INEQUALITIES FOR CERTAIN FUCHSIAN GROUPS

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1. Introduction

In one of the earliest papers on automorphic functions, Poincaré constructed functions automorphic with respect to a Fuchsian group by means of the now well known Poincaré series. If G is a Fuchsian group with ∞ an ordinary point of G, the convergence of the Poincaré series depends upon the convergence of the series

$$\sum(G, z, t) = \sum_{V \in G} |V'(z)|^t$$

where z is any ordinary point of G. In 1882, Poincaré [15, p. 206] showed that this series converges if $t > 1$.

Now suppose that G is finitely generated. If G is of the first kind, then [13, p. 181]

$$\sum(G, z, 1) = +\infty \tag{1.1}$$

whereas if G is of the second kind, then [13, p. 178]

$$\sum(G, z, 1) < +\infty. \tag{1.2}$$

An obvious question, then, is to what extent can (1.2) be improved upon. In this paper we show that (1.2) is best possible when regarded as being a statement applicable to all finitely generated Fuchsian groups of the second kind but nevertheless can be improved upon for any given group. More precisely, we prove the following two theorems.

THEOREM 1. *Given any number t satisfying $t < 1$, there exists a finitely generated Fuchsian group of the second kind with ∞ an ordinary point of G and with*

$$\sum(G, z, t) = +\infty$$

for every ordinary point z.

THEOREM 2. *Let G be a finitely generated Fuchsian group of the second kind with ∞ an ordinary point of G. Then there exists a real number t satisfying $t<1$ and*

$$\sum(G, z, t) < +\infty \tag{1.3}$$

for every ordinary point z.

Now let L be the set of limit points of G, denote by $m(L)$ the linear measure of L and again assume that G is a finitely generated Fuchsian group. If G is of the first kind then obviously $m(L)>0$ whereas if G is of the second kind, then $m(L)=0$ [13, p. 324]. We prove that this latter result is best possible when regarded as a statement applicable to all finitely generated Fuchsian groups of the second kind but can also be improved upon for any group. More precisely, we prove the following two theorems.

THEOREM 3. *Given any number t satisfying $t<1$, there exists a finitely generated Fuchsian group of the second kind with ∞ an ordinary point of G and with L having infinite t-dimensional Hausdorff measure.*

THEOREM 4. *Let G be any finitely generated Fuchsian group of the second kind. Then there exists a real number t satisfying $t<1$ such that L has zero t-dimensional Hausdorff measure.*

The striking parallel between the first two and the last two theorems is explained by the next result.

THEOREM 5. *Let G be a finitely generated Fuchsian group with ∞ an ordinary point of G. If t is a real number such that*

$$\sum(G, z, t) < +\infty \tag{1.4}$$

for some ordinary point z, then L has zero t-dimensional Hausdorff measure.

We note immediately that Theorem 1 is an immediate consequence of Theorems 3 and 5 and that Theorem 4 is an immediate consequence of Theorems 2 and 5; thus we need only prove Theorems 2, 3 and 5. The proofs of Theorems 2 and 3 are long and for the benefit of the reader it seems desirable to discuss these results in a more general context before giving the proofs.

First, we write $m_t(L)$ for the t-dimensional Hausdorff measure of L and use $d(L)$ to denote the Hausdorff dimension of L. This is defined by

$$\mathrm{d}(L) = \inf \{t>0: m_t(L) = 0\}$$

and the details of the construction of the measures m_t can be found, for example, in [5]. Next, we write

$$\delta(G) = \inf\{t>0: \sum(G, z, t) < +\infty\} \tag{1.5}$$

where z is an ordinary point of G. As is well known, $\delta(G)$ is independent of z. An immediate consequence of Theorem 5 is the following (weaker) result.

COROLLARY. *In the above notation,* $d(L) \leqslant \delta(G)$.

The conclusion in Theorem 5 has been proved in the case when G is a Schottky group by Akaza [2], [3] and [4]. Our proof of Theorem 5 is, however, quite different.

Theorem 5 contains two well-known but non-trivial results. If G is a finitely generated group of the first kind, then $m_1(L)>0$ and so using Theorem 5 we can deduce (1.1). If G is a finitely generated group of the second kind, then it is very easy to establish (1.2) and so, using Theorem 5 again, we can deduce that $m_1(L)=0$.

In [9] Dalzell proved that if G is a finitely generated Fuchsian group of the second kind and if G contains no parabolic elements, then

$$\sum_{V\in G} |V'(z)| \log(|V'(z)|^{-1}) < +\infty \tag{1.6}$$

for every ordinary point z. Theorem 2 is clearly an improvement of this result both in that (1.3) is stronger than (1.6) and also that G may contain parabolic elements.

The group G_λ generated by the elements.

$$P(z) = z+\lambda, \quad E(z) = -1/z, \qquad \lambda>0,$$

is called a Hecke group and is of the second kind if $\lambda>2$. In [7] the author studied the function $\delta(G_\lambda)$ as a function of λ (note that the notation in [7] differs from that used here; the $\delta(G)$ used in [7] is twice that defined by (1.5)). In particular, it was proved that (in our present notation) $\delta(G_\lambda)>\frac{1}{2}$, that

$$\delta(G_\lambda) = \tfrac{1}{2} + O(\lambda^{-1})$$

as $\lambda \to +\infty$ and that $\delta(G_\lambda)<1$ if $\lambda \geqslant 2.8\ldots$. The natural conjecture was then made that $\delta(G_\lambda)<1$ if $\lambda>2$ (that is, if G_λ is of the second kind) and we see now from Theorem 2 that this is so.

In [6, p. 734] the author showed that there exists a finitely generated Fuchsian group with $d(L)>\frac{1}{2}$. This is contained in the much stronger Theorem 3 and indeed, Theorems 3 and 4 completely solve the problem of the range of values of $d(L)$ in the case of Fuchsian groups.

In the last few years, several papers have appeared in which there are estimates of $m_t(L)$ for various Fuchsian and Kleinian groups (e.g. [1], [2], [3], [4] and [6]). Some of the results in this paper have been generalized so as to be applicable to Kleinian groups and so generalize some of these results. It is hoped to publish these later.

The theorems stated above are all concerned with finitely generated Fuchsian groups. The results have been stated this way for brevity; the real requirement is the geometrical one that the groups possess a fundamental region having a finite number of sides and it is known that these two conditions are equivalent (e.g. [11], [14]). Indeed, if G is finitely generated, the fundamental region N_0, defined as the set of points hyperbolically closer to a point w than to any other image of w (see [13, p. 146]), has a finite number of sides. This follows from the results contained in [11]. We shall use these facts implicitly throughout this paper.

We can easily see that Theorem 5 is false for infinitely generated Fuchsian groups and we give two counterexamples. First, it is easy to construct an infinitely generated Fuchsian group of the second kind with $m_1(L)>0$. To do this one simply constructs a sequence of hyperbolic elements, each leaving the unit disc invariant and having the isometric circles of all of these elements and their inverses external to each other. This construction can be carried out in such a manner that the images of ∞ under these elements accumulate at a set of positive one-dimensional measure and so if G is the group generated by these elements, G is of the second kind and so $\sum(G, z, t)<+\infty$. By construction, however, $m_1(L)>0$.

A counterexample of a different type is suggested by a remark of Tsuji [17, p. 515]. Here Tsuji suggests the construction of an infinitely generated group of the first kind in which $\sum(G, z, 1)<+\infty$ and again, the existence of such a group shows that Theorem 5 is false for infinitely generated groups.

The remainder of the paper consists of the proofs of Theorems 2, 3 and 5. From now on, and without further mention, we will reserve the symbol G to denote a finitely generated Fuchsian group and the symbol L for the set of limit points of G.

2. The proof of Theorem 5

Let G be a group satisfying the hypotheses of Theorem 5. If $G_1=AGA^{-1}$ for some bilinear transformation A satisfying $A^{-1}\infty\notin L$, then G_1 also satisfies the hypotheses of Theorem 5 and further, $\sum(G, z, t)$, $\sum(G_1, Az, t)$ converge or diverge together. Also, the set of limit points of G_1 is $A(L)$ and it is easily seen that $m_t(L)$ and $m_t(A(L))$ are zero or positive together (this follows as A and A^{-1} satisfy a Lipschitz condition of order 1 on some neighbourhood of L and $A(L)$ respectively). Thus we may consider G_1 rather than G and this implies that without loss of generality we may assume that the unit circle $\{z: |z|=1\}$ is the principal circle of G.

The proof of Theorem 5 depends on a theorem on Diophantine approximation for Fuchsian groups proved by Rankin [16] and Lehner [13, p. 334]. The form of this result given in [13] is not in the form best suited to our needs and it is simpler to deduce a modified version directly from Lehner's generalization [13, p. 181] of a result of Hedlund [12, p. 538]. We need

LEMMA 2.1, [13, p. 181]. *Let G be a finitely generated Fuchsian group with $\{z: |z|=1\}$ as its principal circle. Then there exists a constant m satisfying $0<m<1$ and depending only on G with the following property. If ζ is a limit point of G but not a parabolic vertex, then there exists a sequence of points z_n tending radially to ζ as $n\to\infty$ and a sequence of distinct elements V_n in G with $|V_n^{-1}(z_n)|\leqslant m$.*

With ζ, z_n and V_n as in Lemma 2.1, we have

$$\begin{aligned}|\zeta-V_n(0)| &\leqslant |\zeta-z_n|+|z_n-V_n(0)| = (1-|z_n|)+|z_n-V_n(0)|\\ &=(1-|V_n(0)|)+(|V_n(0)|-|z_n|)+|z_n-V_n(0)|\\ &\leqslant (1-|V_n(0)|)+2|z_n-V_n(0)|. \end{aligned}\tag{2.1}$$

In order to estimate these last two terms, we write

$$V_n(z)=\frac{a_nz+\bar{c}_n}{c_nz+\bar{a}_n},\quad |a_n|^2-|c_n|^2=1$$

and note that

$$(1-|V_n(0)|)\leqslant 1-|V_n(0)|^2=(1+|c_n|^2)^{-1}<|c_n|^{-2}.\tag{2.2}$$

Also, if σ is the straight line segment joining the origin to $V_n^{-1}(z_n)$, then $V_n(\sigma)$ has end-points $V_n(0)$ and z_n and so

$$|z_n-V_n(0)|\leqslant \text{length}\,[V_n(\sigma)]=\int_\sigma |V_n'(z)|\cdot|dz|\leqslant\frac{\text{length}\,(\sigma)}{|c_n|^2\inf_\sigma|z-V_n^{-1}(\infty)|^2}\leqslant[|c_n|(1-m)]^{-2}.$$

Using this together with (2.1) and (2.2) we find that

$$|\zeta-V_n(0)|\leqslant 3(1-m)^{-2}|c_n|^{-2}\tag{2.3}$$

for infinitely many n. If we now write

$$Q(V)=\{z: |z-V(0)|\leqslant 3(1-m)^{-2}|c|^{-2}\}$$

where

$$V(z)=\frac{az+\bar{c}}{cz+\bar{a}},\quad |a|^2-|c|^2=1,$$

is in G, we see from (2.3) that any limit point ζ of G that is not a parabolic vertex lies in infinitely many of the discs $Q(V)$ for V in G.

If (1.4) holds then $t>0$ and

$$\sum_{V\in G,\, V\infty\neq\infty} |c|^{-2t} < +\infty$$

and so

$$\sum_{V\in G,\, V\infty\neq\infty} [\operatorname{diam} Q(V)]^t < +\infty.$$

Thus for any positive ε, we can find a finite subset K of G (including all those V in G for which $V\infty=\infty$) such that

$$\sum_{V\in G-K} [\operatorname{diam} Q(V)]^t < \varepsilon \tag{2.4}$$

(where here and elsewhere the minus sign denotes the set-theoretic difference). If P denotes the set of parabolic vertices of G, then, as we have already seen,

$$L-P \subset \bigcup_{V\in G-K} Q(V)$$

for every finite subset K of G and this together with (2.4) implies that $m_t(L-P)=0$. As P is a countable set, $m_t(P)=0$ for all $t>0$ and so $m_t(L)=0$ if (1.4) holds. The proof is now complete.

In view of the fact that this result is perhaps, the basic result of this paper, the author feels that it is worth giving a second, and completely different, proof of it. The above proof does not depend on the fact that (1.1) holds for groups G of the first kind nor on the fact that $m_1(L)=0$ for groups of the second kind and so gives an alternative proof of these results. If we use the fact that $\sum(G, z, t)$ converges if $t>1$ and diverges for $t=1$ when G is of the first kind we see that Theorem 5 has been proved for groups of the first kind. We thus assume that G satisfies the hypotheses of Theorem 5 and is of the second kind. The fundamental region N_0 constructed with $z=0$ as its centre (by considering a conjugate group we may assume that no element of G other than the identity fixes $z=0$) has a finite number of free sides $s_1, \ldots, s_n$ which we regard as open arcs of $|z|=1$. Clearly $\{z\colon |z|=1\}$ is the disjoint union

$$\{z\colon |z|=1\} = L \cup E \cup \Big(\bigcup_{V\in G} \bigcup_{i=1}^{n} V(s_i)\Big)$$

where L is the set of limit points of G and E is the set of end points of the free sides and their images. As there are only countably many free sides, $m_1(E)=0$. We need the following result.

LEMMA 2.2, [8]. *Let I be an open subset of the interval $J=[0, 1]$ with $m_1(I)=1$. If the components I_n of I have length a_n and if $\sum a_n^{\beta}$ converges for some β with $0<\beta\leqslant 1$, then* $m_\beta(J-I)=0$.

By modifying this result so that it applies to subsets of $\{z: |z|=1\}$ rather than J or by considering a conjugate group to G so that the limit set is contained in J, we see that $m_t(L \cup E)=0$ if

$$\sum_{V\in G}\sum_{i=1}^{n}[\text{length }(V(s_i))]^t < +\infty. \tag{2.5}$$

As

$$\text{length }(V(s_i)) = \int_{s_i} |V'(z)|\,|dz| \leqslant M|c|^{-2}$$

(this holds as the orbit of ∞ cannot accumulate at any point in the closure of any s_i) we see that (1.4) implies (2.5) and hence that $m_t(L)=0$. This completes the second proof of Theorem 5.

3. The proof of Theorem 3

To prove Theorem 3 it is sufficient to consider any number t satisfying $0<t<1$ and to construct a group G with $d(L)\geqslant t$. The group that we shall use is the Hecke group $G[\varepsilon]$ generated by the transformations

$$P(z)=z+2(1+\varepsilon),\;\; E(z)=-1/z \tag{3.1}$$

where ε is a real, positive parameter. The limit set L of this group is an unbounded subset of the real line; thus ∞ is not an ordinary point of $G[\varepsilon]$. Theorem 3 requires that ∞ be an ordinary point of G and this condition is easily met. We shall show that for sufficiently small ε, we have $d(L\cap[-1,1])\geqslant t$. If $AG[\varepsilon]A^{-1}$ is any conjugate group which has ∞ as an ordinary point, then

$$d(A(L)) \geqslant d(A(L\cap[-1,1])) \geqslant d(L\cap[-1,1]) \geqslant t,$$

the second inequality holding as A^{-1} satisfies a Lipschitz condition with exponent 1 in some neighbourhood of $A(L\cap[-1,1])$ (the results contained in the Appendix of [6] show that the first two inequality signs could be replaced by equality signs; we shall not need this however). In any event, $AG[\varepsilon]A^{-1}$ is a finitely generated Fuchsian group of the second kind ($G[\varepsilon]$ is of the second kind) with ∞ as an ordinary point and with $d(A(L))\geqslant t$. We therefore need only prove that

$$d(L\cap[-1,1])\geqslant t \tag{3.2}$$

where L is the set of limit points of $G[\varepsilon]$.

It will be helpful to bear in mind during the proof that the region

$$\{x+iy: |x|<1+\varepsilon,\ x^2+y^2>1\}$$

is a fundamental region for $G[\varepsilon]$ and that the action of E is an inversion in the boundary of the closed disc

$$Q = \{z: |z| \leqslant 1\} \tag{3.3}$$

followed by a reflection in the imaginary axis. We shall use Q to denote the disc (3.3) throughout this proof and without further mention.

For each non-zero integer n, define

$$V_n(z) = EP^n(z)$$

which is in $G[\varepsilon]$, and, for each finite sequence $n_1, \ldots, n_k$ of non-zero integers, define

$$V(n_1, \ldots, n_k)(z) = V_{n_1} \ldots V_{n_k}(z) \tag{3.4}$$

and

$$Q(n_1, \ldots, n_k) = V(n_1, \ldots, n_k)(Q). \tag{3.5}$$

As $V_n(Q) \subset Q$ we see that for any sequence $n_1, \ldots, n_k$,

$$Q \supset Q(n_1) \supset Q(n_1, n_2) \supset \ldots \supset Q(n_1, \ldots, n_k)$$

and also that if $r \neq s$, then

$$Q(n_1, \ldots, n_k, r) \cap Q(n_1, \ldots, n_k, s) = \varnothing. \tag{3.6}$$

These results show that the system of discs

$$\{Q(n_1, \ldots, n_k): k > 0;\ n_1, \ldots, n_k \neq 0\}$$

yields a Cantor-like construction with residual set

$$L_1 = \bigcap_{k=1}^{\infty} \bigcup_{V \in G_k} V(Q) \tag{3.7}$$

where G_k is the set of elements of the form (3.4) for varying $n_1, \ldots, n_k$ but fixed k. We shall need the following elementary result.

LEMMA 3.1. *L_1 is a subset of $L \cap [-1, 1]$.*

Proof. We see from (3.5), (3.6) and (3.7) that the points of L_1 are precisely those points that can be written in the form

$$\bigcap_{k=1}^{\infty} Q(n_1, \ldots, n_k) \tag{3.8}$$

for some fixed infinite sequence $n_1, n_2, \ldots$. As

$$V(n_1, \ldots, n_k)(\infty) \in Q(n_1, \ldots, n_{k-1}) \subset Q$$

we see that the point (3.8) of L_1 is in the closure of the orbit of ∞ (which itself is in L) and hence is in $L \cap [-1, 1]$ provided that

$$|Q(n_1, \ldots, n_k)| \to 0 \tag{3.9}$$

as $k \to \infty$ for every fixed sequence $n_1, n_2, \ldots$ (here and elsewhere in this proof we use $|\Delta|$ to denote the diameter of a disc Δ). Using (3.4), an elementary computation shows that for all non-zero n, $|V_n'(z)| \leqslant (1+2\varepsilon)^{-1}$ on Q and so

$$|Q(n_1, \ldots, n_k)| \leqslant (1+2\varepsilon)^{-1} |Q(n_1, \ldots, n_{k-1})| \leqslant 2(1+2\varepsilon)^{-k}$$

the second inequality following from repeated applications of the preceding one. This establishes (3.9) and completes the proof of Lemma 3.1.

The techniques for estimating the Hausdorff dimension of a set formed from a Cantor-like construction are reasonably well developed (see, for example, [5]). There are, however, two major difficulties to overcome in applying these techniques to our construction. The first is that in passing from one stage of the construction to the next, one replaces, say, $Q(n_1, \ldots, n_k)$ by infinitely many (rather than finitely many) $Q(n_1, \ldots, n_k, n_{k+1})$. This difficulty is overcome by selecting only a finite number of suitable $Q(n_1, \ldots, n_k, n_{k+1})$ at each stage and using only these in the construction. The second difficulty is that the ratios

$$|Q(n_1, \ldots, n_k, n_{k+1})| \cdot |Q(n_1, \ldots, n_k)|^{-1}$$

are not well-behaved in the sense of constructions of this nature. We avoid this difficulty by modifying the above construction of L_1 so as to avoid images of Q under successive applications of V_1 or of V_{-1} (for it is these that give rise to the badly-behaved ratios). Roughly speaking, we replace V_1 and V_{-1} in G_1 by a set of elements $V_1^{(1)}, \ldots, V_1^{(N)}, V_{-1}^{(1)}, \ldots, V_{-1}^{(N)}$ (to be described in detail later) and then use the modified G_1 to generate a semi-group of transformations (each of which will still be of the form (3.4)). The images of Q under the transformations of the semi-group yield a Cantor-like construction with a residual set L_2 which is a subset of L_1. This and Lemma 3.1 imply that

$$d(L_2) \leqslant d(L \cap [-1, 1]) \leqslant 1 \tag{3.10}$$

and then there remains a rather delicate estimation of $d(L_2)$ to show that

$$\lim_{\varepsilon \to 0} \limsup_{N \geqslant 2} d(L_2) = 1 \tag{3.11}$$

which, together with (3.2) gives the required result. We proceed now with the formal proof.

Let ε be a positive number and N an integer satisfying $N \geqslant 2$ (ε and N are the parameters occurring in (3.11) and will be held fixed until just before the end of the proof). Next, let $G[\varepsilon]$ be as previously described in (3.1) and let Γ_1 be the set consisting of the elements

(A) $V_2, V_{-2}, \ldots, V_N, V_{-N}$,

together with the elements $V(n_1, \ldots, n_k, m)$ where $n_1, \ldots, n_k, m$ satisfy one of the following conditions:

(B) $1 \leqslant k \leqslant N$, $n_1 = \ldots = n_k = 1$ and $2 \leqslant |m| \leqslant N$,
(B') $1 \leqslant k \leqslant N$, $n_1 = \ldots = n_k = -1$ and $2 \leqslant |m| \leqslant N$,
(C) $1 \leqslant k \leqslant N$, $n_1 = \ldots = n_k = 1$ and $m = -1$,
(C') $1 \leqslant k \leqslant N$, $n_1 = \ldots = n_k = -1$ and $m = 1$.

We shall refer to these elements as being of type A, B, B', C and C' respectively.

Having defined Γ_1, we now define Γ_n for all positive integers n by the inductive definition

$$\Gamma_{n+1} = \{UV \colon U \in \Gamma_n,\ V \in \Gamma_1\} = \{U_1 \ldots U_{n+1} \colon U_i \in \Gamma_1, i = 1, \ldots, n+1\}$$

and further, define

$$L_2 = \bigcap_{k=1}^{\infty} \bigcup_{V \in \Gamma_k} V(Q). \tag{3.12}$$

It is clear that if $V \in \Gamma_k$ and T_1 and T_2 are in Γ_1 with $T_1 \neq T_2$, then

$$V(Q) \supset VT_1(Q)$$

and

$$VT_1(Q) \cap VT_2(Q) = \varnothing.$$

From these facts and Lemma 3.1 we can easily prove that

$$L_2 \subset L_1 \subset L \cap [-1, 1]$$

and so (3.10) holds. It remains therefore to establish (3.11).

To do this we need the concept of a spherical Cantor set. This is essentially a set constructed in a similar manner to the classical Cantor set but with a little more metrical freedom in the construction. This construction may be carried out (as in our case) in the plane using discs instead of intervals and details of such sets together with estimates of their Hausdorff dimension can be found in [5].

LEMMA 3.2. *In the above notation, L_2 is a spherical Cantor set constructed from the discs* $\{U(Q)\colon U\in\Gamma_n,\ n\geqslant 1\}$.

The construction of L_2 is that of replacing $U(Q)$, $U\in\Gamma_n$ by

$$\bigcup_{V\in\Gamma_1} UV(Q)$$

at each stage of the construction. We now write $\Gamma=\bigcup_n\Gamma_n$ and so, by virtue of Lemma 3.2, we may rewrite [5, Theorem 4(ii), p. 683] in our present notation to give the following result.

LEMMA 3.3. *If θ satisfies $0<\theta<1$ and if*

$$\sum_{V\in\Gamma_1}|UV(Q)|^{\theta}\geqslant|U(Q)|^{\theta} \tag{3.13}$$

for all U in Γ, then $d(L_2)\geqslant\theta$.

The validity of Lemma 3.3 thus depends upon Lemma 3.2 which has yet to be proved. In order to attain continuity of the basic ideas involved in the proof we proceed a little further before proving Lemma 3.2. Our next step is to establish the following simple result.

LEMMA 3.4. *Let k be any integer greater than one and let the positive numbers $\delta_1, \ldots, \delta_k$, δ and s satisfy $0\leqslant\delta_j\leqslant\delta<1$ and*

$$0\leqslant s\leqslant\delta_1+\ldots+\delta_k<1. \tag{3.14}$$

Then

$$\delta_1^{\theta}+\ldots+\delta_k^{\theta}\geqslant 1, \tag{3.15}$$

where

$$\theta=1-(1-s)(1-\delta)^{-1}. \tag{3.16}$$

We shall use Lemma 3.4 by taking the numbers $\delta_1, \ldots, \delta_k$ to be the ratios

$$|UV(Q)|\cdot|U(Q)|^{-1},\quad U\in\Gamma,\ V\in\Gamma_1 \tag{3.17}$$

for then the inequality (3.13) is precisely (3.15). We thus obtain the estimate of $d(L_2)$ given by (3.16) and Lemma 3.3 if we establish Lemmas 3.2 and 3.4 and verify that with the above choice of $\delta_1, \ldots, \delta_k$, the hypotheses of Lemma 3.4 are satisfied. We now begin the task of establishing these results.

Proof of Lemma 3.4. Let ν be the unique positive number satisfying

$$\delta_1^{\nu}+\ldots+\delta_k^{\nu}=1, \tag{3.18}$$

thus $0<\nu<1$. Next,

$$1-\delta_j^{1-\nu}=\int_{\delta_j}^{1} d(x^{1-\nu})=(1-\nu)\int_{\delta_j}^{1} x^{-\nu}dx \geqslant (1-\nu)(1-\delta_j)\geqslant (1-\nu)(1-\delta).$$

Using this inequality together with (3.14) and (3.18) we have

$$1-s\geqslant \sum_{j=1}^{k}\delta_j^{\nu}(1-\delta_j^{1-\nu})\geqslant (1-\nu)(1-\delta)$$

and so using (3.16) we can easily deduce that $\theta \leqslant \nu$. This together with (3.18) yields (3.15) and the proof of Lemma 3.4 is complete.

Proof of Lemma 3.2. The definitions in [5, p. 680] imply that we must establish the existence of positive constants A_1 and A_2 such that (i) for all U in Γ and all V in Γ_1,

$$|UV(Q)|\geqslant A_1|U(Q)| \tag{3.19}$$

and (ii) for all U in Γ and all distinct T_1 and T_2 in Γ_1,

$$\varrho[UT_1(Q),\ UT_2(Q)]\geqslant A_2|U(Q)| \tag{3.20}$$

where ϱ is defined by

$$\varrho(E,F)=\inf\{|e-f|: e\in E,\ f\in F\}.$$

With our choice (3.17) of the $\delta_1, ..., \delta_k$ in Lemma 3.4 the constants s and δ in Lemma 3.4 also become bounds on the ratios (3.17) and so at this point it is advantageous to derive a general distortion theorem for the family Γ. An application of Koebe's distortion theorem [10, p. 175] would give a short proof of Lemma 3.2 but does not, however, seem strong enough to yield useful estimates for the constants appearing in Lemma 3.4. We prove the following result in which the estimates are more explicit.

LEMMA 3.5. *Let $J=[-1, 1]$, let I be any sub-interval of J and let $U\in\Gamma$. Then*

$$(1/5)\,|I|\leqslant\frac{|U(I)|}{|U(J)|}\leqslant(5/4)\,|I|.$$

Also, if $V\in\Gamma_1$ then

$$|UV(J)|\leqslant(5/6)\,|U(J)|. \tag{3.21}$$

We remark that we are using $|I|$ to denote the length of I (a one-dimensional disc). No ambiguity will arise from the two uses of this symbol; indeed as Q has its centre on the real line and as elements of Γ leave the real line invariant, we do have

$$|U(Q)|=|U(J)|. \tag{3.22}$$

The proof of Lemma 3.2 is easily completed. By taking I to be the intersection of $V(Q)$ with the real axis we have from (3.22) and Lemma 3.5 that

$$|UV(Q)| = |U(I)| \geqslant A_1 |U(Q)|$$

where
$$A_1 = (1/5) \min \{ |V(Q)| : V \in \Gamma_1 \}$$

and this is positive as Γ_1 is finite. This establishes (3.19). The proof of (3.20) is similar. The set

$$(-1,1) - \bigcup_{V \in \Gamma_1} V(Q)$$

consists of a finite number of open arcs σ_i and, if T_1 and T_2 are distinct elements of Γ_1, there exists a subarc σ of J lying between $T_1(Q)$ and $T_2(Q)$ with $|\sigma| \geqslant \min |\sigma_i| > 0$. As $U(\sigma)$ lies between $UT_1(Q)$ and $UT_2(Q)$ we may use Lemma 3.5 and (3.22) to deduce that

$$\varrho[UT_1(Q),\, UT_2(Q)] \geqslant |U(\sigma)| \geqslant (1/5)|\sigma| \cdot |U(Q)| \geqslant (1/5)|U(Q)|(\min |\sigma_i|)$$

which established (3.20). This completes the proof of Lemma 3.2 subject to Lemma 3.5. Indeed, the proof of Lemma 3.5 is the only outstanding item in our programme so far.

Proof of Lemma 3.5. Let I, J and U be as in the statement of Lemma 3.5 and put $w = U^{-1}(\infty)$ (thus w is a real number). Our first task is to estimate w. As $U \in \Gamma$, we see that

$$U = U_1 \dots U_k = V(n_1 \dots n_s,\, n_{s+1}) \quad (U_i \in \Gamma_1)$$

where $U_k = V(n_r, \dots, n_{s+1})$, say. If U_k is of type A, then $r = s+1$ and $|n_{s+1}| \geqslant 2$. If U_k is of type B or B', then $r \leqslant s$ and again, $|n_{s+1}| \geqslant 2$. Finally, if U_k is of type C or C', then $r \leqslant s$ and either $n_s = 1$ $n_{s+1} = -1$, or $n_s = -1$, $n_{s+1} = 1$. Noting that

$$w = (EP^{n_1} \dots EP^{n_{s+1}})^{-1}(\infty) = P^{-n_{s+1}}(EP^{-n_s} \dots EP^{-n_1})(0)$$

which belongs to $P^{-n_{s+1}}Q(-n_s, \dots, -n_1)$, we find that

$$w \in P^{-n_{s+1}}Q(-n_s). \tag{3.23}$$

If U_k is of type A, B or B', then $|n_{s+1}| \geqslant 2$ and so replacing (3.23) by the weaker statement:

$$w \in P^{-n_{s+1}}(Q)$$

we find that $|w| \geqslant 3$. If U_k is of type C or C', then (3.23) becomes

$$w \in P(Q(-1)) \cup P^{-1}(Q(1))$$

or, equivalently
$$P^{-1}EP(w) \in J \quad \text{or} \quad PEP^{-1}(w) \in J.$$

This in turn implies that
$$|w| \geqslant \lambda + (1+\lambda)^{-1} \geqslant 7/3$$

where $\lambda = 2+2\varepsilon \geqslant 2$. In all case, then, we have $|w| \geqslant 7/3$. We now put $I = [\alpha, \beta]$ (as we are only concerned with $|I|$ we may assume that I is closed) and note that

$$\frac{|U(I)|}{|U(J)|} = \left(\int_\alpha^\beta |U'(x)|\,dx\right)\left(\int_{-1}^1 |U'(x)|\,dx\right)^{-1} = \left(\int_\alpha^\beta (x-w)^{-2}dx\right)\left(\int_{-1}^1 (x-w)^{-2}dx\right)^{-1}$$

$$= \frac{1}{2}|I|\left\{\frac{(w+1)(w-1)}{(w-\alpha)(w-\beta)}\right\}.$$

Using $-1 \leqslant \alpha \leqslant \beta \leqslant 1$, we find that if $w \geqslant 7/3$, then

$$\frac{1}{2}|I|\left(\frac{w-1}{w-\beta}\right) \leqslant \frac{|U(I)|}{|U(J)|} \leqslant \frac{1}{2}|I|\left(\frac{w+1}{w-\alpha}\right)$$

and so

$$\frac{1}{2}|I|\left(\frac{w-1}{w+1}\right) \leqslant \frac{|U(I)|}{|U(J)|} \leqslant \frac{1}{2}|I|\left(\frac{w+1}{w-1}\right)$$

which gives the required estimate as $w \geqslant 7/3$. A similar argument establishes the result if $w \leqslant -7/3$ and this completes the proof of the first part of Lemma 3.5.

Finally, if $V \in \Gamma_1$, then $V = V(n_1, ..., n_s)$ for some $n_1, ..., n_s$ and we have

$$V(J) \subset V(n_1)(J).$$

An elementary computation shows that

$$|V(n_1)(J)| = (n_1\lambda - 1)^{-1} - (n_1\lambda + 1)^{-1} \leqslant 2/3$$

as $|n_1\lambda| \geqslant 2$. Applying the first inequality in Lemma 3.5 with $I = V(J)$ we find that

$$|UV(J)| \leqslant (5/4)(2/3)|U(J)|$$

and the proof of Lemma 3.5 is complete.

Lemmas 3.2 to 3.5 inclusive have now been proved and, with the choice (3.17) of the $\delta_1, ..., \delta_k$ in Lemma 3.4, we can use Lemma 3.5 to obtain estimates of the constants s and δ occurring in Lemma 3.4. Indeed, the inequality (3.21) in Lemma 3.5 shows directly that we can take δ to be 5/6. Next, we note that

$$\delta_1 + \dots + \delta_k = \sum_{V \in \Gamma_1} |UV(J)| \cdot |U(J)|^{-1}. \tag{3.24}$$

If we now write

$$F = (-1, 1) - \bigcup_{V \in \Gamma_1} V(J) \tag{3.25}$$

then F is an open subset of $(-1, 1)$ and we have

$$|U(J)| = m_1(U(F)) + \sum_{V \in \Gamma_1} |UV(J)|. \tag{3.26}$$

As F is a union of open intervals, we deduce from Lemma 3.5 that $m_1(U(F)) \leqslant (5/4)m_1(F)|U(J)|$ and using this inequality with (3.24) and (3.26) we find that

$$\delta_1 + \ldots + \delta_k = 1 - m_1(U(F))|U(J)|^{-1} \geqslant 1 - (5/4)m_1(F).$$

We have taken δ to be 5/6; we now define s by

$$s = 1 - (5/4)m_1(F).$$

Lemmas 3.3 and 3.4 together with (3.17) and (3.24) enable us to deduce that $d(L_2) \geqslant \theta$ where $\theta = 1 - (15/2)m_1(F)$. This gives

$$d(L_2) \geqslant 1 - 8m_1(F). \tag{3.27}$$

Recalling that in order to prove Theorem 3 it is only necessary to establish (3.11), we now find that it is only necessary to prove that

$$\lim_{\varepsilon \to 0} \limsup_{N \to \infty} m_1(F) = 0$$

where F is defined by (3.25). This is geometrically obvious; however, we prefer an analytic proof. To achieve this, we define a set T by

$$T = I - \bigcup_{|n|=1}^{N} V(n)(J) \tag{3.28}$$

where $I = (-1, 1)$ and also, for convenience, define $u_n = 1$ and $v_n = -1$ for each positive integer n. We then have

$$\begin{aligned} F - T &= \bigcup_{|n|=1}^{N} V(n)(J) - \bigcup_{V \in \Gamma_1} V(J) = \bigcup_{n=-1,1} V(n)(J) - \bigcup_{V \in \Gamma_1} V(J) \\ &= [F \cap V(1)(J)] \cup [F \cap V(-1)(J)] \end{aligned} \tag{3.29}$$

as for every subset K of I, we have

$$K - \bigcup_{V \in \Gamma_1} V(J) = K \cap F.$$

Next, we have

$$\begin{aligned} &F \cap V(u_1, \ldots, u_r)(J) - V(u_1, \ldots, u_r)(T) \\ &= V(u_1, \ldots, u_r)(J) - \bigcup_{V \in \Gamma_1} V(J) - V(u_1, \ldots, u_r)(T) = V(u_1, \ldots, u_r)(J - T) - \bigcup_{V \in \Gamma_1} V(J) \\ &= [V(u_1, \ldots, u_r)(\bigcup_{|n|=1}^{N} V(n)(J))] \cup [V(u_1, \ldots, u_r)\{-1, +1\}] - \bigcup_{V \in \Gamma_1} V(J) \\ &= V(u_1, \ldots, u_r, u_{r+1})(J) \cup V(u_1, \ldots, u_r)\{-1, +1\}, \end{aligned}$$

the penultimate equality following from (3.28) and $J=[-1, 1]$ and the last equality from the definition of Γ_1. This gives

$$m_1[F \cap V(u_1, ..., u_r)(J)] = m_1[V(u_1, ..., u_r)(T)] + m_1[V(u_1, ..., u_{r+1})(J)].$$

A similar equation holds with $u_1, ..., u_{r+1}$ replaced by $v_1, ..., v_{r+1}$ and using these two equations for $r=1, ..., N-1$ and also (3.29) we find that

$$m_1(F) = m_1(T) + \sum_{r=1}^{N-1} m_1[V(u_1, \dots, u_r)(T)] + \sum_{r=1}^{N-1} m_1[V(v_1, \dots, v_r)(T)]$$
$$+ m_1[V(u_1, \dots, u_{N+1})(J)] + m_1[V(v_1, \dots, v_{N+1})(J)].$$

As both T and J are symmetrical with respect to the imaginary axis, we find that $m_1[V(u_1, ..., u_r)(T)] = m_1[V(v_1, ..., v_r)(T)]$ and also that a similar equation holds with T replaced by J. Thus we have

$$m_1(F) \leqslant m_1(T) + 2\sum_{r=1}^{N-1} m_1[V(u_1, \dots, u_r)(T)] + 2m_1[V(u_1, \dots, u_{N+1})(J)]. \tag{3.30}$$

Although it is easy to obtain simple estimates for these terms it does not seem a trivial matter to obtain estimates delicate enough to give the required information when ε tends to zero and N tends to ∞.

We first estimate $m(T)$. To do this note that T consists of the origin together with the images under an inversion in the boundary of Q of the intervals $[\lambda r+1, \lambda(r+1)-1]$ $r=-N, ..., N-1$ and the intervals $(-\infty, -[N\lambda+1])$, $(N\lambda+1, +\infty)$, where, as before, $\lambda = 2+2\varepsilon$. This enables us to compute $m_1(T)$:

$$m_1(T) = \frac{2}{N\lambda+1} + 2\sum_{r=0}^{N-1}\left\{\frac{1}{\lambda r+1} - \frac{1}{\lambda(r+1)-1}\right\} \leqslant (1/N) + 2\sum_{r=0}^{N-1}\frac{2\varepsilon}{(\lambda r)^2} < (1/N) + 3\varepsilon. \tag{3.31}$$

We next estimate $m_1[V(u_1, ..., u_r)(T)]$. If we put

$$V(u_1, \dots, u_r)(z) = \frac{a_r z + b_r}{c_r z + d_r}, \quad a_r d_r - b_r c_r = 1 \tag{3.32}$$

we know that the pole of $V(u_1, ..., u_r)$ lies in $P^{-1}(Q)$ and so $|d_r| > |c_r|$. Thus we can obtain the following estimate:

$$m_1[V(u_1, \dots, u_r)(T)] = \int_T |V(u_1, \dots, u_r)'(z)| \cdot |dz|$$
$$= \int_T |c_r z + d_r|^{-2} |dz| \leqslant (|d_r| - |c_r|)^{-2} m_1(T). \tag{3.33}$$

Our next task is to compute c_r and d_r. By definition we have $V(u_1, ..., u_r) = (EP)^r$ and so if a_r, b_r, c_r and d_r are as in (3.32), we have

$$V(u_1, ..., u_{r+1}) = V(u_1, ..., u_r)\, EP$$

and so
$$\begin{pmatrix} a_{r+1} & b_{r+1} \\ c_{r+1} & d_{r+1} \end{pmatrix} = \begin{pmatrix} a_r & b_r \\ c_r & d_r \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & \lambda \end{pmatrix}.$$

From this we deduce that $c_{r+1} = d_r$ and $d_{r+1} = \lambda d_r - c_r$ with initial conditions $c_1 = 1$, $d_1 = \lambda$. Eliminating c_r and using standard techniques to solve the resulting difference equation (with constant coefficients) we find that

$$c_r = (p^r - q^r)(p-q)^{-1}$$

and
$$d_r = (p^{r+1} - q^{r+1})(p-q)^{-1}$$

where p and q are the roots of $x^2 - \lambda x + 1 = 0$.

We remark that c_r and d_r are only determined to within a factor of -1 (although d_r/c_r is unique) and this corresponds to the two choices of the ordered pair (p, q). If we write

$$p = \tfrac{1}{2}(\lambda + [\lambda^2 - 4]^{\frac{1}{2}}), \quad q = \tfrac{1}{2}(\lambda - [\lambda^2 - 4]^{\frac{1}{2}})$$

we find that $p > q > 0$ and that

$$(p-q)(|d_r| - |c_r|) = p^r(p-1) + q^r(1-q) \geqslant p^r(p-1) \tag{3.34}$$

as $pq = 1$ and $p > q > 0$ implies that $0 < q < 1$. From this we can deduce that

$$\sum_{r=1}^{N-1} (|d_r| - |c_r|)^{-2} \leqslant \sum_{r=1}^{\infty} \left(\frac{p-q}{p-1}\right)^2 p^{-2r} = \left(\frac{p-q}{p-1}\right)^2 (p^2-1)^{-1}.$$

If we now write
$$\mu = \tfrac{1}{2}(\lambda^2 - 4)^{\frac{1}{2}} \geqslant (2\varepsilon)^{\frac{1}{2}} \tag{3.35}$$

we find that $p = 1 + \varepsilon + \mu$, $q = 1 + \varepsilon - \mu$ and so we have

$$\sum_{r=1}^{N-1} (|d_r| - |c_r|)^{-2} \leqslant \frac{4\mu^2}{(\varepsilon+\mu)^3 (2+\varepsilon+\mu)} \leqslant \frac{\lambda^2 - 4}{2\mu^3} \leqslant (4\varepsilon + 2\varepsilon^2)(2\varepsilon)^{-\frac{3}{2}} \leqslant 3/\varepsilon^{\frac{1}{2}}$$

if $\varepsilon < 1$, the penultimate inequality following from (3.35). From this and (3.33) we can deduce that

$$\sum_{r=1}^{N-1} m_1[V(u_1, \dots, u_r)(T)] \leqslant 3 m_1(T)(\varepsilon)^{-\frac{1}{2}}. \tag{3.36}$$

Next, the estimate (3.33) is valid with T replaced by J and this and (3.14) yields

$$m_1[V(u_1, \dots, u_{N+1})(J)] \leqslant \left(\frac{p-q}{p-1}\right)^2 p^{-2(N+1)} m_1(J). \tag{3.37}$$

Using (3.30), (3.31), (3.36) and (3.37) we find that

$$m_1(F) \leqslant m_1(T)[1+6\varepsilon^{-\frac{1}{2}}] + 4\left(\frac{p-q}{p-1}\right)^2 p^{-2(N+1)} \leqslant \left(\frac{1}{N}+3\varepsilon\right)(1+6\varepsilon^{-\frac{1}{2}}) + 4\left(\frac{p-q}{p-1}\right)^2 p^{-2(N+1)}.$$

As p and q depend only on ε and as $p>1$, we deduce that

$$\limsup_{N\to\infty} m_1(F) \leqslant 3\varepsilon + 18\sqrt{\varepsilon}.$$

As L (the limit set of $G[\varepsilon]$) is independent of N, this inequality together with (3.10) and (3.27) implies that

$$d(L \cap [-1, 1]) \geqslant 1 - 8(3\varepsilon + 18\varepsilon^{\frac{1}{2}}).$$

It is now clear that if $t<1$, then for sufficiently small positive ε,

$$d(L \cap [-1, 1]) \geqslant t$$

and the proof of Theorem 3 is complete.

4. The proof of Theorem 2

We begin by proving a lemma on Dirichlet series which will be used later and which does not depend on the notion of a Fuchsian group.

LEMMA 4.1. *Let $A_1, A_2, \dots$ be a sequence of positive numbers such that*

$$A(t) = \sum_{n=1}^{\infty} A_n^t$$

converges if $t>\frac{1}{2}$. If $a_1, a_2, \dots$ is any sequence of numbers satisfying $0 \leqslant a_n \leqslant A_n$ $(n=1, 2, \dots)$ and if t satisfies $5/6 \leqslant t \leqslant 1$, then

$$\sum_{n=1}^{\infty} a_n^t \leqslant \sum_{n=1}^{\infty} a_n + 6(1-t)[A(2/3) + A(4/3)].$$

Proof. If f is defined by

$$f(z) = \sum_{n=1}^{\infty} a_n^z$$

then f is defined and analytic on $\{\text{Re}(z) > \frac{1}{2}\}$ and satisfies

$$|f(z)| \leqslant \sum_{n=1}^{\infty} a_n^x \leqslant A(x) \quad (z = x + iy)$$

there. If $|x-1| < 1/3$, then

$$A_n^x \leqslant A_n^{\frac{2}{3}} + A_n^{\frac{4}{3}}$$

(the inequality holding with one term of the sum according to whether A_n is not greater than or not less than 1) and so $|f(z)| \leqslant A(\frac{2}{3}) + A(\frac{4}{3})$ on $|z-1| < \frac{1}{3}$. Cauchy's inequality implies that $|f'(z)| \leqslant 6[A(\frac{2}{3}) + A(\frac{4}{3})]$ on $|z-1| \leqslant \frac{1}{6}$. Thus if $\frac{5}{6} \leqslant t \leqslant 1$, then $f(t) \leqslant f(1) + |f(t) - f(1)| \leqslant f(1) + 6(1-t)[A(\frac{2}{3}) + A(\frac{4}{3})]$ and this is the required result.

We return now to the theory of Fuchsian groups. As Theorem 2 is known (and easily proved) to be true when G has at most two limit points [7, p. 474] we assume that G has uncountably many limit points. As we have already mentioned in the proof of Theorem 5, it is sufficient to consider a conjugate group AGA^{-1} provided that $A^{-1}(\infty) \notin L$, the set of limit points of G. Without loss of generality, then, we assume that the elements of G preserve the upper half-plane and, of course, the extended real axis which we shall denote by R^1. We note that R^1 is considered as a subset of the extended complex plane and hence contains the single point at infinity.

As is well known, the upper half-plane can be given a hyperbolic metric and a normal (or Dirichlet) polygon N_0 constructed from this metric is a fundamental region for the action of G on the upper half-plane (for details of this, see [13, Chapter IV] where this is done for the disc rather than the half-plane). We now wish to make certain justifiable assumptions on G and N_0. First, by choosing the centre of N_0 outside of some set of plane measure zero we may assume that each parabolic cycle on the boundary of N_0 consists of a single vertex and also that (in the notation of [13, p. 149–151]) there are no accidental vertices of the first kind lying on R^1. We note that as G is finitely generated, N_0 has only finitely many sides ([11], [14]) and so every vertex of N_0 which lies on R^1 is either a parabolic vertex p or an accidental vertex q. Our choice of centre as given above implies that in the former case the sides of N_0 that meet at p are conjugated by a parabolic element of the group whereas in the second case, q is the intersection of a side of N_0 and a (closed) free side of N_0. This means that q is the common end-point of two images of free sides of N_0 and that some neighbourhood of q is covered by the closure of the union of two images of N_0. These properties are preserved under conjugation; thus by considering (if necessary) a conjugate group we may assume that ∞ or one of its images lies in the open set N_0. After relabelling (if necessary) we may assume that $\infty \in N_0$. It is more convenient in this proof to consider the action of G on the extended complex plane rather than on the upper half-plane; thus we modify our notation and from now on denote by N_0 the union of the normal polygon described above as N_0, its reflection in the real axis and its free sides on R^1.

Thus N_0 is a fundamental region for the action of G on the extended plane; it is finite sided, symmetric with respect to the real axis; it contains a neighbourhood of ∞ and its vertices have the properties listed above. The free sides of N_0 are those of the N_0 as originally defined.

In this proof we shall use A', $\bar{A}$ and ∂A to denote the closure, the reflection in the real axis and the boundary respectively of a set A and we shall use $|E|$ to denote the linear measure of a measurable subset E of the real line. We shall also make an attempt to avoid as much as possible of the geometrical argument that is so common in this subject. With this in mind we first construct a function π that is defined on the extended plane, that satisfies a Minimum Principle both there and on the real line and that analytically describes the tesselation of the plane by N_0 and its images under G.

First, we say that two elements U and V in G are adjacent if and only if

$$U(N_0') \cap V(N_0') \neq \varnothing.$$

Next, we put $G_0 = \{I\}$ (I is the identity element in G) and assuming that $G_0, G_1, \ldots, G_n$ have been defined we define G_{n+1} as the set of those V in G satisfying

(a) $V \notin G_0, G_1, \ldots, G_n$ and

(b) V is adjacent to some U in G_n.

From (a) we see that the G_n are mutually disjoint and so we can define a function π^* on $\bigcup_{n=0}^{\infty} G_n$ by $\pi^*(V) = n$ if and only if $V \in G_n$. Our immediate task is to show that $G = \bigcup_{n=0}^{\infty} G_n$. If U and V are adjacent and if $\pi^*(U)$ is defined and equal to n, say, then (b) holds. If (a) holds, then by definition, $V \in G_{n+1}$ and so $\pi^*(V) = n+1$ whereas if (a) fails to hold, then $\pi^*(V)$ is already defined and is not greater than n. Thus if U and V are adjacent, $\pi^*(U)$ and $\pi^*(V)$ are either both defined or both undefined. If V is now any element of G, the hyperbolic line joining ∞ to $V(\infty)$ crosses, in turn, the adjacent regions

$$N_0 = I(N_0),\ V_1(N_0),\ V_2(N_0),\ \ldots,\ V_s(N_0) \quad (V = V_s)$$

and as $\pi^*(I)$ is defined, so is $\pi^*(V_s)$. Thus $G = \bigcup_{n=0}^{\infty} G_n$ and π^* is defined on G. If U is adjacent to V, then V is to U and the above argument shows that in this case

$$|\pi^*(U) - \pi^*(V)| \leq 1. \tag{4.1}$$

We also note that if P conjugates the sides of N_0 ending at a parabolic vertex on ∂N_0, then $P^n \in G_1$ for all non-zero integers n. A similar statement holds for parabolic vertices on the boundaries of the images of N_0.

We are now in a position to define the function π mentioned above. If $z \in V(N_0)$ for some V in G, we define $\pi(z)$ to be $\pi^*(V)$. This defines π on a dense subset of the complex plane and we complete the definition by the requirement that

$$\pi(z) = \liminf_{w \to z} \pi(w) \tag{4.2}$$

the lower limit being taken over w in $\bigcup_{V \in G} V(N_0)$.

The following remarks and lemmas describe some of the basic properties of the function π. First, π only assumes the values $+\infty, 0, 1, 2, \ldots$. Next, although (4.2) was only introduced to define π on a nowhere dense set of points it is, in fact, valid for all z.

LEMMA 4.2. *$\pi(z) < +\infty$ if and only if z is an ordinary point of G or the fixed point of a parabolic element in G.*

LEMMA 4.3 (The Minimum Principle I). *Let γ be a closed Jordan curve in the finite complex plane and let D be the interior of γ. Then*

$$\min_{z \in D} \pi(z) \geqslant \min_{z \in \gamma} \pi(z). \tag{4.3}$$

Further, if $V(N_0) \subset D$ and $w \in V(N_0)$, then

$$\pi^*(V) = \pi(w) > \min_{z \in \gamma} \pi(z). \tag{4.4}$$

LEMMA 4.4 (The Minimum Principle II). *Let a and b be points in $V(N_0') \cap R^1$ with $a < b$. If $x \in (a, b)$ then*

$$\pi(x) \geqslant \min \{\pi(a), \pi(b)\} \tag{4.5}$$

with equality holding if and only if $x \in V(N_0')$.

LEMMA 4.5. *Suppose that $V \in G$ and $z \in V(N_0')$. Then $\pi^*(V) - 1 \leqslant \pi(z) \leqslant \pi^*(V)$.*

LEMMA 4.6. *For $n \geqslant 1$ the set $\Delta_n = \{z : \pi(z) \geqslant n\}$ is open. Further, $\pi = n - 1$* on *$\partial \Delta_n - L$.*

We remark immediately that the upper bound in Lemma 4.5 follows immediately from (4.2) and we shall need this before proving Lemma 4.5. Also, with reference to Lemma 4.6, it is false that $\pi = n - 1$ on $\partial \Delta_n$. This is easily seen as Lemma 4.2 implies that any parabolic vertex p is a point of accumulation of points at which $\pi = \infty$; thus $\pi(p) < +\infty$ and $p \in \partial \Delta_n$ for all n. It is true, however, that $\pi = n - 1$ on the boundary of each component of Δ_n and this will be proved in the proof of Lemma 4.8.

Proof of Lemma 4.2. If z is an ordinary point of G, then $z \in V(N_0')$ for some V in G. The same holds if z is fixed by a parabolic element in G [13, p. 149] and so in both cases $\pi(z) \leqslant \pi^*(V) < +\infty$.

Now suppose that $\pi(z) < +\infty$. The result is obvious if $\pi(z) = 0$, thus we assume that

$\pi(z) \geqslant 1$. The definition of π implies that there exist sequences z_n and V_n with $z_n \in V_n(N_0)$, $z_n \to z$ as $n \to \infty$ and

$$\pi(z) = \pi(z_n) = \pi^*(V_n). \tag{4.6}$$

We first rule out the possibility that there are infinitely many distinct V_n in the sequence. If so, we may consider a subsequence and relabel; equivalently we assume that $V_n \neq V_m$ if $n \neq m$. As $\pi^*(V_n) \geqslant 1$, each V_n is adjacent to some U_n with $\pi^*(U_n) = \pi^*(V_n) - 1$ and so there exists a sequence w_n satisfying

$$w_n \in V_n(N_0') \cap U_n(N_0').$$

As the V_n are distinct and as $\infty \in N_0$, the euclidean diameter of $V_n(N_0)$ tends to zero as $n \to \infty$ and so $w_n \to z$ as $n \to \infty$. This implies the existence of a sequence w_n' with $w_n' \in U_n(N_0)$ and $w_n' \to z$ as $n \to \infty$. Thus

$$\pi(z) \leqslant \lim_{n \to \infty} \pi(w_n') = \pi^*(V_n) - 1$$

which contradicts (4.6). Thus there exists a V in G with $V_n = V$ for infinitely many n. For these n we have $z_n \in V(N_0)$ and $z_n \to z$ as $n \to \infty$. Thus $z \in V(N_0')$ and z is an ordinary point or a fixed point of some parabolic element in G.

Proof of Lemma 4.3. We first establish (4.4). Suppose that $w \in V(N_0)$ and $V(N_0) \subset D$ and, for convenience, put $\pi^*(V) = n$ and $V = V_n$. It follows that V_n is adjacent to some V_{n-1} with $\pi^*(V_{n-1}) = n-1$. This process can be continued and so we construct a sequence of elements

$$I = V_0,\ V_1,\ \dots,\ V_n = V$$

in G with $\pi^*(V_r) = r$ and V_r adjacent to V_{r+1} $(r = 0, \dots, n-1)$. This implies that there exist points $w_0, \dots, w_{n-1}$ such that

$$w_r \in V_r(N_0') \cap V_{r+1}(N_0') \quad (r = 0, \dots, n-1)$$

and so
$$K = [\bigcup_{r=0}^{n} V_r(N_0)] \cup \{w_0, \dots, w_{n-1}\}$$

is arcwise connected. Further, we have $\pi \leqslant n-1$ on

$$K_1 = [\bigcup_{r=0}^{n-1} V_r(N_0)] \cup \{w_0, \dots, w_{n-1}\}$$

as $\pi(w_r) \leqslant r$. Now construct a simple arc τ lying in K and joining w (inside γ) to ∞ (outside γ). It follows that τ meets γ at a point z_1, say, and by our initial assumption $z_1 \in K_1$. Thus

$$\pi^*(V) = n > \pi(z_1) \geqslant \min_{z \in \gamma} \pi(z)$$

and this is (4.4).

To prove (4.3) define

$$D^* = D \cap \bigcup_{V \in G} V(N_0)$$

and note that if $z \in D$, then

$$\pi(z) = \liminf_{w \to z,\ w \in D^*} \pi(w). \tag{4.7}$$

Let $w \in D^*$, then $w \in V(N_0)$ for some V. If $V(N_0) \subset D$ then by (4.4),

$$\pi(w) = \pi^*(V) > \min_{z \in \gamma} \pi(z).$$

The alternative hypothesis, namely $V(N_0) \not\subset D$, implies that $V(N_0) \cap \gamma = \varnothing$ and so

$$\pi(w) = \pi^*(V) \geqslant \min_{z \in \gamma} \pi(z).$$

In any case, then, this last inequality holds and we see that $\pi \geqslant \min_{z \in \gamma} \pi(z)$ on D^* and hence, by (4.7), on D. This establishes (4.3) and completes the proof of Lemma 4.3.

Proof of Lemma 4.4. This follows easily from the Minimum Principle I. We join a to b by a Jordan arc γ_1 which lies entirely in $V(N_0) \cap \{\operatorname{Im}(z) > 0\}$ except for the end-points a and b. Then $\gamma_1 \cup \bar{\gamma}_1$ is a closed Jordan curve γ lying entirely in $V(N_0')$. If $x \in (a, b)$ then x lies inside γ and so by (4.3),

$$\pi(x) \geqslant \min_{z \in \gamma} \pi(z).$$

As $\pi \leqslant \pi^*(V)$ on $V(N_0')$ with equality on $V(N_0)$ we see that

$$\min_{z \in \gamma} \pi(z) = \min \{\pi(a), \pi(b)\} \tag{4.8}$$

and so (4.5) follows. Suppose now that equality holds in (4.5). We can find sequences z_k and V_k such that $z_k \in V_k(N_0)$, $V_k \in G$, $\pi(z_k) = \pi(x)$ and $z_k \to x$ as $k \to \infty$. If $V_k(N_0) \subset D$, then (4.4) (with $V = V_k$ and $w = z_k$) and (4.5) (with equality holding) contradict $\pi(z_k) = \pi(x)$. Thus $V_k(N_0) \not\subset D$ and so, as $V_k(N_0)$ meets D, we see that $V_k(N_0)$ meets γ. This implies that $V_k(N_0)$ meets $V(N_0)$ and so $V_k = V$ for all k. Thus $z_k \in V(N_0)$ and so $x \in V(N_0')$ as required.

Proof of Lemma 4.5. To establish the lower bound we consider two cases.

Case 1. Suppose that $z \notin L$. Then there exists a finite maximal subset $V_1, \ldots, V_s$ of G with, say, $V = V_1$ and a neighbourhood N of z such that

$$z \in \bigcap_{r=1}^{s} V_r(N_0'), \quad N \subset \bigcup_{r=1}^{s} V_r(N_0'). \tag{4.9}$$

This implies that each V_r is adjacent to $V_1 (= V)$ and so by (4.1) and (4.2),

$$\pi^*(V_r) \geqslant \pi^*(V) - 1 \quad (1 \leqslant r \leqslant s)$$

and
$$\pi(z) = \min \{\pi^*(V_1), ..., \pi^*(V_s)\}.$$

The second inequality in this case is a trivial consequence of these last two results.

Case 2. Suppose that $z \in L$. The upper bound for $\pi(z)$ established above together with Lemma 4.2 shows that z is the fixed point of some parabolic element P in G. As $z \in \partial V(N_0)$, there exists a horocycle H (an open disc lying in the upper half-plane and having z on its boundary) with

$$H \cup \bar{H} \subset \bigcup_{r=-\infty}^{+\infty} P^r V(N_0').$$

Next, choose a point x_1 in a free side of $V(N_0)$ and note that the circle γ having the segment with end-points x_1 and z as diameter lies entirely in $V(N_0) \cup \{z\}$. For each integer r, write $\gamma_r = P^r(\gamma)$ and let D_r be the interior of γ_r. For sufficiently large r, say $r = k$, γ_r and γ_{-r} lie on different sides of z (which lies on R^1) and the set

$$D_k \cup D_{-k} \cup \gamma_k \cup \gamma_{-k} \cup [\bigcup_{|r| \leqslant k} P^r V(N_0')]$$

covers a neighbourhood N of z. Now let $T(N_0)$ intersect N. Then either $T = P^r V$ for some r satisfying $|r| \leqslant k$ or $T(N_0)$ is contained in either D_k or D_{-k}. If $T(N_0) \subset D_k$, we deduce from (4.4) that

$$\pi^*(T) \geqslant 1 + \min_{w \in \gamma_k} \pi(w) = 1 + \min \{\pi(z), \pi^*(P^k V)\} \geqslant 1 + \min \{\pi(z), \pi^*(V) - 1\}$$
$$= \min \{1 + \pi(z), \pi^*(V)\}$$

the last inequality holding as for each r, V and $P^r V$ are adjacent (at z). A similar inequality holds if $T(N_0) \subset D_{-k}$ and the same reason shows that if $T = P^r V$ for some r satisfying $|r| \leqslant k$, then

$$\pi^*(T) \geqslant \pi^*(V) - 1.$$

Thus in all cases
$$\pi^*(T) \geqslant \min \{1 + \pi(z), \pi^*(V) - 1\}$$

and so

$$\pi(z) = \liminf_{w \to z,\, w \in E} \pi(w) \geqslant \min \{1 + \pi(z), \pi^*(V) - 1\} = \pi^*(V) - 1 \quad (E = \bigcup_{T \in G} T(N_0) \cap N)$$

as required.

Proof of Lemma 4.6. Let $z \in \Delta_n$, then there exists a neighbourhood N of z such that $\pi \geqslant n$ on $\bigcup_{V \in G} V(N_0) \cap N$. It follows from (4.2) that $\pi \geqslant n$ on N and so Δ_n is open.

Now suppose that z is an ordinary point on $\partial\Delta_n$. Then there exists a maximal finite subset $V_1, \ldots, V_s$ of G satisfying (4.9) and by (4.1) and (4.2),

$$\pi(z) = \min \{\pi^*(V_r): r = 1, \ldots, s\} \geqslant \max \{\pi^*(V_r): r = 1, \ldots, s\} - 1 \geqslant n - 1$$

as $z \in (\Delta_n)'$. As $z \notin \Delta_n$, $\pi(z) \leqslant n-1$ and so $\pi = n-1$ on $\partial\Delta_n - L$.

Our proof of Theorem 2 depends upon a detailed examination of the topological and metrical properties of the sets Δ_n. Clearly the sequence of sets Δ_n is monotonic with n, thus we can label the components of Δ_n as $\Delta(i_1, \ldots, i_n)$ in such a way so that

$$\Delta(i_1, \ldots, i_n) \supset \Delta(i_1, \ldots, i_n, i_{n+1}).$$

For the sake of brevity we introduce the following notation. We denote by $\mathbf{I}_n$ $(n \geqslant 1)$ the set of n-tuples $\mathbf{i} = (i_1, \ldots, i_n)$ for which $\Delta(i_1, \ldots, i_n)$ is defined and rewrite $\Delta(i_1, \ldots, i_n)$ as $\Delta(\mathbf{i})$; we also write $\mathbf{I}$ for $\bigcup_{n=0}^{\infty} \mathbf{I}_n$. Next, we denote by $Z(\mathbf{i})$ the set of 1-tuples $\mathbf{j} = (j)$ for which $\Delta(i_1, \ldots, i_n, j)$ is defined and write $\Delta(\mathbf{i}, \mathbf{j})$ for $\Delta(i_1, \ldots, i_n, j)$ where $\mathbf{j} \in Z(\mathbf{i})$.

In order to proceed with the examination of the sets Δ_n we make the following definitions. We write

$$\Delta_n(\mathbf{i}) = \{z \in \Delta(\mathbf{i}): \pi(z) = n\}$$

and note that $\Delta(\mathbf{i})$ is the disjoint union of the sets

$$\Delta_n(\mathbf{i}), \bigcup_{\mathbf{j} \in Z(\mathbf{i})} \Delta(\mathbf{i}, \mathbf{j}) \tag{4.10}$$

Next, we write $\quad \sigma_n = \Delta_n \cap R^1, \quad \sigma(\mathbf{i}) = \Delta(\mathbf{i}) \cap R^1 \quad$ and $\quad \sigma_n(\mathbf{i}) = \Delta_n(\mathbf{i}) \cap R^1.$

Finally, we denote by $G(\mathbf{i})$ the set of V in G with the properties (i) $V(N_0) \subset \Delta(\mathbf{i})$ and (ii) $\pi^*(V) = n$ where $\mathbf{i} \in \mathbf{I}_n$. If $U \in G$, if

$$U(N_0) \cap \Delta(\mathbf{i}) \neq \varnothing$$

and if $\pi^*(U) = n$, then $U \in G(\mathbf{i})$.

The outline of the remainder of the proof of Theorem 2 is as follows. We see from (4.10) that $\sigma(\mathbf{i})$ consists of the disjoint union of the sets

$$\sigma_n(\mathbf{i}), \bigcup_{\mathbf{j} \in Z(\mathbf{i})} \sigma(\mathbf{i}, \mathbf{j}). \tag{4.11}$$

We first prove three lemmas which describe $G(\mathbf{i})$ and the topological properties of the $\sigma(\mathbf{i})$. After this, we need two more lemmas which give information on the metrical properties of the sets $\sigma(\mathbf{i})$. Then (4.11), these last two lemmas and Lemma 4.1 yield the required result.

LEMMA 4.7. (i) *The* $\sigma(\mathbf{i})$ $(\mathbf{i} \in \mathbf{I}_n)$ *are disjoint open intervals and hence are the components of* σ_n.

(ii) $\Delta_n(\mathbf{i})$ *is the closure relative to* $\Delta(\mathbf{i})$ *of* $\bigcup_{V\in G(\mathbf{i})} V(N_0)$.

(iii) *Each component of* $\sigma_n(\mathbf{i})$ *is either* (a) *a parabolic vertex on* $\partial V(N_0)$ *for some* V *in* $G(\mathbf{i})$ *or* (b) *the closure relative to* $\sigma(\mathbf{i})$ *of a finite union of intervals* $V_r(\tau_r)$ *where* $V_r \in G(\mathbf{i})$ *and the* τ_r *are free sides in* N_0.

LEMMA 4.8. *Each interval* $\sigma(\mathbf{i}, \mathbf{j})$ *abuts* (*and so lies between*) *intervals* $V(\tau)$ *and* $U(\tau_1)$ *where* U *and* V *are in* $G(\mathbf{i})$ *and where each of* τ *and* τ_1 *is a free side in* N_0 *or a parabolic vertex on* ∂N_0.

(i) *If, for some choices of* U *and* V, *we have* $U = V$ *then this choice is unique and* $\Delta(\mathbf{i}, \mathbf{j}) = V(\Delta(\mathbf{r}))$ *for some* $\mathbf{r}$ *in* $\mathbf{I}_1$.

(ii) *If, for all choices of* U *and* V, *we have* $U \neq V$, *then* U *and* V *are unique and* $\Delta(\mathbf{i}, \mathbf{j}) = V(\Sigma)$ *where* Σ *is a component of the complement of* $N_0' \cup (V^{-1}U)(N_0')$. *Further, this latter set contains at least one ordinary point.*

The transformations U and V in Lemma 4.8 are not necessarily unique. However, (i) and (ii) do show that if there is more than one possible pair (U, V), then among all such possible pairs, there is a unique pair with $U = V$. Lemma 4.8 also implies that $\Delta(\mathbf{i}, \mathbf{j})$ is bounded by one (if $U = V$) or two (if $U \neq V$) regions $U(N_0)$ and $V(N_0)$ and that these regions are unique if chosen according to (i) and (ii). Typical situations in cases (i) and (ii) are illustrated in figures 1 and 2 below.

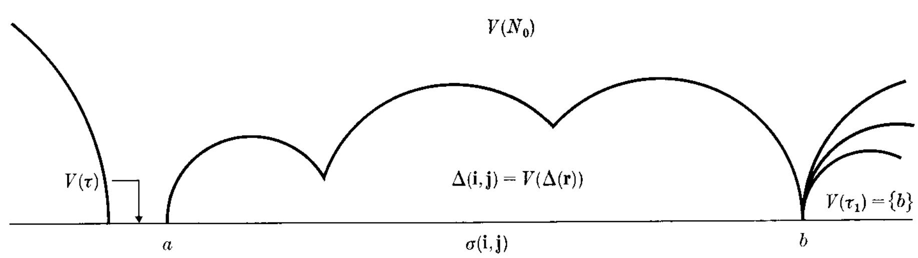

Fig. 1

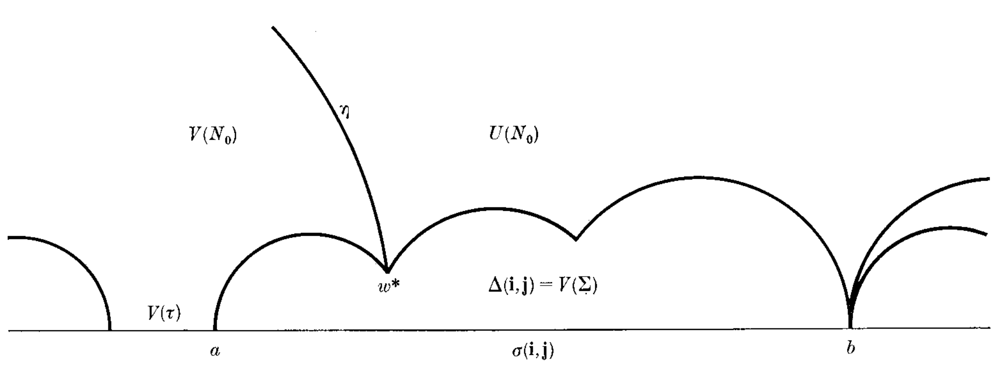

Fig. 2

In the course of the proof of Lemma 4.8 we shall apply the transormation V^{-1} to the situation described in figures 1 and 2. This leads to the situations illustrated in figures 3 and 4 respectively. In particular, these figures illustrate the $\Delta(\mathbf{r})$ and Σ occurring in (i) and (ii) of Lemma 4.8.

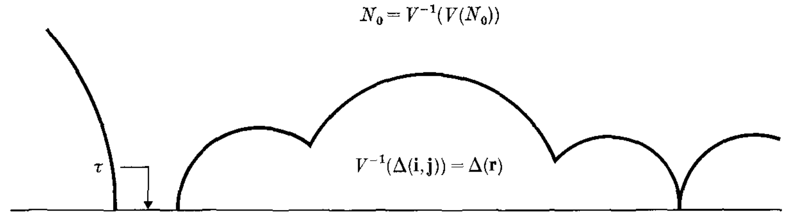

Fig. 3

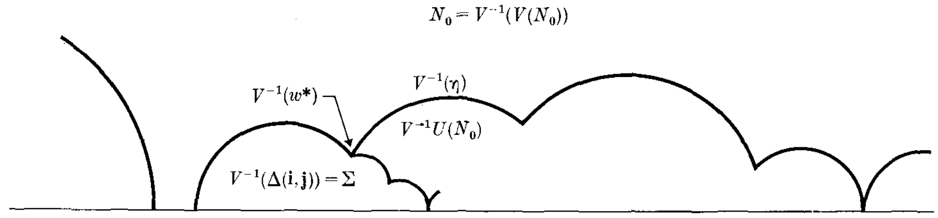

Fig. 4

LEMMA 4.9. *There exists a positive integer K (depending only on G) such that for each $\mathbf{i}$ in $\mathbf{I}$, $G(\mathbf{i})$ in the disjoint union*

$$G(\mathbf{i}) = G^0(\mathbf{i}) \cup \dots \cup G^K(\mathbf{i}) \tag{4.12}$$

where $G^0(\mathbf{i})$ has at most K elements and where $G^r(\mathbf{i})$ $(1 \leqslant r \leqslant K)$ is a subset of a set of the form $\{TP_r^n V_r: n \text{ an integer}\}$ where T, P_r, V_r are all in G, P_r is parabolic and where $P_1, \dots, P_K$, $V_1, \dots, V_K$ depend only on G.

Proof of Lemma 4.7. We note that $\sigma_n = \bigcup_{\mathbf{i} \in \mathbf{I}_n} \sigma(\mathbf{i})$, thus we need only show that the $\sigma(\mathbf{i})$ are connected for they are obviously disjoint relatively open subsets of R^1. Suppose now that a and b are two points in $\sigma(\mathbf{i})$ with $a < b$. Then as $\Delta(\mathbf{i})$ is open and connected we can join a to b by a curve γ in $\Delta(\mathbf{i})$ in such a way that γ consists of only finitely many straight line segments. As $V(N_0)$ is symmetric with respect to the real axis for all V in G we have $\pi(z) = \pi(\bar{z})$ and so Δ_n and the $\Delta(\mathbf{i})$ are also symmetric with respect to the real axis. If we now let $\bar{\gamma}$ be the reflection of γ in the real axis we see that every x in (a, b) either lies on γ or lies inside a closed Jordan curve consisting of an arc of γ and an arc of $\bar{\gamma}$. In the first case $x \in \Delta(\mathbf{i})$; in the second case the Minimum Principle I implies that $\pi(x) \geqslant n$ and so $x \in \Delta_n$. It follows that $[a, b] \subset \Delta_n$ and so $[a, b] \subset \Delta(\mathbf{i})$. Thus $[a, b] \subset \sigma(\mathbf{i})$, $\sigma(\mathbf{i})$ is connected and the proof of (i) is complete.

The proof of (ii) is easy. First, $\pi = n$ $(\mathbf{i} \in \mathbf{I}_n)$ on $E(\mathbf{i}) = \bigcup_{V \in G(\mathbf{i})} V(N_0)$ and so if z lies in the closure of $E(\mathbf{i})$ relative to $\Delta(\mathbf{i})$, then both $\pi(z) \leqslant n$ and $z \in \Delta(\mathbf{i})$. As $z \in \Delta(\mathbf{i})$, $\pi(z) \geqslant n$ and so $\pi(z) = n$, that is $z \in \Delta_n(\mathbf{i})$.

Conversely, suppose that $z \in \Delta_n(\mathbf{i})$. If z does not lie in the closure of $E(\mathbf{i})$ relative to $\Delta(\mathbf{i})$ then there exists a neighbourhood N of z contained in $\Delta(\mathbf{i}) - E(\mathbf{i})$. If $w \in N \cap V(N_0)$ for some V, then $\pi(w) \geqslant n$ and $\pi^*(V) \neq n$ and so $\pi(w) \geqslant n+1$. This implies that $\pi(z) \geqslant n+1$ contrary to our assumption that $z \in \Delta_n(\mathbf{i})$, thus (ii) is proved.

We now prove (iii). Let σ be a component of $\sigma_n(\mathbf{i})$; we again consider two cases.

Case 1. Suppose that $\sigma \cap L \neq \varnothing$. By Lemma 4.2, σ contains the fixed point p of some parabolic element in G. As G is a non-elementary group, there exist limit points of G which are not fixed points of elements of G and which are arbitrarily close to and on both sides of p. Lemma 4.2 implies that $\pi = +\infty$ at these points and so $\sigma = \{p\}$ which is of the required form.

Case 2. Suppose now that σ contains only ordinary points of G and let σ have end-points α and β where $\alpha \leqslant \beta$. We first show that $\alpha < \beta$. If $x \in \sigma$ then x lies in some free side $V(\tau)$ of $V(N_0)$ or is the common end-point of two abutting free sides $V(\tau)$ and $U(\tau_1)$, say. In the first case $\pi^*(V) = \pi(x) = n$ and so $V \in G(\mathbf{i})$ while in the second case, the set $V(N_0') \cup U(N_0')$ contains a neighbourhood of x and so

$$n = \pi(x) = \min \{\pi^*(U), \pi^*(V)\}.$$

It follows that $V(\tau) \subset \sigma$ in the first case while in the second case either $V(\tau) \subset \sigma$ or $U(\tau_1) \subset \sigma$. In any event, $\alpha < \beta$. The same reasoning as in Case 1 shows that $[\alpha, \beta]$ consists entirely of ordinary points and, being compact, therefore meets only a finite number of free sides. If a free side meets σ, then it is contained in σ. Hence σ contains a finite number of free sides and meets no other free side. It is now clear that σ is the closure relative to $\sigma(\mathbf{i})$ of the union of these free sides and the proof of Lemma 4.7 is complete.

Proof of Lemma 4.8. Lemma 4.7 (i) implies that $\sigma(\mathbf{i}, \mathbf{j})$ is an open interval, say (a, b) where $a < b$ and where a and b are not in Δ_{n+1}. Thus $\pi(a) \leqslant n$ and $\pi(b) \leqslant n$. We first show that $\pi(a) = n$ and $\pi(b) = n$. Lemma 4.6 shows that $\pi(a) = n$ if a is an ordinary point of G, thus we assume (by Lemma 4.2) that a is a parabolic vertex of G. The situation at such points is described in the proof of Lemma 4.5 (Case 2) and we use the notation used there. For suitably large $|r|$, there exist points in $P^r V(N_0) \cup P^{-r} V(N_0)$ lying arbitrarily close to and on either side of a. It therefore follows that some of these points lie in $\sigma(\mathbf{i}, \mathbf{j})$ and so for some r, say $r=k$, $\pi^*(P^k V) \geqslant n+1$. As $a \in \partial P^k V(N_0)$, Lemma 4.5 implies that

$$\pi(a) \geqslant \pi^*(P^k V) - 1 \geqslant n$$

and so we see that $\pi(a)=n$. Similarly, we have $\pi(b)=n$.

Next, Lemma 4.7 (iii) implies that there exist U and V in $G(\mathbf{i})$ and τ_a and τ_b either a free side in N_0 or a parabolic vertex on ∂N_0 with $a \in V(\tau_a)$ and $b \in U(\tau_b)$. We now show that

$$\partial\Delta(\mathbf{i}, \mathbf{j}) \subset \partial U(N_0) \cup \partial V(N_0).$$

The argument used to prove Lemma 4.7 (i) together with (4.4) shows that if α and β $(\alpha<\beta)$ lie on the real axis and are also in the closure of $\Delta(\mathbf{i}, \mathbf{j})$, then $\pi \geqslant n+1$ on (α, β). This implies that $a \leqslant \alpha$ and $\beta \leqslant b$ and so

$$\partial\Delta(\mathbf{i}, \mathbf{j}) \cap R^1 = \{a, b\} \subset \partial U(N_0) \cup \partial V(N_0). \tag{4.13}$$

Now let $w \in \partial\Delta(\mathbf{i}, \mathbf{j})$ and suppose that $\operatorname{Im}(w) \neq 0$. Lemma 4.6 implies that $\pi(w)=n$ and so there exists a T in $G(\mathbf{i})$ with $w \in \partial T(N_0)$. Next, let x_0 be any point in a free side of $T(N_0)$. Then, by the hyperbolic convexity of the intersection of the upper half-plane and $T(N_0)$, the circular arc joining w to $\bar{w}$ and passing through x_0 lies entirely in $T(N_0)$ except for its end-points w and $\bar{w}$. Next, join w to $\bar{w}$ by a curve in $\Delta(\mathbf{i}, \mathbf{j})$ that meets the real axis at exactly one point, say y_0. Thus $a<y_0<b$ and, as $\pi=n$ on a neighbourhood of x_0 ($T \in G(\mathbf{i})$ and so $\pi^*(T)=n$) we see that either $x_0<a$ or $x_0>b$. We first assume that $x_0<a$. Then as the interval (x_0, y_0) lies inside the closed curve γ constructed above and passing through x_0, w and $\bar{w}$, we see that the point a lies inside γ. Thus $V(N_0)$ meets the interior of γ. By construction

$$\gamma \subset T(N_0) \cup \{w, \bar{w}\} \cup \Delta(\mathbf{i}, \mathbf{j})$$

and so either $V(N_0)$ meets $T(N_0)$ or $V(N_0)$ lies inside γ. The latter assertion is false as in this case we can deduce from the Minimum Principle I, (4.4), that

$$n = \pi^*(V) > 1 + \min_{z \in \gamma} \pi(z) = 1 + \pi^*(T) = 1 + n$$

which is false; thus $V(N_0)$ meets $T(N_0)$ and $V=T$. If $x_0>b$ a similar argument shows that $U=T$ and we have proved that, relative to the union of the upper and lower half-planes,

$$\partial\Delta(\mathbf{i}, \mathbf{j}) \subset \partial V(N_0) \cup \partial U(N_0). \tag{4.14}$$

Using (4.13) we see that, as asserted above, this is valid relative to the extended complex plane.

Now note that as $\Delta(\mathbf{i}, \mathbf{j})$ is connected, it lies in one component of the complement of $V(N_0') \cup U(N_0')$. Using (4.14) a simple topological argument shows that $\Delta(\mathbf{i}, \mathbf{j})$ is in fact this component and also (as $\partial\Delta(\mathbf{i}, \mathbf{j})$ meets both $\partial V(N_0)$ and $\partial U(N_0)$ and is connected) that there exists a w^* (see figs. 2 and 4) satisfying

$$w^* \in \partial V(N_0) \cap \partial U(N_0). \tag{4.15}$$

We may thus write $\Delta(\mathbf{i}, \mathbf{j}) = V(\Sigma)$ where Σ is a component of the complement of $N_0' \cup (V^{-1}U)(N_0')$ and where N_0' and $V^{-1}U(N_0')$ have a point in common, namely $V^{-1}(w^*)$.

If, for some choices of U and V, $U = V$, then $\sum$ is a component of the complement of N_0' and hence $\Sigma = \Delta(\mathbf{r})$ for some $\mathbf{r}$ in $\mathbf{I}_1$ (this is the definition of $\mathbf{I}_1$). Suppose also that U_1 and V_1 are two other choices of U and V with $U_1 = V_1$. Then $\Delta(\mathbf{i}, \mathbf{j}) = V_1(\Delta(\mathbf{s}))$ for some $\mathbf{s}$ in I_1 and so

$$\Delta(\mathbf{r}) = V^{-1}V_1\Delta(\mathbf{s}).$$

This implies that $V_1 = V$. A similar argument shows that U is unique and this completes the proof of (i).

Now suppose that for all choices of U and V, $U \neq V$. The point w^* occurring in (4.15) cannot be a or b (otherwise we could choose $U = V$), thus by (4.13), Im $(w^*) \neq 0$ and so w^* is an ordinary point. Thus to complete the proof of (ii) we need only show that U and V are unique.

Suppose that $a \in V(N_0') \cap V_1(N_0')$ and $\pi^*(V) = \pi^*(V_1) = n$. Choose points x_0 and x_1 in free sides of $V(N_0)$ and $V_1(N_0)$ respectively and suppose that $V \neq V_1$. Thus $x_0 \neq x_1$ and we may suppose without loss of generality that $x_0 < x_1$. We note also that x_0 and x_1 lie outside (a, b). If $x_0 > b$ the Minimum Principle II applied to the interval (a, x_0) shows that $\pi \geq n+1$ on (a, x_0) except at points in $V(N_0')$. We thus see that $b \in V(N_0')$ and for some choice of U, $U = V$. This possibility has been excluded, thus $x_0 < a$ and similarly, $x_1 < a$. This gives $x_0 < x_1 < a$ and the Minimum Principle II applied to (x_0, a) shows that $\pi(x_1) \geq n+1$ or $x_1 \in V(N_0')$. The inequality is false, thus $V = V_1$. A similar argument shows that U is unique and the proof of Lemma 4.8 is complete.

Proof of Lemma 4.9. Referring to and temporarily using the notation used in Lemma 4.8 (ii), we see that whichever $\sigma(\mathbf{i}, \mathbf{j})$ is chosen, the possible choices of $V^{-1}U$ are finite as $N_0' \cap (V^{-1}U)(N_0')$ contains some ordinary point. Thus the possible choices of the sets Σ as described in Lemma 4.8 (ii) are also finite. We now say that a set is a Σ-set if and only if it is of the form $\Delta(\mathbf{r})$ for some $\mathbf{r}$ in $\mathbf{I}_1$ (see fig. 3) or of the form Σ as described in Lemma 4.8 (ii) (see fig. 4). The point here is that the $\Delta(\mathbf{i})$ can only arise as images of Σ-sets. Indeed, Lemma 4.8 completely characterizes the $\Delta(\mathbf{i})$ for $\mathbf{i}$ in $\mathbf{I}_n$, $n \geq 2$ and we see that in this case

$\Delta(\mathbf{i}) = T(\Sigma_0)$ for some T in G and some Σ-set, say Σ_0. If $\mathbf{i} \in \mathbf{I}_1$, $\Delta(\mathbf{i})$ is itself a Σ-set and we put $T = I$. It now follows that for each $\mathbf{i}$ in $\mathbf{I}$, there exists a Σ-set Σ_0 and a T in G with

$$\Delta(\mathbf{i}) = T(\Sigma_0).$$

Next, as we have seen above, there are only finitely many Σ-sets and these each have a relatively simple structure (see figures 3 and 4). The boundary of each Σ-set consists of finitely many sides and vertices of images of N_0. It follows that there are only finitely many sides of images of N_0 which lie on the boundary of some Σ-set; hence only finitely many images $Q(N_0')$, say at most K_1, which meet one of these sides in an ordinary point. Next, there are only finitely many parabolic vertices, say K_2, that lie on the boundary of some Σ-set. It is more convenient to assume that $K_1 = K_2$ and this can be arranged as follows. If $K_2 \geqslant K_1$ we may replace K_1 by K_2 in the above argument without destroying its validity; we then write $K = K_2$. If $K_1 > K_2$ we put $K = K_1$ and adjoin $(K_1 - K_2)$ other parabolic vertices to our above set of parabolic vertices to give a set $\{p_1, ..., p_K\}$ of parabolic vertices which now includes all those parabolic vertices lying on the boundary of some Σ-set. Thus there exists an integer K and a set $\{p_1, ..., p_K\}$ of parabolic vertices of G (both depending only on G) such that

(a) there are at most K elements Q in G such that the intersection of $Q(N_0')$ and the boundary of some Σ-set is a non-empty set of ordinary points of G and

(b) if p is a parabolic vertex on the boundary of some Σ-set, then $p = p_r$ for some r in $\{1, ..., K\}$.

Now let $\mathbf{i} \in \mathbf{I}_n$ where $n \geqslant 1$. If $V \in G(\mathbf{i})$, then $\pi^*(V) = n$ and V is adjacent to some U with $\pi^*(U) = n-1$. It follows from Lemma 4.5 that $\pi = n-1$ at some point of $\partial V(N_0)$ and so $\partial V(N_0) \cap \partial\Delta(\mathbf{i}) \neq \emptyset$. Recalling that $\Delta(\mathbf{i}) = T(\Sigma_0)$, we see that

$$\partial T^{-1} V(N_0) \cap \partial(\Sigma_0) \neq \emptyset.$$

We denote by $G^0(\mathbf{i})$ the set of V in $G(\mathbf{i})$ for which $T^{-1}V$ is some element Q as described in (a) above. Thus $G^0(\mathbf{i})$ contains at most K elements. If $T^{-1}V(N_0')$ meets $\partial\Sigma_0$ at some limit point, this point must be one of the parabolic vertices $p_1, ..., p_K$ and we denote by $G^r(\mathbf{i})$ the set of V in $G(\mathbf{i})$ for which

$$P_r \in T^{-1} V(N_0) \cap \partial\Sigma_0.$$

If under this classification some V appears in more than one set $G^r(\mathbf{i})$ we merely regard V as being in that set for which r is minimum and not in the others; thus $G(\mathbf{i})$ has the decomposition (4.12) and it remains to establish the description of $G^r(\mathbf{i})$, $1 \leqslant r \leqslant K$ as given in Lemma 4.9.

For each p_r we choose V_r such that p_r is a parabolic vertex on $\partial V_r(N_0)$. Next, we choose P_r to be a parabolic element in G that generates the stabilizer of p_r. Our earlier choice concerning the centre of N_0 implies that the only images of N_0 which have p_r on their boundary are the images $P_r^m V_r(N_0)$ for integral m. It follows that if $V \in G^r(\mathbf{i})$, then (as above) $p_r \in T^{-1}V(N_0)$ and so $T^{-1}V \in \{P_r^m V_r\}$. Thus

$$G^r(\mathbf{i}) \subset \{TP_r^m V_r\text{: integral } m\}$$

where P_r is a parabolic element in G and where $V_r \in G$. This completes the proof of Lemma 4.9.

Before proceeding to the metrical properties of the $\sigma(\mathbf{i})$ we need to introduce our last piece of notation. First, we recall from (4.11) and Lemma 4.7 (iii) that $\sigma(\mathbf{i})$ is the disjoint union of the following sets

(a) $\sigma(\mathbf{i}, \mathbf{j})$, $\mathbf{j} \in Z(\mathbf{i})$,

(*b*) $V(\tau)$ where $V \in G(\mathbf{i})$ and τ is a free side in N_0

and

(c) $E = \bigcup_{V \in G(\mathbf{i})} \partial V(N_0) \cap R^1$.

In fact, E consists only of end-points of the intervals described in (b) and parabolic vertices on $\partial V(N_0)$, $V \in G(\mathbf{i})$; thus E is countable.

Now let λ_V be the smallest interval containing (i.e. the convex hull of) $V(N_0') \cap R^1$. The Minimum Principle II implies that $\lambda_V \subset \sigma(\mathbf{i})$ and also that the λ_V ($V \in G(\mathbf{i})$) are non-overlapping. Note that $\bigcup_{V \in G(\mathbf{i})} \lambda_V$ certainly includes all sets $V(\tau)$ as described in (b) above. It also includes some sets $\sigma(\mathbf{i}, \mathbf{j})$ as described in (a); namely those that are described in Lemma 4.8 (i), for in this case, $\sigma(\mathbf{i}, \mathbf{j}) \subset \lambda_V$. Apart from a subset of E, then, $\sigma(\mathbf{i}) - \bigcup_{V \in G(\mathbf{i})} \lambda_V$ consists of the union of those $\sigma(\mathbf{i}, \mathbf{j})$ described in (a) above which are also described by Lemma 4.8 (ii). In this case we assume that λ_V lies to the left of $\sigma(\mathbf{i}, \mathbf{j})$, λ_U to the right of $\sigma(\mathbf{i}, \mathbf{j})$ and adjoin $\sigma(\mathbf{i}, \mathbf{j})$ to λ_V. The intervals σ_V ($V \in G(\mathbf{i})$) are the intervals λ_V with $\sigma(\mathbf{i}, \mathbf{j})$ adjoined where applicable. More precisely, if the right hand end-point of λ_V, say β_V, is the left-hand end-point of some $\sigma(\mathbf{i}, \mathbf{j})$, then $\sigma_V = \lambda_V \cup \sigma(\mathbf{i}, \mathbf{j})$; otherwise $\lambda_V = \sigma_V$. It follows that

$$\sigma(\mathbf{i}) = \bigcup_{V \in G(\mathbf{i})} \sigma_V$$

and that the σ_V are non-overlapping. Thus

$$|\sigma(\mathbf{i})| = \sum_{V \in G(\mathbf{i})} |\sigma_V|. \tag{4.16}$$

Next, every interval as described in (a) or (b) above is a subset of exactly one σ_V, $V \in G(\mathbf{i})$, and the number of intervals of the type (a) and (b) contained in any one σ_V is bounded above, the bound being independent of V. Another important property of the σ_V is expressed in the following lemma.

LEMMA 4.10. *There exists a positive δ depending only on G such that if $z \in V^{-1}(\sigma_V)$, then*

$$|z - V^{-1}(\infty)| \geqslant \delta > 0. \tag{4.17}$$

Proof. The definition of σ_V shows that $V^{-1}(\sigma_V)$ consists of a union of free sides of N_0 and some sets of the form $\Sigma_0 \cap R^1$ where Σ_0 is a Σ-set as defined in the proof of Lemma 4.9. As $\infty \in N_0$, there exists a positive δ depending only on G such that (4.17) holds whenever z lies in a free side in N_0. Thus we need only establish the existence of such a δ for sets of the form $\Sigma_0 \cap R^1$.

Let $\sigma(\mathbf{i}, \mathbf{j})$ be one of the intervals in σ_V, $V \in G(\mathbf{i})$. Then we may write $\sigma(\mathbf{i}, \mathbf{j}) = (\alpha, \beta)$ and use the notation of Lemma 4.8 where $V(\tau)$ occurs on the left of $\sigma(\mathbf{i}, \mathbf{j})$ and $U(\tau_1)$ on the right of $\sigma(\mathbf{i}, \mathbf{j})$.

If the hypotheses of Lemma 4.8 (ii) hold, that is if $U \neq V$ for all choices of U and V, then a free side of $V(N_0)$ must lie to the left of α (this is proved in the proof of Lemma 4.8 (ii)) and similarly, a free side of $U(N_0)$ must lie to the right of β. Thus, in the obvious sense, ∞ is separated from $\sigma(\mathbf{i}, \mathbf{j})$ by free sides of $U(N_0)$ and $V(N_0)$. Thus $V^{-1}\infty$ is separated from $V^{-1}(\sigma(\mathbf{i}, \mathbf{j}))$ by free sides of N_0 and $V^{-1}U(N_0)$. As $V^{-1}U(N_0)$ abuts N_0 at some ordinary point, we see that there exists a positive δ_1 depending only on G and such that a free side of $V^{-1}U(N_0)$ has length at least δ_1. Thus (4.17) is established in this case.

There remains the case when for some choice of U and V, $U = V$. In this case there may exist free sides in $V(N_0)$ lying both to the left and right of $\sigma(\mathbf{i}, \mathbf{j})$. If so the argument given in the preceding case is valid. If not, we may assume that all free sides of $V(N_0)$ lie to the left of $\sigma(\mathbf{i}, \mathbf{j})$ and certainly there is at least one free side of $V(N_0)$. It follows that β is a parabolic vertex on $\partial V(N_0)$ and so there exists an image $W(N_0)$ abutting $V(N_0)$ on a side through β and such that a free side of $W(N_0)$ lies to the right of β. We now consider the two possibilities (a) $W \neq I$ and (b) $W = I$. In (a), ∞ is separated from $\sigma(\mathbf{i}, \mathbf{j})$ by a free side of $V(N_0)$ and a free side of $W(N_0)$ and these two images of N_0 about along a side. This is precisely the situation we have already considered, namely when $U \neq V$ and so we need only consider (b). In this case $V(N_0)$ abuts N_0 along a side ending at a parabolic vertex and so V is one of a finite number of parabolic generators of G. As $\infty \notin \sigma_V$, $V^{-1}\infty \notin V^{-1}\sigma_V$ and a δ satisfying (4.17) exists in this case too as we are only considering a finite number of possible V. Thus Lemma 4.10 is established.

We now select a compact sub-interval τ^* of $R^1 \cap N_0$ such that $|\tau^*| > 0$. This interval will remain unaltered throughout the remainder of the paper. Our next lemma may be explained intuitively as follows. We have seen that each $\sigma(\mathbf{i})$ is the disjoint union of the sets of types (a), (b) and (c) (these are the sets used in defining the σ_V). There may be infinitely many sets of types (a) and (b) in $\sigma(\mathbf{i})$; however the general form of these sets is determined by $G(\mathbf{i})$ and this is adequately described by Lemma 4.9. If no parabolic elements were present, $G(\mathbf{i})$ would be finite and we could easily complete the proof using a lemma similar to Lemma 3.4. As we are allowing the existence of parabolic elements, we see that $G(\mathbf{i})$ may be infinite. In this case we shall use Lemma 4.1 (a result similar to Lemma 3.4 but applicable to infinite sequences). This requires the sets of types (a) and (b) to satisfy some regularity condition; this is ensured by Lemma 4.9. The next lemma is simply a verification of the required regularity condition. For technical reasons we prefer to discuss images of τ^* rather than images of free sides of N_0. This is done in (i); in (ii) we discuss the regularity of the $\sigma(\mathbf{i}, \mathbf{j})$.

LEMMA 4.11. (i) *There exists a sequence A_n $(n=1, 2, \ldots)$ of positive numbers depending only on G and such that*

(a) $\sum_{n=1}^{\infty} A_n^t < +\infty$ *if* $t > \frac{1}{2}$ *and*

(b) *for all* $\mathbf{i}$ *in* $\mathbf{I}$, *there exists an enumeration* $\{V_n\}$ *of* $G(\mathbf{i})$ *such that* $|V_n(\tau^*)| \leqslant A_n |\sigma(\mathbf{i})|$.

(ii) *There exists a sequence B_n $(n=1, 2, \ldots)$ of positive numbers depending only on G and such that*

(c) $\sum_{n=1}^{\infty} B_n^t < +\infty$ *if* $t > \frac{1}{2}$ *and*

(d) *for all* $\mathbf{i}$ *in* $\mathbf{I}$, *there exists an enumeration* $\{\sigma(n)\}$ *of* $\{\sigma(\mathbf{i}, \mathbf{j}) : \mathbf{j} \in Z(\mathbf{i})\}$ *such that* $|\sigma(n)| \leqslant B_n |\sigma(\mathbf{i})|$.

Temporarily assuming the validity of this lemma we can now complete the proof of Theorem 2. First, we write

$$V(z) = \frac{az+b}{cz+d}, \quad ad-bc=1 \tag{4.18}$$

and note that if $V \neq I$, then

$$V(\tau^*) = \frac{1}{|c|^2} \int_{\tau^*} \frac{dx}{|x - V^{-1}\infty|^2} \geqslant m|c|^{-2} \tag{4.19}$$

where m is positive and depends only on G. Next, Lemma 4.10 implies that

$$|\sigma_V| \leqslant \int_{|z - V^{-1}\infty| > \delta} |V'(z)| \cdot |dz| \leqslant \frac{1}{|c|^2} \left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \right) \left(\frac{dx}{x^2} \right) = \frac{2}{\delta |c|^2}. \tag{4.20}$$

The inequalities (4.19) and (4.20) imply that

$$|\sigma_V| \leq \frac{2}{(\delta m)}|V(\tau^*)| \tag{4.21}$$

and so
$$\sum_{\sigma(\mathbf{i},\mathbf{j})\subset\sigma_V} |\sigma(\mathbf{i},\mathbf{j})| \leq |\sigma_V| - |V(\tau^*)| \leq |\sigma_V|\,(1-\tfrac{1}{2}m\delta).$$

The fact that each $\sigma(\mathbf{i},\mathbf{j})$ is a subset of some σ_V ($V \in G(\mathbf{i})$) together with (4.16) implies that

$$\sum_{\mathbf{j}\in Z(\mathbf{i})} |\sigma(\mathbf{i},\mathbf{j})| \leq (1-\tfrac{1}{2}m\delta) \sum_{V\in G(\mathbf{i})} |\sigma_V| = m_1|\sigma(\mathbf{i})| \tag{4.22}$$

where m_1 depends only on G and satisfies $0<m_1<1$. Lemma 4.1 (with A_n and a_n replaced by B_n and $|\sigma(n)|\cdot|\sigma(\mathbf{i})|^{-1}$ respectively) and Lemma 4.11 show that if t satisfies $5/6 \leq t \leq 1$, then

$$\sum_{\mathbf{j}\in Z(\mathbf{i})} \left(\frac{|\sigma(\mathbf{i},\mathbf{j})|}{|\sigma(\mathbf{i})|}\right)^t \leq \sum_{\mathbf{j}\in Z(\mathbf{i})} \left(\frac{|\sigma(\mathbf{i},\mathbf{j})|}{|\sigma(\mathbf{i})|}\right) + M(1-t)$$

where M depends only on G. An application of (4.22) yields

$$\sum_{\mathbf{j}\in Z(\mathbf{i})} |\sigma(\mathbf{i},\mathbf{j})|^t \leq |\sigma(\mathbf{i})|^t \{m_1 + M(1-t)\} \leq \theta\,|\sigma(\mathbf{i})|^t \tag{4.23}$$

where t is now chosen in the interval $(5/6,\ 1)$ so as to satisfy

$$0<\theta = m_1 + M(1-t) < 1$$

(this is possible as $0<m_1<1$ and $M>0$). From now on the symbol t is reserved for this fixed value that depends only on G. We now deduce from (4.23) that

$$\sum_{\mathbf{k}\in \mathbf{I}_{n+1}} |\sigma(\mathbf{k})|^t = \sum_{\mathbf{i}\in\mathbf{I}_n} \sum_{\mathbf{j}\in Z(\mathbf{i})} |\sigma(\mathbf{i},\mathbf{j})|^t \leq \theta \sum_{\mathbf{i}\in\mathbf{I}_n} |\sigma(\mathbf{i})|^t$$

and so
$$\sum_{\mathbf{i}\in\mathbf{I}} |\sigma(\mathbf{i})|^t = \sum_{n=1}^{\infty} \sum_{\mathbf{i}\in\mathbf{I}_n} |\sigma(\mathbf{i})|^t \leq (1-\theta)^{-1} \sum_{\mathbf{i}\in\mathbf{I}_1} |\sigma(\mathbf{i})|^t < +\infty \tag{4.24}$$

as $\mathbf{I}_1$ is a finite set, the $\Delta(\mathbf{i})$, $\mathbf{i}\in\mathbf{I}_1$ simply being the components of the complement of N_0'.

Next, Lemma 4.1 and Lemma 4.11 (i) yield the inequality

$$\sum_{V\in G(\mathbf{i})} \left(\frac{|V(\tau^*)|}{|\sigma(\mathbf{i})|}\right)^t \leq \sum_{V\in G(\mathbf{i})} \left(\frac{|V(\tau^*)|}{|\sigma(\mathbf{i})|}\right) + M_1(1-t)$$

(where M_1 depends only on G) which in turn yields

$$\sum_{V\in G(\mathbf{i})} |V(\tau^*)|^t \leq |\sigma(\mathbf{i})|^t \{1+M_1\} \tag{4.25}$$

as $t>0$ and as (4.16) implies that

$$\sum_{V\in G(\mathbf{i})}|V(\tau^*)|\leqslant\sum_{V\in G(\mathbf{i})}|\sigma_V|=|\sigma(\mathbf{i})|.$$

If we now combine (4.18), (4.19), (4.24) and (4.25) and note that

$$\{I\}\cup[\bigcup_{\mathbf{i}\in\mathbf{I}}G(\mathbf{i})]=G$$

we deduce that

$$\sum_{V\in G, V\neq I}|c|^{-2t}<+\infty$$

which is well known to be equivalent to the conclusion (1.3) of Theorem 2. It therefore only remains to prove Lemma 4.11.

Proof of Lemma 4.11. The proof of (ii) follows easily from (i) and so we temporarily assume that (i) holds. Let $\sigma(n)$ be an interval as described in (d) and suppose that $\sigma(n)\subset\sigma_{V_k}$ where $V_k\in G(\mathbf{i})$. Then (4.21) and (b) imply that

$$|\sigma(n)|\leqslant|\sigma_{V_k}|\leqslant(2/\delta m)|V_k(\tau^*)|\leqslant(2/\delta m)A_k|\sigma(\mathbf{i})|.$$

We thus define $B_n=(2/\delta m)A_k$ and so (d) holds. Each n determines a unique k and the number of $\sigma(n)$ satisfying $\sigma(n)\subset\sigma_V$ for any given V is at most K_1 where K_1 is the number of components of $V(N_0')\cap R^1$ (or equivalently, of $N_0'\cap R^1$). This implies that

$$\sum_{n=1}^{\infty}B_n^t\leqslant K_1(2/\delta m)\sum_{n=1}^{\infty}A_n^t$$

and so (c) follows from (a). We have now shown that (ii) follows from (i).

The proof of (i) uses the decomposition of $G(\mathbf{i})$ as described in Lemma 4.9, (4.12). First, we may clearly define $A_n=1$ for those V in $G^0(\mathbf{i})$. Next, it is clearly sufficient to establish (i) in the case when $\{V_n\}$ is an enumeration of some $G^r(\mathbf{i})$ rather than of $G(\mathbf{i})$ and this is simpler because Lemma 4.9 gives an explicit characterization of the $G^r(\mathbf{i})$. Finally, it is sufficient to find such a sequence A_n where n assumes all integral values.

We now recall the following facts from Lemma 4.9 and its proof. If $\mathbf{i}\in\mathbf{I}_n$ and $n=1$, then $T=I$. If however $n\geqslant 2$, then we write $\Delta(\mathbf{i})=\Delta(\mathbf{j},\mathbf{k})$, $\mathbf{k}\in Z(\mathbf{j})$ (more explicitly, $\mathbf{i}=(i_1,\ldots,i_n)$, $\mathbf{j}=(i_1,\ldots,i_{n-1})$, $\mathbf{k}=i_n$) and note that T is the V appearing in Lemma 4.8 (i) and (ii) if, in the statement of this lemma, we replace $\mathbf{i}$ and $\mathbf{j}$ by $\mathbf{j}$ and $\mathbf{k}$ respectively and if we assume in (ii) that $V(\tau)$ appears on the left of $\sigma(\mathbf{i},\mathbf{j})$. If $n\geqslant 2$ then, we have $\pi^*(T)=n-1$, $T\in G(\mathbf{j})$ and $\sigma(\mathbf{j},\mathbf{k})\,(=\sigma(\mathbf{i}))$ meets $T(N_0')$ at the left hand end-point of $\sigma(\mathbf{j},\mathbf{k})$. As T is the V appearing in Lemma 4.8 we have the vital relations

$$\sigma(\mathbf{i})\subset\sigma_T \tag{4.26}$$

and
$$\Delta(\mathrm{i}) = T(\Sigma_0) \tag{4.27}$$

where Σ_0 is some Σ-set. The general element V in $G^r(\mathrm{i})$ satisfies

$$V(\tau^*) \subset \sigma(\mathrm{i}) \tag{4.28}$$

and by Lemma 4.9 is of the form $V = TP_r^n V_r$. Writing $\tau^*(n, r) = P_r^n V_r(\tau^*)$ we see from (4.26), (4.27) and (4.28) that

$$\tau^*(n, r) \subset T^{-1}(\sigma_T) \tag{4.29}$$

and also that

$$\frac{|V(\tau^*)|}{|\sigma(\mathrm{i})|} = \frac{|T(\tau^*(n, r))|}{|T(\Sigma_0 \cap R^1)|} \tag{4.30}$$

$$= \left(\int_{\tau^*(n,r)} |x - T^{-1}\infty|^{-2} dx\right) \left(\int_{\Sigma_0 \cap R^1} |x - T^{-1}\infty|^{-2} dx\right)^{-1} \leqslant m|\tau^*(n, r)| \tag{4.31}$$

$$= m|P_r^n V_r(\tau^*)|$$

where m depends only on G. This follows as (4.29) allows us to use Lemma 4.10 in estimating the numerator in (4.31) while the denominator in (4.31) is bounded below by a quantity depending only on G. This estimate has been made under the assumption that $n \geqslant 2$. If $n = 1$, $T = I$ and (4.30) reduces to (4.31), thus (4.31) holds for $n \geqslant 1$. The interval $V_r(\tau^*)$ is one of a finite number of compact intervals in R^1 none of which meet the orbit of ∞ and P_r is one of a finite number of parabolic elements in G. We prove

LEMMA 4.12. *There exists a positive number M (depending only on G) such that*

$$|P_r^n V_r(\tau^*)| \leqslant M n^{-2}. \tag{4.32}$$

With this available we put $A_n = M n^{-2}$ for each integer n for which $TP_r^n V_r$ is in $G^r(\mathrm{i})$. This proves Lemma 4.11 (i) for $G^r(\mathrm{i})$ and hence in general. The last stage, then, of our proof of Theorem 2 is the following proof.

Proof of Lemma 4.12. The set

$$F = V_1(\tau^*) \cup \ldots \cup V_k(\tau^*)$$

is a compact set not meeting the orbit of ∞ and not containing any p_r. Thus for each r and each n

$$|P_r^n V_r(\tau^*)| \leqslant |P_r^n(F)| < +\infty. \tag{4.33}$$

Next, put
$$\varrho = \inf\{|z - p_r| : z \in F,\ r = 1, \ldots, K\}.$$

As
$$P_r^n(z) - p_r = \frac{z - p_r}{(z - p_r)\, k_r n + 1}$$

for some k_r, we have
$$\left|\frac{d}{dz} P_r^n(z)\right| = |(z - p_r)\, n k_r + 1|^{-2}$$

and so for all sufficiently large n,

$$|P_r^n V_r(\tau^*)| \leqslant |P_r^n(F)| \leqslant \frac{|F|}{(n|k_r|\varrho - 1)^2} \leqslant 2|F|(n\varrho|k_r|)^{-2}.$$

Clearly this and (4.33) imply (4.32) and the proof of Theorem 2 is now complete.

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