ON PITMAN EFFICIENCY OF SOME TESTS OF SCALE FOR THE GAMMA DISTRIBUTION

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1. Summary

A comparison is made of several two sample rank tests for scale change in Gamma distributions. A test is studied, the exponential scores test, which offers greater Pitman efficiency than some standard tests (for example, the Wilcoxon) when the shape parameter γ is small.

2. Introduction

This paper considers a two sample testing problem which has arisen in connection with weather control (Neyman and Scott [6], section 5).

We let the random variable X represent the amount of rainfall during a day of nonzero precipitation under natural weather conditions, and let Y represent the rainfall in the same region during a day of rain in which the clouds have been seeded. It is assumed that X has a Gamma distribution and that the effect of seeding is multiplicative. The problem is to test for a positive seeding effect, using m observations on X and n observations on Y.

Formally, let $F_{\gamma,\delta}$ be the distribution on the positive real line with density

$$f_{\gamma,\delta}(x) = \frac{\delta^{\gamma}}{\Gamma(\gamma)} x^{\gamma-1} e^{-\delta x}, \tag{2.1}$$

where $\gamma > 0$, $\delta > 0$. The assumptions imply that when $F_{\gamma,\delta}$ is the distribution of X, then $F_{\gamma,\delta\xi}$ is the distribution of Y, for some $\xi > 0$. The problem becomes one of using samples $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ to test the hypothesis

$$H: \xi \geqq 1 \tag{2.2}$$

against the alternative

$$K: \xi < 1. \tag{2.3}$$

Our primary goal is to determine a nonparametric test which will perform better than standard tests, such as the Wilcoxon, whose efficiency has been found to be unsatisfactory.

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3. The L Test

The problem has been considered by Taha [9], who proposes using the test statistic

$$L = \frac{1}{n} \sum_{i=1}^{n} s_i^2, \tag{3.1}$$

where the s_i are the ranks of the Y in the combined sample. Taha shows that this statistic is superior to the Wilcoxon statistic W (in the sense of asymptotic efficiency) for small values of γ, most notably in the interval $0 < \gamma \leqq 1$.

4. LMP rank test and the exponential scores

For fixed γ and δ, the locally most powerful rank test of H is independent of δ and is easily computed to be based on the "gamma scores" statistic S_γ,

$$S_\gamma = \frac{1}{n} \sum_{i=1}^{n} E[Z^{(s_i)}], \tag{4.1}$$

where $Z^{(1)} < \cdots < Z^{(N)}$ are the order statistics of a sample of size $N = n + m$ from $F_{\gamma,1}$.

Capon [1] has shown this test to be asymptotically efficient, but, unfortunately, it is dependent on γ. However, it may be hoped that for small values of γ a reasonably good test may be based on S_1, the exponential scores, which were apparently first proposed by Savage [8]. The value of S_1 may be computed by using

$$E[Z^{(s)}] = \frac{1}{n} + \cdots + \frac{1}{n - s + 1}, \tag{4.2}$$

and the test may be carried out with the help of the approximation

$$\frac{mS_1}{N - nS_1} \sim F_{2n,2m} \tag{4.3}$$

(for a more detailed discussion of this point, see Cox [3]).

5. Method for determining Pitman efficiencies

For sequences $\{S_N\}$, $\{T_N\}$ of test statistics, suppose that μ_N, ν_N, σ_N^2, and τ_N^2 are functions of ξ such that, when ξ_N is the true parameter value,

$$\frac{S_N - \mu_N(\xi_N)}{\sigma_N(\xi_N)} \quad \text{and} \quad \frac{T_N - \nu_N(\xi_N)}{\tau_N(\xi_N)} \tag{5.1}$$

tend to $N(0, 1)$ for all sequences

$$\xi_N = 1 - kN^{-1/2}, \tag{5.2}$$

where $k \geqq 0$. Then, under suitable regularity conditions (see, for example, Noether [7]), the asymptotic Pitman efficiency of $\{S_N\}$ relative to $\{T_N\}$ is given by

$$e(S, T) = \lim_{N \to \infty} \frac{e_N(S)}{e_N(T)}, \tag{5.3}$$

where

$$e_N(S) = \left[\frac{\mu_N'(1)}{\sigma_N(1)}\right]^2, \tag{5.4}$$

with $e_N(T)$ defined analogously. The quantity $e_N(S)$ is the so called "efficacy" of S_N.

For the results that follow, we may apply the above procedure by assuming that

$$\lambda_N = \frac{m}{n} \to \lambda, \tag{5.5}$$

where $0 < \lambda < 1$.

Since the distribution of any rank statistic depends only on γ and ξ and is independent of the scale parameter δ, it follows that the efficacy of a rank test depends only on γ. For ease of computation, we take $\delta = 1$ and denote the distribution functions of $F_{\gamma,1}$ and $F_{\gamma,\xi}$ by F and G_ξ, respectively.

6. Computation of the efficacies

6.1. *Wilcoxon.* See Taha [9] for the result

$$e_N(W) \sim 12\lambda(1 - \lambda)\, N \left[\int_0^\infty x f^2(x)\, dx\right]^2. \tag{6.1}$$

6.2. *Gamma scores.* Asymptotic normality of S_γ follows from theorems 1 and 2 of Chernoff and Savage [2], with

$$\mu_N(\xi) = \int F^{-1}\left(\frac{m}{N} F + \frac{n}{N} G_\xi\right) dG_\xi \tag{6.2}$$

and

$$\sigma_N^2(\xi) = \text{Var}\,(S_\gamma | \xi). \tag{6.3}$$

Differentiation of (6.2) yields

$$\mu_N'(1) = \frac{m}{N}\gamma. \tag{6.4}$$

It can be shown that under the hypothesis that X and Y come from the same continuous distribution,

$$\text{Var}\,(S_\gamma) \sim \frac{m}{nN}\gamma. \tag{6.5}$$

Therefore,

$$e_N(S_\gamma) \sim \lambda(1 - \lambda)N\gamma. \tag{6.6}$$

6.3. *Exponential scores.* The procedure here is similar to that of 6.2, but we use

$$\mu_N(\xi) = \int F_{1,1}^{-1}\left(\frac{m}{N}F + \frac{n}{N}G_\xi\right) dG_\xi \tag{6.7}$$
$$= -\int \log\left[1 - \left(\frac{m}{N}F + \frac{n}{N}G_\xi\right)\right] dG_\xi.$$

Differentiation yields

$$\mu_N'(1) = \frac{m}{N}\int_0^\infty \frac{xf^2(x)}{1 - F(x)}\, dx, \tag{6.8}$$

which, when combined with (6.5) for $\gamma = 1$, gives

$$e_N(S_1) \sim \lambda(1 - \lambda)\, N\left[\int_0^\infty \frac{xf^2(x)}{1 - F(x)}\, dx\right]^2. \tag{6.9}$$

6.4. *L test.* Taha [9] has obtained

$$e_N(L) \sim 45\lambda(1 - \lambda)N\left[\int_0^\infty xf^2(x)F(x)\, dx\right]^2. \tag{6.10}$$

7. Efficiency results

The results of several cloud seeding experiments (again, see [6], section 5) suggest that the parameter γ regularly lies in the interval (0.45, 0.75). In line with this, efficiencies have been computed for the value $\gamma = 0.50$ and $\gamma = 0.65$, as well as for the case in which the underlying distribution is exponential, $\gamma = 1$. In addition, the behavior of the efficiencies as $\gamma \to 0$ follows from the asymptotic relation

$$-\gamma \log \gamma = O\left[\int_0^\infty \frac{xf^2(x)}{1 - F(x)}\, dx\right], \tag{7.1}$$

which can be established by bounding the integrand appropriately.

TABLE I

EFFICIENCIES

The value 0.82 for $e(L, S_1)$ with $\gamma = 0.65$ was obtained by interpolating in the graph of Taha [9].

γ	$e(W, S_1)$	$e(L, S_1)$	$e(S_\gamma, S_1)$
1.00	0.75	0.87	1.00
0.65	0.67	0.82	1.01
0.50	0.63	0.75	1.03
Limit at zero	0	0	∞

The efficiency results imply that in the interval of relevance to the cloud seeding experiment, the exponential scores test is asymptotically almost as good as the LMP rank test S_γ, and apparently offers an improvement over the L test. The performance of the Wilcoxon test is poor. Presumably equally poor are the performances of other nonparametric tests commonly used in weather control problems, namely the Kolmogorov-Smirnov and median tests ([6], figure 6), but their behavior will not be investigated here.

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