_2, \xi_3, \tau, \eta_1, \eta_2, \eta_3$ are seven homogeneous linear functions of three variables. The expanded form of $\varDelta$ is

$$(\xi_2\xi_3+\xi_3\xi_1+\xi_1\xi_2)\eta_1\eta_2\eta_3+(\eta_2\eta_3+\eta_3\eta_1+\eta_1\eta_2)\xi_1\xi_2\xi_3+ \\ +\tau(\xi_2\xi_3+\xi_3\xi_1+\xi_1\xi_2)(\eta_2\eta_3+\eta_3\eta_1+\eta_1\eta_2),$$

and this is exactly the same form as that obtained in § 27.

The equation of the contact quartic associated with those quadrics of N which touch the solid $l X_2 + m X_3 + n T + p Y_2 + q Y_3 = 0$ is

$$\begin{vmatrix} \xi_1+\xi_2 & \xi_3 & 0 & 0 & 0 & l \\ \xi_1 & \xi_1+\xi_3 & \xi_3 & 0 & 0 & m \\ 0 & \xi_3 & \xi_3+\tau+\eta_2 & \eta_2 & 0 & n \\ 0 & 0 & \eta_2 & \eta_1+\eta_2 & \eta_1 & p \\ 0 & 0 & 0 & \eta_1 & \eta_1+\eta_3 & q \\ l & m & n & p & q & 0 \end{vmatrix} = 0,$$

and we obtain the ∞^4 contact quartics of the system $\mathfrak{S}$ by giving different values to the ratios $l:m:n:p:q$. We have already met with the equations of certain quartics of this system: for example

$$(\xi_2\xi_3 + \xi_3\xi_1 + \xi_1\xi_2)(\eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2) = 0$$

is the contact quartic associated with those quadrics of N which touch Σ; the equation of Σ is $T = 0$, and the contact quartic is obtained by putting $l = m = p = q = 0$ and expanding the determinant. Similarly the other contact quartics arise by giving appropriate values to l, m, n, p, q and expanding the determinant.