Part B

Metarecursion

Metarecursion theory lifts classical recursion theory (CRT) from the natural numbers to the recursive ordinals via definitions in hyperarithmetic terms. It makes precise the vague idea that Π_1^1-ness is analogous to recursive enumerability. As a generalization of classical recursion theory, it is strong enough to carry out the solution of Post's problem and the construction of a maximal set. Thus priority arguments make sense in the context of Π_1^1 sets, and supply results not obtainable by more direct means. As an outgrowth of hyperarithmetic theory, it provides a concrete introduction to the fundamentals of α-recursion beyond ω.

Chapter V
Metarecursive Enumerability

With the aid of a Π_1^1 set of unique notations for recursive ordinals, the fundamental notions of classical recursion are lifted to metarecursion. "Finite" is raised to "metafinite", "recursively enumerable" to "metarecursively enumerable", and "Turing reducible to" "metarecursive in". A set is said to be regular if its intersection with every metafinite set is metafinite. It is shown that each metarecursively enumerable set has the same metadegree as some regular, metarecursively enumerable set.

1. Fundamentals of Metarecursion

Metarecursion theory originated in Kreisel's observation that hyperarithmetic subsets of Π_1^1 sets of natural numbers behave much like finite subsets of recursively enumerable sets. (A similar observation was made independently by Hartley Rogers with respect to Rice's theorem on indices.) The natural enumeration of a Π_1^1 set in ω_1^{CK} steps yields a hyperarithmetic set at each step, just as the enumeration of a recursively enumerable set yields a finite set at each step. If A is Π_1^1, then there is a recursive function f such that

$$(x)[x \in A \leftrightarrow f(x) \in O]$$

(Theorem 5.4.I). Let

$$A_\delta = \{x \mid |f(x)| \leq \delta\}.$$

Then $A = \cup\{A_\delta \mid \delta < \omega_1^{\mathrm{CK}}\}$, and each A_δ is hyperarithmetic (Lemma 2.4.II). Takeuti (1960) was the first to generalize recursion theory from natural numbers to ordinals; he replaced recursive enumerability by a schematic approach equivalent to Σ_1 definability. Kreisel was the first to point out that a generalization of recursion theory ought to pay special attention to the concept of finiteness in addition to that of recursive enumerability. His insight went beyond the idea that finite computations should be replaced by certain infinite ones. For inspiration he drew on model theory. He proved (Kreisel 1961, 1965): a Π_1^1 set A of axioms of ω-logic has a model if every hyperarithmetic subset of A has a model. His result suggested that generalizations of the compactness theorem of first order logic ought to take generalizations of finiteness into account, and that one such generalization, in the context of Π_1^1 sets, was hyperarithmeticity.

Kreisel also maintained that any generalization of recursion theory worthy of investigation should be able to support the priority method of Friedberg and Muchnik. In particular it should yield a positive solution to Post's problem, that is incomparable degrees of recursively enumerable sets. Spector proved what some thought was a negative solution of Post's problem for Π_1^1 sets of numbers, when he showed that every non-hyperarithmetic Π_1^1 set has the same hyperdegree as Kleene's O (Proposition 7.2.II). Kreisel took an opposing view of Spector's result. The set of natural numbers was not the correct domain when recursively enumerable was replaced by Π_1^1. Since finite was akin to hyperarithmetic, it followed that r.e. sets of numbers were akin to Π_1^1 sets of hyperarithmetic sets. With ω replaced by HYP, Spector's negative argument no longer worked.

Another reason for changing the domain came from consideration of Post's theorem: a set is recursive iff both it and its complement are r.e. With domain ω the Post theorem became: a set is hyperarithmetic iff both it and its complement are Π_1^1. Thus with domain ω, recursive became akin to hyperarithmetic, a connection ruled out by the equating of hyperarithmetic with finite.

A further argument against Spector's theorem as a negative solution to Post's problem lay in the difference between Turing and hyperarithmetic reducibility. A Turing reduction procedure P consists of a recursively enumerable sequence of finite computations. A set B is reduced by applying to B those elements of P that fit B, those computations based on membership statements true of B. A hyperarithmetic reduction procedure Q fails to resemble a Π_1^1 sequence of hyperarithmetic computations. Q does not specify computations in advance. If Q is applied to a non-hyperarithmetic B, then the resulting computations are hyperarithmetic in B rather than hyperarithmetic.

Kreisel's idea that Π_1^1 sets of hyperarithmetic sets of numbers correspond to recursively enumerable sets of numbers with respect to priority arguments was sound and marked the beginning of metarecursion theory. Suppose for a moment that a metarecursively enumerable set is defined to be a Π_1^1 set of hyperarithmetic sets of numbers. Then a metarecursive set is a metarecursively enumerable set whose complement with respect to the domain of hyperarithmetic sets is also metarecursively enumerable. And a metafinite set is a hyperarithmetic collection of hyperarithmetic sets. In short meta r.e. means Π_1^1 *on* HYP, metarec. means Δ_1^1 *on* HYP, and metafinite means hyperarithmetic. Kreisel conjectured, rightly as it turned out, that metarecursively enumerable sets would be amenable to priority arguments. The use of hyperarithmetic sets as individuals was much too awkward, and soon gave way to indices, then to notations for recursive ordinals, and finally to recursive ordinals as in the next subsection.

1.1 Definition of Metarecursive. Let Q be a Π_1^1 set of unique notations of recursive ordinals (Theorem 2.4.III). Let $n{:}\,\omega_1^{\mathrm{CK}} \to Q$ take each recursive ordinal to its unique notation. Thus $|n(\beta)| = \beta$.

Assume $A \subseteq \omega_1^{CK}$. A is called *metarecursively enumerable* (Kreisel and Sacks 1963) if $n[A]$ is Π_1^1. ($n[A] = \{n(a) | a \in A\}$.) A is called *metarecursive* if A and $\omega_1^{CK} - A$ are metarecursively enumerable. A is said to be *metafinite* if $n[A]$ is hyperarithmetic.

The choice of Q makes no difference. Suppose Q_1 and Q_2 are Π_1^1 sets of unique notations. Let $\theta\colon Q_1 \to Q_2$ be the unique map such that $|\theta(n)| = |n|$. Then $\theta(x) = y$ iff

(1) $\quad x \in Q_1$ & $y \in Q_2$ & $(\mathrm{E}f)$ [f is an order-preserving map of $Q_1^{<x}$ onto $Q_2^{<y}$].

($Q_i^{<x}$ is $\{z | z \in Q_i \ \&\ z <_O x\}$.) The f of (1), if it exists, is unique, hence hyperarithmetic. Thus the graph of θ is Π_1^1. Let $n_i\colon \omega_1^{CK} \to Q_i$ take each recursive ordinal to its unique notation in Q_i. Then

$$n_1[A] \text{ is } \Pi_1^1 \quad \text{iff} \quad n_2[A] \text{ is } \Pi_1^1,$$

since $n_2 = \theta n_1$.

Let $\Phi \subseteq \omega_1^{CK} \times \omega_1^{CK}$ be a partial function. ϕ is said to be *partial metarecursive* if

$$\{\langle n(\alpha), n(\beta)\rangle | \phi(\alpha) = \beta\}$$

is Π_1^1.

1.2 Proposition. *Let K, $A \subseteq \omega_1^{CK}$*

(*i*) *K is metafinite $\leftrightarrow K$ is metarecursive and bounded above by some $\beta < \omega_1^{CK}$.*

(*ii*) *A is meta r.e. $\leftrightarrow$ A is the range of a partial metarecursive function.*

(*iii*) *If K is metafinite, and ϕ is partial metarecursive and defined on K, then $\phi[K]$ is metafinite.*

(*iv*) *Assume $A \subseteq \omega$. A is metarecursively enumerable $\leftrightarrow A$ is Π_1^1. A is metafinite $\leftrightarrow$ A is hyperarithmetic.*

Proof

(i) Suppose K is metafinite. Then $n[K]$ is hyperarithmetic, $Q - n[K]$ is Π_1^1, so K is metarecursive. By Spector's bounding theorem (5.6.I), there is a $b \in O$ such that K is bounded above by $|b|$.

Suppose K is metarecursive and bounded above by $|b|$. Then $n[K] \subseteq O_b \cap Q = Q_b$ and $\omega - n[K] = (\omega - Q_b) \cup (Q_b - n[K])$. Since $n[K]$ and $Q - n[K]$ are Π_1^1, and Q_b is r.e. (3.5.I), $n[K]$ must be Δ_1^1.

(iii) Let f be the function whose graph is

$$\{\langle n(x), n(y)\rangle | \phi(x) = y\}.$$

Then $n[\phi[K]] = f[n[K]]$. The latter is Δ_1^1 (as in the proof of Proposition 1.7.I), since $n[K]$ is hyperarithmetic and f is partial Π_1^1.

(iv) Assume $A \subseteq \omega$. $n \restriction \omega$ is hyperarithmetic. Hence A is $\Pi_1^1 \leftrightarrow n[A]$ is Π_1^1. $\quad\square$

Proposition 1.2 (i) suggests that *finite*, in the context of recursion theory, is simply *recursive and bounded*. It seems hard to believe that no argument in classical recursion theory needs the fact that a finite set has a greatest element, but this appears to be the case. Proposition 1.2(iii) is the most important principle of metarecursion theory. In Part C it will be seen to be equivalent to the fact that $L(\omega_1^{\mathrm{CK}})$ satisfies Σ_1 replacement, the key principle of α-recursion theory. Proposition 1.2(iv) makes possible the application of metarecursion theory to Π_1^1 sets, and in particular the construction of a maximal Π_1^1 set via a Friedbergian priority argument.

1.3 Bounded Meta r.e. Sets. Lifting arguments from classical recursion theory to metarecursion theory will not in general be routine. It will not as a rule suffice to prefix all key words with "meta". The reader may already have noticed the following significant difference between metarecursion and classical recursion. There exists a bounded, meta r.e. set that is not metarecursive. The simplest example is Kleene's O according to Proposition 1.2(iv). In the setting of classical recursion theory every bounded set is finite, hence recursive. Bounded, non-metarecursive, meta r.e. sets do not belong to the realm of pathology. They are a source of technical problems whose solutions illuminate the workings of recursion theory.

1.4 Theorem (Enumeration). *For each $n \geq 1$ there is a partial metarecursive function $\phi_n(z, x_1, \ldots, x_n)$ such that: for each partial metarecursive $\psi(x_1, \ldots, x_n)$ there is an $e < \omega$ such that*

$$\psi(x_1, \ldots, x_n) \simeq \phi_n(e, x_1, \ldots, x_n).$$

Proof. Let $n = 1$. As in subsection 5.2.I, each Π_1^1 predicate $P_e(x, y)$ can be put in the form

$$(f)\,(\mathrm{Eu})\,T(\bar{f}(u), e, x, y)$$

for some e determined by P. The proof of Kreisel's uniformization Theorem (2.3.II) yields a recursive function g such that $g(e)$ is an index for the Π_1^1 predicate that uniformizes

$$P_e(x, y) \;\;\&\;\; x \in Q \;\;\&\;\; y \in Q.$$

Define $\phi_1(e, x) \simeq y$ by

$$(f)\,(\mathrm{Eu})\,T(\bar{f}(u), g(e), n(x), n(y)).$$

(n is the notation function of subsection 1.1.) $\square$

1.5 Lemma. *There exist metarecursive functions j and k such that: for each metafinite set K, there exists a unique $\delta < \omega_1^{\mathrm{CK}}$ such that*

$$K = \{x \mid x < j(\delta) \;\;\&\;\; k(\delta, x) = 0\}.$$

Proof. Let Q be a Π_1^1 set of unique notations as in subsection 1.1, and let I be the set of all indices of hyperarithmetic subsets of Q. To be precise, $\langle e, b\rangle \in I$ if $b \in Q$ and

$\{e\}^{H_b}$ is a subset of Q. I is Π_1^1. $<$, a well-ordering of I, is defined by:

$$\begin{aligned}\langle e_1, b_1\rangle < \langle e_2, b_2\rangle &\leftrightarrow (|b_1| < |b_2|)\\ &\vee (|b_1| = |b_2| \quad \text{and} \quad e_1 < e_2).\end{aligned}$$

For each $b \in I$, let G_b be the subset of Q indexed by b. I_1, the Π_1^1 set of unique indices for hyperarithmetic subsets of Q, is defined by:

$$b \in I_1 \leftrightarrow b \in I \quad \text{and} \quad (y)[y < b \rightarrow G_y \neq G_b].$$

There exists a partial recursive function t whose domain includes Q, and which maps Q 1-1 onto I_1. t can be thought of as enumerating I_1 without repetitions. t is defined by effective transfinite recursion on Q. For each b, $t(b)$ is the result of selecting a member of $I_1 - t \upharpoonright b$. The selection is made via Kreisel's uniformization Theorem (2.3.II). Let $P(e, x)$ be the e-th Π_1^1 set. Then 2.3.II supplies a partial Π_1^1 function h such that for all e:

$$(\mathrm{E}x)\, P(e, x) \rightarrow h(e) \text{ is defined} \quad \& \quad P(e, h(e)).$$

The recursion equation for t is

$$t(b) = h(e(b)),$$

where e is a recursive function such that

$$\hat{x} P(e(b), x) = I_1 - t \upharpoonright b.$$

Define

$$k(\gamma, x) = 0 \leftrightarrow x \in G_{t(n(\gamma))}$$

and 1 otherwise.

The predicate

$$b, c \in Q \quad \text{and} \quad (x)[x \in G_{t(b)} \rightarrow |x| < |c|]$$

is Π_1^1, so another Kreisel uniformization allows c to be construed as a partial Π_1^1 function g of b. Define

$$j(\gamma) = g(t(n(\gamma))). \quad \square$$

1.6 Indices for Metafinite Sets. Let j and k be the metarecursive functions of 1.5. For each β define

$$K_\beta = \{x \mid x < j(\beta) \quad \text{and} \quad k(\beta, x) = 0\}.$$

According to Lemma 1.5 there is for each metafinite K just one β such that $K = K_\beta$. β is said to be the index of K regarded as a metafinite set. β is intended to be a strong index. Post (1944) distinguished between weak and strong indices for finite sets. A strong index for a finite set is an instruction for recursively enumerating the set together with an instruction for stopping. A weak index consists simply of the first instruction. Strong indices are needed in recursion theory so that finite sets will live on the same level of complexity as their elements.

Let f be a partial function from ω_1^{CK} into the set of metafinite sets. f is said to be partial metarecursive if there exists a partial metarecursive function g such that $f(\delta) \simeq K_{g(\delta)}$ for all δ.

1.7 Proposition. *If H is metafinite, then $\cup \{K_\delta | \delta \in H\}$ is metafinite.*

Proof. Let K be the union in question. Then

$$x \in K \leftrightarrow (E\delta)[\delta \in H \quad \& \quad x < j(\delta) \quad \& \quad k(\delta, x) = 0].$$

The definitions of Subsection 1.1. make it easy to verify that K and $\omega_1^{\mathrm{CK}} - K$ are metarecursively enumerable. Thus K is metarecursive. By Proposition 1.2(iii), $j[H]$ is metafinite, hence bounded, so K is bounded. Then K is metafinite by Proposition 1.2(i). $\square$

Note that the proof of Proposition 1.7 also shows K is a metarecursive function of H in the sense of subsection 1.6.

1.8 Proposition (Transfinite Recursion). *If I is metarecursive, then the unique solution of*

$$f(\delta) = I(f \upharpoonright \delta) \quad (\delta < \omega_1^{\mathrm{CK}})$$

is metarecursive.

Proof. As in set theory, let F be the set of all g such that:

(a) g is a metafinite function;
(b) domain of g is a recursive ordinal;
(c) $g(\delta) = I(g \upharpoonright \delta)$ for all $\delta \in \operatorname{dom} g$.

F is metarecursive because I is. By induction on γ, F is consistent: if $g_1, g_2 \in G$ and $\gamma \in \operatorname{dom} g_1 \cap \operatorname{dom} g_2$, then $g_1(\gamma) = g_2(\gamma)$. Define $f = \cup F$. f is partial metarecursive with domain $d \subseteq \omega_1^{\mathrm{CK}}$. Suppose $d < \omega_1^{\mathrm{CK}}$. Then g, defined by

$$g \upharpoonright d = f \quad \& \quad g(d) = I(f),$$

belongs to F. $\square$

1.9 Proposition. *If A is metarecursively enumerable but not metafinite, then A is the range of a one-one metarecursive function.*

Proof. $n[A]$ is Π_1^1. By Theorem 5.4.I there is a recursive t such that $x \in n[A]$ iff $t(x) \in O$. Let

$$A^\gamma = \{\delta \mid |t(n(\delta))| < \gamma\}.$$

A^γ is a metarecursive function of γ. Define

$$f(\omega \cdot \gamma + m) = n^{-1}(m) \quad \text{if} \quad m \in n[A^\gamma - \cup_{<\gamma} A^\gamma].$$

Let $f(\omega \cdot \gamma + m) = -1$ otherwise. f is metarecursive and enumerates A without repetitions save for the spurious value -1. A transfinite recursion is needed to eliminate -1. Define g by

$$g(\delta) = \mu x[x \geq \sup{}^{+} g[\delta] \quad \& \quad f(x) \neq -1].$$

μx is read "the least ordinal x such that". $\sup^{+} B$ is the least ordinal greater than every ordinal in B. Since $n[A]$ is not hyperarithmetic, there are arbitrarily large x's such that $f(x) \neq -1$. It follows from Proposition 1.8 that g is metarecursive. Let $h(\delta) = f(g(\delta))$. Then h enumerates A without repetitions. □

1.10–1.16 Exercises

1.10. Prove Proposition 1.2(ii).

1.11. Verify that k and j, defined in the proof of Lemma 1.5 are metarecursive.

1.12. Verify that F, defined in the proof of Proposition 1.8, is metarecursive.

1.13. Show that the set of partial metarecursive functions is closed under composition.

1.14. Suppose f is metarecursive and K is metafinite. Show $f \restriction K$ is metafinite.

1.15. Formulate and prove: the class of metarecursive sets is closed under complementation, metafinite unions, and metafinite intersections.

1.16. Formulate and prove: the class of metarecursively enumerable sets is closed under meta r.e. unions and metafinite intersections.

2. *Metafinite Computations*

This section is a sketch of an equation calculus for computing partial metarecursive functions, a calculus that extends Kleene's by allowing certain infinite computations. A sketch is sufficient, because the calculus has little practical use. It is included only to clarify the concepts of metarecursive function and metarecursive reducibility.

Implementation of the calculus gives rise to metafinite computations. This is seen to be necessarily so, if the definition of metarecursive function is viewed in the light of the remarks made concerning the natural enumeration of O in the proof of Theorem 2.2.I. Suppose f is metarecursive. Then

$$f(x) = y \leftrightarrow \langle n(x), n(y) \rangle \in A$$

for some Π_1^1 $A \subseteq O$. Thus the computation of a value of f corresponds to showing some number belongs to O. The verification of $m \in O$ is accomplished by enumerating O up to stage $|m|$. The resulting object is hyperarithmetic, hence metafinite. The object has the tree-like appearance of a wellfounded computation if it is developed backwards, as in the proof of Theorem 3.5.I, where the predecessors of m in O are generated by starting with m and proceeding downward along branches that end with 1.

2.1 An Infinitary Calculus. A calculus of the sort presented here was first devised independently by Tugué (1964), and by Levy (1963) and Machover (1961). Variations were later studied by Kripke (1964) and Platek (1966). All are inspired by the Kleene calculus for classical recursion theory. The sketch below follows Kripke (1964). The primitive symbols are:

numbers $\underline{\gamma}$ for each $\gamma < \omega_1^{\mathrm{CK}}$;
variables $x, y, z, \ldots$ ranging over ω_1^{CK};
n-place function symbols $f, g, h, \ldots$;
and $\exists$ (there exists), $\forall$(for all), $<$ (less than), $=$ (equals), and $'$(successor).

Terms are defined recursively. Numerals and variables are terms. If t is a term, then t' is a term. If $t_1, \ldots, t_n$ are terms and f is an n-place function symbol, then $f(t_1, \ldots, t_n)$ is a term. If t_0 and t_1 are terms with x not occurring freely in t_0, then $(\mathrm{Ex} < t_0)t_1$ and $(\forall x < t_0)t_1$ are terms.

If t_0 and t_1 are terms, then $t_0 = t_1$ is an equation.

As in Kleene (1952) there are finitely many substitution rules for deriving an equation e from a set S of equations. For example $\underline{\gamma + 1}$ is substituted for $(\underline{\gamma})'$, or $\underline{\gamma}$ for t if $t = \underline{\gamma}$ has already been derived. Kleene's rules are finitary in that each operates on a finite set of premises. It follows that his computatons are finite, because a wellfounded tree with finite branching is necessarily finite. The calculus for metarecursion has only one infinitary rule, W. W operates on

$$\{e\} \cup \{t(\underline{\delta}) = \underline{0} \mid \delta < \gamma\}$$

and yields as conclusion the result of substituting $\underline{0}$ for all occurrences of $(\forall x < \gamma)t(x)$ in e. W is seen to be a kind of ω-rule if 0 is taken to mean true.

Let E be a finite set of equations. The set $C(E)$ of all equations computable from E is defined by transfinite recursion.

$$C(0, E) = E.$$

$e \in C(\beta + 1, E)$ if e is an immediate consequence of equations in $C(\beta, E)$ via a rule whose application mentions no ordinals greater than $\beta + 1$.

$$C(\lambda, E) = \cup \{C(\beta, E) \mid \beta < \lambda\} \quad \text{if } \lambda \text{ is a limit.}$$
$$C(E) = C(\omega_1^{\mathrm{CK}}, E).$$

$C(E)$ has a natural enumeration similar to the one given for O in the proof of Theorem 2.2.I. It follows that $C(E)$ is metarecursively enumerable, and that $C(\beta, E)$ is metafinite for each $\beta < \omega_1^{\mathrm{CK}}$. There is no difficulty in metarecursively assigning recursive ordinals to equations as Gödel numbers.

Each equation in $C(E)$ owes its existence to a metafinite computation with roots in E. Abstractly, each computation is simply a metafinite, wellfounded tree. The set Z of all computations can be defined by a recursion that parallels the one that defines C(E) and so is metarecursively enumerable. It is worth noting that Z is not metacursive (Exercise 2.3), since classical recursion theory suggests otherwise.

2.2. Metacomputable Functions. Let $\phi \subseteq \omega_1^{\mathrm{CK}} \times \omega_1^{\mathrm{CK}}$ be a partial function. ϕ is said to be metacomputable if there exists a finite set E of equations with principal function letter f such that for all $\alpha, \beta < \omega_1^{\mathrm{CK}}$:

(1) $$\phi(\alpha) = \beta \leftrightarrow (f(\underline{\alpha}) = \underline{\beta}) \in C(E).$$

The metacomputable partial functions are the same as the metarecursive partial functions. Checking this assertion involves many tedious details but the ideas needed are straightforward and are familiar to readers of Kleene (1952).

Suppose ϕ is metacomputable. By (1) the graph of ϕ is metarecursively enumerable, since $C(E)$ is. Hence ϕ is partial metarecursive according to the definitions of subsection 1.1.

The other direction is more difficult. Suppose ϕ is partial metarecursive. Thus

(2) $$\{\langle n(\alpha), n(\beta)\rangle \mid \phi(\alpha) = \beta\}$$

is Π_1^1, as in subsection 1.1, and so is many-one reducible to O vi a some ordinary recursive function g. To see that O is the range of a metacomputable function, recall the equations that yield the natural enumeration of O. Since g is metacomputable via Kleene's equation calculus, it follows that (2), is the range of a metacomputable function defined by some finite set E of equations. A slight modification of E defines ϕ.

The most important point suppressed above concerns the consistency of E. In general it is simple to find an E that defines a given partial metarecursive ϕ. "Simple" means that

(3) $$\phi(\alpha) = \beta \rightarrow (f(\underline{\alpha}) = \underline{\beta}) \in C(E)$$

is immediate. The converse of (3), the consistency of E, can be troublesome. Kleene (1952) handles the problem in a proof theoretic fashion. He allows substitutions only under certain narrow restrictions, so narrow no inconsistent computations can slip by. For example, consider the usual recursion equations for $+$. The base equation is

$$f(x, \underline{0}) = x. \tag{4}$$

It allows the computation of $f(\underline{m}, \underline{0}) = m$ for any natural number m. What is to prevent the computation of $f(\underline{m}, \underline{0}) = \underline{n}$ for some $m \neq n$? If such a computation existed, it might have the form

$$f(\underline{m}, t) = \underline{n}, \quad t = \underline{0} \vdash f(\underline{m}, \underline{0}) = \underline{n}. \tag{5}$$

But (5) is not allowed by the Kleene rules. The principal function letter f occurs only on the left sides of equations, and the only substitution permitted on the left is that of a numeral for a variable. Hence $f(\underline{m}, \underline{0})$ can be obtained on the left only by starting with (4) or with an equation whose left side is $f(\underline{m}, y)$. But the recursion equations for $+$ whose left sides have variable second arguments have right sides that cannot give rise to $\underline{0}$, for example $(g(x, y))'$. In short the restrictions on Kleene's rules make it possible to trace an inconsistency back to its source.

2.3–2.4 Exercises

2.3. Formulate and prove: the set of all metafinite computations is not metarecursive.

2.4 (Kripke 1964). Show $C(E, \omega_1^{\mathrm{CK}} + 1) = C(E, \omega_1^{\mathrm{CK}})$.

3. *Relative Metarecursiveness*

Classical recursion theory centers on the notion of Turing reducibility (or relative recursiveness). The corresponding notion for metarecursion theory is at first obscure, because it is not clear which formulation of Turing reducibility should be lifted. As it turns out, the choice matters a great deal. One formulation, when lifted, fails to be transitive, and so does not lead to a suitable concept of degree. In general two sets are said to have the same degree if each is reducible to the other. If the given notion of reducibility is transitive (and reflexive), then the derived concept of degree is an equivalence relation. The concept of degree is essential to the study of recursively enumerable sets, and of Post's problem, several generalizations of which are solved in this book. The first is the metarecursive version of Chapter VI.

3.1 Weakly Metarecursive In. One formulation of Turing reducibility for sets $A, B \subseteq \omega$ is as follows. A Turing reduction procedure ϕ is a recursively enumerable set of quadruples $\langle h, j, n, i \rangle$. h and j are finite subsets of ω; and n, $i \in \omega$. cB, the complement of B, is $\omega - B$. A is Turing reducible to B via ϕ if

(1a) $$n \in A \leftrightarrow (\mathrm{Eh})(\mathrm{Ej})[\langle h, j, n, 0\rangle \in \phi \;\&\; h \subseteq B \;\&\; j \subseteq cB]$$

and

(1b) $$n \notin A \leftrightarrow (\mathrm{Eh})(\mathrm{Ej})[\langle h, j, n, 1\rangle \in \phi \;\&\; h \subseteq B \;\&\; j \subseteq cB]$$

hold for all n. To decide $n \in A$ (from B) enumerate the quadruples in ϕ until one is found that satisfies B. Such a quadruple has the form $\langle h, k, n, i\rangle$ and the property that $h \subseteq B$ and $K \subseteq cB$. Then $N \in A$ iff $i = 0$. The associated computation is finite, because only finitely much of ϕ is enumerated, and only finitely much of B is called for. Let $\leq_T$ denote Tuning reducibility.

To see that $\leq_T$ is transitive, suppose $A \leq_T B$ via ϕ, and $B \leq_T C$ via ψ. To decide $n \in A$ (from C) enumerate ϕ and ψ simultaneously. If $n \in A$, then the associated computation, in essence, is:

(2a) $\langle h,j,n,0\rangle \in \phi$

(2b) $(x)_{x \in h}[\langle h_x, j_x, x, 0\rangle \in \psi]$

(2c) $(y)_{y \in j}[\langle h_y, j_y, y, 1\rangle \in \psi]$

(2d) $U\{h_v \mid v \in h \cup j\} \subseteq C, \quad U\{j_v \mid v \in h \cup j\} \subseteq cC.$

By (2a) $n \in A$ if $h \subseteq B$ and $j \subseteq cB$. By (2b) $h \subseteq B$ if $U\{h_x \mid x \in h\} \subseteq C$ and $U\{j_y \mid y \in j\} \subseteq cC$.

If $n \notin A$, then the associated computation begins with $\langle h, k, n, 1\rangle \in \phi$ and otherwise resembles (2).

The only combinatoric principle used above is: a finite union of finite sets is finite. A glance at Proposition 1.7 suggests that the above demonstration lifts to metarecursion theory, but that is not the case. To see why not, consider $A, B \subseteq \omega_1^{\mathrm{CK}}$. cB, the complement of B, is now $\omega_1^{\mathrm{CK}} - B$. Define A as *weakly metarecursive* in B (in symbols $A \leq_w B$) by:

(3a) $$\delta \in A \leftrightarrow (\mathrm{EH})(\mathrm{EJ})[\langle H, J, \delta, 0\rangle \in \phi \;\&\; H \subseteq B \;\&\; J \subseteq cB]$$

and

(3b) $$\delta \notin A \leftrightarrow (\mathrm{EH})(\mathrm{EJ})[\langle H, J, \delta, 1\rangle \in \phi \;\&\; H \subseteq B \;\&\; J \subseteq cB],$$

where ϕ is metarecursively enumerable, H and K are metafinite, and $\delta < \omega_1^{\mathrm{CK}}$. (3) is the most obvious lifting of (1). The attempt to lift (2) breaks down with (2d). In general

(4) $$\cup\{H_v \mid v \in H \cup J\}$$

is not metafinite despite the fact that $H \cup J$ and H_v ($v \in H \cup J$) are metafinite. (4) is not a metafinite union of metafinite sets in the sense required by Proposition 1.7. For each v let $\delta(v)$ be the strong index of H_v. Thus $H_v = K_{\delta(v)}$ as in subsection 1.6. (4) becomes

(5) $$\cup \{K_{\delta(v)} | v \in H \cup J\}.$$

To apply Proposition 1.7 to (5) it is necessary to know

(6) $$\{\delta(v) | v \in H \cup J\} \text{ is metafinite.}$$

The truth of (6) depends on C. $\delta(v)$ is defined by enumerating ϕ and ψ simultaneously and occasionally referring to C. If the definition of $\delta(v)$ for all $v \in H \cup J$ draws only on a metafinite set of membership facts about C, then (6) is true. Nothing in (3) guarantees such limited use of C.

Driscoll (1968) showed that $\leq_w$ fails to be transitive even on the metarecursively enumerable sets. His theorem will be proved in Chapter VI.

3.2 Metarecursive In. The remedy for Driscoll's troublesome counterexample was found by Kreisel. Another formulation of Turing reducibility is:

(1a) $$j \subseteq A \leftrightarrow (\mathrm{Eh})(\mathrm{Ek})[\langle h,k,j,0\rangle \in \phi \ \& \ h \subseteq B \ \& \ k \subseteq cB]$$

and

(1b) $$j \subseteq cA \leftrightarrow (\mathrm{Eh})(\mathrm{Ek})[\langle h,k,j,1\rangle \in \phi \ \& \ h \subseteq B \ \& \ k \subseteq cB]$$

for all finite $j \subseteq \omega$. A, $B \subseteq \omega$, h and k are finite, and ϕ is recursively enumerable. The symmetry of (1) with respect to finite neighborhood conditions makes the transitivity of $\leq_T$ immediate. A finite set of membership questions about A is reduced to a finite set about B. Suppose $A \leq_T B$ via ϕ, and $B \subseteq_T C$ via ψ. To show $A \leq_T C$ enumerate ϕ and ψ simultaneously. If $j \subseteq A$, then the associated computation is:

(2) $$\begin{aligned}&\langle h,k,j,0\rangle \in \phi;\\ &\langle h_0,k_0,h,0\rangle, \langle h_1,k_1,k,1\rangle \in \psi;\\ &h_0 \cup h_1 \subseteq C; k_0 \cup k_1 \subseteq cC.\end{aligned}$$

If $j \subseteq cA$, then the computation begins with $\langle h, k, j, 1\rangle \in \phi$ and otherwise resembles (2).

The only combinatoric principle used above is: the union of two finite sets is finite. Consequently the above demonstration lifts instantly to metarecursion theory.

Let A, B be sets of recursive ordinals, J, H be metafinite and ϕ be metarecursively enumerable. Say A *is metarecursive* in B (in symbols $A \leq_M B$) via ϕ if

(3a) $\quad J \subseteq A \leftrightarrow (\mathrm{EH})(\mathrm{EK})[\langle H, K, J, 0\rangle \in \phi \quad \& \quad H \subseteq B \quad \& \quad K \subseteq cB]$

and

(3b) $J \subseteq cA \leftrightarrow (\mathrm{EH})(\mathrm{EK})[\langle H, K, J, 1\rangle \in \phi \ \& \ H \subseteq B \ \& \ K \subseteq cB]$ hold for all metafinite J. (2) lifts to show $\leq_M$ is transitive.

Two sets of recursive ordinals have the same metadegree if each is *metarecursive* in the other (in symbols $A \equiv_M B$).

3.3 Post's Method. Post in a celebrated paper (1944) liberated classical recursion theory from formal arguments by presenting recursive enumerability as a natural mathematical notion safely handled by informal mathematical procedures. He also stressed what may be called a dynamic view of recursion theory. For example, he proves the existence of a simple set S by giving instructions in ordinary language for the enumeration of S and then verifying that the instructions do in fact produce a simple set. A formal approach to S would refer to formulas or equations from some formal system. A static approach would attempt to define S by some explicit formula. The advantages of Post's informal, dynamic method are considerable. Without it arguments in classical recursion theory would be lengthy and hard to devise.

His method, and its advantages, lift to metarecursion theory. Consider the proposition:

(1) If A is metarecursively enumerable, then the collection of all metafinite subsets of A is metarecursively enumerable. Formal static proof of (1): Recall I_1 defined in the proof of Lemma 1.5. I_1 is a Π_1^1 set of unique indices for the hyperarithmetic subsets of Q. $n[A]$ is Π_1^1, so

$$\langle e, b\rangle \in I_1 \quad \& \quad \{e\}^{H_b} \subseteq n[A]$$

is also Π_1^1, and (1) is proved.

Informal dynamic proof of (1): Metarecursively enumerate A. A^γ (defined in the proof of Proposition 1.9) is that part of A enumerated prior to stage γ. If $K_\delta \subseteq A$, then $K_\delta \subseteq A^{\gamma_0}$ for some $\gamma_0 < \omega_1^{\mathrm{CK}}$. If $K_\delta \subseteq A^{\gamma_0}$, then enumerate δ. Thus the enumeration of A gives rise to a simultaneous enumeration (of the indices) of the metafinite subsets of A. The existence of γ_0 follows from Proposition 1.2(iii). Let

$$h(x) \quad \text{be} \quad \mu\gamma[x \in A^\gamma].$$

h is partial metarecursive and defined for all $x \in K_\delta$. *So* $h[K_\delta]$ is metafinite, hence bounded by some δ_0.

From now on proofs in metarecursion theory will follow the informal dynamic style originated by Post.

3.4 Proposition. *Suppose A and ϕ are metarecursively enumerable, and for all metafinite K,*

$$K \subseteq cA \leftrightarrow (\mathrm{EH})(\mathrm{EJ})[\langle H, J, K\rangle \in \phi \ \& \ H \subseteq B \ \& \ J \subseteq cB].$$

Then $A \leq_M B$.

Proof. According to (1) of subsection 3.3, the set of all metafinite subsets of A is metarecursively enumerable. Thus there exists a metarecursively enumerable ψ such that

$$K \subseteq A \leftrightarrow K \in \psi$$

for all metafinite K. □

The next proposition is a technical fact needed in Section 4.

3.5 Proposition. *Let A be metarecursively enumerable. Then there exists a metarecursively enumerable A^* such that $A \equiv_M A^*$ and*

$$(X)[A^* \leq_w X \leftrightarrow A^* \leq_M X].$$

Proof. Let A^* be $\{\delta | K_\delta \cap A \neq \phi\}$. A^* is metarecursively enumerable since any enumeration of A induces a simultaneous enumeration of A^*. Proposition 3.4 implies $A \leq_M A^*$, since

(1) $$K_\delta \subseteq cA \leftrightarrow \delta \notin A^*.$$

There exists a metarecursive function g such that

$$K_{g(\delta)} = \cup \{K_\gamma | \gamma \in K_\delta\}$$

Hence

(2) $$K_\delta \subseteq cA^* \leftrightarrow K_{g(\delta)} \subseteq cA,$$

and so $A^* \leq_M A$ by Proposition 3.4.

It follows from (1) and (2) that

$$K_\delta \subseteq cA^* \leftrightarrow g(\delta) \notin A^*.$$

Consequently $A^* \leq_w X$ implies $A^* \leq_M X$. □

3.6 Reducibility for Functions. Suppose $\leq_r$ is a notion of reducibility defined for sets, for example $\leq_w$ or $\leq_M$. $\leq_r$ is extended to functions by identifying functions with their graphs. Thus

$$f \leq_r g \leftrightarrow \mathrm{graph}(f) \leq_r \mathrm{graph}(g).$$

(The graph of f is $\{\langle x,y\rangle \mid f(x)=y\}$.) Similarly

$$f \leq_r g \leftrightarrow \text{graph}(f) \leq_r A.$$

Let c_A be the characteristic function of A. As far as reducibility is concerned it is safe to identify A with the graph of c_A. Thus

$$A \leq_r B \leftrightarrow c_A \leq_r c_B.$$

Of special interest is $f \leq_w A$. It is equivalent to: There exists a metarecursively enumerable ϕ such that

$$f(x)=y \leftrightarrow (\mathrm{EH})(\mathrm{EJ})[\langle H,J,x,y\rangle \in \phi \quad \& \quad H \subseteq A \quad \& \quad J \subseteq cA]$$

for all x, $y < \omega_1^{\mathrm{CK}}$.

3.7–3.8 Exercises

3.7. Define g by $K_{g(\delta)} = \cup\{K \mid \gamma \in K_\delta\}$. Show g is metarecursive.

3.8. The equation calculus of Subsection 2.1 is relativized as follows. Let g be an auxillary function letter. For any $B \subseteq \omega_1^{\mathrm{CK}}$ define $\Delta(B)$ to be

$$\{g(\underline{\delta})=\underline{0} \mid \delta \in B\} \cup \{g(\underline{\delta})=1 \mid \delta \notin B\}.$$

Define $C(E, B)$ as $C(E)$ was in subsection 2.1 save that $C(0,E,B)$ is $E \cup \Delta(B)$. Each equation in $C(E,B)$ owes its existence to a computation of recursive ordinal height, abstractly a wellfounded tree with roots in $C(0,E,B)$. (If B is not metarecursive, then some of these computations may not be metafinite objects.) Define $C_m(E,B)$ to be the subset of $C(E,B)$ based on metafinite computations.

Call A *metafinitely computable from* B (in symbols $A \leq_{mc} B$) if for some E with principal function letter f,

$$\delta \in A \leftrightarrow (f(\underline{\delta})=\underline{0}) \in C_m(E,B)$$

and

$$\delta \notin A \leftrightarrow (f(\underline{\delta})=\underline{1}) \in C_m(E,B)$$

for all $\delta < \omega_1^{\mathrm{CK}}$.

Show $A \leq_{mc} B \leftrightarrow A \leq_w B$.

4. *Regularity*

A set $A \subseteq \omega_1^{\mathrm{CK}}$ is said to be *regular if* $A \cap K$ is metafinite for every metafinite K. The main result of this section is: each metarecursively enumerable set has the same metadegree as some regular, metarecursively enumerable set. This theorem greatly facilitates the study of metarecursively enumerable degrees. The definition of regularity is inspired by its frequent, if hidden, use in classical recursion theory, and by Gödel's definition of constructible class, namely a class whose intersection with each constructible set is a constructible set.

A typical use of regularity in classical recursion theory occurs in the proof of the enumeration theorem (Kleene 1952). Finite neighborhood conditions are replaced by finite initial segment conditions in order to obtain an enumeration theorem for partial recursive functions relative to A (uniformly), a result that does not lift to metarecursion theory.

To elaborate let $A \subseteq \omega$, and $\{e\}^A$ be the e-th function partial recursive in A. Initially $\{e\}^A$ is defined as in (1) of subsection 3.1. There is a recursively enumerable ϕ such that $\{e\}^A(m)$ is defined and has the value n iff

$$\langle h, k, m, n \rangle \in \phi \;\;\&\;\; h \subseteq A \;\;\&\;\; k \subseteq cA \tag{1}$$

for some finite h and k. Thus an enumeration of the ϕ's yields an enumeration of the $\{e\}^A$'s. It can happen that ϕ is inconsistent, that $\{e\}^A(m)$ has more than one value. Kleene eliminates inconsistencies by giving preference to the "least" neighborhood condition that A satisfies. Let $\phi(i)$ be the i-th 4-tuple in the recursive enumeration of ϕ. Now define $\{e\}^A(m)$ to have the value n if $\phi(i)$ satisfies (1) but $\phi(j)$ does not for all $j < i$. In order to compute the preferred value of $\{e\}^A(m)$ it is necessary to know the relation A bears to all the finite sets mentioned in $\phi(j)$ $(j \leq i)$. Since only finitely many finite sets are involved, the preferred value is determined by a bounded initial segment of the characteristic function of A. The regularity of A implies that a bounded initial segment of A is equivalent to a single finite neighborhood condition.

The initial segment trick is not applicable to metarecursion theory because a typical $A \subseteq \omega_1^{\mathrm{CK}}$ is not regular. Machtey (1970) and Ohashi (1970) have shown that metarecursion theory lacks an enumeration theorem of the functions partial metarecursive in A (uniformly). Thus there is no substitute for the initial segment trick. But there does remain an enumeration of the many-valued functions partial metarecursive in A (uniformly).

The next proposition expresses the notion of regularity in dynamic form.

4.1 Proposition. *Let A be the range of some metarecursive f. Then A is regular iff for each δ, f eventually enumerates no new member of A less than δ.*

Proof. Suppose A is regular. Then $A \cap \delta$ is metafinite. For each $\gamma \in (A \cap \delta)$, let $g(\gamma)$ be the least σ such that $f(\sigma) = \gamma$. Since g is metarecursive, $g[A \cap \delta]$ is bounded. $\square$

The study of recursively enumerable sets in classical recursion theory makes frequent use of the fact that every recursively enumerable set is regular in the sense of Proposition 4.1. The regular sets theorem (4.3) makes it safe to assume metarecursively enumerable sets are regular modulo their metadegrees.

The next lemma is inspired by an early result of Dekker (1954), and is the source of regularity in the proof of the regular sets theorem. Suppose A is metarecursively enumerable but not metafinite. By Proposition 1.9 there is a metarecursive f that enumerates A without repetitions. Define the deficiency set of f to be

$$D_f = \{\alpha \mid (E\beta)_{\beta > \alpha}(f(\beta) < f(\alpha))\}.$$

4.2 Lemma

(*i*) *D_f is metarecursively enumerable and regular.*
(*ii*) *cD_f is unbounded.*
(*iii*) *If $A(=f[\omega_1^{\mathrm{CK}}])$ is regular, then $f[cD_f]$ is unbounded.*
(*iv*) *If A is regular, then $A \leq_M D_f$.*
(*v*) *$D_f \leq_M A$.*

Proof.

(i) Fix γ to see $D_f \cap \gamma$ is metafinite. Enumerate $D_f \cap \gamma$ without repetitions as follows. A new member α of $D_f \cap \gamma$ is enumerated when there is a σ such that

$$(1) \qquad \sigma > \alpha \ \ \& \ \ f(\sigma) < f(\alpha) \quad \& \quad (\tau)_{\alpha < \tau < \sigma}[f(\tau) \geq f(\alpha)].$$

If there is a bound on the σ's satisfying (1), then $D_f \cap \gamma$ is metafinite. Suppose there is no such bound. If $\gamma < \sigma_1 < \sigma_2$ both satisfy (1), then $f(\sigma_1) > f(\sigma_2)$. Hence there is an infinite, descending sequence of ordinals.

(ii) Fix γ. Let δ be the infimum of $\{f(\tau) \mid \tau > \gamma\}$. Then $f^{-1}(\delta) \in cD_f$.

(iii) follows from (ii) and Proposition 4.1.

(iv) By (iii) $K \subseteq cA$ iff

$$(E\delta)\,(f(\delta) > \sup K \ \ \& \ \ \delta \notin D_f \ \ \& \ \ K \subseteq cf[\delta])$$

for all metafinite K. Now apply Proposition 3.4.

(v) For all metafinite K,

$$K \subseteq cD_f \leftrightarrow \bigcup_{\alpha \in K} (f(\alpha) - f[\alpha]) \subseteq cA. \qquad \square$$

4.3 Theorem (Sacks 1966). *Each metarecursively enumerable set has the same metadegree as some regular, metarecursively enumerable set.*

Proof. Suppose A is metarecursively enumerable and $A \cap \gamma$ is not metafinite. Let $n\colon \omega_1^{\mathrm{CK}} \to \omega$ be metarecursive and one-one. (n could be the notation function of subsection 1.1.) Then $n[A \cap \gamma]$ is metarecursively enumerable but not metafinite. Let f be a one-one metarecursive function whose range is $n[A \cap \gamma]$, and let B be $f[A]$. B is not metarecursive, because A is not and

$$x \in A \leftrightarrow f(x) \in B.$$

Let g be a metarecursive function that enumerates B without repetitions. D_g, the deficiency set of g (defined just before Lemma 4.2), is the desired regular set.

By Lemma 4.2 D_g is metarecursively enumerable and regular. The concept of "*finitely metarecursive in*" (in symbols $\leq_f$) is useful in showing $Dg \equiv_M A$. Say $U \leq_f V$ if $U \leq_w V$ via some ϕ that mentions only finite neighborhood conditions. Note that $U \leq_f V$ and $V \leq_w W$ imply $U \leq_w W$.

Observe that

$$x \notin D_g \leftrightarrow (g(x) - g[x]) \subseteq cB$$

and that $g(x)$ is finite. It follows from Proposition 3.4 that

$$D_g \leq_f B. \tag{1}$$

$$B \leq_w A \tag{2}$$

is a consequence of:

$$x \notin B \leftrightarrow (x \notin \text{range} f) \vee (x \in \text{range} f \;\&\; f^{-1}(x) \notin A);$$

$$(x \notin \text{range} f) \leftrightarrow x \notin n[\gamma] \vee (x \in n[\gamma] \;\&\; n^{-1}(x) \notin A).$$

From (1) and (2), $D_g \leq_w A$. Hence $D_g \leq_M A$, because every metafinite subset of cD_g is finite. To prove the latter note that the restriction of g to cD_g is strictly increasing. Hence the ordertype of cD_g is the same as that of $g[cD_g]$, namely ω. In addition, cD_g is an unbounded subset of ω_1^{CK} by Lemma 4.2.

To check $A \leq_M D_g$, it suffices to show $A \leq_w D_g$, since Proposition 3.5 makes it safe to assume

$$(X)[A \leq_w X \leftrightarrow A \leq_M X].$$

$A \leq_f B$ and $\leq_f$ is transitive, so it remains only to show $B \leq_f D_g$. But that is the case, because

$$m \notin B \leftrightarrow (E\alpha)[\alpha \notin D_g \quad \& \quad g(\alpha) > m \quad \& \quad m \notin g[\alpha]]$$

for all $m < \omega$. $\square$

The above proof, much cleaner than the original, is due to S. Simpson. The only non-effective step is the choice of γ. W. Maass has shown that choice to be eliminable. To be precise he has proved: there exists a metarecursive function t such that for each δ, $R_\delta \equiv_M R_{t(\delta)}$ and $R_{t(\delta)}$ is regular. (R_δ is the δ-th metarecursively enumerable set in a standard enumeration.)

Theorem 4.3 is very much a result about metarecursively enumerable degrees. It fails for some metadegrees below that of Kleene's O. In fact there is a $T \subseteq \omega$ such that T is Turing reducible to O and no set of the same metadegree as T is regular (Macintyre 1973).

It will be seen in Part C that Theorem 4.3 generalizes readily from ω_1^{CK} to an arbitrary Σ_1 admissible ordinal α.

4.4 ω-sets. A subset of ω_1^{CK} is said to be an ω-set (Owings 1969) if its complement is unbounded and of ordertype ω. Note that the complement of an ω-set cannot be metafinite. A typical ω-set is D_g in the proof of Theorem 4.3. ω-sets will figure significantly in the proof of Simpson's dichotomy, Section 3.VI. If C is a metarecursively enumerable ω-set, then

$$(X)[C \leq_w X \leftrightarrow C \leq_M X],$$

since every metafinite subset of cC is finite.

A is said to be *metacomplete* if A is metarecursively enumerable and

$$(B)[B \text{ meta r.e.} \rightarrow B \leq_M A].$$

4.5 Proposition. *A is metacomplete iff $O \leq_w A$.*

Proof. Suppose $O \leq_w A$ and B is meta r.e. By Proposition 3.5, it is safe to assume

$$B \leq_w A \leftrightarrow B \leq_M A.$$

Let $n\colon \omega_1^{\mathrm{CK}} \rightarrow \omega$ be one-one and metarecursive. Then $n[B]$ is Π_1^1, hence $n[B] \leq_f O$ by Theorem 5.4.I. But $B \leq_f n[B]$, so $B \leq_w A$. $\square$

4.6 Theorem (Owings 1969). *If D is a metarecursively enumerable ω-set, then there exists a metacomplete Π_1^1 set C such that $C \leq_f D$.*

Proof. Let n be a one-one, metarecursive map of ω_1^{CK} onto O. Define

$$t(x) = \mu\delta[\delta \notin D \ \& \ n(\delta) > x] \quad (x < \omega),$$
$$B = \{\delta \mid \delta < tn(\delta)\}, \quad \text{and} \quad C = n[B].$$

t is not metarecursive, so an approximation argument is needed to show C is Π_1^1. Let D be the range of some metarecursive function f. Define

$$D^\sigma = \{f(\tau) \mid \tau < \sigma\}, \quad \text{and}$$
$$t(\sigma, x) = \mu\delta[\delta \notin D^\sigma \ \& \ n(\delta) > x].$$

$t(\sigma, x)$ is metarecursive. More importantly, for each x, $t(\sigma, x)$ is a non-decreasing function of σ whose limit is $t(x)$. Consequently

$$B = \{\delta \mid (E\sigma)[\delta < t(\sigma, n(\delta))]\},$$

hence B is metarecursively enumerable and C is Π_1^1.

Every metafinite subset of cD is finite, so $t \leq_f D$ by Exercise 4.7. It follows that $C \leq_f D$, since for all $x < \omega$,

$$x \notin C \leftrightarrow x \notin n[t(x)].$$

For the metacompleteness of C, assume for the moment that B is an ω-set. By Proposition 4.5 it is enough to show $O \leq_w C$. Recall $x \in O$ iff $2^x \in O$. If $x \in O$, then

$$n^{-1}[\{x, 2^x, 2^{2^x}, \ldots\}].$$

is metafinite, hence touches B because B is an ω-set. Consequently C touches $\{x, 2^x, \ldots\}$. Conversely, if C touches $\{x, 2^x, \ldots\}$, then $x \in O$ because $C \subseteq O$.

To see that B is an ω-set, suppose there is a $\gamma < \omega_1^{\mathrm{CK}}$ such that $\gamma - B$ is infinite. Then

$$\{\delta \mid \delta < \gamma \ \& \ \delta \geq tn(\delta)\}$$

is infinite, and so $\{x \mid x < \omega \ \& \ t(x) < \gamma\}$ is infinite. Fix x so that $t(x) < \gamma$ and $x \geq \max(n[\gamma - D])$. Such an x exists because D is an ω-set. $t(x) \in \gamma - D$, so $x \geq nt(x)$. But the definition of t implies $nt(x) > x$. □

4.7–4.8 Exercises

4.7. Suppose C is metarecursively enumerable and every metafinite subset of cC is finite. Show $X \leq_w C$ implies $X \leq_f C$. ($\leq_f$ is defined in the proof of Theorem 4.3.)

4.8. Call A *simple* if A is metarecursively enumerable, cA is unbounded, and every metarecursively enumerable subset of cA is bounded. Show each non-metarecursive, meta r.e. set has the same metadegree as some simple set.