Chapter VIII
Countable Usls

The countable ideals of $\mathscr{D}$ are characterized in this chapter. In particular, we show that if $\mathscr{L}$ is a countable usl with least element, then $\mathscr{L} \hookrightarrow^* \mathscr{D}$. This characterization of countable ideals of $\mathscr{D}$ is applied to answer questions about automorphisms, elementary equivalence, and definability over $\mathscr{D}$. Results proved in Appendix A and Appendix B.3 are used in this chapter.

1. Countable Ideals of $\mathscr{D}$

We show that every countable usl with least element is isomorphic to an ideal of $\mathscr{D}$. We first state the theorem which provides the necessary table, and then introduce the new kinds of trees needed for the construction. We conclude by characterizing the countable ideals of $\mathscr{D}$. Notation and definitions are carried over from Chap. VII.

1.1 Definition. Let $\{\mathscr{L}_i: i \in N\}$ be given such that for each $i \in N$, $\mathscr{L}_i = \langle L_i, \leqslant_i, \vee_i \rangle$ is a usl and $\mathscr{L}_0 \subseteq \mathscr{L}_1 \subseteq \cdots$. We define the usl $\mathscr{L} = \bigcup\{\mathscr{L}_i: i \in N\} = \langle L, \leqslant, \vee \rangle$ by letting $L = \bigcup\{L_i: i \in N\}$, defining $a \leqslant b$ for $a, b \in L$ if for some $i \in N$, $a, b \in L_i$ and $a \leqslant_i b$, and defining $a \vee b$ for $a, b \in L$ to be the element c such that $c = a \vee_i b$ where i is the least element of N such that $a, b \in L_i$.

If each L_i in Definition 1.1 is finite and has a least element, then each $\mathscr{L}_i$ is a lattice, since every finite usl with least element is a lattice.

Let $\mathscr{L} = \langle L, \leqslant, \vee \rangle$ be a usl, and let $\mathscr{L}_i = \langle L_i, \leqslant_i, \vee_i \rangle$ be a finite usl such that $\mathscr{L}_i \subseteq \mathscr{L}$. Let $a \in L - L_i$ be given, and let $\mathscr{L}^* = \langle L^*, \leqslant^*, \vee^* \rangle$ be the smallest usl such that $L \cup \{a\} \subseteq L^*$ and $\mathscr{L}^* \subseteq \mathscr{L}$. Then $\mathscr{L}^*$ is finite since each element b of L^* is expressible as $b = \bigvee\{d: d \in M\}$ for some $M \subseteq L \cup \{a\}$. Thus we note the following fact:

1.2 Remark. Let $\mathscr{L} = \langle L, \leqslant, \vee \rangle$ be a countable usl with least and greatest elements u_0 and u_1 respectively. Then there is a sequence $\{\mathscr{L}_i: i \in N\}$ of finite lattices such that for each $i \in N$, $\mathscr{L}_i = \langle L_i, \leqslant_i, \vee_i \rangle$, $L_i = \{u_0, \ldots, u_{f(i)}\}$, u_0, $u_1 \in L_0$, and

(i) $\quad \forall i \in N(\mathscr{L}_i \subseteq \mathscr{L}_{i+1}) \qquad$ (as a usl).

(ii) $\quad \mathscr{L} = \bigcup\{\mathscr{L}_i: i \in N\}$.

The passage from embedding finite lattices as initial segments of $\mathscr{D}$ to embedding countable usls with least and greatest elements as initial segments of $\mathscr{D}$ requires approximating to the usl as in Remark 1.2, and using tables which mesh nicely for the approximation. In order to pass from a table for $\mathscr{L}_i$ to a table for $\mathscr{L}_{i+1}$, the table for $\mathscr{L}_i$ must be large enough to contain the restriction of a table for $\mathscr{L}_{i+1}$, etc. Hence we will need a sequence of successively larger tables for each $\mathscr{L}_i$ in order to have the table for $\mathscr{L}_i$ contain restrictions of tables for larger and larger lattices. We first define an inclusion relation for tables of different size tuples, and then define the type of table which will be needed.

1.3 Definition. Let $\mathscr{L}_0, \mathscr{L}_1$ be finite lattices such that $\mathscr{L}_0$ is a subusl of $\mathscr{L}_1$. Let $\mathscr{L}_0$ have universe $L_0 = \{u_0, \ldots, u_k\}$ and let $\mathscr{L}_1$ have universe $L_1 = \{u_0, \ldots, u_m\}$ where $m > k$. For $i = 0, 1$, let Θ_i be a usl table for $\mathscr{L}_i$. We say that $\Theta_1 \subseteq_k \Theta_0$ if $\{\langle n_0, \ldots, n_k \rangle \in N^{k+1}: \langle n_0, \ldots, n_m \rangle \in \Theta_1\} \subseteq \Theta_0$.

1.4 Definition. Let $\mathscr{L}$ be a countable usl with least and greatest elements, and let $\{\mathscr{L}_i : i \in N\}$ be as in Remark 1.2. A double array $\{\Theta_{i,j}: i, j \in N\}$ is said to be a *uniform sequential lattice table* for $\{\mathscr{L}_i : i \in N\}$ (see Fig. 1.1) if there is a strictly increasing function $h: N \to N$ such that:

(i) $\forall i, j \in N(\Theta_{i,j} \subseteq N^{f(i)+1} \; \& \; \Theta_{i,j}$ is finite).

(ii) For all $i \in N$, there is an increasing function $k_i: N \to N$ such that $k_i(0) \geqslant h(i)$, $\{\Theta_{i,k_i(j)}: j \in N\}$ is a recursive weakly homogeneous sequential lattice table for $\mathscr{L}_i$, and for all $j, m \in N$, if $k_i(j) \leqslant m < k_i(j+1)$ then $\Theta_{i,m} = \Theta_{i,k_i(j)}$.

(iii) $\forall i \in N \; \forall j \geqslant h(i+1)(\Theta_{i+1,j} \subseteq_{f(i)} \Theta_{i,j})$.

(iv) $\forall i \in N(c_i(j) = |\Theta_{i,j}|$ is a recursive function).

$$\begin{array}{llll}
\Theta_{0,0} & & & \\
\subseteq\downarrow & & & \\
\Theta_{0,h(1)} \supseteq_{f(0)} & \Theta_{1,h(1)} & & \\
\subseteq\downarrow & \downarrow\subseteq & & \\
\Theta_{0,h(2)} \supseteq_{f(0)} & \Theta_{1,h(2)} \supseteq_{f(1)} & \Theta_{2,h(2)} & \cdots \\
\subseteq\downarrow & \downarrow\subseteq & \downarrow\subseteq & \\
\vdots & \vdots & \vdots &
\end{array}$$

Fig. 1.1

The idea of the table is to let k_i specify the places where we can embed a larger table for $\mathscr{L}_{i+1}$ into the current table for $\mathscr{L}_i$. $h(i)$ tells us where to begin such an embedding for $\mathscr{L}_{i+1}$.

The following theorem is proved as Theorem 3.27 of Appendix B. $\mathscr{L}$ and $\mathscr{L}_i$ as in Remark 1.2 are fixed for the remainder of the section.

1.5 Theorem. *$\{\mathscr{L}_i : i \in N\}$ has a uniform sequential lattice table.*

For the remainder of this section, fix a uniform sequential lattice table $\{\Theta_{i,j}: i,j \in N\}$ for $\{\mathscr{L}_i: i \in N\}$ and let the functions f, h, k_i and c_i be as in Definition 1.4.

In order to build an initial segment of $\mathscr{D}$ which is isomorphic to $\mathscr{L}$, we will have to satisfy requirements of the type which appear in Chap. VII. Each requirement mentions only finitely many elements of L, so there is an i such that all the elements of L which are mentioned by a given requirement lie in L_i. We will try to satisfy such a requirement by forcing it with an *$\mathscr{L}_j$-tree* for some $j \geqslant i$. Thus we will be faced with the problem of starting with an $\mathscr{L}_i$-tree T, and having to find an *$\mathscr{L}_j$-subtree* T^* of T for some $j \geqslant i$. T is determined by $\{\Theta_{i,k}: k \geqslant h(i)\}$ and T^* by $\{\Theta_{j,k}: k \geqslant h(j)\}$. Since $u_1 \in L_0$, it follows from VI.1.2(ii) that the restriction operation of 1.4(iii) from $\Theta_{j,k}$ to $\Theta_{i,k}$ must be one-one. Thus we can construct T^* by extending $T(\emptyset)$ until the level of T corresponding to $h(j)$, and then thinning T out from that point on by lopping off those branches which do not correspond to tuples in the image of the restriction operation. (We are viewing the strings on T corresponding to $\Theta_{i,k}$ as being coded by the first coordinates, $\alpha^{[1]}$, of tuples in $\Theta_{i,k}$.) It is thus convenient to define the domain of an $\mathscr{L}_i$-tree in the following way. (Recall that 0_m is the string of length m consisting only of 0's.)

1.6 Definition. An *$\mathscr{L}_i$-tree* is a map T from

$$\mathscr{T}_i = \{\emptyset\} \cup \{\sigma \in \mathscr{S}: 0_{h(i)} \subset \sigma \,\&\, \forall j(h(i) \leqslant j < \mathrm{lh}(\sigma) \rightarrow$$
$$(\exists \alpha \in \Theta_{i,j})(\sigma(j) = \alpha^{[1]}))$$

into $\mathscr{S}$ which satisfies the following conditions:

(i) $\forall \sigma, \tau \in \mathscr{T}_i(\sigma \subseteq \tau \rightarrow T(\sigma) \subseteq T(\tau))$.

(ii) $\forall \sigma, \tau \in \mathscr{T}_i(\sigma \mid \tau \rightarrow T(\sigma) \mid T(\tau))$.

If T is an $\mathscr{L}_i$-tree and $j > i$, then we define the $\mathscr{L}_j$-tree $T^* = \mathrm{Exp}(T, \mathscr{L}_j)$, *the expansion of T for $\mathscr{L}_j$*, by $T^*(\emptyset) = T(0_{h(j)})$ and for all $\sigma \in \mathscr{T}_j$, $T^*(\sigma) = T(\sigma)$. We note that since $h(i) < h(j)$, it follows from 1.4(iii) that $\mathscr{T}_j \subseteq \mathscr{T}_i$ so $\mathrm{Exp}(T, \mathscr{L}_j)$ is well-defined. Thus we have shown:

1.7 Lemma. *Let T be a uniform $\mathscr{L}_i$-tree and let $j > i$ be given. Then there is a uniform $\mathscr{L}_j$-tree $T^* \subseteq T$ such that for all $h^*: N \rightarrow N$, if T is recursive in h^* then T^* is recursive in h^*.*

Let g be a branch of the $\mathscr{L}_i$-tree T, and let $T^* \subseteq T$ be an $\mathscr{L}_j$-tree. Assume that $g \subset T^*$, and fix $k \in N$ such that $f(i) < k \leqslant f(j)$ (and hence $u_k \in L_j - L_i$). Suppose that $g(x) = y$. Then it is possible that there is no $\alpha \in \bigcup\{\Theta_{i,m}: m \geqslant h(j)\}$ such that $\alpha^{[1]} = y$. Hence the function g_k obtained from g and which corresponds to u_k under the isomorphism of Lemma VI.1.4 may no longer be defined. We thus modify the definition of g_k by fixing the first tree T in the construction which is an $\mathscr{L}_j$-tree for some j such that $u_k \in L_j$, and setting $\mathscr{L}_k(x) = 0$ for all x such that $x \neq \mathrm{lh}(T(\sigma)) + 1$ for any $\sigma \in \mathscr{T}_j$. In order to make sure that g_k is non-degenerate, we impose the additional condition on subtrees that if $T^*(\xi) = T(\sigma)$ then $T^*(\xi * j) \supseteq T(\sigma * j)$; when this extra condition holds, we write $T^* \subseteq^* T$. We note that all subtrees defined in the previous two chapters have this property. In particular, the differentiating trees and splitting trees defined in Chap. VII exist in this setting, and have the same properties as before.

We begin the construction with the identity tree, and use expansion trees, differentiating trees, and splitting trees to force the satisfaction of all requirements. As forcing conditions, we use the set of all recursive trees which are $\mathscr{L}_i$-trees for some $i \in N$, and order the conditions by $\subseteq^*$. By Lemma VI.2.7, all trees constructed in the previous chapter will be uniform and will force the requirement which they were constructed to force. We take expansion subtrees to enable us to use an $\mathscr{L}_i$-tree for a large enough i so that all elements of L mentioned in the requirement are in L_i. Hence the methods of Chap. VII can be applied to prove the following theorem.

1.8 Theorem. $\mathscr{L} = \bigcup\{\mathscr{L}_i : i \in N\} \hookrightarrow^* \mathscr{D}$.

We leave the proof of Theorem 1.8, as well as the remaining results of this section to the reader. A local version of Theorem 1.8 can also be proved. The proof involves the use of e-total trees, together with the observation that if $\{\mathscr{L}_i : i \in N\}$ is $\mathbf{0}^{(2)}$-presentable, then the table $\{\Theta_{i,j} : i, j \in N\}$ for $\mathscr{L}$ together with all the functions mentioned in the definition of table are recursive in $\mathbf{0}^{(2)}$.

1.9 Theorem. *If* $\{\mathscr{L}_i : i \in N\}$ *is* $\mathbf{0}^{(2)}$*-presentable, then* $\mathscr{L} \hookrightarrow^* \{\mathbf{d} : \mathbf{d}^{(2)} = \mathbf{0}^{(2)}\}$.

Theorem 1.8 provides us with a characterization of the countable ideals of $\mathscr{D}$.

1.10 Corollary. *The isomorphism types of countable ideals of* $\mathscr{D}$ *are exactly the isomorphism types of countable usls with least elements.*

1.11 Remarks. Theorem 1.8 was proved by Lachlan and Lebeuf [1976]. The bound produced in Theorem 1.9 was obtained by Richter [1979]. A similar bound for the case of countable distributive lattices had been obtained by Jockusch and Solovay [1977].

1.12–1.17 Exercises

1.12 Let $\mathscr{L}$ be a countable usl with least element and let $\mathbf{d} \in \mathbf{D}$ be given. Show that $\mathbf{d}$ has $2^{\aleph_0}$ distinct $\mathscr{L}$-covers.

1.13 Let $\mathscr{L}$ be a countable usl with least element and let $\mathbf{c} \in \mathbf{D}$ be given. Assume that $\mathscr{L}$ is $\mathbf{c}^{(2)}$-presentable. Show that there is an $\mathscr{L}$-cover $\mathbf{a}$ of $\mathbf{c}$ such that $\mathbf{a} \in \mathbf{L}_2(\mathbf{c}) - \mathbf{L}_1(\mathbf{c})$.

1.14 Let $\mathscr{L}$ be a countable usl with least element, and let $\mathbf{I}$ be a countable ideal of $\mathscr{D}$. Show that $\mathbf{I}$ has $2^{\aleph_0}$ distinct $\mathscr{L}$-covers.

***1.15** Let $\mathbf{b}, \mathbf{d} \in \mathbf{D}$ be given such that $\mathbf{d} \geqslant \mathbf{b}^{(2)}$ and let $\mathscr{L}$ be a $\mathbf{d}$-presentable lattice. Show that $\mathscr{L} \hookrightarrow^* \mathscr{D}[\mathbf{b}, \mathbf{d}]$.

1.16 Let $\mathscr{L} = \bigcup\{\mathscr{L}_i : i \in N\}$ be a $\mathbf{0}^{(3)}$-presentable usl. Show that $\mathscr{L} \hookrightarrow^* \mathscr{D}[\mathbf{0}, \mathbf{0}^{(2)}]$. (*Hint*: Note that by Appendix B.3.28, the usl table $\{\Theta_{i,j} : i \leqslant i_0 \,\&\, j \in N\}$ can be used as part of the table for any countable $\mathscr{L}$ with least element for which $\mathscr{L}_{i_0} \subseteq \mathscr{L}$. Since $\{\mathscr{L}_i : i \in N\}$ is Δ_2^0 over $\mathbf{0}^{(2)}$, there is a sequence of usls $\{\mathscr{L}_{i,j} : i, j \in N\}$ such that for each $i \in N$, $\mathscr{L}_i = \lim_j \mathscr{L}_{i,j}$, and this sequence is recursive in $\mathbf{0}^{(2)}$. Construct the function g directly such that $\mathscr{D}[\mathbf{0}, \mathbf{g}]$ is the desired initial segment, i.e., do not use the forcing approach. Use priorities to change the trees you are working on whenever the approximation to $\mathscr{L}$ changes. Thus if T_{i+1} is a subtree of T_i chosen respectively for $\mathscr{L}_{i+1,j}$ and $\mathscr{L}_{i,j}$ and $\mathscr{L}_{i+1,j+1} \neq \mathscr{L}_{i+1,j}$ but $\mathscr{L}_{i,j+1} = \mathscr{L}_{i,j}$, T_i is kept but T_{i+1} is redefined.)

1.17 (Simpson) Let $\{\mathbf{c}_i : i \in N\}$ be a sequence of degrees such that $\mathbf{0}^{(2)} = \mathbf{c}_0 < \mathbf{c}_1 < \cdots$. Show that there is a sequence $\mathbf{E} = \{\mathbf{e}_i : i \in N\}$ such that $\mathbf{0} = \mathbf{e}_0 < \mathbf{e}_1 < \cdots$, $\mathbf{E}$ is an initial segment of $\mathscr{D}$, and for all $i \in N$, $\mathbf{e}_i^{(2)} = \mathbf{c}_i$. (*Hint*: Use a tree of trees to construct the initial segment, after adjoining a greatest element to the original sequence. The uniformity allows us to pick $\mathbf{e}_1$, then $\mathbf{e}_2$ independently, and continue in this way. The paths through the tree are chosen as in the proof of V.2.12.)

2. Jump Preserving Isomorphisms

We now turn our attention to questions about isomorphisms of cones of degrees, a special case of which is the characterization of the automorphisms of $\mathscr{D}$. A complete characterization of such isomorphisms and automorphisms has not yet been found. But Theorem 1.8 can be used to obtain partial answers to these questions.

The first question which we consider asks: For which $\mathbf{a}, \mathbf{b} \in \mathbf{D}$ is $\mathscr{D}[\mathbf{a}, \infty) \simeq \mathscr{D}[\mathbf{b}, \infty)$? A special case is the Homogeneity Problem posed by Rogers [1967]. Rogers noticed that most theorems proved by constructing degrees with various properties relativize to theorems about $\mathscr{D}[\mathbf{a}, \infty)$ for every $\mathbf{a} \in \mathbf{D}$. His *Homogeneity Problem* was: Is it true that for all $\mathbf{a} \in \mathbf{D}$, $\mathscr{D} \simeq \mathscr{D}[\mathbf{a}, \infty)$? If the assertion of the Homogeneity Problem is true, then relativized versions of theorems would follow as corollaries of the isomorphism theorem, and would not require new proofs. Rogers asked the same question for $\mathscr{D}'$, known as the *Strong Homogeneity Problem*: Is it true that for all $\mathbf{a} \in \mathbf{D}$, $\mathscr{D}' \simeq \mathscr{D}'[\mathbf{a}, \infty)$?

We will show, in this section, that the assertion made in the Strong Homogeneity Problem is false. The key to the proof, strangely enough, is the same relativization phenomenon which motivated the problem. The Homogeneity Problem will be solved in Section 5.

We begin with a theorem which is proved in Appendix A.1.1. Another proof can be given which uses only distributive lattices. Thus all the results of this section can be obtained from the characterization of countable distributive ideals of $\mathscr{D}$ in place of Theorem 1.8.

2.1 Theorem. *Let* $\mathbf{a} \in \mathbf{D}$ *be given. Then there is a countable lattice* $\mathscr{L}_a = \langle L, \leqslant, \vee, \wedge \rangle$ *such that* $\langle L, \leqslant \rangle$ *has a presentation of degree* $\mathbf{a}$, *and every presentation of* $\langle L, \leqslant \rangle$ *has degree* $\geqslant \mathbf{a}$.

The results which deal with jump preserving isomorphisms involve the study of bounded initial segments of $\mathscr{D}[\mathbf{b}, \infty)$ for $\mathbf{b} \in \mathbf{D}$. The next theorem tells us about degrees of presentations of segments of degrees.

2.2 Theorem. *Let* $\mathbf{a}, \mathbf{b} \in \mathbf{D}$ *be given such that* $\mathbf{b} \leqslant \mathbf{a}$. *Then* $\mathscr{D}[\mathbf{b}, \mathbf{a}]$ *is* $\mathbf{a}^{(3)}$*-presentable.*

Proof. It is easily checked by writing down the natural formula that for any set A of degree $\mathbf{a}$, $\{\langle i,j \rangle \in N^2 : \Phi_i^A \leqslant_T \Phi_j^A\} \in \Sigma_3^A$. Fix a set B of degree $\mathbf{b}$. Let $h: N \to N$ be an enumeration, in order of magnitude, of the set C which is determined by the following conditions: We place $i \in C$ if all the conditions hold.

(1) $i \in \mathrm{Tot}(A)$.

(2) $B \leqslant_T \Phi_i^A$.

(3) $\forall j \leqslant i(j \in \mathrm{Tot}(A) \leftrightarrow \Phi_j^A \not\equiv_T \Phi_i^A)$.

By Lemma IV.3.2, $\mathrm{Tot}(A) \leqslant_T A^{(2)}$. Thus by the previous paragraph, $h \in \Delta_4^A$, so $\mathbf{h} \leqslant \mathbf{a}^{(3)}$ by Post's Theorem. Thus we can give a presentation of $\mathscr{D}[\mathbf{b}, \mathbf{a}]$ of degree $\leqslant \mathbf{a}^{(3)}$ as follows: The domain of the presentation is N, and for $i, j \in N$, we define $i \leqslant^* j$ if and only if $\Phi_{h(i)}^A \leqslant_T \Phi_{h(j)}^A$. We now see that $\langle N, \leqslant^* \rangle \simeq \mathscr{D}[\mathbf{b}, \mathbf{a}]$. ▯

Shore [1981] shows that for all $\mathbf{a} \geqslant \mathbf{0}'$, every presentation of $\mathscr{D}[\mathbf{0}, \mathbf{a}]$ has degree $\geqslant \mathbf{a}^{(3)}$, so Theorem 2.2 is best possible. Theorem 2.2 combines with Theorem 2.1 to yield the following corollary.

2.3 Corollary. *If* $\mathscr{D}'[\mathbf{a}, \infty) \simeq \mathscr{D}'[\mathbf{b}, \infty)$ *then* $\mathbf{a}^{(2)} \leqslant \mathbf{b}^{(3)}$.

Proof. Let $\mathscr{L}_{\mathbf{a}^{(2)}}$ be the lattice for $\mathbf{a}^{(2)}$ specified in Theorem 2.1. By Exercise 1.13, there is a $\mathbf{c} \in \mathbf{D}$ such that $\mathbf{c} > \mathbf{a}$ $\mathscr{D}[\mathbf{a}, \mathbf{c}] \simeq \mathscr{L}_{\mathbf{a}^{(2)}}$ and $\mathbf{c} \in \mathbf{L}_2(\mathbf{a})$. Hence $\mathbf{a}^{(2)} = \mathbf{c}^{(2)}$. Let $\mathbf{f}: \mathscr{D}'[\mathbf{a}, \infty) \to \mathscr{D}'[\mathbf{b}, \infty)$ be a jump preserving isomorphism. Then $\mathbf{f(a)} = \mathbf{b}$, and

$$(\mathbf{f(c)})^{(2)} = \mathbf{f}(\mathbf{c}^{(2)}) = \mathbf{f}(\mathbf{a}^{(2)}) = (\mathbf{f(a)})^{(2)} = \mathbf{b}^{(2)}.$$

By Theorem 2.2, $\mathscr{D}[\mathbf{b}, \mathbf{f(c)}]$ is $(\mathbf{f(c)})^{(3)} = \mathbf{b}^{(3)}$-presentable. Hence $\mathscr{L}_{\mathbf{a}^{(2)}}$ is $\mathbf{b}^{(3)}$-presentable. By Theorem 2.1, we must have $\mathbf{b}^{(3)} \geqslant \mathbf{a}^{(2)}$. ▯

The result of Corollary 2.3 is the best known except in special cases. Hence there is no known characterization of the degrees $\mathbf{a}, \mathbf{b}$ such that $\mathscr{D}'[\mathbf{a}, \infty) \simeq \mathscr{D}'[\mathbf{b}, \infty)$. It is not even known whether there are degrees $\mathbf{a} \neq \mathbf{b}$ with $\mathscr{D}[\mathbf{a}, \infty) \simeq \mathscr{D}[\mathbf{b}, \infty)$. Corollary 2.3 is sufficiently strong, however, to provide a negative solution to the Strong Homogeneity Problem.

2.4 Corollary. $\mathscr{D}' \not\simeq \mathscr{D}'[\mathbf{0}^{(2)}, \infty)$.

Proof. We obtain a contradiction under the assumption that $\mathscr{D}' \simeq \mathscr{D}'[\mathbf{0}^{(2)}, \infty)$. By Corollary 2.3, $\mathbf{0}^{(4)} = (\mathbf{0}^{(2)})^{(2)} \leqslant \mathbf{0}^{(3)}$, contradicting Theorem III.2.3(ii). ▯

Corollary 2.3 can also be used to produce other results asserting the non-existence of jump preserving isomorphisms. One such result is Corollary 2.5. Others appear in the exercises at the end of this section.

2.5 Corollary. *For all* $n > 0$, $\mathscr{D}' \not\simeq \mathscr{D}'[\mathbf{0}^{(n)}, \infty)$.

Proof. If $n \geqslant 2$, proceed as in Corollary 2.4. Let $n = 1$ and assume that $\mathbf{f}: \mathscr{D}' \to \mathscr{D}'[\mathbf{0}', \infty)$ is an isomorphism, for the sake of obtaining a contradiction. Then $\mathbf{f}^2 = \mathbf{f} \circ \mathbf{f}: \mathscr{D}' \to \mathscr{D}'[\mathbf{0}^{(2)}, \infty)$ is an isomorphism, contradicting Corollary 2.4. ▯

One of the most natural problems to consider in the study of $\mathscr{D}$ and $\mathscr{D}'$ as algebraic structures is the problem of characterizing the automorphisms of $\mathscr{D}$ and

$\mathcal{D}'$. Again, Rogers [1967] was the first to draw attention to this problem, asking whether $\mathcal{D}$ and $\mathcal{D}'$ have any automorphisms other than the identity. This question is still unanswered. Two directions have been taken towards resolving this problem. The first is to show that any automorphism of $\mathcal{D}$ or $\mathcal{D}'$ has many fixed points. We show this for $\mathcal{D}'$ now and for $\mathcal{D}$ in Sect. 5. The key result is, again, Corollary 2.3. The other direction has already been discussed, namely, the study of automorphism bases for $\mathcal{D}$ and $\mathcal{D}'$. If one could show, for instance, that $\mathcal{D}$ or $\mathcal{D}'$ has a countable automorphism base, then one could reduce the upper bound on the number of automorphisms of $\mathcal{D}$ or $\mathcal{D}'$ from 2^c to $2^{\aleph_0}$, where $c = 2^{\aleph_0}$.

2.6 Corollary. *Let* $\mathbf{f}$ *be an automorphism of* $\mathcal{D}'$. *Then* $\mathbf{f(a)} = \mathbf{a}$ *for all* $\mathbf{a} \geqslant \mathbf{0}^{(3)}$.

Proof. Let $\mathbf{c} \geqslant \mathbf{0}^{(3)}$ be given, and let $\mathbf{d} = \mathbf{f(c)}$. By Corollary III.4.5, there is an $\mathbf{a} \in \mathbf{D}$ such that $\mathbf{a}^{(3)} = \mathbf{a} \cup \mathbf{0}^{(3)} = \mathbf{c}$. Fix such an $\mathbf{a}$, and let $\mathbf{b} = \mathbf{f(a)}$. Then

$$\mathbf{b}^{(3)} = (\mathbf{f(a)})^{(3)} = \mathbf{f}(\mathbf{a}^{(3)}) = \mathbf{f}(\mathbf{a} \cup \mathbf{0}^{(3)}) = \mathbf{f(a)} \cup \mathbf{f}(\mathbf{0}^{(3)}) = \mathbf{b} \cup \mathbf{0}^{(3)}.$$

Furthermore, $\mathbf{b}^{(3)} = \mathbf{f}(\mathbf{a}^{(3)}) = \mathbf{f(c)} = \mathbf{d}$. Since $\mathcal{D}'[\mathbf{a}, \infty)$ and $\mathcal{D}'[\mathbf{b}, \infty)$ must be isomorphic, by Corollary 2.3, $\mathbf{b}^{(2)} \leqslant \mathbf{a}^{(3)}$. Hence $\mathbf{d} = \mathbf{b} \cup \mathbf{0}^{(3)} \leqslant \mathbf{a}^{(3)} = \mathbf{c}$. By symmetry using $\mathbf{f}^{-1}$ in place of $\mathbf{f}$, we conclude that $\mathbf{c} \leqslant \mathbf{d}$. Hence $\mathbf{c} = \mathbf{d} = \mathbf{f(c)}$. ▯

2.7 Remarks. Feiner [1970] solved the Strong Homogeneity Problem. Jockusch found another method to resolve the Strong Homogeneity Problem, and the proof we give is along the lines of Yates [1972]. Yates used a weaker version of Theorem 2.2; the version we present was proved by Richter [1979]. Richter proved Theorem 2.1 and Theorem 2.2 in order to prove Corollary 2.6. A weaker version of Corollary 2.6 which asserted that all degrees $\geqslant \mathbf{0}^{(4)}$ are fixed by all jump preserving automorphisms was proved by Jockusch and Solovay [1977]. Epstein [1979] found a proof of Theorem 2.1 using only distributive lattices, thus allowing the use of the Lachlan [1968] characterization of countable distributive initial segments of $\mathcal{D}$ to obtain the corollaries of this section.

2.8–2.12 Exercises

***2.8** Show that for all $A \subseteq N$, $\{\langle i, j\rangle \in N^2 : \Phi_i^A \leqslant_T \Phi_j^A\} \in \Sigma_3^A$.

2.9 Show that for all $\mathbf{a} \in \mathbf{D}$, if $\mathbf{a}^{(2)} > \mathbf{0}^{(3)}$ then $\mathcal{D}' \not\simeq \mathcal{D}'[\mathbf{a}, \infty)$.

2.10 Show that for all $\mathbf{a} \in \mathbf{D}$, if $\mathbf{a} \geqslant \mathbf{0}'$ then $\mathcal{D}' \not\simeq \mathcal{D}'[\mathbf{a}, \infty)$.

2.11 Show that for all $\mathbf{a} \in \mathbf{D}$, if $\mathbf{a} \in \mathbf{H}_1$ then $\mathcal{D}' \not\simeq \mathcal{D}'[\mathbf{a}, \infty)$.

2.12 Let $\mathbf{d} \in \mathbf{D}$ be given. Show that the following conditions imply that $\mathcal{D}'[\mathbf{a}, \infty) \not\simeq \mathcal{D}'[\mathbf{d}, \infty)$.

(i) $\mathbf{a}^{(2)} > \mathbf{d}^{(3)}$.

(ii) $\mathbf{a} \geqslant \mathbf{d}'$.

(iii) $\mathbf{a} \in \mathbf{H_n(d)}$ for some $n \in N$.

3. *The Degree of* Th($\mathscr{D}$)

We continue our study of Th($\mathscr{D}$). In studying a theory, we first try to determine whether or not it is decidable. Corollary VI.4.6 asserts that Th($\mathscr{D}$) is undecidable. There are now two directions to pursue. One direction is to find natural decidable classes of Th($\mathscr{D}$). This was done in Theorem VII.4.4, and a bound on the complexity of decidable classes of sentences was provided by Corollary VII.4.6. Thus $\mathrm{Th}(\mathscr{D}) \cap \forall_2$ is decidable, but $\mathrm{Th}(\mathscr{D}) \cap \forall_3$ is undecidable. The other direction is to determine how complicated Th($\mathscr{D}$) is by determining its degree. We pursue this direction both for Th($\mathscr{D}$) and the theories of various subusls of $\mathscr{D}$ in this section. The main result states that Th($\mathscr{D}$) has the same degree as second order arithmetic.

Several languages will be referred to in the proof of the main theorem. To talk about $\mathscr{D}$, we use the language $\mathscr{L}_b$, the language of the predicate calculus with one binary relation symbol $\leqslant$ (interpreted as the partial ordering of **D**). An intermediate second order language $\mathscr{L}_I$ will be used to talk about *second order lattices*, i.e., structures $\mathscr{P} = \langle P, \mathscr{I}_P, \leqslant, \in\rangle$, where $\mathscr{I}_P$ is the set of all ideals of $\mathscr{P}$. This language is an expansion of $\mathscr{L}_b$, having second order quantifiers which will range over the countable ideals of a second order lattice used to interpret the language, and an additional binary relation symbol $\in$ (which is to be interpreted as *is an element of* over $P \times 2^P$). Finally, we will use $\mathscr{L}_a$ to talk about the structure $\mathscr{N} = \langle N, 2^N, +, \times, \leqslant_a, \in\rangle$, and call Th($\mathscr{N}$) in the language $\mathscr{L}_a$, *second order arithmetic*. $\mathscr{L}_a$ will be the language of the predicate calculus together with two ternary first order relation symbols $+$ and $\times$ (to be interpreted, respectively, as addition and multiplication over N), a binary relation symbol $\leqslant_a$ (to be interpreted as the ordering of N), together with second order quantifiers ranging over 2^N and a binary relation symbol $\in$ (to be interpreted over $N \times 2^N$ as *is an element of*).

Part of the translation of second order arithmetic into Th($\mathscr{N}$) is carried out in Appendix A.3. We now state Theorem A.3.4, and refer the reader to the appendix for a proof.

3.1 Theorem. *There is a sentence σ of $\mathscr{L}_I$ and an effective translation taking any sentence θ of second order arithmetic into the sentence θ_I of $\mathscr{L}_I$ such that*

$$\mathscr{N} \models \theta \Leftrightarrow \forall \mathscr{P}(\mathscr{P} \models \sigma \rightarrow \mathscr{P} \models \theta_I).$$

Under this translation, the integers are interpreted by an $\exists_2$-definable subset $P^* \subseteq P$ and $\leqslant_P$ is interpreted by an $\exists_2$-formula of $\mathscr{L}_b$. Furthermore, there is a recursively presentable lattice which satisfies σ and whose corresponding P^* is recursive.

The sentence σ whose existence is asserted in Theorem 3.1 allows us to pick out lattices which faithfully interpret arithmetic. The lattices which we will be interested in are of the form $\mathscr{D}[\mathbf{0}, \mathbf{d}]$ for some $\mathbf{d} \in \mathbf{D}$. Hence it will be convenient to relate the property of being a faithful translation of arithmetic to degrees.

3.2 Definition. Let σ be the sentence described in Theorem 3.1. We say that the second order lattice $\mathscr{P} = \langle P, \mathscr{I}_P, \leqslant, \in\rangle$ *codes a standard model of arithmetic* if $\mathscr{P} \models \sigma$. Given any subusl of $\mathscr{D}$, we identify it with the corresponding second order structure interpreting $\mathscr{L}_I$ where second order quantifiers range over ideals. We say that $\mathbf{d} \in \mathbf{D}$ *codes a standard model of arithmetic* if $\mathscr{D}[\mathbf{0}, \mathbf{d}]$ codes a standard model of arithmetic.

The final step in the translation of second order arithmetic into Th($\mathscr{D}$) is to pass from $\mathscr{L}_I$ to $\mathscr{L}_b$, and to find an effective correspondence between a given sentence θ of $\mathscr{L}_I$ with a sentence θ^* of $\mathscr{L}_b$ such that $\mathscr{D} \models \theta_b$ if and only if every second order lattice which satisfies σ also satisfies θ. This translation relies on the characterization of the countable ideals of $\mathscr{D}$ (Theorem 1.8 and Theorem 1.9) and the Exact Pair Theorem (Theorem II.4.8 and Exercise III.3.13). The local versions of these theorems are used to obtain results about some subusls of $\mathscr{D}$. We restate these results here for the convenience of the reader.

3.3 Theorem. *Let $\mathscr{L}$ be a countable lattice. Then $\mathscr{L} \hookrightarrow^* \mathscr{D}$. Furthermore, if $\mathscr{L}$ is recursively presentable, then $\mathscr{L} \hookrightarrow^* \{\mathbf{d} : \mathbf{d}^{(2)} = \mathbf{0}^{(2)}\}$.*

3.4 Theorem. *Let $\mathbf{c} \in \mathbf{D}$, a set C of degree $\mathbf{c}$, a function $f: N \to N$, and an ideal $\mathbf{I}$ of $\mathscr{D}[\mathbf{0}, \mathbf{c}]$ be given such that for all $\mathbf{d} \in \mathbf{D}$,*

$$\mathbf{d} \in \mathbf{I} \Leftrightarrow \exists e \in N(\mathbf{d} \leqslant \Phi^C_{\mathbf{f}(\mathbf{e})}).$$

Then there are $\mathbf{a}, \mathbf{b} \in \mathbf{D}$ such that $\mathbf{a}, \mathbf{b} \leqslant \mathbf{c}' \cup \mathbf{f}$, and for all $\mathbf{d} \in \mathbf{D}$,

$$\mathbf{d} \in \mathbf{I} \Leftrightarrow \mathbf{d} \leqslant \mathbf{a} \,\&\, \mathbf{d} \leqslant \mathbf{b}.$$

We have now laid the groundwork for determining the degree of Th($\mathscr{D}$).

3.5 Theorem. Th($\mathscr{D}$) *and* Th($\mathscr{N}$) *have the same degree.*

Proof. We note that for all $A \subseteq N$, $\{\langle i,j \rangle \in N^2 : \Phi_i^A \leqslant_T \Phi_j^A\} \in \Sigma_3^A$. Given a sentence θ of $\mathscr{L}_b$, form the sentence θ_a of $\mathscr{L}_a$ by letting the quantifiers in θ become second order quantifiers (ranging over 2^N) in θ_a, and replacing all occurrences of $\leqslant$ in θ with $\leqslant_T$ (Turing reducibility), a definable relation over $\mathscr{N}$ in the language $\mathscr{L}_a$. Then for all $\theta \in \mathscr{L}_b$, $\mathscr{D} \models \theta \Leftrightarrow \mathscr{N} \models \theta_a$.

Conversely, let θ be a sentence of $\mathscr{L}_I$ and let σ be as in Theorem 3.1. Let d be a variable of $\mathscr{L}_b$. Form the formula θ_1 from θ by restricting all first order variables occurring in θ to elements $\leqslant d$, and then replacing all second order quantifiers $\exists I$ and $\forall I$ of $\mathscr{L}_I$ with first order quantifiers $\exists y \exists z$ and $\forall y \forall z$ of $\mathscr{L}_b$ respectively, and replacing all occurrences of the formula $x \in I$ with $x \leqslant y \,\&\, x \leqslant z$. We now let θ^* be obtained from θ_1 by letting θ^* be the sentence $\forall d(\sigma_1 \to \theta_1)$. By Theorem 3.1, it suffices to show that for all sentences θ of $\mathscr{L}_I$ and all second order lattices $\mathscr{P}$ satisfying σ, $\mathscr{P} \models \theta \Leftrightarrow \mathscr{D} \models \theta^*$.

Let $\mathscr{P} = \langle P, \mathscr{I}_P, \leqslant, \in \rangle$ be a second order lattice satisfying σ. By Theorem 3.3, there is a degree $\mathbf{d}$ such that $\langle P, \leqslant \rangle \simeq \mathscr{D}[\mathbf{0}, \mathbf{d}]$. Hence $\mathscr{P} \models \theta \Leftrightarrow \mathscr{D}[\mathbf{0}, \mathbf{d}] \models \theta$. But by Theorem 3.4, $\mathscr{D}[\mathbf{0}, \mathbf{d}] \models \theta \Leftrightarrow \mathscr{D} \models \theta^*$. ▯

The techniques used in Theorem 3.5 can be used to characterize the degrees of certain subusls of $\mathscr{D}$ such as $\mathscr{D}_{\mathbf{arith}}$, the usl of arithmetical degrees introduced in Chap. V.5.1. The local facts mentioned in Theorems 3.1, 3.3 and 3.4 will come into play in these proofs.

3.6 Definition. Let $\mathbf{C} \subseteq \mathbf{D}$ be given. Then $\mathbf{C}$ is *closed under jump* if whenever $\mathbf{c} \in \mathbf{C}$, then it is also the case that $\mathbf{c}' \in \mathbf{C}$. $\mathbf{C}$ is a *jump ideal* if $\mathbf{C}$ is an ideal of $\mathscr{D}$ which is closed under jump.

3.7 Definition. Let $\mathbf{d} \in \mathbf{D}$ be given such that $\mathbf{d}$ codes a standard model of arithmetic. The pair $\langle \mathbf{a}, \mathbf{b} \rangle \in \mathbf{D}$ *codes* $W \subseteq N$ *for* $\mathbf{d}$ if $\forall n(n \in W \leftrightarrow \mathbf{d_n} \leqslant \mathbf{a} \,\&\, \mathbf{d_n} \leqslant \mathbf{b})$. (Here $\mathbf{d_n}$ is the interpretation of the integer n under the coding of arithmetic into $\mathscr{D}[\mathbf{0}, \mathbf{d}]$.)

Since the embedding in Theorem 3.3 can be taken into the arithmetical degrees, a careful examination of the proof of Theorem 3.5 will yield the following result.

3.8 Remark. Let $\mathbf{C}$ be a jump ideal of $\mathscr{D}$ such that for all $\mathbf{d} \in \mathbf{C}$ which code standard models of arithmetic, the following conditions hold:

(1) If $\mathbf{a}, \mathbf{b} \in \mathbf{C}$ and $\langle \mathbf{a}, \mathbf{b} \rangle$ codes W for $\mathbf{d}$, then $\mathbf{W} \in \mathbf{C}$.

(2) Given $V \subseteq N$ such that $\mathbf{V} \in \mathbf{C}$, then there are $\mathbf{a}, \mathbf{b} \in \mathbf{C}$ such that $\langle \mathbf{a}, \mathbf{b} \rangle$ codes V for $\mathbf{d}$.

Let $\mathscr{C} = \{A \subseteq N : \mathbf{A} \in \mathbf{C}\}$. (Thus by (1) and (2), the sets in $\mathscr{C}$ are exactly those which are coded by pairs $\langle \mathbf{a}, \mathbf{b} \rangle$ for $\mathbf{d}$.) Then $\mathrm{Th}(\langle N, \mathscr{C}, +, \times, \leqslant_a, \in \rangle)$ and $\mathrm{Th}(\langle \mathbf{C}, \leqslant \rangle)$ have the same degree.

In order to apply Remark 3.8, we must be able to determine where exact pairs can be found. Theorem 3.4 gives some information, but we still must be able to determine the degrees of the functions f which are mentioned in Theorem 3.4. We begin to obtain such information after the next definition.

3.9 Definition. Let $\mathbf{d} \in \mathbf{D}$ code a standard model for arithmetic. We then let $\mathbf{D}_\mathbf{d}^* = \{\mathbf{d_j} : j \in N\}$ denote the interpretation of N in $\mathscr{D}[\mathbf{0}, \mathbf{d}]$, with $\mathbf{d_j}$ interpreting the integer j.

3.10 Remark. In order to find the degree of f, we must translate sentences from $\mathscr{L}_b$ to the language we have used for recursion theory. There is a direct translation when we are working within $\mathscr{D}[\mathbf{0}, \mathbf{d}]$ and B is a set of degree $\mathbf{d}$. We take a sentence σ of $\mathscr{L}_b$, let the variables range over N, and replace all subformulas of the form $x \leqslant y$ with $\Phi_x^B \leqslant_T \Phi_y^B$. Since $\leqslant_T$ is Σ_3^B definable over $\mathscr{D}[\mathbf{0}, \mathbf{d}]$, an $\exists_n$-sentence of $\mathscr{L}_b$ is thus faithfully translated into a sentence which is in Σ_{n+3}^B.

3.11 Lemma. *Let* $\mathbf{d} \in \mathbf{D}$ *code a standard model of arithmetic and let* B *be a set of degree* $\mathbf{d}$. *Then there is a function* $f\colon N \to N$ *such that* $\mathbf{D}_\mathbf{d}^* = \{\mathbf{\Phi}_{\mathbf{f(j)}}^{\mathbf{B}} : j \in N\}$ *and for which for all* $j \in N$, $\Phi_{f(j)}^B$ *has degree* $\mathbf{d_j}$ *and* $\mathbf{f} \leqslant \mathbf{d}^{(5)}$.

Proof. Let $E \subseteq N$ be defined by $e \in E \leftrightarrow \mathbf{\Phi}_\mathbf{e}^\mathbf{B} \in \mathbf{D}_\mathbf{d}^*$. By Theorem 3.1, $\mathbf{D}_\mathbf{d}^*$ has an $\exists_2$ definition in $\mathscr{L}_b$, so by Remark 3.10, $E \in \Sigma_5^B$. Define $E_1 \subseteq E$ by

$$e \in E_1 \leftrightarrow e \in E \,\&\, \forall i \in N(\Phi_i^B \equiv_T \Phi_e^B \to e \leqslant i).$$

Then $E_1 \in \Sigma_5^B$ and E_1 has exactly one index for each $\mathbf{d_j} \in \mathbf{D}_\mathbf{d}^*$. Define $f(j)$ to be the index for $\mathbf{d_j}$ in E_1 for each $j \in N$.

By Theorem 3.1, $\leqslant_a$ is interpreted by an $\exists_2$-definable relation in $\mathscr{L}_b$, over the universe $\mathbf{D}_\mathbf{d}^*$; we let $\leqslant^*$ be this relation. By Remark 3.10, there is a Σ_5^B-definable relation $\leqslant_1$ which interprets $\leqslant^*$ on domain E_1. By Theorem III.2.6(i), both E_1 and $\leqslant_1$ are recursively enumerable in a set C of degree $\mathbf{d}^{(4)}$. Let $h: N \to N^2$ be a function of degree $\leqslant \mathbf{d}^{(4)}$ with range $\leqslant_1$, and let $\leqslant_1^s$ be a maximal linear ordering which extends $\leqslant_1^{s-1}$ and all of whose ordering relationships are specified by $\{h(i): i \leqslant s\}$. If several choices for $\leqslant_1^s$ are possible, we choose the one whose domain is the smallest lexicographically. Then $\leqslant_1^s$ is recursive uniformly in $\mathbf{d}^{(4)}$. Note that for all $j \in N, f(j)$ is the jth element of E_1 under the ordering $\lim_s \leqslant_1^s$. Hence by the Limit Lemma, $\mathbf{f} \leqslant \mathbf{h}' \leqslant \mathbf{d}^{(5)}$. ▯

We are now ready to characterize the sets of integers which can be coded by exact pairs lying in a jump ideal of $\mathscr{D}$.

3.12 Lemma. *Let $W \subseteq N$ and $\mathbf{a}, \mathbf{b}, \mathbf{d} \in \mathbf{D}$ be given such that $\mathbf{d}$ codes a standard model of arithmetic. Let $f: N \to N$ be given such that $\mathbf{D}_\mathbf{d}^* = \{\Phi_{f(j)}^B : j \in N\}$ and for all $j \in N$, $\mathbf{\Phi}_{\mathbf{f(j)}}^{\mathbf{B}} = \mathbf{d}_\mathbf{j}$ and $\mathbf{f} \leqslant \mathbf{d}^{(5)}$. Then*:

(i) *If $\langle \mathbf{a}, \mathbf{b} \rangle$ codes W for $\mathbf{d}$, then $\mathbf{W} \leqslant \mathbf{f} \cup (\mathbf{a} \cup \mathbf{b} \cup \mathbf{d})^{(3)}$.*

(ii) *If $V \subseteq N$ is given such that $V \leqslant_T W$ and $\mathbf{f} \cup \mathbf{d}' \leqslant \mathbf{W}$ then there are $\mathbf{a}, \mathbf{b} \leqslant \mathbf{W}$ such that $\langle \mathbf{a}, \mathbf{b} \rangle$ codes V for $\mathbf{d}$.*

Proof. (i) Fix A, B, $D \subseteq N$ having degree $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{d}$ respectively. Then $e \in W \leftrightarrow \Phi_{f(e)}^D \leqslant_T A \,\&\, \Phi_{f(e)}^D \leqslant_T B$. Hence to decide whether $e \in W$, we must compute $f(e)$ and then ask a $\Sigma_3^{A \oplus B \oplus D}$ question, which can be answered by an oracle of degree $(\mathbf{a} \cup \mathbf{b} \cup \mathbf{d})^{(3)}$. We now see that $\mathbf{W} \leqslant \mathbf{f} \cup (\mathbf{a} \cup \mathbf{b} \cup \mathbf{d})^{(3)}$.

(ii) Fix a set D of degree $\mathbf{d}$, and let e_0 be the least integer such that $\Phi_{e_0}^D = N$. Define $g: N \to N$ by

$$g(0) = \begin{cases} e_0 & \text{if } 0 \notin V \\ f(0) & \text{if } 0 \in V \end{cases}$$

and

$$g(n+1) = \begin{cases} g(n) & \text{if } n+1 \notin V \\ \text{the least index for } \Phi_{g(n)}^D \oplus \Phi_{f(n+1)}^D & \text{if } n+1 \in V. \end{cases}$$

Then $g \leqslant_T f \oplus V$ and by Theorem 3.4, there are $\mathbf{a}, \mathbf{b} \in \mathbf{D}$ such that $\langle \mathbf{a}, \mathbf{b} \rangle$ codes V for $\mathbf{d}$ and $\mathbf{a}, \mathbf{b} \leqslant \mathbf{g} \cup \mathbf{d}' \leqslant \mathbf{V} \cup \mathbf{f} \cup \mathbf{d}' \leqslant \mathbf{W}$. For the ideal induced by g contains $\{\mathbf{\Phi}_{\mathbf{f(e)}}^{\mathbf{D}} : e \in V\}$, and since $\mathbf{D}_\mathbf{d}^*$ consists of a set of independent degrees, this ideal does not contain $\mathbf{\Phi}_{\mathbf{f(e)}}^{\mathbf{D}}$ if $e \notin V$. ▯

We can now obtain a sharper version of Theorem 3.5.

3.13 Theorem. *Let $\mathbf{C}$ be a jump ideal of $\mathscr{D}$ and let $\mathscr{C} = \{A \subseteq N : \mathbf{A} \in \mathbf{C}\}$. Then* Th($\langle N, \mathscr{C}, +, \times, \leqslant_a, \in \rangle$ *and* Th($\langle \mathbf{C}, \leqslant \rangle$) *have the same degree.*

Proof. By Remark 3.8, it suffices to show that (1) and (2) are satisfied. Since C is a jump ideal of $\mathscr{D}$, (1) follows from Lemma 3.12(i) and Lemma 3.11, and (2) follows from Lemma 3.12(ii) and Lemma 3.11. ▯

Transfinite iterates of the jump operator are useful for locating degrees of theories.

3.14 Definition. Let $B \subseteq N$ be given. Define $B^{(\omega)} \subseteq N^2$ by $(B^{(\omega)})^{[n]} = B^{(n)}$ for all $n \in N$, and let $B^{(\omega)}$ have degree $\mathbf{b}^{(\omega)}$ where $\mathbf{b}$ is the degree of B. $\mathbf{b}^{(\omega)}$ is called the

ω-jump of $\mathbf{b}$. The jump operation can be iterated through the recursive ordinals in a well-defined way. Thus $\mathbf{b}^{(\omega+\omega)} = (\mathbf{b}^{(\omega)})^{(\omega)}$.

The following corollaries can be drawn from Theorem 3.13.

3.15 Corollary. Th($\mathscr{D}_{\mathbf{arith}}$) *has degree* $\mathbf{0}^{(\omega+\omega)}$.

Proof. By Theorem 3.13, Th($\mathscr{D}_{\mathbf{arith}}$) has the same degree as Th($\langle N, \mathscr{A}, +, \times, \leqslant_a, \in\rangle$) where $\mathscr{A} = \{A \subseteq N: A$ is arithmetical$\}$. The latter theory has degree $\mathbf{0}^{(\omega+\omega)}$. ▯

Theorem 3.13 also provides us with another proof of Corollary V.5.14.

3.16 Corollary. $\mathscr{D}_{\mathbf{arith}}$ *and* $\mathscr{D}$ *are not elementarily equivalent.*

Proof. Th($\mathscr{D}_{\mathbf{arith}}$) and Th($\mathscr{D}$) have different degrees. ▯

The proof of Theorem 3.5 can be extended to yield information about the definability of various degrees and classes of degrees over $\mathscr{D}$ from a parameter. The parameter appears because we are translating formulas, rather than sentences, between languages. A sample theorem is proved below. This topic is pursued further in the exercises.

3.17 Theorem. *Let* $\mathbf{B} \subseteq \mathbf{C}$ *be given such that* $\mathbf{C}$ *is a jump ideal of* $\mathscr{D}$ *and* $\mathbf{B}$ *is closed downwards and under jump. Let* $\mathscr{C} = \{A \subseteq N: \mathbf{A} \in \mathbf{C}\}$. *Then* $\mathbf{B}$ *is definable over* $\langle \mathbf{C}, \leqslant, \mathbf{0}^{(2)}\rangle$ *if and only if* $\mathbf{B}$ *is definable over* $\mathscr{N}_{\mathscr{C}} = \langle N, \mathscr{C}, +, \times, \leqslant_a, \in\rangle$.

Proof. Note that $\mathbf{B}$ is definable over $\mathscr{N}_{\mathscr{C}}$ if and only if $\mathscr{B} = \{A \subseteq N: \mathbf{A} \in \mathbf{B}\}$ is definable over $\mathscr{N}_{\mathscr{C}}$. Let θ be a formula of $\mathscr{L}_b$ which defines $\mathbf{B}$ over $\langle \mathbf{C}, \leqslant, \mathbf{0}^{(2)}\rangle$. Define the formula θ_a of $\mathscr{L}_a$ by letting the quantifiers in θ become second order quantifiers (ranging over $\mathscr{C}$) in $\mathscr{L}_a$, replacing all occurrences of $\leqslant$ in θ with $\leqslant_T$ (Turing reducibility), and replacing the parameter $\mathbf{0}^{(2)}$ with a set of degree $\mathbf{0}^{(2)}$ (which must be definable in $\mathscr{L}_a$ over $\mathscr{N}_{\mathscr{C}}$ since $\mathscr{C}$ is a jump ideal of $\mathscr{D}$). Then for all $\mathbf{c} \in \mathbf{C}$ and all sets A of degree $\mathbf{c}$, $\langle \mathbf{C}, \leqslant, \mathbf{0}^{(2)}\rangle \models \theta(c) \Leftrightarrow \mathscr{N}_{\mathscr{C}} \models \theta_a(A)$.

Let τ be a formula of $\mathscr{L}_a$ having one free second order variable such that $Y \in \mathscr{B} \Leftrightarrow \mathscr{N}_{\mathscr{C}} \models \tau(Y)$. For each formula θ of $\mathscr{L}_a$, form the formula θ_1 as in the proof of Theorem 3.5, and let σ and σ_1 be as in the proof of Theorem 3.5. Note that τ_1 will have three free variables, d, w_1 and w_2. We claim that $\mathbf{y} \in \mathbf{B} \Leftrightarrow \langle \mathbf{C}, \leqslant, \mathbf{0}^{(2)}\rangle \models \tau^*(\mathbf{y})$ where $\tau^*(y)$ is the formula

$$\forall d \leqslant 0^{(2)}(\sigma_1(d) \to \forall w_1, w_2 \leqslant y \cup 0^{(2)}(\tau_1(d, w_1, w_2))).$$

To verify this claim, first assume that $\mathbf{y} \in \mathbf{B}$. Fix any $\mathbf{d} \leqslant \mathbf{0}^{(2)}$ such that $\langle \mathbf{C}, \mathscr{I}_{\mathbf{C}}, \leqslant, \in\rangle \models \sigma_1(\mathbf{d})$, noting that by Theorem 3.1 and Theorem 3.3, such a $\mathbf{d}$ must exist. Fix degrees $\mathbf{w}_1, \mathbf{w}_2 \leqslant \mathbf{y} \cup \mathbf{0}^{(2)}$. By Lemma 3.11 and Lemma 3.12(i), $\langle \mathbf{w}_1, \mathbf{w}_2\rangle$ codes a set W for $\mathbf{d}$ such that $\mathbf{W} \leqslant \mathbf{0}^{(7)} \cup (\mathbf{y} \cup \mathbf{0}^{(2)})^{(3)} \leqslant \mathbf{y}^{(7)}$. Since $\mathbf{B}$ is closed downwards (anything $\leqslant$ an element of $\mathbf{B}$ is in $\mathbf{B}$) and under jump, $W \in \mathscr{B}$. Thus by the proof of Theorem 3.5, $\langle \mathbf{C}, \leqslant\rangle \models \tau_1(\mathbf{d}, \mathbf{w}_1, \mathbf{w}_2)$, so $\langle \mathbf{C}, \leqslant, \mathbf{0}^{(2)}\rangle \models \tau^*(\mathbf{y})$.

Conversely, assume that $\langle \mathbf{C}, \leqslant, \mathbf{0}^{(2)}\rangle \models \tau^*(\mathbf{y})$. Let $\mathbf{d} \leqslant \mathbf{0}^{(2)}$ be given such that $\mathbf{d} \in \mathbf{GL}_2$, $\langle \mathbf{C}, \mathscr{I}_{\mathbf{C}}, \leqslant, \in\rangle \models \sigma_1(\mathbf{d})$, and the function f specified in Lemma 3.11 is recursive. Such a $\mathbf{d}$ exists by Theorem 3.1 and the proof of Theorem 3.3. By Lemma 3.12(ii), any set $Y \in \mathbf{y}$ is coded by a pair $\langle \mathbf{w}_1, \mathbf{w}_2\rangle$ for $\mathbf{d}$ with $\mathbf{w}_1, \mathbf{w}_2 \leqslant \mathbf{y} \cup \mathbf{0}^{(2)}$.

Thus $\langle \mathbf{C}, \mathscr{I}_{\mathbf{C}}, \leqslant, \in \rangle \models \tau_1(\mathbf{d}, \mathbf{w}_1, \mathbf{w}_2)$ and so by the proof of Theorem 3.5, $\mathscr{N}_{\mathscr{C}} \models \tau(Y)$. Hence $Y \in \mathscr{B}$. ▯

3.18 Remarks. The first results dealing with definability over degree-theoretic structures were obtained by Jockusch and Simpson [1976] who considered the structure $\mathscr{D}'$. Theorem 3.5 was proved by Simpson [1977]. Simpson [1977] also proved some of the corollaries and definability results for $\mathscr{D}'$ which we mentioned in this section for $\mathscr{D}$. Simpson produced a direct coding of Th($\mathscr{N}$) into Th($\mathscr{D}$), bypassing the intermediate languages and theories. Our proofs closely follow those of Nerode and Shore [1979], [1980], and involve the structure $\mathscr{D}$ with an additional parameter. Corollary 3.16 was proved by Jockusch [1973] in a different way (see V.5.14).

We now state some further definability results which are due to Nerode and Shore [1980] and Jockusch and Shore [1983a]. Weaker versions of some of these results can be found in Jockusch and Simpson [1976] and Simpson [1977]. The reader is referred to Rogers [1967] for a definition of the Δ^1_n sets.

3.19 Further Results

(i) *Let* $\mathscr{D}_{\mathbf{arith}}(\mathbf{b})$ *be the degrees arithmetical relative to* $\mathbf{b}$. *Then* $\mathscr{D}_{\mathbf{arith}} \simeq \mathscr{D}_{\mathbf{arith}}(\mathbf{b})$ *only if* $\mathbf{b}$ *is arithmetical.*

(ii) *Let* $\mathbf{A} \subseteq \mathbf{B} \subseteq \mathbf{C}$ *be jump ideals of* $\mathscr{D}$. *Then* $\mathbf{B}$ *is definable over* $\langle \mathbf{C}, \leqslant, \mathbf{A} \rangle$ (*i.e., a predicate picking out the elements of A is introduced into the language*) *if and only if* $\mathscr{B}^*$ *is definable over* $\mathscr{N}_{\mathscr{C}}$ *from a predicate for* $\mathscr{A}^*$.

(iii) *For all* $n \geqslant 1$, *the relation* $\mathbf{a}$ *is* Δ^1_n *in* $\mathbf{b}$ *is definable over both* $\langle \mathbf{D}, \leqslant, \mathbf{D}_{\mathbf{arith}} \rangle$ *and* $\langle \mathbf{D}, \leqslant, \mathbf{0}^{(2)} \rangle$.

(iv) *The relation* $\mathbf{a} = \mathbf{b}^{(\omega)}$ *is definable over both* $\langle \mathbf{D}, \leqslant, \mathbf{D}_{\mathbf{arith}} \rangle$ *and* $\langle \mathbf{D}, \leqslant, \mathbf{0}^{(2)} \rangle$.

(v) *A relation* $R \leqslant (\mathbf{D}[\mathbf{0}^{(\omega)}, \infty))^n$ *is definable over* $\mathscr{N}$ *if and only if it is definable over* $\langle \mathbf{D}, \leqslant, \mathbf{D}_{\mathbf{arith}} \rangle$. *Also,* $\langle \mathbf{D}, \leqslant, \mathbf{D}_{\mathbf{arith}} \rangle$ *can be replaced with* $\langle \mathbf{D}, \leqslant, \mathbf{0}^{(2)} \rangle$ *in this statement.*

(vi) *Let* $\mathbf{C}$ *be a jump ideal of* $\mathscr{D}$ *and let* $\mathbf{b} \in \mathbf{C}$ *be given such that* $\mathbf{b} \geqslant \mathbf{0}^{(7)}$. *Then* $\mathbf{b}$ *is definable over* $\mathscr{N}$ *if and only if* $\mathbf{b}$ *is definable over* $\langle \mathbf{C}, \leqslant, \mathbf{0}^{(2)} \rangle$.

3.20 Remarks. The idea of looking at definability from parameters was also studied by Epstein [1979]. Harrington and Shore [1981] have shown that there is a jump ideal of $\mathscr{D}$ which is captured somewhere between the arithmetical and hyper-arithmetical degrees and which is definable over $\mathscr{D}$. They use this ideal instead of $\mathscr{D}_{\mathbf{arith}}$ to obtain definability results over $\mathscr{D}$; it eliminates the necessity of adding a parameter or set to the language. The methods of proof of that theorem are substantially different from those used in this book, so we will not prove that result. The Harrington and Shore result was improved upon by Jockusch and Shore [1983a] who showed that the set of arithmetical degrees is a jump ideal of $\mathscr{D}$ which is definable over $\mathscr{D}$. These results are useful for giving simpler proofs than we have given for the results presented in the next two sections, and sharpening the statement of some of those theorems. We refer the reader to Shore [1981a] for statements and proofs of some of these results.

4. Elementary Equivalence over $\mathscr{D}'$

We showed in Sect. 2 that the Strong Homogeneity Problem has a negative solution. Other homogeneity problems are considered in the next two sections. In this section, we show that we still get a negative answer if we weaken the problem, and ask for elementary equivalence instead of isomorphism. Thus we show that it is not the case that for all $\mathbf{b} \in \mathbf{D}$, $\mathscr{D}' \equiv \mathscr{D}'[\mathbf{b}, \infty)$. In the next section, we consider homogeneity problems over $\mathscr{D}$ rather than $\mathscr{D}'$.

The idea of the proof is to use definability results, and to show that we can differentiate between theories by using a sentence which asserts that there is an exact pair below the double jump of the least element of the structure which codes a set of degree $\geqslant \mathbf{0}^{(5)}$. This sentence cuts down sharply on the set of $\mathbf{b} \in \mathbf{D}$ such that $\mathscr{D}'[\mathbf{b}, \infty) \equiv \mathscr{D}'$.

4.1 Theorem. *If* $\mathscr{D}' \equiv \mathscr{D}'[\mathbf{b}, \infty)$ *then* $\mathbf{b}^{(2)} \leqslant \mathbf{0}^{(5)}$.

Proof. The language used for $\mathscr{D}'$ is $\mathscr{L}'_b$, an expansion of $\mathscr{L}_b$ by a unary function symbol which is to be interpreted as the jump operator. Assume that $\mathbf{b}^{(2)} \nleqslant \mathbf{0}^{(5)}$. We note that we have a definable constant m in our language which is interpreted by $\mathbf{c}$ in $\mathscr{D}'[\mathbf{c}, \infty)$ for all $\mathbf{c}$. m just satisfies the sentence asserting that it is the least element of the structure. We show that the sentence τ mentioned above differentiates between $\mathscr{D}'$ and $\mathscr{D}'[\mathbf{b}, \infty)$ as the sets which can be coded by exact pairs in these structures are different.

Consider the sentence which asserts that there are degrees $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{d}$ and a set $V \subseteq N$ such that $\mathbf{d}^{(2)} = \mathbf{m}^{(2)}$, $\mathbf{d}$ codes a standard model of arithmetic, $\mathbf{a}_1, \mathbf{a}_2 \leqslant \mathbf{m}^{(2)}$, $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle$ codes V for $\mathbf{d}$, and $\mathbf{V} \nleqslant \mathbf{0}^{(5)}$. This sentence can be written as a sentence of $\mathscr{L}_a$ which is faithfully interpreted over $\mathscr{N}$. Hence by the proof of Theorem 3.5, this sentence is interpreted faithfully by a sentence τ of $\mathscr{L}'_b$. By Lemma 3.11 and Lemma 3.12(i), $\mathscr{D}' \nvDash \tau$. By Theorem 3.1, a relativized version of Theorem 3.3, and Lemma 3.12(ii), if $V \subseteq N$ and $\mathbf{V} \leqslant \mathbf{b}^{(2)}$, then there is a pair $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle$ and a degree $\mathbf{d} \geqslant \mathbf{b}$ such that $\mathbf{d}^{(2)} = \mathbf{b}^{(2)}$, $\mathbf{d}$ codes a standard model of arithmetic, $\mathbf{b} \leqslant \mathbf{a}_j \leqslant \mathbf{b}^{(2)}$ for $j = 1, 2$ and $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle$ codes V for $\mathbf{d}$. Pick such a set V of degree $\mathbf{b}^{(2)}$. Then $\mathbf{V} \nleqslant \mathbf{0}^{(5)}$, so $\mathscr{D}'[\mathbf{b}, \infty) \models \tau$. ∎

The following corollary is now immediate.

4.2 Corollary. *There is a degree* $\mathbf{b} \in \mathbf{D}$ *such that* $\mathscr{D}' \not\equiv \mathscr{D}'[\mathbf{b}, \infty)$.

The methods of Chap. XII will allow the condition $\mathbf{b}^{(2)} \leqslant \mathbf{0}^{(5)}$ to be replaced with the condition $\mathbf{b}^{(3)} = \mathbf{0}^{(3)}$. The proof of Theorem 4.1 makes heavy use of the definability of the degree $\mathbf{0}$ over $\mathscr{N}$. Shore [1981a] proves a result which replaces the degree $\mathbf{0}$ with any definable degree $\mathbf{a}$ and works for $\mathscr{D}$ in place of $\mathscr{D}'$. Jockusch and Shore [1983a] improved this result, showing that for such $\mathbf{a}$, if $\mathscr{D}[\mathbf{a}, \infty) \equiv \mathscr{D}[\mathbf{b}, \infty)$ then $\mathbf{a}$ and $\mathbf{b}$ have the same arithmetical degree. The definability of $\mathbf{a}$ is not necessary if $\cong$ replaces $\equiv$.

4.3 Remarks. The first theorem similar to Theorem 4.1 was proved by Simpson [1977] with the conclusion being that $\mathbf{b}^{(\omega)} = \mathbf{0}^{(\omega)}$. Simpson then obtained Corollary 4.2. Theorem 4.1 as stated and the exercises below were proved by Nerode and

Shore [1980]. The improvement of the condition to $\mathbf{b}^{(3)} = \mathbf{0}^{(3)}$ is due to Shore [1982].

4.4–4.5 Exercises

4.4 Show that if $\mathbf{a} \in \mathbf{D}$ is definable over $\mathcal{N}$ and $\mathcal{D}'[\mathbf{b}, \infty) \equiv \mathcal{D}'[\mathbf{a}, \infty)$ then $\mathbf{b}^{(2)} \leqslant \mathbf{a}^{(5)}$.

4.5 For each $n \in N$ and each jump ideal $\mathbf{C}$ of $\mathcal{D}$, let $\langle \mathbf{C}, \leqslant, {}^{(\mathbf{n})} \rangle$ be the structure which interprets the unary function symbol of $\mathcal{L}'_b$ as the nth jump. Fix a jump ideal $\mathbf{C}$ of $\mathcal{D}$ and $k, n \in N$ such that $k \neq n$. Show that $\langle \mathbf{C}, \leqslant, {}^{(\mathbf{n})} \rangle \not\equiv \langle \mathbf{C}, \leqslant, {}^{(\mathbf{k})} \rangle$. (*Hint*: A sentence which differentiates between the structures asserts that there is a standard model of arithmetic coded by a degree $\mathbf{d} \leqslant \mathbf{m}^{(2)}$ ($\mathbf{m}$ is, again, the least element of the structure and $\mathbf{m}^{(2)}$ denotes two consecutive applications of the operation of the structure; hence over $\langle \mathbf{C}, \leqslant, {}^{(\mathbf{n})} \rangle$, $\mathbf{m}^{(2)}$ is really the degree of the element $\mathbf{m}^{(2\mathbf{n})}$ under the true interpretation of the jump operator) and a pair of elements below $\mathbf{m}^{(6)}$ which codes a set whose degree is not $\leqslant \mathbf{0}^{(6\mathbf{n}+3)}$.)

5. *Isomorphisms Between Cones of Degrees*

The homogeneity problems deal with questions about cones of degrees. (A cone of degrees is a class of degrees of the form $\mathcal{D}[\mathbf{b}, \infty)$ for some $\mathbf{b} \in \mathbf{D}$.) Given $\mathbf{a}, \mathbf{b} \in \mathbf{D}$, Rogers [1967] asks if $\mathcal{D}[\mathbf{a}, \infty) \simeq \mathcal{D}[\mathbf{b}, \infty)$. This question is known as the Homogeneity Problem. A variation of this problem also appears in the literature, namely, for $\mathbf{a}, \mathbf{b} \in \mathbf{D}$, is $\mathcal{D}[\mathbf{a}, \infty) \equiv \mathcal{D}[\mathbf{b}, \infty)$? We show that in both cases, there is a choice of $\mathbf{a}$ and $\mathbf{b}$ for which there is a negative answer to the problem.

The homogeneity problem is solved by showing that any isomorphism between cones of degrees has a cone of fixed points (hence any automorphism of $\mathcal{D}$ must also have a cone of fixed points). We then show that $\mathcal{D} \not\simeq \mathcal{D}[\mathbf{b}, \infty)$ where $\mathbf{b}$ is chosen sufficiently large so that it is the base of a cone of minimal covers.

The following lemma allows us to compute the vertex of a cone of fixed points for any given isomorphism between cones of degrees. It shows that the set of degrees

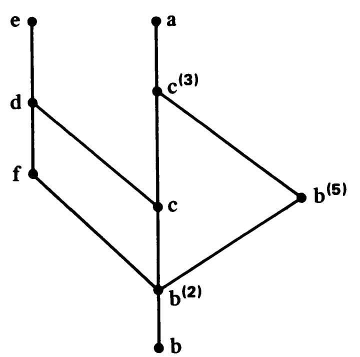

Fig. 5.1

$\mathbf{d} \geqslant \mathbf{b}$ such that $\mathscr{D}[\mathbf{b}, \mathbf{d}]$ has a sufficiently nice presentation is squeezed between two known classes of degrees. The lemma will thus allow us to compute bounds on the degrees in that set. The comparability relationships between the degrees mentioned in the proof are pictured in Fig. 5.1.

5.1 Lemma. *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{D}$ *be given such that* $\mathbf{b}^{(2)} \leqslant \mathbf{c}$ *and* $\mathbf{a} \geqslant \mathbf{c}^{(3)}$. *Let* $\mathbf{F_b}[\mathbf{c}, \mathbf{a}] = \mathbf{D}[\mathbf{c}, \mathbf{a}] \cap \{\mathbf{d} \geqslant \mathbf{b} : \mathbf{d}^{(3)} \leqslant \mathbf{a}\}$ *and* $\mathbf{E_b}[\mathbf{c}, \mathbf{a}] = \{\mathbf{d} \geqslant \mathbf{c} : \exists \mathbf{e} \in \mathbf{D}(\mathscr{D}[\mathbf{b}, \mathbf{d}]$ *is* $\mathbf{e}$*-presentable* $\& \, \mathbf{e} \leqslant \mathbf{a})\}$. *Then* $\mathbf{F_b}[\mathbf{c}, \mathbf{a}] \subseteq \mathbf{E_b}[\mathbf{c}, \mathbf{a}] \subseteq \mathbf{D}[\mathbf{c}, \mathbf{a}]$.

Proof. It follows from the fact that $\mathscr{D}[\mathbf{b}, \mathbf{d}]$ is $\mathbf{d}^{(3)}$-presentable for all $\mathbf{d} \geqslant \mathbf{b}$ that $\mathbf{F_b}[\mathbf{c}, \mathbf{a}] \subseteq \mathbf{E_b}[\mathbf{c}, \mathbf{a}]$. Suppose that $\mathbf{d} \geqslant \mathbf{c}$ and $\mathbf{d} \nleqslant \mathbf{a}$. By Theorem 1.1 of Appendix A, there is a lattice of degree $\mathbf{d}$, i.e., a lattice $\mathscr{L}$ which has a presentation of degree $\mathbf{d}$ and such that any presentation of $\mathscr{L}$ has degree $\geqslant \mathbf{d}$. By Exercise 1.15 and since $\mathbf{b}^{(2)} \leqslant \mathbf{c} \leqslant \mathbf{d}$, there is an $\mathbf{f} \leqslant \mathbf{d}$ such that $\mathscr{L} \simeq \mathscr{D}[\mathbf{b}, \mathbf{f}]$. If $\mathbf{e}$ is the degree of a presentation of $\mathscr{D}[\mathbf{b}, \mathbf{d}]$, then since $\mathscr{D}[\mathbf{b}, \mathbf{f}]$ is a sublattice of $\mathscr{D}[\mathbf{b}, \mathbf{d}]$ with least and greatest elements, $\mathscr{D}[\mathbf{b}, \mathbf{f}]$ must also be $\mathbf{e}$-presentable. Hence by choice of $\mathscr{L}$, $\mathbf{d} \leqslant \mathbf{e}$. Since $\mathbf{d} \nleqslant \mathbf{a}$, $\mathbf{e} \nleqslant \mathbf{a}$ so $\mathbf{d} \notin \mathbf{E_b}[\mathbf{c}, \mathbf{a}]$. ▯

We now show that every isomorphism between cones of degrees has a cone of fixed points. We prove the theorem by choosing one of the cones to be the set of all degrees, and note later that the result relativizes. Figure 5.2 is useful for following the proof.

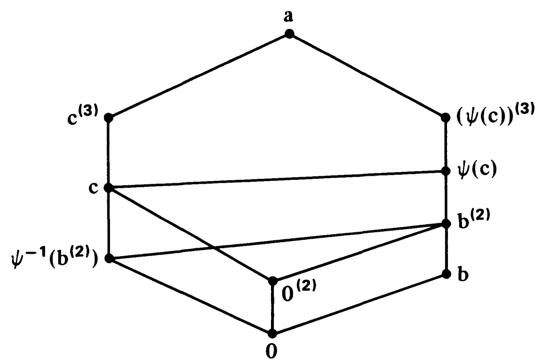

Fig. 5.2

5.2 Theorem. *Let* $\mathbf{b} \in \mathbf{D}$ *be given and let* $\psi : \mathscr{D} \to \mathscr{D}[\mathbf{b}, \infty)$ *be an isomorphism with inverse* ψ^{-1}. Let $\mathbf{c} = \psi^{-1}(\mathbf{b}^{(2)}) \cup \mathbf{0}^{(2)}$ *and let* $\mathbf{a} \geqslant \mathbf{c}^{(3)} \cup (\psi(\mathbf{c}))^{(3)}$ *be given. Then* $\psi(\mathbf{a}) = \mathbf{a}$.

Proof. We note that $\sup(\mathbf{D}[\mathbf{c}, \mathbf{a}]) = \mathbf{a}$. Also, by Exercise IV.4.16, $\sup(\mathbf{F_0}[\mathbf{c}, \mathbf{a}]) = \mathbf{a}$. By Lemma 5.1, $\mathbf{F_0}[\mathbf{c}, \mathbf{a}] \subseteq \mathbf{E_0}[\mathbf{c}, \mathbf{a}] \subseteq \mathbf{D}[\mathbf{c}, \mathbf{a}]$ so $\sup(\mathbf{E_0}[\mathbf{c}, \mathbf{a}]) = \mathbf{a}$. Let $\psi(\mathbf{E_0}[\mathbf{c}, \mathbf{a}])$ be the image of $\mathbf{E_0}[\mathbf{c}, \mathbf{a}]$ under ψ. Then $\psi(\mathbf{E_0}[\mathbf{c}, \mathbf{a}]) = \{\mathbf{d} \geqslant \psi(\mathbf{c}) : \exists \mathbf{e} \in \mathbf{D}(\mathscr{D}[\mathbf{b}, \mathbf{d}]$ is $\mathbf{e}$-presentable $\& \, \mathbf{e} \leqslant \mathbf{a})\} = \mathbf{E_b}[\psi(\mathbf{c}), \mathbf{a}]$. We note that $\sup(\mathbf{D}[\psi(\mathbf{c}), \mathbf{a}]) = \mathbf{a}$, and by Exercise IV.4.16, $\sup(\mathbf{F_b}[\psi(\mathbf{c}), \mathbf{a}]) = \mathbf{a}$. By Lemma 5.1 (which we can apply since $\psi(\mathbf{c}) \geqslant \mathbf{b}^{(2)}$), $\mathbf{F_b}[\psi(\mathbf{c}), \mathbf{a}] \subseteq \mathbf{E_b}[\psi(\mathbf{c}), \mathbf{a}] \subseteq \mathbf{D}[\psi(\mathbf{c}), \mathbf{a}]$ so $\sup(\mathbf{E_b}[\psi(\mathbf{c}), \mathbf{a}]) = \mathbf{a}$. Since ψ is an isomorphism, $\psi(\mathbf{a}) = \psi(\sup(\mathbf{E_0}[\mathbf{c}, \mathbf{a}])) = \sup(\psi(\mathbf{E_0}[\mathbf{c}, \mathbf{a}])) = \sup(\mathbf{E_b}[\psi(\mathbf{c}), \mathbf{a}]) = \mathbf{a}$. ▯

Theorem 5.2 relativizes to yield the following result.

5.3 Corollary. *Let* $\mathbf{b}, \mathbf{d} \in \mathbf{D}$ *be given, and let* $\psi: \mathscr{D}[\mathbf{b}, \infty) \to \mathscr{D}[\mathbf{d}, \infty)$ *be an isomorphism. Then there is a* $\mathbf{c} \in \mathbf{D}$ *such that* $\psi(\mathbf{a}) = \mathbf{a}$ *for all* $\mathbf{a} \geqslant \mathbf{c}$.

Theorem 5.2 also immediately tells us that every automorphism of $\mathscr{D}$ has a cone of fixed points.

5.4 Corollary. *Let* ψ *be an automorphism of* $\mathscr{D}$. *Then there is a* $\mathbf{c} \in \mathbf{D}$ *such that* $\psi(\mathbf{a}) = \mathbf{a}$ *for all* $\mathbf{a} \geqslant \mathbf{c}$.

A vertex for the cone of Corollary 5.4 can be computed from Theorem 5.2, and depends on the degrees of various images and inverse images of ψ. Jockusch and Shore [1983a] have used other methods to show that the vertex of such a cone can be chosen to be the degree $\mathbf{0}^{(\omega)}$, independently of ψ.

The solution to the Homogeneity Problem uses two facts proved earlier in the book. We restate these facts in the following remark for the reader's convenience. The theorems are proved as Theorem V.5.3 and Theorem V.5.12 respectively.

5.5 Remark. For all $n \in N$, $\mathbf{0}^{(n)}$ is not a minimal cover. (This result is applied in its relativized form: For all $n \in N$ and $\mathbf{d} \in \mathbf{D}$, $\mathbf{d}^{(n)}$ is not a minimal cover for any $\mathbf{b} \geqslant \mathbf{d}$.) Also, there is a cone of minimal covers. In fact, Jockusch and Shore [1983a] compute the vertex of such a cone as $\mathbf{0}^{(\omega)}$. (This result is also applied in its relativized form: For all $\mathbf{b} \in \mathbf{D}$, $\mathscr{D}[\mathbf{b}, \infty)$ contains a cone of minimal covers of degrees $\geqslant \mathbf{b}$.)

We now have enough computational information to show that the cone of degrees above the vertex of a cone of minimal covers above $\mathbf{0}^{(2)}$ is not isomorphic to $\mathscr{D}$. For such an isomorphism ψ would have the property that $\psi(\mathbf{0}^{(2)})$ has an eighth jump which is the vertex of a cone of minimal covers above $\psi(\mathbf{0}^{(2)})$.

5.6 Theorem. *There is an* $\mathbf{e} \in \mathbf{D}$ *such that for all* $\mathbf{b} \geqslant \mathbf{e}$, $\mathscr{D}[\mathbf{b}, \infty) \not\simeq \mathscr{D}$. ($\mathbf{e}$ *can be chosen as any vertex of a cone of minimal covers above* $\mathbf{0}^{(2)}$, *so by Remark 5.5, we can choose* $\mathbf{e} = \mathbf{0}^{(\omega)}$.)

Proof. By Remark 5.5, we can fix a degree $\mathbf{e}$ which is the vertex of a cone of minimal covers above $\mathbf{0}^{(2)}$, (i.e., for all $\mathbf{b} \geqslant \mathbf{e}$, $\mathbf{b}$ is a minimal cover of a degree $\geqslant \mathbf{0}^{(2)}$). Let $\mathbf{b} > \mathbf{e}$ be given. Assume that there is an isomorphism $\psi: \mathscr{D} \to \mathscr{D}[\mathbf{b}, \infty)$ in order to obtain a contradiction. Let $\mathbf{c} = \psi^{-1}(\mathbf{b}^{(2)}) \cup \mathbf{0}^{(2)}$. By Theorem 1.1 of Appendix A and Exercise 1.15, there is an $\mathbf{f} \leqslant \mathbf{c}$ and a lattice $\mathscr{L} \simeq \mathscr{D}[\mathbf{0}, \mathbf{f}]$ such that for all $\mathbf{d} \in \mathbf{D}$, if $\mathscr{L}$ is $\mathbf{d}$-presentable then $\mathbf{c} \leqslant \mathbf{d}$. ($\mathscr{L}$ is obtained as in the proof of Lemma 5.1.) Since ψ is an isomorphism, $\mathscr{L} \simeq \mathscr{D}[\mathbf{b}, \psi(\mathbf{f})]$. Note that $\mathscr{D}[\mathbf{b}, \psi(\mathbf{f})]$ is $(\psi(\mathbf{f}))^{(3)}$-presentable, so $\mathbf{c} \leqslant (\psi(\mathbf{f}))^{(3)} \leqslant (\psi(\mathbf{c}))^{(3)}$.

By choice of $\mathbf{c}$, $\psi(\mathbf{c}) = \mathbf{b}^{(2)} \cup \psi(\mathbf{0}^{(2)})$. Since $\mathbf{0} < \mathbf{0}^{(2)}$, $\mathbf{b} = \psi(\mathbf{0}) < \psi(\mathbf{0}^{(2)})$, so $\psi(\mathbf{c}) \leqslant (\psi(\mathbf{0}^{(2)}))^{(2)}$. Hence $\mathbf{c} \leqslant (\psi(\mathbf{c}))^{(3)} \leqslant (\psi(\mathbf{0}^{(2)}))^{(5)}$. By Theorem 5.2, $\mathbf{a} = (\psi(\mathbf{0}^{(2)}))^{(8)}$ is the vertex of a cone of fixed points. Since $\mathbf{a} \geqslant \mathbf{b} \geqslant \mathbf{e}$, $\mathbf{a}$ is also the vertex of a cone of minimal covers above $\mathscr{D}[\psi(\mathbf{0}^{(2)}), \infty)$. But by Remark 5.5, $\mathbf{a} = (\psi(\mathbf{0}^{(2)}))^{(8)}$ is not a minimal cover of any degree above $\psi(\mathbf{0}^{(2)})$, a contradiction. ∎

The last homogeneity problem is to decide whether for all $\mathbf{c} \in \mathbf{D}$, $\mathscr{D} \equiv \mathscr{D}[\mathbf{c}, \infty)$. The answer, again, is a negative one. A formula $\sigma(x)$ in the language $\mathscr{L}_b$ can be defined having the property that if there are $\mathbf{b}, \mathbf{c} \in \mathbf{D}$ such that $\mathscr{D}[\mathbf{c}, \infty) \models \sigma(\mathbf{b})$ then $\mathscr{D}[\mathbf{b}, \infty) \simeq \mathscr{D}[\mathbf{0}^{(5)}, \infty)$. This reduces the elementary equivalence problem to an isomorphism problem which Theorem 5.6 has already solved. The formula $\sigma(x)$ asserts that there is an isomorphism between $\mathscr{D}[\mathbf{x}, \infty)$ and $\mathscr{D}[\mathbf{0}^{(5)}, \infty)$. Since $\mathbf{0}^{(5)}$ is definable in second order arithmetic, such an assertion can be made over $\mathscr{N}$. However, as in the proof of Theorem 3.17, a straightforward translation of this formula into $\mathscr{L}_b$ will introduce parameters. A more careful translation must therefore be given.

5.7 Theorem. *There is an* $\mathbf{e} \in \mathbf{D}$ *such that for all* $\mathbf{c} \geqslant \mathbf{e}$, $\mathscr{D} \not\equiv \mathscr{D}[\mathbf{c}, \infty)$. ($\mathbf{e}$ *can be chosen to be* $\mathbf{0}^{(\omega)}$.)

Proof. We apply the translation given in the proof of Theorem 3.5 to formulas, and note that since $\emptyset^{(5)}$ is definable over $\mathscr{N}$, there is a formula $G(x, y, d)$ of $\mathscr{L}_b$ such that for all $\mathbf{b} \in \mathbf{D}$ and every $\mathbf{d} \in \mathbf{D}[\mathbf{b}, \infty)$ which codes a standard model of arithmetic in $\mathscr{D}[\mathbf{b}, \infty)$,

$\mathscr{D}[\mathbf{b}, \infty) \models G(\mathbf{x}, \mathbf{y}, \mathbf{d}) \Leftrightarrow$ the set $\emptyset^{(5)}$ is recursive in the set W coded by the exact pair $\langle \mathbf{x}, \mathbf{y} \rangle$ for $\mathbf{d}$.

In other words, $G(x, y, d)$ holds in $\mathscr{D}[\mathbf{b}, \infty)$ if $\emptyset^{(5)}$ is recursive in a set W coded by an exact pair (over $\mathscr{D}[\mathbf{b}, \infty)$) for $\mathbf{d}$. Similarly, there is a formula $L(b, x, y, d, a)$ of $\mathscr{L}_b$ such that for all $\mathbf{c} \in \mathbf{D}$ and $\mathbf{d} \in \mathbf{D}[\mathbf{c}, \infty)$ for which $\mathbf{d} \leqslant \mathbf{a}$ and $\mathbf{d}$ codes a standard model of arithmetic in $\mathscr{D}[\mathbf{c}, \infty)$

$$\mathscr{D}[\mathbf{c}, \infty) \models L(b, x, y, d, a) \Leftrightarrow \mathbf{b} = \sup(\mathbf{R})$$

where

(1) $\mathbf{R} = \{\mathbf{u} \geqslant \mathbf{a} : \forall \mathbf{v}, \mathbf{z} \leqslant \mathbf{u}$ (if $\langle \mathbf{v}, \mathbf{z} \rangle$ codes S for $\mathbf{d}$ and $\langle \mathbf{x}, \mathbf{y} \rangle$ codes W for $\mathbf{d}$ then $S \leqslant_T W)\}$.

In other words, $L(\mathbf{b}, \mathbf{x}, \mathbf{y}, \mathbf{d}, \mathbf{a})$ says that $\mathbf{b}$ is the sup of the degrees $\mathbf{u}$ such that only sets recursive in W (coded by $\langle \mathbf{x}, \mathbf{y} \rangle$) are coded by exact pairs below $\mathbf{u}$.

By Theorem 3.1 and the proof of Theorem 3.3, we can fix $\mathbf{d}^*$ such that $\mathbf{d}^{*(2)} = \mathbf{0}^{(2)}$ and $\mathbf{d}^*$ codes a standard model of arithmetic in which the function f of Lemma 3.11 is recursive. Let $\mathbf{R}$ be as in (1) but with $\mathbf{0}^{(2)}$ and $\mathbf{d}^*$ in place of $\mathbf{a}$ and $\mathbf{d}$ respectively. Let $\mathbf{x}, \mathbf{y} \in \mathbf{D}$ be given such that $\langle \mathbf{x}, \mathbf{y} \rangle$ codes a set W of degree $\geqslant \mathbf{0}^{(5)}$ for $\mathbf{d}^*$, and let W have degree $\mathbf{w}$. Let $\mathbf{Q} = \{\mathbf{u} \in \mathbf{D} : \mathbf{u} \geqslant \mathbf{0}^{(2)} \,\&\, \mathbf{u}^{(3)} \leqslant \mathbf{w}\}$ and let $\mathbf{T} = \mathbf{D}[\mathbf{0}, \mathbf{w}]$. The proof depends upon the following fact:

(2) $\mathbf{Q} \subseteq \mathbf{R} \subseteq \mathbf{T} \,\&\, \sup(\mathbf{Q}) = \sup(\mathbf{T}) = \mathbf{w}$.

We now verify (2). First note that since $\mathbf{w} \geqslant \mathbf{0}^{(5)}$, by Exercise IV.4.16, there are $\mathbf{x}_0, \ldots, \mathbf{x}_n \in \mathbf{Q}$ such that $\cup\{\mathbf{x}_i : i \leqslant n\} = \mathbf{w}$. Hence $\mathbf{w} = \sup(\mathbf{Q})$. Clearly $\mathbf{w} = \sup(\mathbf{T})$. If $\mathbf{u} \in \mathbf{Q}$, then by Lemma 3.12(i), any set S coded by a pair $\leqslant \mathbf{u}$ for $\mathbf{d}^*$ has degree $\leqslant \mathbf{f} \cup (\mathbf{u} \cup \mathbf{d}^*)^{(3)}$. Since f is recursive, $\mathbf{d}^* \leqslant \mathbf{0}^{(2)}$ and $\mathbf{u} \in \mathbf{Q}$, $\mathbf{f} \cup (\mathbf{u} \cup \mathbf{d}^*)^{(3)} \leqslant \mathbf{u}^{(3)} \leqslant \mathbf{w}$. Hence $\mathbf{Q} \subseteq \mathbf{R}$. Since f is recursive and $\mathbf{d}^{*(2)} = \mathbf{0}^{(2)}$, it follows from Lemma 3.12(ii) that if $\mathbf{0}^{(2)} \leqslant \mathbf{u} \nleqslant \mathbf{w}$ and $\mathbf{u} \in \mathbf{R}$, then there is an exact pair below $\mathbf{u}$ which codes a set of degree $\mathbf{u}$ for $\mathbf{d}^*$. Hence $\mathbf{R} \subseteq \mathbf{T}$. Thus (2) holds.

Let $\mathscr{D} \models G(\mathbf{x}, \mathbf{y}, \mathbf{d}^*)$. Then by (1) and (2),

(3) $$\mathscr{D} \models L(\mathbf{b}, \mathbf{x}, \mathbf{y}, \mathbf{d}^*, \mathbf{0}^{(2)}) \Leftrightarrow \mathbf{b} = \sup(\mathbf{R}) \Leftrightarrow \mathbf{w} = \mathbf{b}.$$

The isomorphism from $\mathscr{D}[\mathbf{0}^{(5)}, \infty)$ to $\mathscr{D}[\mathbf{b}, \infty)$ which is produced is a map from $\{W \subseteq N : \emptyset^{(5)} \leqslant_T W\}$ onto $\mathscr{D}[\mathbf{b}, \infty)$ which is invariant on degrees and one-one, onto, and order preserving when viewed as a map on degrees. This isomorphism takes W coded by $\langle \mathbf{x}, \mathbf{y} \rangle$ for $\mathbf{d}$ such that $G(\mathbf{x}, \mathbf{y}, \mathbf{d})$ onto the degree $\mathbf{b}$ such that $L(\mathbf{b}, \mathbf{x}, \mathbf{y}, \mathbf{d}, \mathbf{a})$. It follows from (3) that for $\mathbf{a} = \mathbf{0}^{(2)}$ and $\mathbf{d} = \mathbf{d}^*$, this map induces the identity map on $\mathscr{D}[\mathbf{0}^{(5)}, \infty)$. We now write down a formula which asserts that the map discussed above is an isomorphism of $\mathscr{D}[\mathbf{0}^{(5)}, \infty)$ with $\mathscr{D}[\mathbf{x}, \infty)$.

$$\begin{aligned}
&\sigma(x) \equiv \exists a, d < x(d < a \,\&\, d \text{ codes a standard model of arithmetic} \\
&\&\, \forall x_1, y_1(G(x_1, y_1, d) \to \exists b \geqslant x(L(b, x_1, y_1, d, a))) \,\& \\
&\forall b \geqslant x \,\exists x_1, y_1(G(x_1, y_1, d) \,\&\, L(b, x_1, y_1, d, a)) \,\& \\
&\forall x_1, y_1, x_2, y_2 \,\forall b_1, b_2 \geqslant x(G(x_1, y_1, d) \,\&\, G(x_2, y_2, d) \,\&\, L(b_1, x_1, y_1, d, a) \\
&\&\, L(b_2, x_2, y_2, d, a) \to (b_1 \leqslant b_2 \leftrightarrow \text{the set coded by } \langle x_1, y_1 \rangle \\
&\text{for } d \text{ is recursive in the set coded by } \langle x_2, y_2 \rangle \text{ for } d))).
\end{aligned}$$

It easily follows that $\mathscr{D} \models \sigma(\mathbf{0}^{(5)})$. Furthermore, by the definitions of G and L and since every set of degree $\geqslant \mathbf{0}^{(5)}$ is coded by an exact pair in any standard model of arithmetic coded by $\mathbf{d}$ for $\mathscr{D}[\mathbf{x}, \infty)$, we conclude that if there are $\mathbf{b}, \mathbf{c} \in \mathbf{D}$ such that $\mathscr{D}[\mathbf{c}, \infty) \models \sigma(\mathbf{b})$ then $\mathscr{D}[\mathbf{b}, \infty) \simeq \mathscr{D}[\mathbf{0}^{(5)}, \infty)$.

Let $\mathbf{c} \in \mathbf{D}$ be given such that $\mathscr{D} \equiv \mathscr{D}[\mathbf{c}, \infty)$. Since $\mathscr{D} \models \exists x(\sigma(x))$, there is a $\mathbf{b} \geqslant \mathbf{c}$ such that $\mathscr{D}[\mathbf{c}, \infty) \models \sigma(\mathbf{b})$. By the previous paragraph, $\mathscr{D}[\mathbf{b}, \infty) \simeq \mathscr{D}[\mathbf{0}^{(5)}, \infty)$. Hence if $\mathbf{e}$ is chosen as in Theorem 5.6 relativized to $\mathscr{D}[\mathbf{0}^{(5)}, \infty)$, then it follows that $\mathbf{c} \ngeqslant \mathbf{e}$. ▯

The definability of $\emptyset^{(5)}$ over $\mathscr{N}$ seems to be crucial to any proof of Theorem 5.7. For Martin has shown using the Axiom of Projective Determinateness that there is a degree $\mathbf{b}$ such that for all $\mathbf{c} \geqslant \mathbf{b}$, $\mathscr{D}[\mathbf{c}, \infty) \equiv \mathscr{D}[\mathbf{b}, \infty)$ (see Yates [1970]).

5.8 Remarks. The results of this section are due to Shore [1979], [1982]. In fact, Shore [1982] contains a proof of a stronger version of Theorem 5.7, namely, that if $\mathscr{D} \equiv \mathscr{D}[\mathbf{c}, \infty)$ then $\mathbf{c}^{(3)} = \mathbf{0}^{(3)}$. Jockusch and Shore [1983a] also prove that if $\mathbf{c}$ is definable over $\mathscr{N}$ and $\mathscr{D}[\mathbf{b}, \infty) \equiv \mathscr{D}[\mathbf{c}, \infty)$, then $\mathbf{b}$ and $\mathbf{c}$ have the same arithmetical degree. Thus if we assume the Axiom of Constructibility, then $\mathscr{D}[\mathbf{c}, \infty) \equiv \mathscr{D}[\mathbf{b}, \infty)$ implies that $\mathbf{b}$ and $\mathbf{c}$ have the same arithmetical degree.

5.9 Exercise. Show that for every $\mathbf{d} \in \mathbf{D}$ there is a $\mathbf{c} \geqslant \mathbf{d}$ such that $\mathscr{D}[\mathbf{b}, \infty) \not\simeq \mathscr{D}[\mathbf{c}, \infty)$.