CR deformation of cyclic quotient surface singularities

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Abstract. We compute the first order CR deformation of cyclic quotient surface singularities.

Introduction

The purpose of this paper is to fix the first order CR deformation of cyclic quotient surface singularities $A_{n,q}$ (cf. Theorem 3.13). Although the first order deformation of $A_{n,q}$ was computed in [**Ri1**] by an algebraic way, deformation of $A_{n,q}$ is still interesting and a new duality phenomenon is recently discovered (cf. [**Ri2**]). On the other hand, after establishing general CR deformation theory of normal isolated singularities in [**B-E**] and [**M1**], CR analysis on the 3-sphere was applied to describe deformations of rational quotient singularities; [**B**] for $A_{n,1}$ and [**K**] for $A_{n,n-1}$ $(n \geq 2)$, D_{n+2} $(n \geq 2)$, E_6, E_7, E_8. In this paper, we compute the first order CR deformation of the remaining $A_{n,q}$.

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1. CR deformation of normal isolated singularities

In this section, we recall the formalism of the CR deformation of normal isolated singularities in [**M1**]. Since we are concentrated in the first order deformations, we will pay no attention to the obstruction to higher order deformations.

1.1. CR structure. A CR structure is given by a sub-bundle $\overline{S_M} \subset \mathbb{C}TM$ such that

(*i*) $S_M \bigcap \overline{S_M} = \{0\}$ with denoting $S_M = \overline{\overline{S_M}}$,
(*ii*) $\overline{S_M}$ is involutive; that is, $[X, Y] \in \Gamma(M, \overline{S_M})$ holds for any $X, Y \in \Gamma(M, \overline{S_M})$.

We fix a sub bundle $\mathbb{C}F_M \subset \mathbb{C}TM$ such that $\mathbb{C}F_M \simeq \mathbb{C}TM/(S_M \oplus \overline{S_M})$ holds. Then we have type decompositions of $\mathbb{C}TM$ and $\mathbb{C}T^*M$, respectively:

$$\mathbb{C}TM = \mathbb{C}F_M \oplus S_M \oplus \overline{S_M},$$
$$\mathbb{C}T^*M = \mathbb{C}F_M^* \oplus S_M^* \oplus \overline{S_M}^*.$$

We denote $T'M = \mathbb{C}F_M \oplus S_M$.

If we denote $A_M^{0,q} := \Gamma(M, \wedge^q \overline{S_M}^*)$, the above type decompositions induce the tangential Cauchy-Riemann complexes;

$$0 \longrightarrow A_M^0 \xrightarrow{\overline{\partial}_b} A_M^{0,1} \xrightarrow{\overline{\partial}_b} \cdots, \tag{1.1}$$

$$0 \longrightarrow A_M^0(T'M) \xrightarrow{\overline{\partial}_{T'}} A_M^{0,1}(T'M) \xrightarrow{\overline{\partial}_{T'}} \cdots. \tag{1.2}$$

1.2. CR deformation. Let $(V, 0)$ be a germ of a reduced normal Stein space in $\mathbf{C}^N$ satisfying $\mathrm{Sing}(V) = \{0\}$. We denote $f : V \to \mathbf{C}^N$ the natural embedding and $h_1(w_1, \cdots, w_N) = \cdots = h_m(w_1, \cdots, w_N) = 0$ the defining equation of V. We fix a strongly pseudo-convex domain $0 \in \Omega \subset \mathbf{C}^N$ so that V and $\partial\Omega$ intersect transversely. We denote $M := V \bigcap \partial\Omega$.

A (formal) CR deformation of $(V, 0)$ is given by a $(\phi(t), g(t), k(t)) \in K_M^1[[t_1, \ldots, t_d]]$ (where $K_M^1 = A_M^{0,1}(T'M) \oplus A_M^0(T^{1,0}\mathbb{C}^N{}_{|M}) \oplus H^0(M)^m$) satisfying

$$(\phi(0), g(0), k(0)) = (0, 0, 0), \tag{1.3}$$

$$\big(\bar{\partial}_{T'}\phi(t) - R(\phi(t)),\, (\bar{\partial}_b - \phi(t))(f + g(t)),\, (h + k) \circ (f + g(t))\big) = (0, 0, 0), \tag{1.4}$$

where $R(\phi)$ is a non-linear partial differential operator (cf. [**M1**]).

1.3. Deformation complex. Let $K_M^{\bullet,\bullet}$ be the following double-complex;

$$\begin{array}{ccccc}
0 & \longrightarrow & H^0(M, T^{1,0}\mathbb{C}^N{}_{|M}) & \xrightarrow{H} & H^0(M)^m \\
\downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\
K_M^{0,0} := A_M^0(T'M) & \xrightarrow{F} & A_M^0(T^{1,0}\mathbb{C}^N{}_{|M}) & \xrightarrow{H} & (A_M^0)^m \\
\downarrow{\scriptstyle \bar{\partial}_{T'}} & & \downarrow{\scriptstyle \bar{\partial}_b} & & \downarrow{\scriptstyle \bar{\partial}_b} \\
A_M^{0,1}(T'M) & \xrightarrow{F} & A_M^{0,1}(T^{1,0}\mathbb{C}^N{}_{|M}) & \xrightarrow{H} & (A_M^{0,1})^m \\
\downarrow{\scriptstyle \bar{\partial}_{T'}} & & \downarrow{\scriptstyle \bar{\partial}_b} & & \downarrow{\scriptstyle \bar{\partial}_b} \\
A_M^{0,2}(T'M) & \xrightarrow{F} & A_M^{0,2}(T^{1,0}\mathbb{C}^N{}_{|M}) & \xrightarrow{H} & (A_M^{0,2})^m \\
\downarrow{\scriptstyle \bar{\partial}_{T'}} & & \downarrow{\scriptstyle \bar{\partial}_b} & & \downarrow{\scriptstyle \bar{\partial}_b} \\
\vdots & & \vdots & & \vdots
\end{array}$$

where $H^0(M, T^{1,0}\mathbb{C}^N{}_{|M})$ (resp. $H^0(M)$) denote the space of CR sections of $T^{1,0}\mathbb{C}^N{}_{|M}$ (resp. the space of CR functions) and $F := \rho^{1,0} \circ df$ and H denotes the homomorphism given by $H(v) = (v(h_1), \ldots, v(h_m))$ for $v \in T^{1,0}\mathbb{C}^N$, and i denote the natural inclusion map.

We denote $(K_M^\bullet, d)$ its total simple complex. That is

$$K_M^q := K_M^{0,q} \oplus K_M^{1,q-1} \oplus K_M^{2,q-2},$$
$$d(a_q, b_{q-1}, c_{q-2}) := (\bar{\partial}_{T'} a_q, \bar{\partial}_b b_{q-1} + (-1)^q F a_q, \bar{\partial}_b c_{q-2} + (-1)^{q-1} H b_{q-1})$$

where we denote $\bar{\partial}_b b_{-1} := i b_{-1}$, $\bar{\partial}_b c_{-1} := i c_{-1}$.

Theorem 1.1. *(*[**B-E**], [**M1**]*) The first order CR deformation space is* $H_d^1(K_M^\bullet)$.

2. CR deformation of cyclic quotient singularities

Let ζ_n be a primitive n-th root of 1 and $V_{n,q} := \mathbf{C}^2/G_{n,q}$ with $0 < q < n$ and $(n, q) = 1$ where $G_{n,q}$ is a cyclic group generated by the action $(z, w) \to (\zeta_n z, \zeta_n^q w)$. If $M_{n,q} := S^3/G_{n,q}$, then $M_{n,q}$ is a strongly pseudo-convex boundary of a Stein domain of $V_{n,q}$ with only isolated singularity at the origin.

Since the CR analysis on $M_{n,q}$ is treated as a CR analysis on S^3 which is invariant under $G_{n,q}$-action, we will describe CR-deformations of $V_{n,q}$ by means of invariant CR structures on S^3.

2.1. CR structure on S^3. Let $S^3 \subset \mathbb{C}^2$ be the unit 3-sphere defined by the equation $|z|^2 + |w|^2 = 1$ then the complex structure of $\mathbb{C}^2$ induces a CR structure on S^3 by

$$\overline{S} := \mathbb{C}TS^3 \bigcap T^{0,1}\mathbb{C}^2{}_{|S^3}.$$

We denote this canonical CR structure on S^3 by $^\circ T''$ and its complex conjugate by $^\circ T'$. Then, $^\circ T''$ and $^\circ T'$ are C^∞ trivial line bundle generated by $\overline{Z}$ and Z, respectively, where

$$\overline{Z} := w\frac{\partial}{\partial \overline{z}} - z\frac{\partial}{\partial \overline{w}},\ Z := \overline{w}\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial w}.$$

Let

$$T := \mathrm{Im}(z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w})$$

and $\mathbb{C}F$ be a C^∞ sub-bundle of $\mathbb{C}TS^3$ generated by T. We use the abbreviations T' and $A^{0,q}$ for $T'S^3$ and $A^{0,q}_{S^3}$, respectively. Then we have

LEMMA 2.1. (1) *$\bar{\partial}_b f = (\bar{Z}f) \otimes \bar{Z}^*$ for $f \in C^\infty(S^3)$,*
(2) *$\bar{\partial}_{T'}(\phi Z + \psi T) = (\bar{Z}\phi)Z \otimes \bar{Z}^* - 2\sqrt{-1}\phi T \otimes \bar{Z}^* + (\bar{Z}\psi)T \otimes \bar{Z}^*$ for $\phi Z + \psi T \in A^0(T')$.*

PROOF. (1) is trivial.
(2) Since $[\bar{Z}, Z] = -2\sqrt{-1}T$ and $[\bar{Z}, T] = 0$, $\bar{\partial}_{T'}(\phi Z + \psi T)(\bar{Z}) = (\bar{Z}\phi)Z + \phi[\bar{Z}, Z] + (\bar{Z}\psi)T + \psi[\bar{Z}, T] = (\bar{Z}\phi)Z - 2\sqrt{-1}\phi T + (\bar{Z}\psi)T.$ □

2.2. CR analysis on S^3. A differentiable function $f \in C^\infty(S^3)$ is called a spherical harmonic of bidegree (p, q) if it is the restriction on the sphere S^3 of a harmonic polynomial of holomorphic degree p and anti-holomorphic degree q on the ambient space $\mathbf{C}^2$; that is, $f = \tilde{f}_{|M}$ with

$$\tilde{f} = \sum_{\alpha+\beta=p,\, \gamma+\delta=q} c_{\alpha,\beta,\gamma,\delta} z^\alpha w^\beta \overline{z^\gamma}\,\overline{w^\delta} \quad \text{and} \quad \Delta\tilde{f} = 0.$$

We will abbreviate it as

$$f = \sum_{\alpha+\beta=p,\, \gamma+\delta=q} c_{\alpha,\beta,\gamma,\delta} z^\alpha w^\beta \overline{z^\gamma}\,\overline{w^\delta}.$$

Then there exists an orthonormal bases of $L^2(S^3)$ consisting of the harmonic polynomials. We denote $H^{p,q}$ the space of all harmonic polynomials of bidegree (p, q). Clearly,

- $H^{p,0} = \{f = \sum_{\alpha+\beta=p} c_{\alpha,\beta,0,0} z^\alpha w^\beta\}$,
- $H^{0,q} = \{f = \sum_{\gamma+\delta=q} c_{0,0,\gamma,\delta} \overline{z^\gamma}\,\overline{w^\delta}\}$,

and $\dim_{\mathbf{C}} H^{p,0} = p + 1$, $\dim_{\mathbf{C}} H^{0,q} = q + 1$.

LEMMA 2.2. *([**Ru**], Proposition 18.3.3)*

(1) *Z maps $H^{p,q}$ isomorphically onto $H^{p-1,q+1}$ if $p \geq 1$.*
(2) *$\bar{Z}$ maps $H^{p,q}$ isomorphically onto $H^{p+1,q-1}$ if $q \geq 1$.*
(3) *T maps $H^{p,q}$ into itself and all functions in $H^{p,q}$ are eigen functions of T with the eigen value $\sqrt{-1}(p-q)$.*

3. Computation of $H^1(K^{\bullet}_{M_{n,q}})$

Let $V_{n,q} := \mathbf{C}^2/G_{n,q}$ be a cyclic quotient singularity as at the beginning of the previous section.

With the Hirzeburch-Jung continued fraction

$$\frac{n}{n-q} = a_2 - \cfrac{1}{a_3 - \cfrac{1}{\cdots - \cfrac{1}{a_{e-1}}}} \quad (a_2 \geq 2,\, a_3 \geq 2,\, \ldots,\, a_{e-1} \geq 2),$$

the first order deformation space of $V_{n,q}$ was computed in [**Ri1**] as follows:

$$\dim_{\mathbf{C}} \mathrm{Ext}^1(\Omega^1_{V_{n,q}}, \mathcal{O}_{V_{n,q}}) = \begin{cases} (\sum_{\epsilon=2}^{e-1} a_\epsilon) - 2 \ (e \geq 4) \\ a_2 - 1 \ (e = 3) \end{cases} \tag{3.1}$$

where e is the dimension of the minimal embedding of $V_{n,q}$.

Since we have the following isomorphism (cf. [**M1**])

$$\begin{aligned} \mathrm{Ext}^1(\Omega^1_{V_{n,q}}, \mathcal{O}_{V_{n,q}}) &\simeq H^1(K^{\bullet}_{M_{n,q}}) \\ &\simeq \mathrm{Ker}\{H^1(M_{n,q}, T'M_{n,q}) \to H^1(M_{n,q}, T^{1,0}\mathbb{C}^N{}_{|M_{n,q}})\}, \end{aligned} \tag{3.2}$$

CR description of the first order deformation of $V_{n,q}$ is to fix a canonical basis of the subspace $\mathrm{Ker}\{H^1(M_{n,q}, T'M_{n,q}) \to H^1(M_{n,q}, T^{1,0}\mathbb{C}^N{}_{|M_{n,q}})\}$ of $H^1(M_{n,q}, T'M_{n,q})$.

We denote the $G_{n,q}$-action by

$$g : (z, w) \mapsto (\zeta_n z, \zeta_n^q w).$$

First, we remark that

PROPOSITION 3.1. *For $p \geq 0$,*

(1) *$\Delta Z^p(z^\alpha w^\beta) = \Delta \bar{Z}^p(\overline{z^\alpha w^\beta}) = 0$,*
(2) *$\Delta(g^* Z^p(z^\alpha w^\beta)) = \Delta(g^* \bar{Z}^p(\overline{z^\alpha w^\beta})) = 0$.*

PROOF. (1) is trivial.
(2) follows from (1) and the following lemma. □

LEMMA 3.2. *For $f \in C^\infty(S^3)$,*

(1) *$g^* Z(f) = \zeta_n^{-q-1} Z(g^* f)$,*
(2) *$g^* \bar{Z}(f) = \zeta_n^{q+1} \bar{Z}(g^* f)$.*

PROOF. (1) $g^* Z(f) = \overline{\zeta_n}^q \overline{w} g^* \frac{\partial f}{\partial z} - \overline{\zeta_n} \overline{z} g^* \frac{\partial f}{\partial w} = \overline{\zeta_n}^{-q-1} Z(g^* f)$.
(2) follows from (1). □

Hence, $\{Z^p(z^\alpha w^\beta)\}_{\alpha+\beta=s}$ (resp. $\{\bar{Z}^p(\overline{z^\alpha w^\beta})\}_{\alpha+\beta=s}$) forms a basis of $H^{s-p,p}$ (resp. $H^{p,s-p}$).

LEMMA 3.3. (1) $g_*^{-1}Z = \overline{\zeta}_n^{1+q}Z$,
(2) $g_*^{-1}T = T$,
(3) $g^*\bar{Z}^* = \overline{\zeta}_n^{1+q}\bar{Z}^*$.

PROOF. (1) $g_*^{-1}Z = g_*^{-1}(\overline{w}\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial w}) = \overline{\zeta}_n^{1+q}Z$.
(2) Since $g_*^{-1}(z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}) = (z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w})$, we have $g_*^{-1}T = T$.
(3) $g^*(\overline{w}d\overline{z} - \overline{z}d\overline{w}) = \overline{\zeta}_n^{1+q}\bar{Z}^*$. □

PROPOSITION 3.4. *Let us consider the natural projection $S^3 \to M_{n,q}$.*

(1) $f^{s,t} = \sum_{\alpha+\beta=s+t} f_{\alpha,\beta,0,0}Z^t(z^\alpha w^\beta) \in C^\infty(S^3)$ *is pullback of a function on $M_{n,q}$ if and only if*
$$f_{\alpha,\beta,0,0} = 0 \ \text{ for } (\alpha - t) + (\beta - t)q \neq 0 \bmod n.$$
(2) $f^{s,t} = \sum_{\gamma+\delta=s} f_{0,0,\gamma,\delta}\bar{Z}^s(\overline{z^\gamma w^\delta}) \in C^\infty(S^3)$ *is pullback of a function on $M_{n,q}$ if and only if*
$$f_{0,0,\gamma,\delta} = 0 \ \text{ for } (\gamma - s) + (\delta - s)q \neq 0 \bmod n.$$
(3) $\phi^{s,t}\bar{Z}^* = \sum_{\alpha+\beta=s+t} \phi_{\alpha,\beta,0,0}Z^t(z^\alpha w^\beta)\bar{Z}^* \in A^{0,1}_{S^3}$ *is pullback of a tangential (0,1)-form on $M_{n,q}$ if and only if*
$$\phi_{\alpha,\beta,0,0} = 0 \ \text{ for } (\alpha - t - 1) + (\beta - t - 1)q \neq 0 \bmod n.$$
(4) $\phi^{s,t}\bar{Z}^* = \sum_{\gamma+\delta=s} \phi_{0,0,\gamma,\delta}\bar{Z}^s(\overline{z^\gamma w^\delta})\bar{Z}^* \in A^{0,1}_{S^3}$ *is pullback of a tangential (0,1)-form on $M_{n,q}$ if and only if*
$$\phi_{0,0,\gamma,\delta} = 0 \ \text{ for } (\gamma - s + 1) + (\delta - s + 1)q \neq 0 \bmod n.$$
(5) $\phi^{s,t}Z + \psi^{s,t}T \in A^0_{S^3}(T')$, *where* $\phi^{s,t} = \sum_{\alpha+\beta=s} \phi_{\alpha,\beta,0,0}Z^t(z^\alpha w^\beta)$ *and* $\psi^{s,t} = \sum_{\alpha+\beta=s} \psi_{\alpha,\beta,0,0}Z^t(z^\alpha w^\gamma)$, *is pullback of a tangent vector field on $M_{n,q}$ if and only if*
$$\phi_{\alpha,\beta,0,0} = 0 \ \text{ for } (\alpha - t - 1) + (\beta - t - 1)q \neq 0 \bmod n$$
and
$$\psi_{\alpha,\beta,0,0} = 0 \ \text{ for } (\alpha - t) + (\beta - t)q \neq 0 \bmod n.$$
(6) $\phi^{s,t}Z + \psi^{s,t}T \in A^0_{S^3}(T')$, *where* $\phi^{s,t} = \sum_{\gamma+\delta=s} \phi_{0,0,\gamma,\delta}\bar{Z}^s(\overline{z^\gamma w^\delta})$ *and* $\psi^{s,t} = \sum_{\gamma+\delta=s} \psi_{0,0,\gamma,\delta}\bar{Z}^s(\overline{z^\gamma w^\delta})$, *is pullback of a tangent vector field on $M_{n,q}$ if and only if*
$$\phi_{0,0,\gamma,\delta} = 0 \ \text{ for } (\gamma - s + 1) + (\delta - s + 1)q \neq 0 \bmod n$$
and
$$\psi_{0,0,\gamma,\delta} = 0 \ \text{ for } (\gamma - s) + (\delta - s)q \neq 0 \bmod n.$$
(7) $\phi^{s,t}Z\otimes\bar{Z}^* + \psi^{s,t}T\otimes\bar{Z}^* \in A^{0,1}_{S^3}(T')$, *where* $\phi^{s,t} = \sum_{\alpha+\beta=s} \phi_{\alpha,\beta,0,0}Z^t(z^\alpha w^\gamma)$ *and* $\psi^{s,t} = \sum_{\alpha+\beta=s} \psi_{\alpha,\beta,0,0}Z^t(z^\alpha w^\gamma)$, *is pullback of a T'-valued tangential (0,1)-form on $M_{n,q}$ if and only if*
$$\phi_{\alpha,\beta,0,0} = 0 \ \text{ for } (\alpha - t - 2) + (\beta - t - 2)q \neq 0 \bmod n$$
and
$$\psi_{\alpha,\beta,0,0} = 0 \ \text{ for } (\alpha - t - 1) + (\beta - t - 1)q \neq 0 \bmod n.$$

(8) *$\phi^{s,t} Z \otimes \bar{Z}^* + \psi^{s,t} T \otimes \bar{Z}^* \in A^{0,1}_{S^3}(T')$, where $\phi^{s,t} = \sum_{\gamma+\delta=s} \phi_{0,0,\gamma,\delta} \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta})$ and $\psi^{s,t} = \sum_{\gamma+\delta=s} \psi_{0,0,\gamma,\delta} \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta})$, is pullback of a T'-valued tangential (0,1)-form on $M_{n,q}$ if and only if*

$$\phi_{0,0,\gamma,\delta} = 0 \quad \textit{for } (\gamma - s + 2) + (\delta - s + 2)q \neq 0 \bmod n$$

and

$$\psi_{0,0,\gamma,\delta} = 0 \quad \textit{for } (\gamma - s + 1) + (\delta - s + 1)q \neq 0 \bmod n.$$

Proof. We will prove (2), (4), (6) and (8). The other part will be proved by similar calculations.

(2) By Lemma 3.2 (2), we have

$$g^* \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}) = \zeta_n^{1+q} s \bar{Z}^s g^*(\overline{z^\gamma}\overline{w^\delta}) = \bar{\zeta}_n^{(\gamma-s)+(\delta-s)q} \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}).$$

Since $\{\bar{Z}^s(\overline{z^\gamma}\overline{w^\delta})\}_{\gamma+\delta=s+t}$ is linearly independent, $g^* f^{s,t} = f^{s,t}$ holds if and only if $f_{0,0,\gamma,\delta} = 0$ holds for $(\gamma - s) + (\delta - s)q \neq 0 \bmod n$.

(4) By the same calculation as in the proof of (2) and by Lemma 3.3 (3),

$$g^* \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}) g^*(\overline{Z}^*) = \bar{\zeta}_n^{(\gamma-s+1)+(\delta-s+1)q} \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}) \overline{Z}^*.$$

Hence, $g^* \phi^{s,t} \overline{Z}^* = \phi^{s,t} \overline{Z}^*$ if and only if $\phi_{0,0,\gamma,\delta} = 0$ for $(\gamma - s + 1) + (\delta - s + 1)q \neq 0 \bmod n$.

(6) By the same calculation as in the proof of (2) and by Lemma 3.3 (1) and (2),

$$g^* \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}) g_*^{-1} Z = \bar{\zeta}_n^{(\gamma-s)+(\delta-s)q} \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}) \bar{\zeta}_n^{1+q} Z,$$
$$g^* \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}) g_*^{-1} T = \bar{\zeta}_n^{(\gamma-s)+(\delta-s)q} \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}) T.$$

Hence

$$\sum_{\gamma+\delta=s} \phi_{0,0,\gamma,\delta} g^* \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}) g_*^{-1} Z = \sum_{\gamma+\delta=s} \phi_{0,0,\gamma,\delta} \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}) Z \quad \text{and}$$
$$\sum_{\gamma+\delta=s} \psi_{0,0,\gamma,\delta} g^* \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}) g_*^{-1} T = \sum_{\gamma+\delta=s} \psi_{0,0,\gamma,\delta} \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}) T$$

hold if and only if

$$\phi_{0,0,\gamma,\delta} = 0 \quad \text{for } (\gamma - s + 1) + (\delta - s + 1)q \neq 0 \bmod n \quad \text{and}$$
$$\psi_{0,0,\gamma,\delta} = 0 \quad \text{for } (\gamma - s) + (\delta - s)q \neq 0 \bmod n.$$

(8) By the same calculation as in the proof of (2) and by Lemma 3.3,

$$\begin{aligned} g^* \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}) g_*^{-1} Z \otimes g^* \bar{Z}^* &= \bar{\zeta}_n^{(\gamma-s)+(\delta-s)q} \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}) \bar{\zeta}_n^{2+2q} Z \otimes \bar{Z}^* \\ &= \bar{\zeta}_n^{(\gamma-s+2)+(\delta-s+2)q} \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}) Z \otimes \bar{Z}^* \end{aligned}$$

and

$$\begin{aligned} g^* \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}) g_*^{-1} T \otimes g^* \bar{Z}^* &= \bar{\zeta}_n^{(\gamma-s)+(\delta-s)q} \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}) \bar{\zeta}_n^{1+q} T \otimes \bar{Z}^* \\ &= \bar{\zeta}_n^{(\gamma-s+1)+(\delta-s+1)q} \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}) T \otimes \bar{Z}^*. \end{aligned}$$

Hence

$$\sum_{\gamma+\delta=s} \phi_{0,0,\gamma,\delta} g^* \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}) g_*^{-1} Z \otimes g^* \bar{Z}^* = \sum_{\gamma+\delta=s} \phi_{0,0,\gamma,\delta} \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}) Z \otimes \bar{Z}^* \quad \text{and}$$
$$\sum_{\gamma+\delta=s} \psi_{0,0,\gamma,\delta} g^* \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}) g_*^{-1} T \otimes g^* \bar{Z}^* = \sum_{\gamma+\delta=s} \psi_{0,0,\gamma,\delta} \bar{Z}^s(\overline{z^\gamma}\overline{w^\delta}) T \otimes \bar{Z}^*$$

hold if and only if

$$\phi_{0,0,\gamma,\delta} = 0 \ \text{ for } (\gamma - s + 2) + (\delta - s + 2)q \neq 0 \bmod n \ \text{ and}$$
$$\psi_{0,0,\gamma,\delta} = 0 \ \text{ for } (\gamma - s + 1) + (\delta - s + 1)q \neq 0 \bmod n.$$

□

We note that $\bar{\partial}_b$ and $\bar{\partial}_{T'}$ commute with the pullbacks.

Next, we consider the embedding $M_{n,q} \hookrightarrow \mathbf{C}^N$.

Set $\Lambda_{n,q} := \{(\alpha, \beta) \,|\, 0 \leq \alpha \leq n,\ 0 \leq \beta \leq n,\ \alpha + \beta q \equiv 0 \bmod n\}$ and $N := \#\Lambda_{n,q}$.

$q_{n,q} : \mathbf{C}^2 \to \mathbf{C}^N$ be a holomorphic map given by

$$X_{\alpha,\beta} = z^\alpha w^\beta \ ((\alpha, \beta) \in \Lambda_{n,q}).$$

We denote $q_{n,q|S^3} : S^3 \to \mathbf{C}^N$ by the same symbol $q_{n,q}$.

The tangent map $\rho^{1,0} \circ dq_{n,q} : T' \to q_{n,q}^* T^{1,0}\mathbf{C}^N$ is given by

$$g_{\alpha,\beta} = \phi Z(z^\alpha w^\beta) + \sqrt{-1}(\alpha + \beta)\psi z^\alpha w^\beta \tag{3.3}$$

if $\rho^{1,0} \circ dq_{n,q}(\phi Z + \psi T) = \sum_{(\alpha,\beta)\in\Lambda_{n,q}} g_{\alpha,\beta} \frac{\partial}{\partial X_{\alpha,\beta}}$.

By Lemmas 2.2 and 2.1,

Lemma 3.5. *For* $\phi = \sum_{t\geq 0} \phi^{0,t} Z \otimes \bar{Z}^* + \sum_{t \geq 1} \psi^{0,t} T \otimes \bar{Z}^*$ $(\phi^{0,t},\ \psi^{0,t} \in H^{0,t})$,

$$\phi \in \bar{\partial}_{T'} A^0_{M_{n,q}}(T'M_{n,q}) \textit{ holds if and only if } \phi = 0.$$

Proposition 3.6. *Let*

$$\phi = \sum_{t\geq 0} \phi^{0,t} Z \otimes \bar{Z}^* + \sum_{t\geq 1} \psi^{0,t} T \otimes \bar{Z}^* \in A^{0,1}_{M_{n,q}}(T'M_{n,q})$$

and $\phi^{0,t} = \sum_{\gamma+\delta=t} \phi_{0,0,\gamma,\delta} \overline{z^\gamma}\overline{w^\delta}$ *and* $\psi^{0,t} = \sum_{\gamma+\delta=t} \psi_{0,0,\gamma,\delta} \overline{z^\gamma}\overline{w^\delta}$.

Suppose

$$\phi_{0,0,\gamma,\delta} = 0 \ \textit{ for } (\gamma + 2) + (\delta + 2)q \not\equiv 0 \bmod n,$$
$$\psi_{0,0,\gamma,\delta} = 0 \ \textit{ for } (\gamma + 1) + (\delta + 1)q \not\equiv 0 \bmod n$$

hold. Then, if $\rho^{1,0} \circ dq_{n,q}\phi \in \bar{\partial}_b A^0_{M_{n,q}}(T^{1,0}\mathbf{C}^N_{|M_{n,q}})$*, the following equations hold:*

$$(\alpha + \beta + c + d + 1)(\alpha d - \beta c)\phi_{0,0,\alpha+c-1,\beta+d-1} + \sqrt{-1}(\alpha + \beta)(\alpha + c)(\beta + d)\psi_{0,0,\alpha+c,\beta+d} = 0$$
$$\textit{for all } (\alpha,\, \beta) \in \Lambda_{n,q} \ \textit{ and } c \geq 0,\ d \geq 0 \ \textit{ such that } \alpha + c \geq 1,\ \beta + d \geq 1, \tag{3.4}$$

$$\psi_{0,0,\alpha+c,0} = \psi_{0,0,0,\beta+d} = 0$$
$$\textit{for all } (\alpha,\, 0),\ (0,\, \beta) \in \Lambda_{n,q} \ \textit{ and } c \geq 0,\ d \geq 0. \tag{3.5}$$

PROOF. Let $\rho^{1,0} \circ dq_{n,q}\phi = \sum_{(\alpha,\,\beta)\in\Lambda_{n,q}} g_{\alpha,\,\beta} \frac{\partial}{\partial X_{\alpha,\beta}} \otimes \bar{Z}^*$. Then, by (3.3),

$$g_{\alpha,\,\beta} = \sum_{t\geq 0}\sum_{\gamma+\delta=t} \phi_{0,0,\gamma,\delta}\overline{z^\gamma}\overline{w^\delta}(\alpha z^{\alpha-1}w^\beta\overline{w} - \beta z^\alpha w^{\beta-1}\overline{z}) + \sqrt{-1}(\alpha+\beta)\sum_{t\geq 1}\sum_{\gamma+\delta=t}\psi_{0,0,\gamma,\delta}\overline{z^\gamma}\overline{w^\delta}z^\alpha w^\beta.$$

We note that $\rho^{1,0} \circ dq_{n,q}\phi \in \bar{\partial}_b A^0_{M_{n,q}}(T^{1,0}\mathbf{C}^N_{|M_{n,q}})$ implies

$$< g_{\alpha,\,\beta},\, \overline{z^c}\overline{w^d} >= 0 \ \text{ for all } c \geq 0,\, d \geq 0.$$

For the case of $\alpha + c \geq 1$, $\beta + d \geq 1$;

$$\begin{aligned} < g_{\alpha,\,\beta},\, \overline{z^c}\overline{w^d} > &= \phi_{0,0,\alpha+c-1,\beta+d-1}(\alpha||\overline{z^{\alpha+c-1}}\overline{w^{\beta+d}}||^2 - \beta||\overline{z^{\alpha+c}}\overline{w^{\beta+d-1}}||^2) \\ &\quad + \sqrt{-1}(\alpha+\beta)\psi_{0,0,\alpha+c,\beta+d}||\overline{z^{\alpha+c}}\overline{w^{\beta+d}}||^2 \\ &= \frac{(\alpha+c-1)!(\beta+d-1)!(\alpha d - \beta c)}{(\alpha+\beta+c+d)!}\phi_{0,0,\alpha+c-1,\beta+d-1} \\ &\quad + \sqrt{-1}\frac{(\alpha+\beta)(\alpha+c)!(\beta+d)!}{(\alpha+\beta+c+d+1)!}\psi_{0,0,\alpha+c,\beta+d}. \end{aligned}$$

For the case of $\beta + d = 0$;

$$< g_{\alpha,\,0},\, \overline{z^c} >= \sqrt{-1}\alpha\psi_{0,0,\alpha+c,0}||\overline{z^{\alpha+c}}||^2.$$

For the case of $\alpha + c = 0$;

$$< g_{0,\,\beta},\, \overline{w^d} >= \sqrt{-1}\beta\psi_{0,0,0,\beta+d}||\overline{w^{\beta+d}}||^2.$$

Hence, we have the lemma. □

For $\mathbf{e} := (\alpha,\,\beta) \in \Lambda_{n,q}$, we denote

$$\begin{cases} X_\phi(\mathbf{e}) := \phi_{0,0,\alpha-2,\beta-2} \\ X_\psi(\mathbf{e}) := \psi_{0,0,\alpha-1,\beta-1}. \end{cases}$$

Note that $X_\psi((1,1)) = 0$ in the case of $(n,\,q) = (n,\,n-1)$.

Then, the equation (3.4) and (3.5) are written as follows:

$$(\alpha+\beta+\gamma+\delta-1)(\alpha\delta-\beta\gamma-\alpha+\beta)X_\phi(\mathbf{e}+\mathbf{e}') + \sqrt{-1}(\alpha+\beta)(\alpha+\gamma-1)(\beta+\delta-1)X_\psi(\mathbf{e}+\mathbf{e}') = 0 \quad (3.6)$$

for all $\mathbf{e} = (\alpha,\,\beta)$, $\mathbf{e}' = (\gamma,\,\delta) \in \Lambda_{n,q}$ satisfying $\gamma \geq 1$, $\delta \geq 1$ and $\alpha+\gamma \geq 2$, $\beta+\delta \geq 2$,

$$X_\phi(\mathbf{e}) = X_\psi(\mathbf{e}) = 0 \quad (3.7)$$

for $\mathbf{e} = (\alpha,\,\beta)$ with $\alpha = 0$ or $\beta = 0$, and

$$X_\phi(\mathbf{e}) = 0 \quad (3.8)$$

for $\mathbf{e} = (\alpha,\,\beta)$ with $\alpha = 1$ or $\beta = 1$.

We compute $X_\phi(\mathbf{e})$ and $X_\psi(\mathbf{e})$.

By the Hirzeburch-Jung algorithm, we obtain a minimal generators of $\Lambda_{n,q}$ as follows (cf. [**Ri1**]). Let

$$\frac{n}{n-q} = a_2 - \cfrac{1}{a_3 - \cfrac{1}{\cdots - \cfrac{1}{a_{e-1}}}} \quad (a_2 \geq 2,\, a_3 \geq 2,\, \dots,\, a_{e-1} \geq 2)$$

be the continued fractional expansion. Then, $i_1 = n > i_2 = n-q > i_3 > \cdots > i_{e-1} = 1 > i_e = 0$ and $j_1 = 0 < j_2 = 1 < j_3 < \cdots < j_{e-1} < j_e = n$ are defined by

$$i_\epsilon + j_\epsilon q \equiv 0 \bmod n \quad (\epsilon = 1, \cdots, e), \tag{3.9}$$

$$i_{\epsilon-1} = a_\epsilon i_\epsilon - i_{\epsilon+1} \quad (\epsilon = 2, \cdots, e-1), \tag{3.10}$$

$$j_{\epsilon-1} = a_\epsilon j_\epsilon - j_{\epsilon+1} \quad (\epsilon = 2, \cdots, e-1). \tag{3.11}$$

We denote $\mathbf{e}_\epsilon = (i_\epsilon,\, j_\epsilon)$.

DEFINITION 3.7. Let $\mathbf{e} := (\alpha,\, \beta) \in \Lambda_{n,q}$.

(1) $\mathbf{e}$ is inside-decomposable if there exist $\mathbf{e}' := (\alpha',\, \beta')$, $\mathbf{e}'' := (\alpha'',\, \beta'') \in \Lambda_{n,q}$ such that
  (i) $\alpha' \geq 1$, $\beta' \geq 1$, $\alpha'' \geq 1$, $\beta'' \geq 1$,
  (ii) $\mathbf{e}'$, $\mathbf{e}''$ are linearly independent over $\mathbf{R}$,
  (iii) $\mathbf{e} = \mathbf{e}' + \mathbf{e}''$.
(2) $\mathbf{e}$ is edge-decomposable if $\mathbf{e} = \mathbf{e}_2 + m\mathbf{e}_1$ or $\mathbf{e} = \mathbf{e}_{e-1} + m\mathbf{e}_e$ $(m \geq 1)$.
(3) $\mathbf{e}$ is proportional if $\mathbf{e} = m\mathbf{e}_\epsilon$ $(m \geq 1)$.

REMARK 3.8. There may be elements which are inside-decomposable and proportional, while there exists no element which is edge- and inside-decomposable or edge-decomposable and proportional.

PROPOSITION 3.9. (1) *$X_\phi(m\mathbf{e}_1) = X_\psi(m\mathbf{e}_1) = 0$ $(m \geq 1)$, $X_\phi(m\mathbf{e}_e) = X_\psi(m\mathbf{e}_e) = 0$ $(m \geq 1)$.*
(2) *$X_\phi(\mathbf{e}_2) = 0$, $X_\phi(\mathbf{e}_{e-1}) = 0$.*
(3) *If $\mathbf{e}$ is inside-decomposable, $X_\phi(\mathbf{e}) = X_\psi(\mathbf{e}) = 0$.*
(4) *If $\mathbf{e}$ is edge-decomposable, $X_\phi(\mathbf{e}) = X_\psi(\mathbf{e}) = 0$.*

PROOF. (1) and (2) are clear from (3.7) and (3.8).

(3) Let $\mathbf{e} = \mathbf{e}' + \mathbf{e}''$ where $\mathbf{e}' = (\alpha,\, \beta)$, $\mathbf{e}'' = (\gamma,\, \delta)$ with $\alpha \geq 1$, $\beta \geq 1$, $\gamma \geq 1$, $\delta \geq 1$ and $\alpha\delta - \beta\gamma \neq 0$.

By (3.6) and (3.7), we have

$$(\alpha+\beta+\gamma+\delta-1)(\alpha\delta-\beta\gamma-\alpha+\beta)X_\phi(\mathbf{e}) + \sqrt{-1}(\alpha+\beta)(\alpha+\gamma-1)(\beta+\delta-1)X_\psi(\mathbf{e}) = 0,$$

and

$$(\alpha+\beta+\gamma+\delta-1)(\beta\gamma-\alpha\delta-\gamma+\delta)X_\phi(\mathbf{e}) + \sqrt{-1}(\gamma+\delta)(\alpha+\gamma-1)(\beta+\delta-1)X_\psi(\mathbf{e}) = 0.$$

Since

$$\begin{vmatrix} \alpha\delta-\beta\gamma-\alpha+\beta & \alpha+\beta \\ \beta\gamma-\alpha\delta-\gamma+\delta & \gamma+\delta \end{vmatrix} = (\alpha\delta-\beta\gamma)(\alpha+\beta+\gamma+\delta-2) \neq 0,$$

$X_\phi(\mathbf{e}) = X_\psi(\mathbf{e}) = 0$.

(4) Let $\mathbf{e} = \mathbf{e}_2 + m_1\mathbf{e}_1$ or $\mathbf{e} = \mathbf{e}_{e-1} + m_e\mathbf{e}_e$. Then $X_\phi(\mathbf{e}) = X_\psi(\mathbf{e}) = 0$ follows by the condition (3.7). □

PROPOSITION 3.10. (1) *All elements in $\Lambda_{n,q}$ are classified into the above three types; inside-decomposable, edge-decomposable and proportional.*

(2) *Proportional elements which are not inside- nor edge-decomposable are;*
for the case of $e \geq 4$, $\lambda\mathbf{e}_\epsilon$ $(3 \leq \epsilon \leq e-2,\ 1 \leq \lambda \leq a_\epsilon - 1;\ \epsilon = 2$ or $e-1,\ 1 \leq \lambda \leq a_\epsilon;\ \epsilon = 1$ or $e,\ \lambda \geq 1)$,
for the case of $e = 3$, $\lambda\mathbf{e}_\epsilon$ $(\epsilon = 2,\ 1 \leq \lambda \leq a_2 + 1;\ \epsilon = 1$ or $3,\ \lambda \geq 1)$.

By Propositions 3.9 and 3.10,

PROPOSITION 3.11. (1) *For $e \geq 4$,*

$$X_\phi(\mathbf{e}) = X_\psi(\mathbf{e}) = 0$$

unless $\mathbf{e} = \lambda\mathbf{e}_\epsilon$ for $3 \leq \epsilon \leq e-2,\ 1 \leq \lambda \leq a_\epsilon - 1;\ \epsilon = 2$ or $e-1,\ 1 \leq \lambda \leq a_\epsilon$, and

$$X_\phi(\mathbf{e}_2) = X_\phi(\mathbf{e}_{e-1}) = 0.$$

(2) *For $e = 3$,*

$$X_\phi(\mathbf{e}) = X_\psi(\mathbf{e}) = 0$$

unless $\mathbf{e} = \lambda\mathbf{e}_2$ for $1 \leq \lambda \leq a_2 + 1$ and

$$X_\phi(\mathbf{e}_2) = 0.$$

PROPOSITION 3.12. (1) *If $e \geq 4$, $X_\phi(\mathbf{e}) = X_\psi(\mathbf{e}) = 0$ for $\mathbf{e} = a_\epsilon\mathbf{e}_\epsilon$ $(\epsilon = 2,\ e-1)$.*

(2) *If $e = 3$, $X_\phi((a_2+1)\mathbf{e}_2) = 0$ and $X_\psi(\lambda\mathbf{e}_2) = 0$ $(\lambda \geq 1)$.*

PROOF. (1) Recall the relation $a_2\mathbf{e}_2 = \mathbf{e}_1 + \mathbf{e}_3$.

By applying (3.6) and (3.7) to $\mathbf{e} := a_2\mathbf{e}_2$ and $\mathbf{e} := \mathbf{e}_1 + \mathbf{e}_3$, we have

$$(a_2 i_2 + a_2 j_2 - 1)(j_2 - i_2)X_\phi(\mathbf{e}) + \sqrt{-1}(i_2 + j_2)(a_2 i_2 - 1)(a_2 j_2 - 1)X_\psi(\mathbf{e}) = 0,$$

$$(i_1 + i_3 + j_1 + j_3 - 1)(i_1 j_3 - j_1 i_3 - i_1 + j_1)X_\phi(\mathbf{e}) + \sqrt{-1}(i_1 + j_1)(i_1 + i_3 - 1)(j_1 + j_3 - 1)X_\psi(\mathbf{e}) = 0.$$

Since $\begin{vmatrix} j_2 - i_2 & i_2 + j_2 \\ i_1 j_3 - j_1 i_3 - i_1 + j_1 & i_1 + j_1 \end{vmatrix} = 2\begin{vmatrix} i_1 & j_1 \\ i_2 - i_3 & j_2 - j_3 \end{vmatrix} = 2n(1-n) \neq 0$, we have $X_\phi(a_2\mathbf{e}_2) = X_\psi(a_2\mathbf{e}_2) = 0$.

$X_\phi(a_{e-1}\mathbf{e}_{e-1}) = X_\psi(a_{e-1}\mathbf{e}_{e-1}) = 0$ follows by a similar argument.

(2) First, we apply (3.4) to $\mathbf{e} := \lambda\mathbf{e}_2$.

$$\sqrt{-1}(i_2 + j_2)(\lambda i_2 - 1)(\lambda j_2 - 1)X_\psi(\mathbf{e}) = 0.$$

Next, we use the relation $(a_2+1)\mathbf{e}_2 = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$.

By applying (3.4) to $\mathbf{e} := \mathbf{e}_1 + (\mathbf{e}_2 + \mathbf{e}_3)$, we have

$$(i_1 + i_2 + i_3 + j_1 + j_2 + j_3 - 1)(i_1(j_2 + j_3) - j_1(i_2 + i_3) - i_1 + j_1)X_\phi(\mathbf{e}) + \sqrt{-1}(i_1 + j_1)(i_1 + i_2 + i_3 - 1)(j_1 + j_2 + j_3 - 1)X_\psi(\mathbf{e}) = 0.$$

Therefore we infer $X_\phi(\mathbf{e}) = 0$ from $X_\psi((a_2+1)\mathbf{e}_2) = 0$. □

Taking account of (3.1) and (3.2) and by Propositions 3.11 and 3.12 and (3.6), we have

THEOREM 3.13. *We have the following basis of* $\mathrm{Ker}\{H^1(M_{n,q}, T'M_{n,q}) \to H^1(M_{n,q}, T^{1,0}\mathbb{C}^N{}_{|M_{n,q}})\}$*:*

(1) *(The case of* $e \geq 4$*)*

$$\overline{z}^{\lambda i_\epsilon - 2}\overline{w}^{\lambda j_\epsilon - 2} Z \otimes \bar{Z}^* + \sqrt{-1}\frac{(\lambda i_\epsilon + \lambda j_\epsilon - 1)(j_\epsilon - i_\epsilon)}{(i_\epsilon + j_\epsilon)(\lambda i_\epsilon - 1)(\lambda j_\epsilon - 1)}\overline{z}^{\lambda i_\epsilon - 1}\overline{w}^{\lambda j_\epsilon - 1} T \otimes \bar{Z}^*$$

$$(\epsilon = 2, \dots, e-1,\ \lambda = 2, \dots, a_\epsilon - 1)$$

$$\overline{z}^{i_\epsilon - 2}\overline{w}^{j_\epsilon - 2} Z \otimes \bar{Z}^*\ (\epsilon = 3, \dots, e-2),\quad \overline{z}^{i_\epsilon - 1}\overline{w}^{j_\epsilon - 1} T \otimes \bar{Z}^*\ (\epsilon = 2, \dots, e-1)$$

(2) *(The case of* $e = 3$*, cf.* [**K**]*)*

$$\overline{z}^{\lambda i_2 - 2}\overline{w}^{\lambda j_2 - 2} Z \otimes \bar{Z}^*\ (\lambda = 2, \dots, a_2)$$

Bibliography

[B] Bland, J., *CR deformations for rational homogeneous surface singularities*, Science in China Series A: Mathematics **48**, Suppl. 1, 74–85.

[B-E] ——— and Epstein, C. L., *Embeddable CR-structures and deformations of pseudo-convex surfaces, Part I: Formal deformations*, J. Alg. Geom. **5** (1996), 277–368.

[K] Kodama, M., *CR description of the infinitesimal deformations of simple singularities*, preprint.

[M1] Miyajima, K., *CR construction of the flat deformations of normal isolated singularities*, J. Alg. Geom. **8** (1999), 403–470.

[Ri1] Riemenschneider, O., *Deformationen von Quotientensingularitäten (nach zyklischen Gruppen)*, Math. Ann. **209**, (1974), 211–248.

[Ri2] ———, *A note on the toric duality between the cyclic quotient surface singularities* $A_{n,q}$ *and* $A_{n,n-q}$, preprint.

[Ru] Rudin, W., *Function theory in the unit ball of* $\mathbb{C}^n$, Grundlehren der mathematischen Wissenschaften 241, Springer Verlag, 1980.

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