THE MEAN-VALUE OF THE RIEMANN ZETA FUNCTION.

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1. It was shown by Hardy and Littlewood[1] that an analogue of the ordinary mean-value theorem for Dirichlet series exists for the Riemann zeta function on the critical line. Writing

$$I(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^2 \, dt$$

they showed that

$$I(T) \sim T \log T \tag{1.1}$$

as $T \to \infty$.

A substantial advance was made by Littlewood[2] who proved that

$$I(T) = T \log T - T(1 + \log 2\pi - 2\gamma) + O(T^{\frac{3}{4}+\epsilon}) \tag{1.2}$$

by means of results connected with the approximate functional equation for $\zeta(s)$. Using improved forms of this equation Ingham[3] and Titchmarsh[4] were able to reduce the power of T in the error term in (1.2) to $\frac{1}{2}$ and $^5/_{12}$ respectively.

The problem has more than superficial affinities with a well-known divisor problem[5], namely the behaviour of

$$\sum_{n<x} d(n). \tag{1.3}$$

Here (1.2) is the analogue of Dirichlet's classical formula and its later refinements. My object in this paper is to establish what corresponds, in the case

[1] G. H. HARDY and J. E. LITTLEWOOD, Acta math. 41 (1918), 119—196.

[2] J. E. LITTLEWOOD, Proc. London Math. Soc. (2), 20 (1922), Records, XXII—XXVIII.

[3] A. E. INGHAM, Proc. London Math. Soc. (2), 27 (1926), 273—300.

[4] E. C. TITCHMARSH, Quart. J. of Math. (Oxford), 5 (1934), 195—210.

[5] The connection is to some extent apparent in my paper, Quart. J. of Math. (Oxford), 10 (1939), 122—128. I hope to go in this question more deeply in a subsequent paper.

45—48173. *Acta mathematica.* 81. Imprimé le 1 juillet 1949.

of the mean-value of the zeta function, to Voronoi's summation formula[1] for (1.3). Actually two groups of new terms emerge, the number of terms in each group depending on T, so that the result falls into the category of approximate functional equations. I prove the following

Theorem: *For* $T > 1$, $AT < N < A'T$,

$$I(T) = T\log T - T(1 + \log 2\pi - 2\gamma) + \frac{1}{\sqrt{2}}\sum_{n\le N}(-1)^n \frac{d(n)}{\sqrt{n}}\left(\sinh^{-1}\sqrt{\frac{\pi n}{2T}}\right)^{-1}\cdot$$

$$\cdot\left(\frac{T}{2\pi n}+\frac{1}{4}\right)^{-\frac{1}{4}}\sin\left\{2T\sinh^{-1}\sqrt{\frac{\pi n}{2T}} + \sqrt{(2\pi n T + \pi^2 n^2)} + \frac{\pi}{4}\right\} -$$

$$- 2\sum_{n\le N'}\frac{d(n)}{\sqrt{n}}\left(\log\frac{T}{2\pi n}\right)^{-1}\sin\left(T\log\frac{T}{2\pi n} - T - \frac{\pi}{4}\right) + O(\log^2 T),$$

where

$$N' = \frac{T}{2\pi} + \frac{N}{2} - \sqrt{\left(\frac{N^2}{4} + \frac{NT}{2\pi}\right)},$$

and A, A' *are any constants such that* $0 < A < A'$.

It is worthy of remark that the first set of oscillating terms on the right-hand side approximate, for low values of n, to the leading terms in Voronoi's summation formula, apart from an oscillating sign. The second set of such terms would be obtained by integrating formally a suitable expansion of $\zeta(s)\zeta(1-s)$.

The argument may be summarized as follows. In the formula, valid when $\sigma > 1$,

$$\frac{1}{2T}\int_{-T}^{T}|\zeta(\sigma+it)|^2\,dt = \zeta(2\sigma) + \sum_{\substack{m=1\\ m\neq n}}^{\infty}\sum_{n=1}^{\infty}(mn)^{-\sigma}\sin\left(T\log\frac{m}{n}\right)\left(T\log\frac{m}{n}\right)^{-1}$$

it is found expedient to classify the terms according to the value of $m - n$. We are thus led to consider, instead of $\zeta(u)$ or $\zeta(u)\zeta(v)$, the function

$$f(u, v) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} r^{-u}(r+s)^{-v}. \tag{1.3}$$

This double series is convergent when $\Re\{u+v\} > 2$, $\Re\{v\} > 1$; imposing the additional condition $\Re\{u\} < 0$ we can apply Poisson's summation formula to

[1] See, for instance, A. Oppenheim, Proc. London Math. Soc. (2), 26 (1927), 295—350, and references there given.

the r-sum over the range $r = 0, 1, \ldots \infty$. We thus derive a single series expansion for

$$g(u, v) = f(u, v) - \frac{\Gamma(u + v - 1)\,\Gamma(1 - u)}{\Gamma(v)}\,\zeta(u + v - 1) \tag{1.4}$$

valid when $\Re\{u\} < 0$, $\Re\{v\} > 1$, and $\Re\{u + v\} > 0$. This series may be continued inside the critical strip by means of Voronoi's summation formula, giving what is substantially the approximate functional equation for $g(u, 1 - u)$. The resulting expression for

$$\zeta(\tfrac{1}{2} + it)\,\zeta(\tfrac{1}{2} - it)$$

may then be integrated over $\frac{1}{2} - iT$, $\frac{1}{2} + iT$ to produce (4.4). The proof is completed by approximating to the trigonometric integrals involved; this is accomplished by the saddle-point method, on which I prove a general result, namely Lemma 1 of this paper.

We use the symbol A throughout to denote a positive absolute constant, not necessarily the same one at each occurrence. Where desirable, in order to avoid confusion, other such constants will be denoted by A', A''.

2. We start from the identity, valid when $\Re\{u\} > 1$, $\Re\{v\} > 1$,

$$\zeta(u)\,\zeta(v) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-u} n^{-v} = \zeta(u + v) + f(u, v) + f(v, u) \tag{2.1}$$

where $f(u, v)$ is defined by (1.3). We show first that $f(u, v)$ is a meromorphic function of u and v when $\Re\{u + v\} > 0$. We have, taking $\Re\{v\} > 1$, denoting the integral part of x by $[x]$, and writing

$$\mu(x) = x - [x] - \tfrac{1}{2}, \quad \mu_1(x) = \int_1^x \mu(y)\,dy$$

the relations

$$\begin{aligned}\sum_{s=1}^{\infty}(r + s)^{-v} &= v\int_r^{\infty}([x] - r)\,x^{-v-1}\,dx \\ &= r^{1-v}(v - 1)^{-1} - \tfrac{1}{2}r^{-v} - v\int_r^{\infty}\mu(x)\,x^{-v-1}\,dx \\ &= r^{1-v}(v - 1)^{-1} - \tfrac{1}{2}r^{-v} - v(v + 1)\int_r^{\infty}\mu_1(x)\,x^{-v-2}\,dx \\ &= r^{1-v}(v - 1)^{-1} - \tfrac{1}{2}r^{-v} + O\{v(v + 1)\,r^{-v-1}\}\end{aligned}$$

since $\mu_1(x)$ is bounded. Hence

$$f(u, v) = (v - 1)^{-1} \sum_{r=1}^{\infty} r^{1-u-v} - \tfrac{1}{2} \sum_{r=1}^{\infty} r^{-u-v} + O\left\{ v(v+1) \sum_{r=1}^{\infty} |r^{-u-v-1}| \right\}.$$

It follows that

$$f(u, v) - (v - 1)^{-1} \zeta(u + v - 1) + \tfrac{1}{2} \zeta(u + v)$$

is regular for $\Re\{u + v\} > 0$. Thus (2.1) holds when both u and v lie in the critical strip, apart from the poles at $v = 1$, $u + v = 1, 2$.

3. We consider next the case $\Re\{u\} < 0$, $\Re\{u + v\} > 2$. We have then, by Poisson's formula,

$$\sum_{r=0}^{\infty} r^{-u}(r + s)^{-v} = \int_0^{\infty} x^{-u}(x + s)^{-v}\, dx + 2 \sum_{m=1}^{\infty} \int_0^{\infty} x^{-u}(x + s)^{-v} \cos 2\pi m x\, dx$$

$$= s^{1-u-v} \left\{ \int_0^{\infty} y^{-u}(1 + y)^{-v}\, dy + 2 \sum_{m=1}^{\infty} \int_0^{\infty} y^{-u}(1 + y)^{-v} \cos 2\pi m s y\, dy \right\}.$$

Summing with respect to s, and using (1.4) and the result

$$\int_0^{\infty} y^{-u}(1 + y)^{-v}\, dy = \Gamma(u + v - 1)\, \Gamma(1 - u)\, \{\Gamma(v)\}^{-1}$$

this gives

$$g(u, v) = 2 \sum_{s=1}^{\infty} s^{1-u-v} \sum_{m=1}^{\infty} \int_0^{\infty} y^{-u}(1 + y)^{-v} \cos 2 m s y\, dy. \tag{3.1}$$

To investigate the convergence of this expression we note that for $\Re\{u\} < 1$, $\Re\{u + v\} > 0$, and $n \geq 1$,

$$2 \int_0^{\infty} y^{-u}(1 + y)^{-v} \cos 2\pi n y\, dy = n^{u-1} \int_0^{\infty} y^{-u}(1 + y/n)^{-v} (e^{2\pi i y} + e^{-2\pi i y})\, dy$$

$$= n^{u-1} \int_0^{i\infty} y^{-u}(1 + y/n)^{-v} e^{2\pi i y}\, dy + n^{u-1} \int_0^{-i\infty} y^{-u}(1 + y/n)^{-v} e^{-2\pi i y}\, dy = O\left(\frac{n^{u-1}}{u - 1}\right)$$

uniformly for bounded u and v. It follows that the double series (3.1) is absolutely convergent for $\Re\{u\} < 0$, $\Re\{v\} > 1$, $\Re\{u + v\} > 0$, by comparison with

$$\sum_{s=1}^{\infty} |s^{-v}| \sum_{m=1}^{\infty} |m^{u-1}|$$

and that it represents an analytic function of both variables in this region. Hence (3.1) holds throughout this region and then

$$g(u, v) = 2\sum_{n=1}^{\infty} \sigma_{1-u-v}(n) \int_0^{\infty} y^{-u}(1+y)^{-v} \cos 2\pi n y \, dy \tag{3.2}$$

where $\sigma_\alpha(n)$ denotes the sum of the α-th powers of the divisors of n.

Combining this with the result of § 2 we see that when $0 < \Re\{u\} < 1$, $0 < \Re\{v\} < 1$, $u + v \neq 1$,

$$\zeta(u)\zeta(v) = \zeta(u+v) + \zeta(u+v-1)\Gamma(u+v-1)\left\{\frac{\Gamma(1-u)}{\Gamma(v)} + \frac{\Gamma(1-v)}{\Gamma(u)}\right\} + \\ + g(u, v) + g(v, u) \tag{3.3}$$

where $g(u, v)$ is the analytic continuation of the function given by (3.2). It is however the exceptional case, $u + v = 1$, which we need. By § 2 the function $g(u, v)$ is here continuous, and the contribution of the remaining terms may by obtained by considerations of continuity. Write $u + v = 1 + \delta$, where $|\delta| < \frac{1}{2}$. The first two terms on the right-hand side of (3.3) give us

$$\zeta(1+\delta) + \zeta(\delta)\Gamma(\delta)\left\{\frac{\Gamma(1-u)}{\Gamma(1-u+\delta)} + \frac{\Gamma(u-\delta)}{\Gamma(u)}\right\}$$

$$= \zeta(1+\delta) + \zeta(1-\delta)(2\pi)^\delta \tfrac{1}{2} \sec \tfrac{1}{2}\pi\delta \left\{\frac{\Gamma(1-u)}{\Gamma(1-u+\delta)} + \frac{\Gamma(u-\delta)}{\Gamma(u)}\right\}$$

$$= \frac{1}{\delta} + \gamma + \left(\frac{1}{\delta} - \gamma\right)(1 + \delta \log 2\pi)\tfrac{1}{2}\left\{1 - \delta\frac{\Gamma'(1-u)}{\Gamma(1-u)} + 1 - \delta\frac{\Gamma'(u)}{\Gamma(u)}\right\} + O(\delta)$$

$$= \tfrac{1}{2}\left\{\frac{\Gamma'(1-u)}{\Gamma(1-u)} + \frac{\Gamma'(u)}{\Gamma(u)}\right\} + 2\gamma - \log 2\pi + O(\delta).$$

Hence we have, for $0 < \Re\{u\} < 1$,

$$\zeta(u)\zeta(1-u) = \tfrac{1}{2}\left\{\frac{\Gamma'(1-u)}{\Gamma(1-u)} + \frac{\Gamma'(u)}{\Gamma(u)}\right\} + \\ + 2\gamma - \log 2\pi + g(u, 1-u) + g(1-u, u). \tag{3.4}$$

4. From now on we confine ourselves to the case $u + v = 1$, with a view to the eventual application $u = \frac{1}{2} + it$, $v = \frac{1}{2} - it$. We have then, if $\Re\{u\} < 0$,

$$g(u, 1-u) = 2\sum_{n=1}^{\infty} d(n) \int_0^{\infty} y^{-u}(1+y)^{u-1} \cos 2\pi n y \, dy.$$

We need an analytic continuation of $g(u, 1-u)$ valid when $\Re\{u\} = \frac{1}{2}$. The most convenient way of obtaining this seems to be to use the summation formula for $d(n)$. We have[1], if x is not an integer

[1] See for instance OPPENHEIM, loc. cit.

$$\sum_{n<x} d(n) = D(x) + \Delta(x),$$

where

$$D(x) = x(\log x + 2\gamma - 1) + \tfrac{1}{4}$$

and

$$\Delta(x) = \frac{x^{\frac{1}{4}}}{\pi\sqrt{2}} \sum_{n=1}^{\infty} \frac{d(n)}{n^{3/4}} \left\{\cos\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right) - \frac{3}{32\pi\sqrt{nx}} \sin\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right)\right\} + O(x^{-3/4}) \quad (4.1)$$

$$= O(x^{1/3} \log x), \quad (4.2)$$

the series being boundedly convergent in any finite x-interval.

Let now N be a positive integer, and let $h(u, x)$ denote the expression

$$2\int_0^{\infty} y^{-u}(1 + y)^{u-1} \cos 2\pi xy\, dy. \quad (4.3)$$

Then

$$\sum_{n>N} d(n) h(u, n) = -\sum_{n>N} d(n) \int_n^{\infty} \frac{\partial}{\partial x} h(u, x)\, dx = -\int_{N+\frac{1}{2}}^{\infty} \sum_{N<n<x} d(n) \frac{\partial}{\partial x} h(u, x)\, dx$$

$$= -\{D(N + \tfrac{1}{2}) + \Delta(N + \tfrac{1}{2})\} h(u, N + \tfrac{1}{2}) - \int_{N+\frac{1}{2}}^{\infty} (D(x) + \Delta(x)) \frac{\partial}{\partial x} h(u, x)\, dx$$

$$= -\Delta(N + \tfrac{1}{2}) h(u, N + \tfrac{1}{2}) + \int_{N+\frac{1}{2}}^{\infty} D'(x) h(u, x)\, dx - \int_{N+\frac{1}{2}}^{\infty} \Delta(x) \frac{\partial}{\partial x} h(u, x)\, dx.$$

Hence

$$g(u, 1 - u) = \sum_{n \leq N} d(n) h(u, n) - \Delta(N + \tfrac{1}{2}) h(u, N + \tfrac{1}{2}) + \int_{N+\frac{1}{2}}^{\infty} (\log x + 2\gamma) h(u, x)\, dx -$$

$$-\int_{N+\frac{1}{2}}^{\infty} \Delta(x) \frac{\partial}{\partial x} h(u, x)\, dx = g_1(u) - g_2(u) + g_3(u) - g_4(u),$$

say. Here $g_1(u)$ and $g_2(u)$ are analytic functions in the region $\Re\{u\} < 1$, since the expression (4.3) for $h(u, x)$ is analytic in this region.

Take next $g_4(u)$; we need an order result for $\frac{\partial}{\partial x} h(u, x)$. We have

$$h(u, x) = \int_0^{i\infty} y^{-u}(1 + y)^{u-1} e^{2\pi i x y}\, dy + \int_0^{-i\infty} y^{-u}(1 + y)^{u-1} e^{-2\pi i x y}\, dy,$$

so that

$$\frac{\partial}{\partial x} h(u, x) = 2\pi i \int_0^{i\infty} y^{1-u} (1 + y)^{u-1} e^{2\pi i x y}\, dy - 2\pi i \int_0^{-i\infty} y^{1-u} (1 + y)^{u-1} e^{-2\pi i x y}\, dy$$

$$= 2\pi i x^{u-2} \int_0^{i\infty} y^{1-u} (1 + y/x)^{u-1} e^{2\pi i y}\, dy -$$

$$- 2\pi i x^{u-2} \int_0^{-i\infty} y^{1-u} (1 + y/x)^{u-1} e^{-2\pi i y}\, dy = O(x^{u-2})$$

for $\Re\{u\} \leq 1$ and bounded u. Hence, by (4.2),

$$\Delta(x) \frac{\partial}{\partial x} h(u, x) = O(x^{u-5/3} \log x)$$

so that the integral defining $g_4(u)$ is an analytic function of u at any rate when $\Re\{u\} < \frac{2}{3}$.

It remains to consider $g_3(u)$. For brevity we denote $N + \frac{1}{2}$ by X. Then

$$g_3(u) = \int_X^\infty (\log x + 2\gamma)\, dx \left\{ \int_0^{i\infty} y^{-u} (1 + y)^{u-1} e^{2\pi i x y}\, dy + \int_0^{-i\infty} y^{-u} (1 + y)^{u-1} e^{-2\pi i x y}\, dy \right\}.$$

But, if $\Re\{u\} < 0$, $\displaystyle\int_X^\infty (\log x + 2\gamma)\, dx \int_0^{i\infty} y^{-u} (1 + y)^{u-1} e^{2\pi i x y}\, dy$

$$= \left[(\log x + 2\gamma) \int_0^{i\infty} y^{-u} (1 + y)^{u-1} \frac{e^{2\pi i x y}}{2\pi i y}\, dy \right]_X^\infty - \int_X^\infty \frac{dx}{x} \int_0^{i\infty} y^{-u} (1 + y)^{u-1} \frac{e^{2\pi i x y}}{2\pi i y}\, dy$$

$$= - \frac{1}{2\pi i} (\log X + 2\gamma) \int_0^{i\infty} y^{-u-1} (1 + y)^{u-1} e^{2\pi i X y}\, dy -$$

$$- \frac{1}{2\pi i} \int_X^\infty dx \int_0^{i\infty} y^{-1-u} (x + y)^{u-1} e^{2\pi i y}\, dy =$$

$$= - \frac{1}{2\pi i} (\log X + 2\gamma) \int_0^{\infty} y^{-u-1} (1 + y)^{u-1} e^{2\pi i X y}\, dy + \frac{1}{2\pi i u} \int_0^{i\infty} y^{-1-u} (X + y)^{u} e^{2\pi i y}\, dy.$$

The latter integral may now be taken over $0, \infty$ and the variable changed from y to y/X. We treat the integral over $0, -i\infty$ similarly and combine the results. Hence

$$g_3(u) = -\frac{\log X + 2\gamma}{\pi}\int_0^\infty y^{-u-1}(1+y)^{u-1}\sin 2\pi Xy\,dy + \\ + \frac{1}{\pi u}\int_0^\infty y^{-u-1}(1+y)^u \sin 2\pi Xy\,dy.$$

These integrals are uniformly convergent for $\Re\{u\} \leq 1 - \varepsilon < 1$. We have therefore obtained an expression for $g(u, 1-u)$ which exists when $\Re\{u\} = \frac{1}{2}$.

We now derive the corresponding expression for $I(T)$. We have, by (3.4),

$$2iI(T) = \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \zeta(u)\zeta(1-u)\,du$$

$$= \log\frac{\Gamma(\frac{1}{2}+iT)}{\Gamma(\frac{1}{2}-iT)} + 2iT(2\gamma - \log 2\pi) + 2\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} g(u, 1-u)\,du.$$

Hence, using Stirling's formula,

$$I(T) = T\log T - T(1 + \log 2\pi - 2\gamma) + \frac{1}{i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} g(u, 1-u)\,du + O(1)$$

$$= T\log T - T(1 + \log 2\pi - 2\gamma) + I_1 - I_2 + I_3 - I_4 + O(1), \quad (4.4)$$

where

$$I_\nu = \frac{1}{i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} g_\nu(u)\,du, \quad (\nu = 1, 2, 3, 4)$$

so that

$$I_1 = 4\sum_{n\leq N} d(n)\int_0^\infty \frac{\sin\left(T\log\frac{1+y}{y}\right)\cos 2\pi ny}{y^{\frac{1}{2}}(1+y)^{\frac{1}{2}}\log\frac{1+y}{y}}\,dy, \quad (4.5)$$

$$I_2 = 4\Delta(X)\int_0^\infty \frac{\sin\left(T\log\frac{1+y}{y}\right)\cos 2\pi Xy}{y^{\frac{1}{2}}(1+y)^{\frac{1}{2}}\log\frac{1+y}{y}}\,dy, \quad (4.6)$$

$$I_3 = -2\left(\frac{\log X + 2\gamma}{\pi}\right)\int\limits_0^\infty \frac{\sin\left(T\log\frac{1+y}{y}\right)\sin 2\pi X y}{y^{3/2}(1+y)^{1/2}\log\frac{1+y}{y}}\,dy +$$

$$+\frac{1}{\pi i}\int\limits_0^\infty \frac{\sin 2\pi X y}{y}\,dy \int\limits_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\left(\frac{1+y}{y}\right)^u \frac{du}{u}, \qquad (4.7)$$

and lastly

$$I_4 = \frac{1}{i}\int\limits_X^\infty \Delta(x)\,dx \int\limits_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{\partial}{\partial x} h(u, x)\,du,$$

where N is a positive integer and $X = N + \frac{1}{2}$. We shall restrict N to a range of values of the form $A T < N < A' T$.

We may derive a more explicit formula for I_4 as follows. We have

$$\int\limits_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{\partial}{\partial x} h(u, x)\,du = 4i\frac{\partial}{\partial x}\int\limits_0^\infty \frac{\sin\left(T\log\frac{1+y}{y}\right)\cos 2\pi x y}{y^{\frac{1}{2}}(1+y)^{\frac{1}{2}}\log\frac{1+y}{y}}\,dy$$

$$= 4i\frac{\partial}{\partial x}\int\limits_0^\infty \frac{\sin\left(T\log\frac{x+y}{y}\right)\cos 2\pi y}{y^{\frac{1}{2}}(x+y)^{\frac{1}{2}}\log\frac{x+y}{y}}\,dy = 4i\int\limits_0^\infty \frac{\cos 2\pi y}{y^{1/2}(x+y)^{3/2}\log\frac{x+y}{y}}\cdot$$

$$\cdot\left\{T\cos\left(T\log\frac{x+y}{y}\right) - \sin\left(T\log\frac{x+y}{y}\right)\left(\frac{1}{2} + \left(\log\frac{x+y}{y}\right)^{-1}\right)\right\}dy.$$

Hence

$$I_4 = 4\int\limits_X^\infty \frac{\Delta(x)}{x}\,dx\int\limits_0^\infty \frac{\cos 2\pi x y}{y^{\frac{1}{2}}(1+y)^{3/2}\log\frac{1+y}{y}}\cdot$$

$$\cdot\left\{T\cos\left(T\log\frac{1+y}{y}\right) - \sin\left(T\log\frac{1+y}{y}\right)\left(\frac{1}{2} + \left(\log\frac{1+y}{y}\right)^{-1}\right)\right\}dy. \qquad (4.8)$$

5. We approximate to these integrals by the saddle-point method. All we need on this point is contained in the following

Lemma 1. Let $f(z)$, $\varphi(z)$ be two functions of the complex variable z, and (a, b) a real interval, such that (i) for $a \leq x \leq b$, $f(x)$ is real and $f''(x) > 0$, (ii) for a certain positive differentiable function $\mu(x)$, defined in (a, b), $f(z)$ and $\varphi(z)$ are analytic for

$$a \leq x \leq b, \quad |z - x| \leq \mu(x)$$

(iii) there are positive functions $F(x)$, $\Phi(x)$ defined in (a, b) such that

$$\varphi(z) = O\{\Phi(x)\},\ f'(z) = O\{F(x)(\mu(x))^{-1}\},\ \{f''(z)\}^{-1} = O\{(\mu(x))^2 (F(x))^{-1}\}$$

for

$$a \leq x \leq b, \quad |z - x| \leq \mu(x),$$

the constants implied in these order results being absolute.

Let k be any real number, and if $f'(x) + k$ has a zero in (a, b) denote it by x_0. Let the values of $f(x)$, $\varphi(x)$, etc., at a, x_0, b be characterised by the suffixes a, 0, and b respectively. Then

$$\int_a^b \varphi(x) \exp 2\pi i\{f(x) + kx\}\, dx = \varphi_0 f_0''^{-\frac{1}{2}} e^{2\pi i (f_0 + k x_0) + \frac{1}{4}\pi i} +$$

$$+ O\left\{\int_a^b \Phi(x) \exp\{-A|k|\mu(x) - A F(x)\}(dx + |d\mu(x)|)\right\} +$$

$$+ O(\Phi_0 \mu_0 F_0^{-3/2}) + O\left(\frac{\Phi_a}{|f_a' + k| + f_a''^{\frac{1}{2}}}\right) + O\left(\frac{\Phi_b}{|f_b' + k| + f_b''^{\frac{1}{2}}}\right).$$

If $f'(x) + k$ has no zero in (a, b) then the terms involving x_0 are to be omitted.

We take the case in which $f'(x) + k$ has a zero in (a, b). Denote by $\lambda(x)$ the function $\alpha\mu(x)$ where α is a positive absolute constant, less than $\frac{1}{2}$, to be fixed later.

We deform the path of integration into a contour joining the points a, $a - \lambda_a(1 + i)$, $x_0 - \lambda_0(1 + i)$, $x_0 + \lambda_0(1 + i)$, $b + \lambda_b(1 + i)$, b. Denoting the corresponding parts of the integrals by $J_1, \ldots, J_5$, respectively, we take J_1, J_3, and J_5 along straight lines, while, J_2, J_4 are to be taken along the loci of the points $x \pm \lambda(x)(1 + i)$, as the case may be.

We have then, for $z = x + (1 + i)y$, $-\lambda(x) \leq y \leq \lambda(x)$, $a \leq x \leq b$,

$$f(z) + kz = f(x) + kx + (1 + i)y(f'(x) + k) + iy^2 f''(x) + \theta(y),$$

say, where, by Taylor's theorem[1],

$$|\theta(x)| < A\,F(x)\,|y|^3(\mu(x))^{-3}$$

Hence by taking α sufficiently small we may make

$$|\theta(y)| < \tfrac{1}{2}\,y^2 f''(x).$$

We have then

$$\Re\,\{2\,\pi\,i\,(f(z) + k\,z)\} < -\,2\,\pi\,y\,(f'(x) + k) - \pi\,y^2 f''(x). \tag{5.1}$$

This gives

$$J_1 = O\left\{\int\limits_0^{\lambda_a} \Phi_a \exp\left(-\,2\,\pi\,y\,|f'_a + k)| - \pi\,y^2 f''_a\right) d\,y\right\} = O\left\{\frac{\Phi_a}{|f'_a + k| + f''^{1/2}_a}\right\}$$

with a similar bound for J_5.

We have also, by the same argument,

$$J_2 = O\left\{\int\limits_a^{x_0} \Phi(x) \exp\left(-\,2\,\pi\,\lambda(x))\,|f'(x) + k| - \pi\,(\lambda(x))^2 f''(x)\right)(d\,x + |\,d\,\lambda(x)|)\right\}.$$

Now if

$$|k| \leq 2\,|f'(x)|$$

then

$$\lambda(x)\,|f'(x) + k| = O(\lambda(x) f'(x)) = O(F(x)),$$

and also

$$k\,\mu(x) = O(f'(x)\,\mu(x)) = O(F(x)),$$

while if

$$k \geq 2\,|f'(x)|$$

then

$$\lambda(x)\,|f'(x) + k| \geq \lambda(x)\,(|k| - |f'(x)|) \geq A\,|k|\,\mu(x).$$

Furthermore

$$(\lambda(x))^2 f''(x) > A\,F(x),$$

and hence, in any event,

$$-\,2\,\pi\,\lambda(x)\,|f'(x) + k| - \pi\,(\lambda(x))^2 f''(x) < -\,A\,|k|\,\mu(x) - A\,F(x).$$

This gives

$$J_2 = O\left\{\int\limits_a^{x_0} \Phi(x) \exp\left(-\,A\,|k|\,\mu(x) - A\,F(x)\right)(d\,x + |\,d\,\mu(x)|)\right\},$$

with a corresponding bound for J_4.

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[1] See for instance 'Modern Analysis', (E. T. WHITTAKER and G. N. WATSON, 4th edition Cambridge, 1927), § 5.4.

It remains to estimate J_3. We write

$$J_3 = \int\limits_{-\lambda_0(1+i)}^{\lambda_0(1+i)} \varphi(x_0+y)\, e^{2\pi i\{f(x_0+y)+k(x_0+y)\}}\,dy$$
$$= \int\limits_{-\lambda_0(1+i)}^{-v(1+i)} + \int\limits_{-v(1+i)}^{v(1+i)} + \int\limits_{v(1+i)}^{\lambda_0(1+i)} = J_{31} + J_{32} + J_{33},$$

say, where v is a number less than λ_0. We take in fact

$$v = \lambda_0(1 + F_0^{1/3})^{-1}.$$

We have then, by (5.1),

$$J_{33} = O\left\{\Phi_0 \int\limits_v^\infty e^{-\pi y^2 f_0''}\,dy\right\} = O\{\Phi_0 (v f_0'')^{-1} e^{-\pi v^2 f_0''}\}.$$

Hence

$$v^2 f_0'' = \lambda_0^2 f_0'' (1 + F_0^{1/3})^{-2} > A F_0 (1 + F_0^{1/3})^{-2}$$

and also

$$(v f_0'')^{-1} = v(v^2 f_0'')^{-1} = O(\mu_0 F_0^{-1}(1 + F_0^{1/3})).$$

Hence, for $F_0 \geq 1$,

$$J_{33} = O\{\Phi_0 \mu_0 F_0^{-2/3} e^{-A F_0^{1/3}}\},$$

while for $F_0 \leq 1$

$$J_{33} = O\{\Phi_0 \mu_0 F_0^{-1}\},$$

and so, in either case,

$$J_{33} = O\{\Phi_0 \mu_0 F_0^{-3/2}\}$$

A similar bound clearly holds for J_{31}.

In J_{32} we write

$$e^{2\pi i\{f(x_0+y)+k(x_0+y)\}} = \exp 2\pi i \left\{\sum_{\nu=0}^{3} \frac{1}{\nu!} y^\nu f_0^{(\nu)} + O(y^4 F_0 \mu_0^{-4})\right\}$$
$$= \exp\{2\pi i(f_0 + k x_0) + \pi i y^2 f_0''\}\{1 + \tfrac{1}{6} y^3 f_0''' + O(y^6 F_0^2 \mu_0^{-6}) + O(y^4 F_0 \mu_0^{-4})\}.$$

Since

$$\int\limits_{-v(1+i)}^{v(1+i)} y^{2\nu} e^{-\pi y^2 f_0''}\,dy = O(f_0''^{\,-\nu-\frac{1}{2}}) = O(\mu_0^{2\nu+1} F_0^{-\nu-\frac{1}{2}})$$

the contribution of the two error terms in (5.2) is

$$O(\Phi_0 \mu_0 F_0^{-3/2}).$$

As regards the remainder of (5.2) we have

$$\varphi(x_0+y)(1 + \tfrac{1}{6} y^3 f_0''') = \varphi_0 + y\varphi_0 + \tfrac{1}{6} y^3 \varphi_0 f_0''' + O(y^2 \Phi_0 \mu_0^{-2}) + O(y^4 \Phi_0 F_0 \mu_0^{-4})$$

and hence, since integrals involving odd powers of y vanish,

$$\int_{-v(1+i)}^{v(1+i)} \varphi(x_0 + y)\left(1 + \tfrac{1}{6} y^3 f_0'''\right) e^{\pi i y^2 f_0''}\, dy = \varphi_0 \int_{-v(1+i)}^{v(1+i)} e^{\pi i y^2 f_0''}\, dy +$$

$$+ O\left(\Phi_0 \mu_0^{-2} f_0''^{-3/2}\right) + O\left(\Phi_0 F_0 \mu_0^{-4} f_0''^{-5/2}\right) = \varphi_0 \int_{-\infty(1+i)}^{\infty(1+i)} e^{\pi i y^2 f_0''}\, dy +$$

$$+ O\left(\varphi_0 \int_{v(1+i)}^{\infty(1+i)} e^{\pi i y^2 f_0''}\right) + O\left(\Phi_0 \mu_0 F_0^{-3/2}\right) = \varphi_0 f_0''^{-\frac{1}{2}} e^{\frac{1}{4}\pi i} +$$

$$+ O\left\{\Phi_0 (v f_0'')^{-1} e^{-2\pi v^2 f_0''}\right\} + O\left(\Phi_0 \mu_0 F_0^{-3/2}\right),$$

and the first error term on the right-hand side has already been shown to be of lower order than the second. Hence

$$J_3 = \varphi_0 f_0''^{-\frac{1}{2}} e^{2\pi i (f_0 + k x_0) + \frac{1}{4}\pi i} + O\left(\Phi_0 \mu_0 F_0^{-3/2}\right).$$

This completes the proof of Lemma 1 for the case in which $f'(x) + k$ has a zero in (a, b). In other cases we take the contour through the points a, $a \pm \lambda_a(1 + i)$, $b \pm \lambda_b(1 + i)$, b, the $+$ or $-$ sign being taken according as $f'(x) \geq 0$ or ≤ 0 in (a, b); there is then no term corresponding to J_3.

6. Our first application of Lemma 1 is to the type of integral occurring in I_1, I_2, I_3 and I_4. We have

Lemma 2. *Let α, β, γ, a, b, k, T be real numbers such that α, β, γ are positive and bounded, $\alpha \neq 1$, $0 < a < \frac{1}{2}$, $a < \dfrac{T}{8\pi k}$, $b \geq T$, $k \geq 1$, $T \geq 1$. Then*

$$\int_a^b y^{-\alpha}(1 + y)^{-\beta}\left(\log \frac{1 + y}{y}\right)^{-\gamma} \exp i\left\{T \log \frac{1 + y}{y} + 2\pi k y\right\} dy$$

$$= \frac{1}{2k}\sqrt{\frac{T}{\pi}}\, V^{-\gamma} U^{-\frac{1}{2}} \left(U - \tfrac{1}{2}\right)^{-\alpha}\left(U + \tfrac{1}{2}\right)^{-\beta} \exp i\left\{T V + 2\pi k U - \pi k + \frac{\pi}{4}\right\} +$$

$$+ O\left(a^{1-\alpha} T^{-1}\right) + O\left(b^{\gamma-\alpha-\beta} k^{-1}\right) + R(T, k)$$

uniformly for $|\alpha - 1| > \varepsilon > 0$, where

$$U = \sqrt{\left(\frac{T}{2\pi k} + \frac{1}{4}\right)}, \quad V = 2 \sinh^{-1}\sqrt{\frac{\pi k}{2T}}$$

and

$$R(T, k) = O\left(T^{\frac{1}{2}(\gamma-\alpha-\beta) - \frac{1}{4}} k^{-\frac{1}{2}(\gamma-\alpha-\beta) - 5/4}\right), \quad (1 \leq k \leq T),$$
$$= O\left(T^{-\frac{1}{2}-\alpha} k^{\alpha-1}\right), \qquad (k \geq T).$$

A similar result holds for the corresponding integral with $-k$ *in place of* k, *except that here the explicit term on the right-hand side is to be omitted.*

We apply Lemma 1 with

$$\varphi(x) = x^{-\alpha}(1+x)^{-\beta}\left(\log\frac{1+x}{x}\right)^{-\gamma}, \quad f(x) = \frac{T}{2\pi}\log\frac{1+x}{x},$$

$$\Phi(x) = x^{-\alpha}(1+x)^{\gamma-\beta}, \quad F(x) = \frac{T}{1+x}, \quad \mu(x) = \tfrac{1}{2}x.$$

We dispose first of the terms involving a and b. We have

$$f'(x) = -\frac{T}{2\pi x(1+x)}$$

and so, if $a < \max\left\{\frac{1}{2}, \frac{T}{8\pi k}\right\}$,

$$f'(x) + k < -A\frac{T}{a}$$

so that

$$\frac{\Phi(a)}{f'(a)+k} = O(a^{1-\alpha}T^{-1}).$$

Again, if $b \geq T$,

$$\Phi(b) = O(b^{\gamma-\alpha-\beta}), \; f'(b) = O(T^{-1})$$

so that

$$\frac{\Phi(b)}{f'(b)+k} = O(b^{\gamma-\alpha-\beta}k^{-1}).$$

We consider the error-term integral of Lemma 1. This gives

$$O\left(\int_a^1 x^{-\alpha}e^{-Akx-AT}\,dx\right) + O\left(\int_1^b x^{\gamma-\alpha-\beta}e^{-Akx-AT/x}\,dx\right).$$

The integral over $(a, 1)$ gives, for $|\alpha - 1| > \varepsilon > 0$, $1 \leq k \leq T$,

$$\int_a^1 x^{-\alpha}e^{-Akx-AT}\,dx = O\left(e^{-AT}\int_a^1 x^{-\alpha}\,dx\right) = O\left(e^{-AT}(a^{1-\alpha}+1)\right),$$

while for $k \geq T$ we get

$$\int_a^1 x^{-\alpha}e^{-Akx-AT}\,dx = O\left(e^{-AT}\int_a^{T/k} x^{-\alpha}\,dx\right) + O\left(e^{-AT}\left(\frac{k}{T}\right)^{\alpha}\int_{T/k}^{\infty} e^{-Akx}\,dx\right)$$

$$= O\left\{e^{-AT}(a^{1-\alpha} + T^{1-\alpha}k^{\alpha-1} + T^{-\alpha}k^{\alpha-1})\right\}.$$

The integral over $(1, b)$ gives

$$O\left\{e^{-A\sqrt{kT}}\int_1^T x^{\gamma-\alpha-\beta}\,dx\right\} + O\left\{(b^{\gamma-\alpha-\beta} + T^{\gamma-\alpha-\beta})\int_T^b e^{-Akx}\,dx\right\}$$
$$= O\left\{e^{-A\sqrt{kT}}(T^{1+\gamma-\alpha-\beta} + T)\right\} + O\left\{e^{-AkT}k^{-1}(b^{\gamma-\alpha-\beta} + T^{\gamma-\alpha-\beta})\right\}.$$

All these error terms are of lower order than those given in Lemma 2.

It remains to consider the terms involving x_0. We have

$$x_0(x_0 + 1) = \frac{T}{2\pi k}$$

so that

$$x_0 = U - \tfrac{1}{2}$$

in the notation of Lemma 2. Hence

$$f_0'' = \frac{T(2x_0 + 1)}{2\pi x_0^2(x_0 + 1)^2} = \frac{4\pi k^2 U}{T},$$

$$\log\frac{x_0 + 1}{x_0} = \log\frac{U + \frac{1}{2}}{U - \frac{1}{2}} = 2\sinh^{-1}\sqrt{\frac{\pi k}{2T}} = V,$$

so that

$$f_0 + kx_0 = \frac{TV}{2\pi} + k(U - \tfrac{1}{2}).$$

Hence the main term is

$$\varphi_0 f_0''^{-\frac{1}{2}} e^{2\pi i(f_0 + kx_0) + \frac{1}{4}\pi i} = (U - \tfrac{1}{2})^{-\alpha}(U + \tfrac{1}{2})^{-\beta} V^{-\gamma}\frac{1}{2k}\sqrt{\frac{T}{\pi}}\,U^{-\frac{1}{2}}\cdot$$
$$\cdot\exp i\left(TV + 2\pi kU - \pi k + \frac{\pi}{4}\right).$$

Lastly we deal with the error term involving x_0. Take first the case $1 \leq k \leq T$. We have then

$$A \leq A\sqrt{\frac{T}{k}} < x_0 < A'\sqrt{\frac{T}{k}}, \quad \Phi_0 = O(x^{\gamma-\alpha-\beta}),$$
$$\mu_0 = O(x_0), \quad A\sqrt{kT} < F_0 < A'\sqrt{kT}.$$

The error term in question then gives us, for $1 \leq k \leq T$,

$$O\left(T^{\frac{1}{2}(\gamma-\alpha-\beta) - \frac{1}{4}} k^{-\frac{1}{2}(\gamma-\alpha-\beta) - 5/4}\right).$$

Finally, for $k \geq T$,

$$A\frac{T}{k} < x_0 < A'\frac{T}{k}, \quad \Phi_0 = O(x_0^{-\alpha}), \quad \mu_0 = O(x_0), \quad F_0 > AT,$$

so that in this case the error term gives us

$$O(T^{-\frac{1}{2}-\alpha}k^{\alpha-1}).$$

This completes the proof of Lemma 2 for $k \geq 1$. For $k \leq -1$ the argument differs only in that the terms in x_0 do not occur.

7. *Estimation of I_1 and I_2.* If, in Lemma 2, $0 < \alpha < 1$, $\alpha + \beta > \gamma$, we may make $a \to 0$, $b \to \infty$. Hence, if $\frac{1}{2} < \alpha < \frac{3}{4}$, $1 \leq k < AT$,

$$\int_0^\infty \frac{\sin\left(T\log\frac{1+y}{y}\right)\cos 2\pi k y}{y^\alpha(1+y)^{\frac{1}{2}}\log\frac{1+y}{y}}\,dy = \frac{1}{4k}\sqrt{\frac{T}{\pi}}\,\frac{\sin\left(TV + 2\pi k U - \pi k + \frac{\pi}{4}\right)}{V U^{1/2}(U-\frac{1}{2})^\alpha(U+\frac{1}{2})^{1/2}} +$$

$$+ O(T^{-\frac{1}{2}\alpha}k^{\frac{1}{2}\alpha-\frac{3}{2}}),$$

and since this result holds uniformly in α we may put $\alpha = \frac{1}{2}$. Hence, substituting in (4.5) we get

$$I_1 = \frac{1}{\sqrt{2}}\sum_{n\leq N}\frac{(-1)^n d(n)}{\sqrt{n}}\,\frac{\sin\left\{2T\sinh^{-1}\sqrt{\frac{\pi n}{2T}} + \sqrt{(2\pi n T + \pi^2 n^2)} + \frac{\pi}{4}\right\}}{\left(\sinh^{-1}\sqrt{\frac{\pi n}{2T}}\right)\left(\frac{T}{2\pi n} + \frac{1}{4}\right)^{1/4}} +$$

$$+ O(T^{-1/4}), \qquad (7.1)$$

taking $AT < N < A'T$. Similarly, from (4.6),

$$I_2 = O\{\Delta(X)X^{-\frac{1}{2}}\} = O(T^{-1/6}\log T), \qquad (7.2)$$

since

$$\Delta(x) = O(x^{1/3}\log x), \quad X = N + \tfrac{1}{2}.$$

8. *Estimation of I_3.*

We write (4.7) in the form

$$I_3 = -2\frac{\log X + 2\gamma}{\pi}I_{31} + \frac{1}{\pi i}I_{32}$$

and consider first I_{31}. We divide the range of integration at Y, where $Y = (2X)^{-1}$. The integral over $(0, Y)$ gives

$$I'_{31} = \int\limits_0^Y \frac{\sin\left(T \log \frac{1+y}{y}\right)}{y(1+y)} \cdot \frac{\sin 2\pi X y}{y} \cdot \frac{y^{\frac12}(1+y)^{1/2}}{\log \frac{1+y}{y}}\, dy.$$

Now of the expressions

$$\frac{\sin 2\pi X y}{y}, \quad \frac{y^{\frac12}(1+y)^{\frac12}}{\log \frac{1+y}{y}},$$

the first is monotonic decreasing in $(0, Y)$, the second monotonic increasing. Hence, by repeated-use of the second mean-value theorem,

$$I'_{31} = 2\pi X \int\limits_0^\xi \frac{\sin\left(T \log \frac{1+y}{y}\right)}{y(1+y)} \cdot \frac{y^{\frac12}(1+y)^{\frac12}}{\log \frac{1+y}{y}}\, dy$$

$$= 2\pi X \xi^{\frac12}(1+\xi)^{\frac12} \left(\log \frac{1+\xi}{\xi}\right)^{-1} \int\limits_\eta^\xi \frac{\sin\left(T \log \frac{1+y}{y}\right)}{y(1+y)}\, dy$$

$$= 2\pi X \xi^{\frac12}(1+\xi)^{\frac12} \left(\log \frac{1+\xi}{\xi}\right)^{-1} \left[\frac{1}{T} \cos\left(T \log \frac{1+y}{y}\right)\right]_\eta^\xi$$

for some η, ξ such that $0 \leq \eta \leq \xi \leq Y$. Hence

$$I'_{31} = O(T^{-\frac12}).$$

To the integral over (Y, ∞) we apply Lemma 2, taking $a = Y$, and making $b \to \infty$. We treat the main term on the right-hand side of Lemma 2 as an error term and get

$$\int\limits_Y^\infty \frac{\sin\left(T \log \frac{1+y}{y}\right) \sin 2\pi X y}{y^{3/2}(1+y)^{1/2} \log \frac{1+y}{y}}\, dy = O(T^{-\frac12})$$

so that

$$I_{31} = O(T^{-\frac14}).$$

Take next I_{32}. We divide the range of integration with respect to y at $y = 1$, and term I'_{32}, I''_{32} the contributions of the intervals $(0, 1)$, $(1, \infty)$. Taking first I'_{32} we have, for $0 < y \leq 1$,

$$\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \left(\frac{1+y}{y}\right)^u \frac{du}{u} = 2\pi i - \left(\int_{\frac{1}{2}+iT}^{-\infty+iT} + \int_{-\infty-iT}^{\frac{1}{2}-iT}\right)\left(\frac{1+y}{y}\right)^u \frac{du}{u}.$$

But

$$\int_{\frac{1}{2}+iT}^{-\infty+iT} \left(\frac{1+y}{y}\right)^u \frac{du}{u} = O\left\{T^{-1}\int_{\frac{1}{2}}^{-\infty}\left(\frac{1+y}{y}\right)^\sigma d\sigma\right\} = O(T^{-1}y^{-\frac{1}{2}}),$$

and similarly for the integral over $(-\infty - iT, \frac{1}{2} - iT)$. Hence

$$I'_{32} = 2\pi i \int_0^1 \frac{\sin 2\pi X y}{y} dy + O\left(T^{-1}\int_0^1 |\sin 2\pi X y|\, y^{-3/2} dy\right).$$

But

$$\int_0^1 \frac{\sin 2\pi X y}{y} dy = \frac{\pi}{2} + O(X^{-1}),$$

$$\int_0^{X^{-1}} |\sin 2\pi X y|\, y^{-3/2} dy = O\left(X\int_0^{X^{-1}} y^{-\frac{1}{2}} dy\right) = O(X^{\frac{1}{2}}),$$

$$\int_{X^{-1}}^1 |\sin 2\pi X y|\, y^{-3/2} dy = O\left(\int_{X^{-1}}^1 y^{-3/2} dy\right) = O(X^{\frac{1}{2}}).$$

Hence

$$I'_{32} = \pi^2 i + O(T^{-\frac{1}{2}}).$$

Next

$$I''_{32} = \int_1^\infty \frac{\sin 2\pi X y}{y} dy \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \left(\frac{1+y}{y}\right)^u \frac{du}{u}$$

$$= \left[-\frac{\cos 2\pi X y}{2\pi X y}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\left(\frac{1+y}{y}\right)^u \frac{du}{u}\right]_1^\infty - \int_1^\infty \frac{\cos 2\pi X y}{2\pi X y^2} dy \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\left(\frac{1+y}{y}\right)^u \frac{du}{u} -$$

$$- \int_1^\infty \frac{\cos 2\pi X y}{2\pi X y} dy \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\left(\frac{1+y}{y}\right)^{u-1} \frac{du}{y^2}. \quad (8.1)$$

Now for $y \geq 1$,

$$\int\limits_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \left(\frac{1+y}{y}\right)^u \frac{du}{u} = O\left(\int\limits_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \left|\frac{du}{u}\right|\right) = O(\log T),$$

and hence the first term in (8.1) is $O(T^{-1}\log T)$, and similarly the second term is

$$O\left(T^{-1}\log T\int\limits_1^\infty \frac{dy}{y^2}\right) = O(T^{-1}\log T).$$

The third term may be written

$$\int\limits_1^\infty \frac{\cos 2\pi X y}{2\pi X y^3}\left[\left(\frac{1+y}{y}\right)^{u-1}\left(\log\frac{1+y}{y}\right)^{-1}\right]_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} dy = O\left(T^{-1}\int\limits_1^\infty \frac{dy}{y^2}\right) = O(T^{-1}).$$

Hence

$$I''_{32} = O(T^{-1}\log T)$$

and so, altogether,

$$I_3 = \pi + O(T^{-\frac{1}{2}}\log T). \tag{8.2}$$

9. It remains to consider I_4, as given by (4.8). We estimate first of all the inner integrals, for which purpose we use Lemma 2, making $a \to 0$, $b \to \infty$. We have then, in the notation of Lemma 2, for $k = x > AT$,

$$\int\limits_0^\infty \frac{\cos\left(T\log\frac{1+y}{y}\right)\cos 2\pi x y}{y^{1/2}(1+y)^{3/2}\log\frac{1+y}{y}}\, dy =$$

$$= \frac{1}{4x}\sqrt{\frac{T}{\pi}}\,\frac{\cos\left(TV + 2\pi x U - \pi x + \frac{\pi}{4}\right)}{V U^{1/4}(U-\frac{1}{2})^{1/2}(U+\frac{1}{2})^{3/2}} + O(T^{-1}x^{-\frac{1}{2}})$$

and similarly, for $\nu = 1, 2$,

$$\int\limits_0^\infty \frac{\sin\left(T\log\frac{1+y}{y}\right)\cos 2\pi x y}{y^{1/2}(1+y)^{3/2}\left(\log\frac{1+y}{y}\right)^\nu}\, dy =$$

$$= O\{T^{\frac{1}{2}}(U-\tfrac{1}{2})^{-\frac{1}{2}}x^{-1}\} + O(T^{-1}x^{-\frac{1}{2}}) = O(x^{-\frac{1}{2}}).$$

Hence

$$I_4 = \int_X^\infty \frac{\Delta(x)}{x} \cdot$$

$$\cdot \left\{ \frac{T \cos\left\{2\,T \sinh^{-1}\sqrt{\dfrac{\pi x}{2\,T}} + \sqrt{(2\,\pi x\,T + \pi^2 x^2)} - \pi x + \dfrac{\pi}{4}\right\}}{\sqrt{2x}\,\sinh^{-1}\sqrt{\dfrac{\pi x}{2\,T}}\left\{\sqrt{\left(\dfrac{T}{2\,\pi x} + \dfrac{1}{4}\right)} + \dfrac{1}{2}\right\}\left(\dfrac{T}{2\,\pi x} + \dfrac{1}{4}\right)^{1/4}} + O(x^{-\frac{1}{2}}) \right\} d x .$$

The error term here gives, by (4.2),

$$O\left(\int_X^\infty |\Delta(x)|\, x^{-3/2} d x\right) = O(X^{-1/6} \log X) = O(T^{-1/6} \log T).$$

In the rest of the integral we use (4.1). The error term $O(x^{-3/4})$ gives here

$$O\left(T \int_X^\infty x^{-9/4}\, d x\right) = O(T^{-1/4}).$$

Hence, changing the variable from x to $\sqrt{x}$,

$$I_4 = \frac{T}{\pi}\sum_{n=1}^\infty \frac{d(n)}{n^{3/4}} \int_{\sqrt{X}}^\infty \frac{\cos\left\{2\,T \sinh^{-1}\left(x\sqrt{\dfrac{\pi}{2\,T}}\right) + \sqrt{(2\,\pi x^2\,T + \pi^2 x^4)} - \pi x^2 \dfrac{\pi}{4}\right\}}{x^{3/2} \sinh^{-1}\left(x\sqrt{\dfrac{\pi}{2\,T}}\right)\left\{\sqrt{\left(\dfrac{T}{2\,\pi x^2} + \dfrac{1}{4}\right)} + \dfrac{1}{2}\right\}\left(\dfrac{T}{2\,\pi x^2} + \dfrac{1}{4}\right)^{1/4}} \cdot$$

$$\cdot \left\{\cos\left(4\,\pi x \sqrt{n} - \frac{\pi}{4}\right) - \frac{3}{32\,\pi x \sqrt{n}} \sin\left(4\,\pi x\sqrt{n} - \frac{\pi}{4}\right)\right\} d x + O(T^{-1/6} \log T)$$

$$= \frac{T}{\pi}\sum_{n=1}^\infty \frac{d(n)}{n^{3/4}} J_n + O(T^{-1/6} \log T). \tag{9.1}$$

say.

We estimate this expression by means of the following

Lemma 3. For $A\sqrt{T} < a < A'\sqrt{T}$, $\alpha > 0$,

$$\int_a^\infty \frac{\exp i\left\{4\,\pi x\sqrt{n} - 2\,T \sinh^{-1}\left(x\sqrt{\dfrac{\pi}{2\,T}}\right) - \sqrt{(2\,\pi x^2\,T + \pi^2 x^4)} + \pi x^2\right\}}{x^\alpha \sinh^{-1}\left(x\sqrt{\dfrac{\pi}{2\,T}}\right)\left\{\sqrt{\left(\dfrac{T}{2\,\pi x^2} + \dfrac{1}{4}\right)} + \dfrac{1}{2}\right\}\left(\dfrac{T}{2\,\pi x^2} + \dfrac{1}{4}\right)^{1/4}} d x$$

$$= \frac{4\pi}{T} n^{\frac{1}{2}(\alpha-1)} \left(\log \frac{T}{2\pi n}\right)^{-1} \left(\frac{T}{2\pi} - n\right)^{3/2-\alpha} \exp i\left(T - T\log\frac{T}{2\pi n} - 2\pi i n + \frac{1}{4}\pi i\right) +$$
$$+ O\left\{T^{-\frac{1}{2}\alpha} \min\left(1, \left|2\sqrt{n} + a - \sqrt{\left(a^2 + \frac{2T}{\pi}\right)}\right|^{-1}\right)\right\} + O\left\{n^{\frac{1}{2}(\alpha-1)} \left(\frac{T}{2\pi} - n\right)^{1-\alpha} T^{-3/2}\right\},$$

provided that

$$n \geq 1, \ n < \frac{T}{2\pi}, \ \frac{1}{n}\left(\frac{T}{2\pi} - n\right)^2 > a^2.$$

If the last two restrictions on n are not satisfied, or if $\sqrt{n}$ is replaced by $-\sqrt{n}$, then the main term and the last error term on the right-hand side are to be omitted.

We apply Lemma 1 with a, b as limits of integration, where $b > T$, and

$$\varphi(x) = x^{-\alpha}\left\{\sinh^{-1}\left(x\sqrt{\frac{\pi}{2T}}\right)\right\}^{-1}\left\{\sqrt{\left(\frac{T}{2\pi x^2} + \frac{1}{4}\right)} + \frac{1}{2}\right\}^{-1}\left(\frac{T}{2\pi x^2} + \frac{1}{4}\right)^{-1/4},$$
$$f(x) = \frac{1}{2}x^2 - \sqrt{\left(\frac{Tx^2}{2\pi} + \frac{x^4}{4}\right)} - \frac{T}{\pi}\sinh^{-1}\left(x\sqrt{\frac{\pi}{2T}}\right).$$

We have then

$$f'(x) = x - \sqrt{\left(x^2 + \frac{2T}{\pi}\right)}, \ f''(x) = 1 - x\left(x^2 + \frac{T}{2\pi}\right)^{-1/2},$$

so that we may take

$$\mu(x) = \frac{1}{2}x, \ \Phi(x) = x^{-\alpha}, \ F(x) = T.$$

We dispose first of the error terms in a and b. We have

$$\Phi(a)(|f'_a + 2\sqrt{n}| + f''^{-\frac{1}{2}}_a)^{-1} = O\left\{T^{-\frac{1}{2}\alpha}\min\left(1, \left|2\sqrt{n} + a - \sqrt{\left(a^2 + \frac{2T}{\pi}\right)}\right|^{-1}\right)\right\},$$

and

$$\Phi(b)(f'_b + 2\sqrt{n})^{-1} = O\{b^{-\alpha}(\sqrt{n} + O(Tb^{-1}))^{-1}\}$$

which tends to 0 as b tends to ∞.

The error-term integral of Lemma 1 gives here

$$O\left(\int_a^b x^{-\alpha} e^{-Ax\sqrt{n} - AT} dx\right) = O(e^{-A\sqrt{nT} - AT}).$$

We are left to consider the terms in x_0, where x_0 is given by

$$f'(x_0) + 2\sqrt{n} = 0, \ x_0 = n^{-\frac{1}{2}}\left(\frac{T}{2\pi} - n\right).$$

Hence if $\sqrt{n} \leq -1$, or $n > \frac{T}{2\pi}$, or $n^{-1}\left(\frac{T}{2\pi} - n\right)^2 \leq a^2$, there will be no terms in x_0, and the lemma is proved. In other cases we have

$$f_0'' = 2n\left(\frac{T}{2\pi} + n\right)^{-1}, \quad \left(\frac{T}{2\pi x^2} + \frac{1}{4}\right) = \frac{1}{4}\left(\frac{T}{2\pi} + n\right)^2\left(\frac{T}{2\pi} - n\right)^{-2}$$

$$\sqrt{\left(\frac{T}{2\pi x_0^2} + \frac{1}{4}\right)} - \frac{1}{2} = n\left(\frac{T}{2\pi} - n\right)^{-1}, \quad \sqrt{\left(\frac{T}{2\pi x_0^2} + \frac{1}{4}\right)} + \frac{1}{2} = \frac{T}{2\pi}\left(\frac{T}{2\pi} - n\right)^{-1},$$

and

$$\sinh^{-1}\left(x_0\sqrt{\frac{T}{2\pi}}\right) = \frac{1}{2}\log\frac{T}{2\pi n}.$$

We have also

$$2x_0\sqrt{n} + f_0 = \frac{1}{2}x_0^2 - \frac{T}{2\pi}\log\frac{T}{2\pi n} - \frac{1}{2}x_0\sqrt{\left(\frac{2T}{\pi} + x_0^2\right)} + 2x_0\sqrt{n}$$
$$= \frac{T}{2\pi} - n - \frac{T}{2\pi}\log\frac{T}{2\pi n}.$$

Hence the main term is

$$\varphi_0 f_0''^{-\frac{1}{2}} \exp\left\{2\pi i f_0 + 4\pi i x_0\sqrt{n} + \frac{1}{4}\pi i\right\} =$$
$$= \left\{n^{-\frac{1}{2}}\left(\frac{T}{2\pi} - n\right)\right\}^{-\alpha}\left(\frac{1}{2}\log\frac{T}{2\pi n}\right)^{-1}\left(\frac{T}{2\pi}\right)^{-1}\left(\frac{T}{2\pi} - n\right)\left\{\frac{1}{4}\left(\frac{T}{2\pi} + n\right)^2\left(\frac{T}{2\pi} - n\right)^{-2}\right\}^{-\frac{1}{2}}.$$
$$\cdot\left\{2n\left(\frac{T}{2\pi} + n\right)^{-1}\right\}^{-\frac{1}{2}}\exp i\left(T - 2\pi n - T\log\frac{T}{2\pi n} + \tfrac{1}{4}\pi\right)$$
$$= \frac{4\pi}{T}n^{\frac{1}{2}(\alpha-1)}\left(\frac{T}{2\pi} - n\right)^{3/2-\alpha}\left(\log\frac{T}{2\pi n}\right)^{-1}\exp i\left(T - 2\pi n - T\log\frac{T}{2\pi n} + \frac{1}{4}\pi\right).$$

We complete the proof of Lemma 3 by dealing with the error term involving x_0. We have in fact

$$\Phi_0 x_0 F_0^{-3/2} = O(x_0^{1-\alpha}T^{-3/2})$$
$$= O\left\{n^{\frac{1}{2}(\alpha-1)}\left(\frac{T}{2\pi} - n\right)^{1-\alpha}T^{-3/2}\right\}.$$

Lemma 3 is thus proved.

10. *Estimation of I_4.*

By lemma 3, if $\frac{1}{n}\left(\frac{T}{2\pi} - n\right)^2 > X$, $n < \frac{T}{2\pi}$, that is to say if

$$n < \left(\frac{T}{2\pi} + \frac{X}{2}\right) - \sqrt{\left(\frac{X^2}{4} + \frac{XT}{2\pi}\right)}, \tag{10.1}$$

then, in the notation of (9.1),

$$J_n = \frac{2\pi}{T} n^{1/4} \left(\log \frac{T}{2\pi n}\right)^{-1} \sin\left(T \log \frac{T}{2\pi n} - T - 2\pi n - \frac{\pi}{4}\right) +$$
$$+ O\left\{n^{\frac{1}{4}} T^{-1} \left(\frac{T}{2\pi} - n\right)^{-1} \left(\log \frac{T}{2\pi n}\right)^{-1}\right\} +$$
$$+ O\left\{T^{-\frac{3}{4}} \min\left(\left|2\sqrt{n} + \sqrt{X} - \sqrt{\left(X + \frac{2T}{\pi}\right)}\right|^{-1}, 1\right)\right\} + O\left\{n^{\frac{1}{4}} \left(\frac{T}{2\pi} - n\right)^{-\frac{1}{2}} T^{-3/2}\right\}.$$

Denoting the expression (10.1) by Z we have

$$I_4 = 2 \sum_{1 \le n < Z} \frac{d(n)}{\sqrt{n}} \left(\log \frac{T}{2\pi n}\right)^{-1} \sin\left(T \log \frac{T}{2\pi n} - T - \frac{\pi}{4}\right) + O\left(\sum_{1 \le n < Z} \frac{d(n)\, n^{-\frac{1}{2}}}{T - 2\pi n}\right) +$$
$$+ O\left\{T^{1/4} \sum_{n=1}^{\infty} \frac{d(n)}{n^{3/4}} \min\left(\left|2\sqrt{n} + \sqrt{X} - \sqrt{\left(X + \frac{2T}{\pi}\right)}\right|^{-1}, 1\right)\right\} +$$
$$+ O\left(T^{-\frac{1}{2}} \sum_{1 \le n < Z} \frac{d(n)\, n^{-\frac{1}{4}}}{\sqrt{(T - 2\pi n)}}\right) = I_{41} + I_{42} + I_{43} + I_{44},$$

say. Since $AT < X < A'T$ we have

$$Z = O(T), \quad \frac{T}{2\pi} - Z > AT.$$

Hence

$$I_{42} = O\left(T^{-1} \sum_{1 \le n < Z} d(n)\, n^{-1/2}\right)$$
$$= O(T^{-\frac{1}{2}} \log T),$$

and similarly

$$I_{44} = O(T^{-\frac{1}{4}} \log T).$$

Lastly we have

$$I_{43} = O\left\{T^{1/4} \sum_{n=1}^{\infty} \frac{d(n)}{n^{3/4}} \min\left(|\sqrt{n} - \sqrt{Z}|^{-1}, 1\right)\right\}.$$

We split this sum up at $\frac{1}{2}Z$, $Z - \sqrt{Z}$, $Z + \sqrt{Z}$, $2Z$. We get, using partial summation and the asymptotic formula for $\sum_{n<x} d(n)$,

$$\sum_{n\le\frac{1}{2}Z} d(n)\,n^{-3/4}(\sqrt{Z}-\sqrt{n})^{-1} = O\Big(Z^{-\frac{1}{2}}\sum_{n\le\frac{1}{2}Z} d(n)\,n^{-3/4}\Big)$$
$$= O(T^{-1/4}\log T),$$
$$\sum_{n>2Z} d(n)\,n^{-3/4}(\sqrt{n}-\sqrt{Z})^{-1} = O\Big(\sum_{n>2Z} d(n)\,n^{-5/4}\Big)$$
$$= O(T^{-1/4}\log T),$$
$$\sum_{\frac{1}{2}Z<n\le Z-\sqrt{Z}} d(n)\,n^{-3/4}(\sqrt{Z}-\sqrt{n})^{-1} = O\Big(Z^{-1/4}\sum_{\frac{1}{2}Z<n\le Z-\sqrt{Z}} d(n)\,(Z-n)^{-1}\Big)$$
$$= O(T^{-1/4}\log^2 T)$$

and similarly for the sum over $Z+\sqrt{Z}$, $2Z$. Lastly

$$\sum_{Z-\sqrt{Z}<n\le Z+\sqrt{Z}} d(n)\,n^{-3/4} = O(T^{-1/4}\log T).$$

Hence

$$I_4 = 2\sum_{1\le n<Z}\frac{d(n)}{\sqrt{n}}\left(\log\frac{T}{2\pi n}\right)^{-1}\sin\left(T\log\frac{T}{2\pi n}-T-\frac{\pi}{4}\right)+O(\log^2 T), \qquad (10.2)$$

and here the upper limit Z can be replaced by

$$\frac{T}{2\pi}+\frac{N}{2}-\sqrt{\left(\frac{N^2}{4}+\frac{NT}{2\pi}\right)}$$

which differs from Z by an amount $O(1)$.

Substituting (7.1), (7.2), (8.2), and (10.2), in (4.4) we complete the proof of the Theorem.