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INTEGRABLE CURVES AND SURFACES

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Abstract. The surfaces in three dimensional Euclidean space $\mathbb{R}^3$ obtained through the use of the soliton techniques are called integrable surfaces. Integrable equations and their Lax equations possess certain symmetries. Infinitesimal versions of these symmetries are deformations which are responsible in constructing the integrable surfaces. There are four different types of deformations. The spectral parameter, the gauge, the generalized symmetries and integration parameters deformations. We shall present here how these deformations generate surfaces in $\mathbb{R}^3$ and also in three-dimensional Minkowski $\mathbb{M}_3$ space. The key point here is to start with an integrable equation and its Lax representation. In this work we assume that the Lax equations of integrable equations are given in terms of a group G and its algebra $\mathfrak{g}$ valued functions. The surfaces in $\mathbb{R}^3$ are also represented via respect $\mathfrak{g}$ valued functions. In constructing integrable surfaces we need the solutions of both the integrable equations and their corresponding Lax equations. In this work we use the one soliton solutions of the integrable equations. We solve the Lax equations for one soliton solutions of the integrable equations. Then choosing a deformation one can construct several types of surfaces. After obtaining these surfaces the next is to search for their properties. Most of these surfaces are Weingarten surfaces, Willmore-like surfaces and surfaces which are derivable from a variational principle. We give sketches of the interesting surfaces of Korteweg-de Vries (KdV), modified Korteweg-de Vries (mKdV) and Nonlinear Schrödinger (NLS), sine Gordon (SG) equations.

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CONTENTS

1. Introduction

Differential geometry of curves and surfaces in the three dimensional Euclidean space $\mathbb{R}^3$ is a natural source of nonlinear partial differential equations [10], [11]. Motions of curves and surfaces in $\mathbb{R}^3$ or in $\mathbb{M}_3$ (three dimensional Minkowski space) are responsible for some integrable nonlinear partial differential equations such as NLS equation [20], KdV and mKdV equations [18], [27], [39].

Surface theory in $\mathbb{R}^3$ is widely used in different branches of science, particularly mathematics (differential geometry, topology, Partial Differential Equations (PDEs)), theoretical physics (string theory, general theory of relativity), and biology [4], [8], [33], [38], [51], [52]. There are some special subclasses of surfaces which arise in the branches of science aforementioned. For the classification of surfaces in $\mathbb{R}^3$, particular conditions are imposed on the Gaussian and mean curvatures. These conditions are sometimes given as algebraic relations between curvatures and sometimes given as differential equations for these two curvatures.

Here are some examples of some subclasses of surfaces:

i) Minimal surfaces: $H = 0$.
ii) Surfaces with constant mean curvature: $H = \mathrm{const}$.
iii) Surfaces with constant positive Gaussian curvature: $K = \mathrm{const} > 0$.
iv) Surfaces with constant negative Gaussian curvature: $K = \mathrm{const} < 0$.

v) Surfaces with harmonic inverse mean curvature: $\nabla^2(1/H) = 0$.

vi) Bianchi surfaces: $\nabla^2(1/\sqrt{K}) = 0$ and $\nabla^2(1/\sqrt{-K}) = 0$, for positive Gaussian curvature and negative Gaussian curvature, respectively.

vii) Weingarten surfaces: $f(H, K) = 0$. For example: linear Weingarten surfaces, $c_1 H + c_2 K = c_3$, and quadratic Weingarten surfaces: $c_4 H^2 + c_5 H K + c_6 K^2 + c_7 H + c_8 K = c_9$, where c_j are constants, $j = 1, 2, ..., 9$.

viii) Willmore surfaces: $\nabla^2 H + 2 H(H^2 - K) = 0$.

ix) Surfaces that solve the shape equation of lipid membranes

$$k_c \nabla^2(2H) + k_c(2H + c_0)(2H^2 - c_0 H - 2K) + p - 2\,\omega H = 0$$

where p, ω, k_c, and c_0 are constants.

Here, H and K are the mean and Gaussian curvatures of the surface, respectively.

On the other hand soliton equations play a crucial role for the construction of surfaces. The theory of nonlinear soliton equations was developed in 1960s. Lax representation of integrable equations should exist in order to apply inverse scattering method for finding solutions of these integrable equations. For details of integrable equations one may look at [1] and [9], and references therein. Lax representation of nonlinear PDEs consists of two linear equations which are called Lax equations

$$\Phi_x = U\,\Phi, \qquad \Phi_t = V\,\Phi \tag{1}$$

and their compatibility condition

$$U_t - V_x + [U, V] = 0 \tag{2}$$

where x and t are independent variables. Here U and V are the so called Lax pairs. They depend on independent variables x and t, and a spectral parameter λ. Hereafter the subscripts x and t denote the partial derivatives of the object with respect to x and t, respectively. For our cases, U and V will be 2×2 matrices and they are in a given Lie algebra $\mathfrak{g}$. Equation (2) is called also the zero curvature condition. Integrable equations arise as the compatibility conditions, equation (2), of the Lax equations (1). Since Gauss-Mainardi-Coddazi (GMC) equations are compatibility conditions of Gauss-Weingarten (GW) equations, there is a close relationship between surfaces and Lax equations. GW equations and Lax equations play similar roles but they are not exactly the same. While Lax equations depend on spectral parameters, GW equations do not. Moreover GW equations are written in terms of 3×3 matrices whereas Lax pairs are 2×2 matrices. The former problem can be solved easily by inserting spectral parameters in GW equations using the one dimensional symmetry group of GW equations. The latter problem was solved by Sym [42]. By making use of the isomorphism $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$, he rewrote the GW equations in terms of 2×2 matrices. So for integrable surfaces, GW equations can be written in terms of 2×2 matrices using the conformal parametrization.

Surfaces and integrable equations can be related by the analogy between GW equations and Lax equations. Such a relation is established by the use of Lie groups and Lie algebras. Using this relation, soliton surface theory was first developed by Sym [40–42]. He studied the surface theory in both directions: from geometry to solitons and from solitons to geometry. In the first direction, he obtained some well known soliton equations as a consequence of GMC equations. In the second direction, he obtained the following formula using the deformation of Lax equations for integrable equations

$$F = \Phi^{-1}\frac{\partial \Phi}{\partial \lambda}$$

which gives a relation between a family of immersions (F) into the Lie algebra and the Lax equations for given Lax pairs. Fokas and Gelfand [12] generalized Sym's formula as

$$F = \alpha_1 \Phi^{-1} U\, \Phi + \alpha_2 \Phi^{-1} V\, \Phi + \alpha_3 \Phi^{-1}\frac{\partial \Phi}{\partial \lambda} + \alpha_4\, x\, \Phi^{-1} U\, \Phi \\ + \alpha_5\, t\, \Phi^{-1} V\, \Phi + \Phi^{-1} M\, \Phi$$

where α_i, $i = 1, 2, 3, 4, 5$ and $M \in \mathfrak{g}$ are constants. So by this technique, which is called *the soliton surface technique*, using the symmetries of the integrable equations and their Lax equations we can find a large class of soliton surfaces for given Lax pairs. One may find surfaces developed by soliton surface technique, which belong to subclasses of the surfaces, mentioned in i)- ix) on page 3, in the references [3–7], [12–19], [22–25], [28], [40–46].

On the other hand, there are some surfaces that arise from a variational principle for a given Lagrange function, which is a polynomial of degree less than or equal to two in the mean curvature of the surfaces. Examples of this type are minimal surfaces, constant mean curvature surfaces, linear Weingarten surfaces, Willmore surfaces, and surfaces solving the shape equation for the Lagrange functions. By taking more general Lagrange function of the mean and Gaussian curvatures of the surface, we may find more general surfaces that solve the generalized shape equation (see [21], [29], [33–35], [47–50]). Examples for this type of surfaces can be found in [14–16], [43–46].

Examples of some of these surfaces like Bianchi surfaces, surfaces for which the inverse of the mean curvature is harmonic [4], and the Willmore surfaces [51], [52] are very rare. The main reason is the difficulty of solving the corresponding differential equations. For this purpose, some indirect methods [3–7], [12–19], [22–25], [28], [40–46] have been developed for the construction of surfaces in $\mathbb{R}^3$ and in $\mathbb{M}_3$. Among these methods, soliton surface technique is very effective. In this method, one mainly uses the deformations of the Lax equations of the integrable equations. This way, it is possible to construct families of surfaces corresponding to some integrable equations such as SG, KdV, mKdV and NLS equations [5], [12–18], [40–46] belonging to the afore mentioned subclasses of two-surfaces in a three dimensional flat geometry. In particular, using the symmetries of the integrable equations and their Lax equation, we arrive at different classes of surfaces. There are many attempts in this direction and examples of new two surfaces. Konopelchenko uses generalized Weierstrass formulae for inducing surfaces [22–25]. Konopelchenko gives connection between linear Schrödinger equation and the KdV equation with the surfaces of revolution immersed into the three dimensional space S^3 of constant curvature [25]. He also has used Davey-Stewartson hierarchy [24], mKdV equation [23] to induce surfaces Throogh generalized Weierstrass formulae.

This work is a collection mainly of the authors' publications on surfaces and curves, in particular on soliton surfaces [5], [14], [15], [17–19], [43–46].

2. A Brief Introduction to Curves and Surfaces in $\mathbb{R}^3$

Curves in $\mathbb{R}^3$: Let us define a curve in $\mathbb{R}^3$ as a map $\alpha : I \to \mathbb{R}^3$, where I is an interval in $\mathbb{R}$. Every smooth curve has a defined and differentiable tangent line. Here $\alpha(t)$ denotes the position vector at every point of curve for $t \in I$. At every point of the curve, the tangent vector is defined as

$$\mathbf{t} = \frac{\mathrm{d}\alpha}{\mathrm{d}t}\cdot$$

We assume also that t is the arc length parameter. In this case, the length of the tangent vector is one, i.e., $|\mathbf{t}| = 1$. Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ defines a triad at every point of curve and forms a base at that point. Here $\mathbf{t}, \mathbf{n}$, and $\mathbf{b}$ denote tangent, normal, and binormal vectors, respectively. If $\langle\, , \rangle$ defines the standard inner product in $\mathbb{R}^3$, then the triad forms an orthonormal basis with respect to this inner product i.e.,

$$\langle \mathbf{t}, \mathbf{t}\rangle = \langle \mathbf{n}, \mathbf{n}\rangle = \langle \mathbf{b}, \mathbf{b}\rangle = 1.$$

Other combinations of the inner product are zero. This triad is called *Serret-Frenet* (SF) triad and its change with respect to t is defined by the following SF equations

$$\dot{\mathbf{t}} = k\mathbf{n}, \qquad \dot{\mathbf{n}} = -k\mathbf{t} - \tau\mathbf{b}, \qquad \dot{\mathbf{b}} = \tau\mathbf{n}. \tag{3}$$

Here k and τ are curvature and torsion functions, respectively, which characterize the curve. The dot on top of the letters denotes the derivative with respect to variable t. The functions k and τ define the curve explicitly (we assume rotational and translational symmetric surfaces as same, *isometric surfaces*). Now we give more closed form of SF equations that we are going to use later. Let E and Ω be defined, respectively, as

$$E = (\mathbf{t}, \mathbf{n}, \mathbf{b})^T, \qquad \Omega = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}. \tag{4}$$

Here Ω is an antisymmetric and traceless matrix. We can write the SF equations in terms of E and Ω as follows

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \Omega E.$$

Since we are working with smooth curves, we assume that k and τ are infinitely differentiable functions. This condition might be weaken, but we assume that they are sufficiently differentiable functions. Since the torsion is zero ($\tau = 0$) for the plane curves, the SF equations (3) become more simple, namely

$$\dot{\mathbf{t}} = k\mathbf{n}, \qquad \dot{\mathbf{n}} = -k\mathbf{t}.$$

As we will see in the following sections for many of the integrable equations, plane curves will be enough.

If instead of $\mathbb{R}^3$, we take three dimensional Minkowski space ($\mathbb{M}_3$), the SF equations will be different. Let $\langle\,,\rangle$ be inner product in $\mathbb{M}_3$. The orthonormal base $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ satisfy the following orthogonal conditions

$$\langle \mathbf{t}, \mathbf{t}\rangle = 1, \qquad \langle \mathbf{n}, \mathbf{n}\rangle = -1, \qquad \langle \mathbf{b}, \mathbf{b}\rangle = -1.$$

With this orthogonal conditions SF equations take the following form

$$\dot{\mathbf{t}} = k\mathbf{n}, \qquad \dot{\mathbf{n}} = k\mathbf{t} - \tau\mathbf{b}, \qquad \dot{\mathbf{b}} = \tau\mathbf{n}.$$

As we will see that the curves corresponding to some differential equations will not be in $\mathbb{R}^3$. For that reason signature change will be very important. For the relation between soliton equations and SF equations in different three dimensional geometries ($\mathbb{R}^3$ or $\mathbb{M}_3$) with different signature see [18].

Surfaces in $\mathbb{R}^3$: Let us define a surface in $\mathbb{R}^3$ as a map $\mathbf{Y} : \mathcal{O} \to \mathbb{R}^3$, where $\mathcal{O}$ is an open set in $\mathbb{R}^2$. Position vector of the surface at every point is defined as $\mathbf{Y}(x,t) = (y^1(x,t), y^2(x,t), y^3(x,t))$, where $(x,t) \in \mathcal{O}$. Since we will work with smooth surfaces, similar to SF triad for curves, we can define a triad $\{\mathbf{Y}_x, \mathbf{Y}_t, \mathbf{N}\}$ which is defined at every point of surface and forms a basis for $\mathbb{R}^3$ at these points. Here $\mathbf{Y}_{,x}$ and $\mathbf{Y}_{,t}$ are the tangent vectors of the surface defined at at all points of the surface. The subscripts x and t denote the partial derivatives with respect to the variables x and t, respectively. $\mathbf{N}$ is a unit normal vector which is differentiable at every point of the surface. For the smooth surfaces $\mathbf{N}$ is given as

$$\mathbf{N} = \frac{\mathbf{Y}_{,x} \times \mathbf{Y}_{,t}}{|\mathbf{Y}_{,x} \times \mathbf{Y}_{,t}|}.$$

The equations which gives the change of this triad is called Gauss-Wengarten (GW) equations and they are given as

$$\mathbf{Y}_{,ij} = \Gamma^k_{ij}\,\mathbf{Y}_{,k} + h_{ij}\,\mathbf{N}, \qquad \mathbf{N}_{,i} = -g^{kl}h_{li}\,\mathbf{Y}_{,k} \tag{5}$$

where $\mathbf{Y}_{,i}$ are tangent vectors of the surface, $i = 1, 2$, $x^1 = x$ and $x^2 = t$. In this work, we use Einstein's summation convention on repeated indices $i, j, k, l = 1, 2$.

Here g_{ij} and h_{ij} denote the coefficients of the first and second fundamental forms, respectively. We can find the fundamental forms as

$$g_{ij} = \langle \mathbf{Y}_{,i}, \mathbf{Y}_{,j} \rangle, \qquad h_{ij} = \langle \mathbf{N}, \mathbf{Y}_{,ij} \rangle = -\langle \mathbf{N}_{,i}, \mathbf{Y}_{,j} \rangle. \tag{6}$$

As we see in equations (6), both of the coefficient matrices are symmetric in indices i and j. Here $\langle \, , \rangle$ is a standard inner product in $\mathbb{R}^3$. g^{ij} is the elements of the inverse matrix (g^{-1}) of the matrix g. The matrix g is also called as metric tensor. We can obtain all the interior local properties of the surface using the matrix g. The Christoffel symbol Γ^i_{jk} is defined as

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (g_{lj,k} + g_{lk,j} - g_{jk,l}). \tag{7}$$

GW equations for surfaces are equivalent to SF equations for curves. Since SF equations are ordinary differential equations, they do not have compatibility problem. On the other hand GW equations are partial differential equations. For that reason we need to check the compatibility of the equations. When we take the derivative of the GW equations (5) and take the antisymmetric parts give us two new equations class. These equations are called Gauss-Codazzi (GC) equations and they have the form

$$R^i_{jkl} = g^{im}(h_{mk}h_{jl} - h_{ml}h_{jl}), \qquad h_{ij,k} - \Gamma^m_{ik} h_{mj} = h_{ik,j} - \Gamma^m_{ij} h_{mk} \tag{8}$$

where R^i_{jkl} are components of the Riemann curvature tensor. In the (8), first equation is known as Gauss equation and the second equation is known as Codazzi equations. We do not have additional compatibility equations to equations (8). The Gaussian and mean curvatures of a surface in $\mathbb{R}^3$ are given as

$$K = \det(g^{-1}h), \qquad H = \frac{1}{2}\mathrm{trace}(g^{-1}h). \tag{9}$$

Now we give the following local proposition.

Proposition 1. *Let $\mathbf{Y}_{,x}(x,t)$ and $\mathbf{Y}_{,t}(x,t)$ be two independent differentiable vectors in $\mathbb{R}^3$. If $\mathbf{Y}_{,xt} = \mathbf{Y}_{,tx}$, then there exist a unique surface that accept these vectors as its tangent vectors at every point of it.* (*It is unique except isometric ones.*)

The importance of this proposition is that without knowing the position vector $\mathbf{Y}(x,t)$ we can find local properties of a surface. We can find fundamental forms using equations (6), and the Gaussian and mean curvatures using equation (9). These are enough to obtain local properties of the surface. First and second fundamental forms are given respectively by

$$\mathrm{d}s_I^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j, \qquad \mathrm{d}s_{II}^2 = h_{ij}\mathrm{d}x^i\mathrm{d}x^j.$$

As an example, if we consider the first fundamental form as

$$\mathrm{d}s_I^2 = \sin^2\theta \mathrm{d}x^2 + \cos^2\theta \mathrm{d}t^2$$

the Gaussian curvature satisfy the following equation

$$\theta_{xx} - \theta_{tt} = \frac{1}{2} K \sin(2\theta).$$

If we take the first fundamental form as

$$\mathrm{d}s_I^2 = \mathrm{d}u^2 + \mathrm{d}v^2 - 2\cos\theta \mathrm{d}u\mathrm{d}v$$

the Gaussian curvature satisfy respectively the equation

$$\theta_{uv} = K \sin\theta. \tag{10}$$

Even though the surfaces characterized by these two first fundamental forms looks different, actually they are isometric to each other (through the transformation $x = u+v, t = u-v$). When the Gaussian curvature is constant, we have different forms of the sine-Gordon equations. The condition for these surfaces to be in $\mathbb{R}^3$ is finding the second fundamental form such that they satisfy GC compatibility equations. Classical example for these type surfaces is sphere. In this case h and g are equal to each other and for the unit sphere Gaussian and mean curvatures are one. The relation between Sine-Gordon and surface of sphere is a classical example. This relation goes back to the mathematicians such as Bianchi and Bäcklund. For the history see [10].

Because we are going to use it later, we express vectors in $\mathbb{R}^3$ in terms of 2×2 matrix representation of $\mathfrak{su}(2)$ Lie algebra. In order to do that we use Pauli-sigma matrices given as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{11}$$

We can write $\mathfrak{su}(2)$ valued representation of a vector Y in $\mathbb{R}^3$ as

$$F(x,t) = \mathrm{i} \sum_{k=1}^{3} Y^k \sigma_k \tag{12}$$

where Y^k, $k = 1,2,3$ are components of the vector F. We can write the vector F given in equation (12) more explicitly as

$$F = \mathrm{i} \begin{pmatrix} Y^3 & Y^1 - \mathrm{i}Y^2 \\ Y^1 + \mathrm{i}Y^2 & -Y^3 \end{pmatrix}.$$

In this representation, the inner product of two vectors is defined as

$$\langle F, G \rangle = -\frac{1}{2}\ \mathrm{trace}(FG)$$

where $F, G \in \mathfrak{su}(2)$. The length of the vector is defined as

$$\|F\| = \sqrt{|\langle F, F\rangle|}.$$

If F is the $\mathfrak{su}(2)$ representation of the position vector $\mathbf{Y}(x,t)$, then F_x and F_t are the $\mathfrak{su}(2)$ representation of the tangent vectors $\mathbf{Y}_{,x}$ and $\mathbf{Y}_{,t}$. If we let $\mathfrak{su}(2)$ representation of unit normal vector N as Z, we find it as

$$Z = \frac{[F_x, F_t]}{\|[F_x, F_t]\|}.$$

Here $[.\,,.]$ denotes the usual commutator. Hence we can give the $\mathfrak{su}(2)$ representation of a triad defined at every point of a surface as

$$\{F_x, F_t, Z\}.$$

Because of Proposition 1 finding this triad means that obtaining the surface. In soliton theory, surfaces are developed in this way. Fundamental forms are given in the following way

$$g = \begin{pmatrix} \langle F_x, F_x\rangle & \langle F_x, F_t\rangle \\ \langle F_x, F_t\rangle & \langle F_t, F_t\rangle \end{pmatrix}, \qquad h = -\begin{pmatrix} \langle F_x, Z_x\rangle & \langle F_x, Z_t\rangle \\ \langle F_t, Z_x\rangle & \langle F_t, Z_t\rangle \end{pmatrix}$$

and the Gaussian and mean curvatures take the following forms

$$K = \det(g^{-1}h), \qquad H = \frac{1}{2}\mathrm{trace}(g^{-1}h).$$

3. Integrable Equations

There is no unique definition of *integrability* in the literature. Everyone uses his own definition. We will give some of these. If a given PDE satisfy one of the following condition it is called *integrable*

i) It has a Lax representation
ii) It has Painleve property
iii) It has zero curvature representation
iv) It has Bäcklund transformation
v) There exist infinitely many conserved quantities
vi) It has a recursion operator.

In this section, we will give examples for Lax representation. Lax equations have different formulations depending on the algebra of the Lax operator. For example, Lax operator can be in pseudo-differential operator algebra, polynomial algebra, or matrix algebra. In this work, we will consider the Lax representation in the matrix algebra. We need to use higher order matrices for the system of PDEs. Since we will work with a single PDE, 2×2 matrices will be enough.

Definition 2 (Lax Equations). *Let* $\Phi(x,t,\lambda)$ *be* $\mathrm{SU}(2)$ *valued function such that* $(x,t) \in \mathcal{O} \subset \mathbb{R}^2$*, and* $\lambda \in \mathbb{C}$ *is spectral parameter. Lax equations are defined as*

$$\Phi_x = U\Phi, \qquad \Phi_t = V\Phi \tag{13}$$

where $U(x,t,\lambda)$ *and* $V(x,t,\lambda)$ *are* $\mathfrak{su}(2)$ *valued functions and they satisfy the following equation*

$$U_t - V_x + [U,V] = 0. \tag{14}$$

equation (14) *is the compatibility condition of the equation* (13)*. The matrices* U *and* V *are known as Lax pairs.*

Equation (13) defines a $\mathfrak{su}(2)$ valued connection and equation (14) shows that the curvature of this connection is flat. In differential geometry, equation (14) is also called as zero curvature condition.

Example 1 (Sine-Gordon Equation). *If we consider the following Lax pairs*

$$U = \frac{\mathrm{i}}{2}(-u_x\sigma_1 + \lambda\sigma_3), \qquad V = \frac{\mathrm{i}}{2\lambda}(\sin(u)\sigma_2 - \cos(u)\sigma_3) \tag{15}$$

then the function $u(x,t)$ *satisfy the sine-Gordon equation*

$$u_{xt} = \sin(u) \tag{16}$$

where λ *is the spectral parameter. The* Sine-Gordon *equation* (16)*, is a result of the compatibility condition or the zero curvature condition equation* (14)*. In other words, in order to be compatible the Lax pairs given in equation* (15)*, and* Sine-Gordon *equation, equation* (16) *should be satisfied. Using the Lax equations of the sine-Gordon equation, we can find some of its properties such as Bäcklund transformation, N-soliton solution, infinitely many conservation laws. For more details see* [1], [9], [19].

Example 2 (mKdV Equation). *The Lax pairs of* mKdV *equation are given as*

$$U = \frac{\mathrm{i}}{2}(\lambda\sigma_3 - u\sigma_1), \qquad V = -\frac{\mathrm{i}}{2}((\lambda^3 - \frac{1}{2}\lambda u^2)\sigma_3 + v_1\sigma_1 + v_2\sigma_2) \tag{17}$$

where $v_1 = u_{xx} + u^3/2 - \lambda^2 u$*,* $v_2 = -\lambda u_x$*. Here the function* $u(x,t)$ *satisfies the* mKdV *equation*

$$u_t = u_{xxx} + \frac{3}{2}u^2 u_x. \tag{18}$$

Same as the sine-Gordon equation using the above Lax pairs, we can obtain most of the fundamental properties of the mKdV *equation.*

4. Differential Equations, Curves and Surfaces

We shall be interested in curves where their curvature k and torsion τ satisfy certain coupled nonlinear partial differential equations. These equations are in general nonlinear. From the fundamental theorem of the local theory of curves, there exists, up to isometries, a unique curve for the given functions k and τ. Hence every distinct solution of the partial differential equations satisfied by k and τ define a unique, up to isometries, a unique curve in $\mathbb{R}^3$ or in $\mathbb{M}_3$. For illustration we shall consider here only the plane curves. In this case, given the curvature function k , up to isometries, we can determine the corresponding curve uniquely. As an example let $k = 1/\cosh^2 s$, then the corresponding curve is a catenary parameterized by $\alpha = (s, \cosh s)$. In this work, we would like to study the relation between the soliton equations and the plane curves. Hence we shall not give further examples to obtain curves from the given curvature functions. In the next section we shall see how the solutions of the soliton equations, such as the Korteweg-de Vries equation satisfied by k are related to plane curves.

From Curves to Differential Equations: SF equations defines how the SF triad, $E = (\mathbf{t}, \mathbf{n}, \mathbf{b})^T$, defined at every point of the curve moves along the curve. If the curve moves on the surface S, at the same time, we should be able to write how it changes in the direction of the movement. In order to do that, first we parameterize the surface. Let the surface be parameterized as $(s, t) \in \mathcal{O} \to S$, such that s is arc length parameter and t is the second parameter of the surface S. The movement of the curve is defined in terms of the derivative of the SF triad with respect to variable t as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \Gamma\, E. \tag{19}$$

In addition to the SF equation, the equation (19) determines the change in the t direction. Here Γ is a traceless 3×3 matrix. The entries of this matrix is not free. As we mentioned earlier, SF equations can be written in the following form

$$\frac{\mathrm{d}E}{\mathrm{d}s} = \Omega E$$

where Ω is given by equation (4). The compatibility condition of SF equations and the equation (19) defines the t change and gives the equation

$$\Omega_t - \Gamma_s + \Omega\,\Gamma - \Gamma\,\Omega = 0. \tag{20}$$

Using the equation (20), we can find the entries of Γ in terms of the curvature k and the torsion τ. At the same time, for special choices, the differential relation arise between k and τ. To explain it better, we consider plane curves. We will find it by using the change of the position vector α with respect to s and t. They are given by

$$\frac{\mathrm{d}\alpha}{\mathrm{d}s} = \mathbf{t}, \qquad \frac{\mathrm{d}\alpha}{\mathrm{d}t} = p\mathbf{n} + w\mathbf{t} \tag{21}$$

where p and w are some functions of s and t. The compatibility conditions of the equations (21) have as a result the equations

$$w_s = k\,p, \qquad \mathbf{t}_t = (p_s + kw)\mathbf{n}. \tag{22}$$

The second equation in (22) gives the t change of the tangent vector. If we consider the compatibility conditions of this equation with SF equation $\mathrm{d}\mathbf{t}/\mathrm{d}s = k\,\mathbf{n}$, we obtain the equations

$$k_t = (p_s + w\,k)_s, \qquad \mathbf{n}_t = -(p_s + kw)\mathbf{t}. \tag{23}$$

We do not obtain any further new conditions from the second equation as the compatibility condition and s derivative of the normal vector from SF equation $\mathrm{d}\mathbf{n}/\mathrm{d}s = -k\mathbf{t}$. Hence the entries of the Γ matrix are found. On the other hand the first equation in (23) implies that the curvature k should satisfy a differential equation. If we substitute w given by equation (22) into this equation, we obtain

$$k_t = D^2p + k^2p + k_s \int kp\,\mathrm{d}s \tag{24}$$

where $D = \partial/\partial s$. This equation reminds us the recursion operator $\mathcal{R}$ of the mKdV equation. The above equation takes the appropriate form with a simple calculation

$$k_t = \mathcal{R}\,p, \qquad \mathcal{R} = D^2 + k^2 + k_s D^{-1} k \tag{25}$$

where $\mathcal{R}$ is the recursion operator of mKdV equation and D^{-1} is the integral operator. For example, if we take $p = k_s$, k satisfies the mKdV equation

$$k_t = k_{sss} + \frac{3}{2}k^2 k_s. \tag{26}$$

But in general, $p(s,t)$ is a free function and if we take $p = \mathcal{R}^n k_s$, the equation (25) provides the mKdV hierarchy, where n is positive integer. Every solution of the mKdV equation, especially soliton solution, gives different curves in the plane. As far as we know, this side of the problem has not been worked that much. In other words, the local and general properties of the plane curves that correspond to mKdV equation and its hierarchy has not been studied. It is also possible to plot these surfaces with the computer's aid.

Another point that we should mention is arbitrary choice of the function p results different curves whereas the equation satisfied by k does not have to be integrable. For example, if we choose $p = \mathrm{e}^k$, the equation we obtain from equation (25) is not integrable.

Here we used plane curves as an application. Similarly it can be done for $\mathbb{R}^3$. In $\mathbb{R}^3$, there will be a separate equation for the torsion function τ. Following the similar approach, a coupled nonlinear PDEs are obtained for k and τ. For more details see [18] . Furthermore, Minkowski plane curves are also studied in that article. In this way, some new equations are obtained those could not be obtained from plane curves in $\mathbb{R}^3$. As an example, Let us consider a three dimensional general space with the signature $1+2\epsilon$ (ϵ is 1 and -1 for $\mathbb{R}^3$ and $\mathbb{M}_3$, respectively). We can find mKdV equation with both signature on plane curves as $k_t = k_{sss} + (3/2)\epsilon k^2\, k_s$ using the method described above.

Additionally for obtaining the NLS equation one can look [20] and [26]. Hasimoto is trying to find relationship between tornado in nature and solutions of the NLS equation.

5. Theory of Soliton Surfaces

Soliton surface technique is a method to construct surfaces in $\mathbb{R}^3$ and in $\mathbb{M}_3$. In this technique, the main tool is the deformations of Lax equations of integrable equations. In the literature, there are certain surfaces corresponding to certain integrable equations like SG, sinh-Gordon, KdV, mKdV, and NLS equations [3–5], [12–18], [28], [40–46]. Symmetries of the integrable equations for given Lax pairs play the crucial role in this method which was first developed by Sym [40–42] and then it was generalized by Fokas and Gel'fand [12], Fokas *et al* [13], [5] and Cieśliński [7]. Now by considering surfaces in a Lie group and in the corresponding Lie algebra, we give the general theory.

Let G be a Lie group and $\mathfrak{g}$ be the corresponding Lie algebra. We give the theory for dim $\mathfrak{g} = 3$ but it is possible to generalize it for any finite dimension n. Assume that there exists an inner product $\langle\, ,\rangle$ on a Lie algebra $\mathfrak{g}$ such that for g_1, $g_2 \in \mathfrak{g}$ as $\langle g_1, g_2\rangle$. Let $\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\}$ be the orthonormal basis in $\mathfrak{g}$ such that $\langle \mathrm{e}_i, \mathrm{e}_j\rangle = \delta_{ij}$ $(i, j = 1, 2, 3)$, where δ_{ij} is the Kronecker delta.

Let Φ be a G valued differentiable function of x, t, and λ for every $(x, t) \in \mathcal{O} \subset \mathbb{R}^2$ and $\lambda \in \mathbb{R}$. So a map can be defined from tangent space of G to the Lie algebra $\mathfrak{g}$ as

$$\Phi_x\, \Phi^{-1} = U, \qquad \Phi_t\, \Phi^{-1} = V \tag{27}$$

where Φ_x and Φ_t are the tangent vectors of Φ, U and V are functions of x, t and λ, and take values in $\mathfrak{g}$.

The function Φ defined by equation (27) exists if and only if U and V satisfy the following equation

$$U_t - V_x + [U, V] = 0 \tag{28}$$

where $[\, ,]$ is the Lie algebra commutator such that $[\mathrm{e}_i, \mathrm{e}_j] = c_{ij}^k\, \mathrm{e}_k$, $i, j = 1, 2, 3$, and c_{ij}^k are structural constants of $\mathfrak{g}$. Repeated indices are summed up from 1 to 3.

Indeed, Φ exists if and only if the equations given in equation (27) are compatible. To prove that, we differentiate the first and second equations in equation (27) with respect to t and x, respectively and we obtain the following equations

$$\Phi_{xt}\,\Phi^{-1} = U_t + \Phi_x\,\Phi^{-1}\,\Phi_t\,\Phi^{-1}, \qquad \Phi_{tx}\,\Phi^{-1} = V_x + \Phi_t\,\Phi^{-1}\,\Phi_x\,\Phi^{-1}. \quad (29)$$

Since left hand sides of these equations are equal, equating the right hand sides of these equations and using equation (27) we obtain the equation (28).

So Φ is a surface in G defined by equation (27) with the compatibility condition given by equation (28). Now let us introduce a surface S in the Lie algebra $\mathfrak{g}$. Let F be a $\mathfrak{g}$ valued differentiable function of x, t, and λ for every $(x,t) \in \mathcal{O} \subset \mathbb{R}^2$ and $\lambda \in \mathbb{R}$. The first and second fundamental forms of S are defined as

$$\begin{aligned} \mathrm{d}s_I^2 &\equiv g_{ij}\mathrm{d}x^i\mathrm{d}x^j = \langle F_x, F_x\rangle \mathrm{d}x^2 + 2\langle F_x, F_t\rangle \mathrm{d}x\mathrm{d}t + \langle F_t, F_t\rangle \mathrm{d}t^2 \\ \mathrm{d}s_{II}^2 &\equiv h_{ij}\mathrm{d}x^i\mathrm{d}x^j = \langle F_{xx}, N\rangle \mathrm{d}x^2 + 2\langle F_{xt}, N\rangle \mathrm{d}x\mathrm{d}t + \langle F_{tt}, N\rangle \mathrm{d}t^2 \end{aligned} \quad (30)$$

where g_{ij} and h_{ij} are the components of the first and second fundamental forms, respectively. Here $i,j = 1,2$, $x^1 = x$ and $x^2 = t$, and $N \in \mathfrak{g}$ is defined as

$$\langle N, N\rangle = 1, \qquad \langle F_x, N\rangle = \langle F_t, N\rangle = 0.$$

Here $\{F_x, F_t, N\}$ forms a frame at each point of the surface S.

We are working in a finite dimensional Lie algebra $\mathfrak{g}$. Therefore the latter has a matrix representation by Ado's theorem. We use matrices, so the adjoint map is of the form $\Phi^{-1}\,A\,\Phi$, for $\Phi \in$ G and $A \in \mathfrak{g}$.

By using the adjoint representation, we can relate the surfaces in G to the surfaces in $\mathfrak{g}$ as

$$F_x = \Phi^{-1}\,A\,\Phi, \qquad F_t = \Phi^{-1}\,B\,\Phi \quad (31)$$

where A and B are $\mathfrak{g}$ valued differentiable functions of x, t, and λ for every $(x,t) \in \mathcal{O} \subset \mathbb{R}^2$ and $\lambda \in \mathbb{R}$.

The equations given in (31) define a surface S if and only if A and B satisfy the following equation

$$A_t - B_x + [A, V] + [U, B] = 0. \quad (32)$$

Indeed, the equations (31) have no meaning unless they are compatible. In other words equation (32) is the compatibility condition of the equations in (31), i.e., $F_{,xt} = F_{,tx}$.

The normal vector $\mathbf{N}$ of S can appear also in the following form by using the adjoint representation

$$N = \Phi^{-1}\,C\,\Phi \quad (33)$$

where $C \in \mathfrak{g}$.

Since inner product is invariant under adjoint representation, using equations (31) and (33) we can find the first and second fundamental forms of the surface S. Using equations (30) we obtain the components of the first and the second fundamental forms as

$$
\begin{aligned}
&g_{11}=\langle A, A\rangle, && g_{12}=g_{21}=\langle A, B\rangle, && g_{22}=\langle B, B\rangle \\
&h_{11}=\langle A_x+[A,U], C\rangle, && h_{12}=h_{21}=\langle A_t+[A,V], C\rangle, && h_{22}=\langle B_t+[B,V], C\rangle
\end{aligned}
\tag{34}
$$

where

$$C=\frac{[A,B]}{\|[A,B]\|}, \qquad \|A\|=\sqrt{|\langle A, A\rangle|}. \tag{35}$$

The following theorem summarizes the results given above.

Theorem 3. *Let U, V, A, and B be $\mathfrak{g}$ valued differentiable functions of x, t, and λ for every $(x,t)\in\mathcal{O}\subset\mathbb{R}^2$ and $\lambda\in\mathbb{R}$. Assume that U, V, A, and B satisfy the following equations*

$$U_t-V_x+[U,V]=0 \tag{36}$$

and

$$A_t-B_x+[A,V]+[U,B]=0. \tag{37}$$

Then the following equations

$$\Phi_x=U\,\Phi, \qquad \Phi_t=V\,\Phi \tag{38}$$

and

$$F_x=\Phi^{-1}\,A\,\Phi, \qquad F_t=\Phi^{-1}\,B\,\Phi \tag{39}$$

define the surfaces $\Phi\in G$ and $F\in\mathfrak{g}$, respectively. The first and the second fundamental forms of the surface F are respectively

$$\mathrm{d}s_I^2\equiv g_{ij}\mathrm{d}x^i\mathrm{d}x^j, \qquad \mathrm{d}s_{II}^2\equiv h_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{40}$$

where $i,j=1,2$, $x^1=x$, $x^2=t$, g_{ij} and h_{ij} are of the form that appear in equations (34) *and* (35)*. The Gaussian and the mean curvatures of the surface are given by*

$$K=\det(g^{-1})h, \qquad H=\frac{1}{2}\,\mathrm{trace}(g^{-1}h) \tag{41}$$

where g and h denote the matrices (g_{ij}) and (h_{ij}), respectively, and g^{-1} stands for the inverse of the matrix g.

For a differential equation which has Lax representation, we find A and B matrices and then we find the first and second fundamental forms. Using the fundamental forms we easily find the Gaussian and mean curvatures of the surface. If it is possible, we will try to find also the position vector F. As a result of these we study the properties of the surfaces. The matrices A and B relates the differential equations and surfaces by equations given in equation (39). In general, solving the equation given by equation (37) to find A and B and expressing the position vector are difficult. In order to overcome this difficulty we define an operator δ.

Definition 4. *Let δ be a operator acts on differentiable functions. If δ satisfy the following conditions*

$$\delta \partial_x = \partial_x \delta, \qquad \delta \partial_t = \partial_t \delta$$
$$\delta(f\,g) = g\delta(f) + f\delta(g), \qquad \delta(af + bg) = a\delta(f) + b\delta(g).$$

Here f and g are differentiable functions, and a and b are constants. We call such operators as deformation operators.

The following proposition gives a solution for finding A and B matrices.

Proposition 5. *Let Φ, U, and V are the matrices satisfying the equations given in* (36) *and* (38)*, A and B defines as $A = \delta U$ and $B = \delta V$, respectively. Equation for A and B in equation* (37) *is automatically satisfied and we have the following equations*

$$(\Phi^{-1}\delta\Phi)_x = \Phi^{-1}A\Phi, \qquad (\Phi^{-1}\delta\Phi)_t = \Phi^{-1}B\Phi.$$

The Proposition 5 gives a family of surfaces for every deformation operator δ. In the following proposition, we give the relation that directly connects deformation operators and surfaces. We also obtain the expression for the position vectors of the surface.

Proposition 6. *Let F be $\mathfrak{g}$ valued position vector. The position vector F and its partial derivatives are given as*

$$F = \Phi^{-1}\delta\Phi, \qquad F_x = \Phi^{-1}A\Phi, \qquad F_t = \Phi^{-1}B\Phi. \tag{42}$$

To prove it, it is enough to check the compatibility condition i.e., $(F_x)_t = (F_t)_x$. That is satisfied by Proposition 5. By Proposition 6, we can find one or more surfaces (depending on the deformation operator δ) that corresponds to a differential equation.

Now finding deformation operators in soliton theory and hence determining the matrices A and B becomes an important step. The following proposition answers the question how to find A and B without solving the equation in constructing the surfaces. When we talk about symmetries of soliton equations, we do not mean just symmetries of integrable equations but also symmetries of Lax equations.

Proposition 7. *The followings are the deformation operators of soliton equations.*

a) *Nonlinear integrable equations are invariant under spectral parameter deformation. In this case, the deformation operator is $\delta = \partial/\partial\lambda$. Hence A and B matrices are given as*

$$A = \frac{\partial U}{\partial \lambda}, \qquad B = \frac{\partial V}{\partial \lambda}$$

and position vector of the surface and its derivatives take the following forms

$$F = \Phi^{-1}\frac{\partial \Phi}{\partial \lambda}, \qquad F_x = \Phi^{-1}\frac{\partial U}{\partial \lambda}\Phi, \qquad F_t = \Phi^{-1}\frac{\partial V}{\partial \lambda}\Phi.$$

This type of relation first studied by Sym [40–42].

b) *Another deformations is Gauge symmetries of the Lax equations. Under Gauge transformation* Φ*,* U*, and* V *change as*

$$\Phi' = S\Phi, \qquad U' = SUS^{-1} + S_x S^{-1}, \qquad V' = SVS^{-1} + S_t S^{-1}$$

but Lax equations keep their form. These Gauge transformations define a new δ *operator. If we let* $S = I + \epsilon M$ *such that* $\epsilon^2 = 0$*, then we get* $\delta\Phi = M\Phi$*. Here* M *is any traceless* 2×2 *matrix. The matrices* A *and* B *are obtained by*

$$A = \delta U = \frac{\partial M}{\partial x} + [M, U], \qquad B = \delta V = \frac{\partial M}{\partial t} + [M, U]$$

and the position vector of the surface is given as

$$F = \Phi^{-1} M \Phi. \tag{43}$$

More information about this deformation can be found in [5], [7], [12], [13].

c) *Symmetries of the nonlinear integrable equations are another type of deformation. These symmetries are two types. First one is classical Lie symmetries which preserve the differential equation. The second is the generalized symmetries of nonlinear integrable equations. The latter transformation maps solutions to solutions. Deformation operator for these symmetries is taken as Freche't derivative* (*see* [12] *and* [5]). *In other words, for a differentiable function* F*,* $\delta F(x)$ *is defined as*

$$\delta F(x) = \lim_{\epsilon \to 0} \frac{\mathrm{d} f(x + \epsilon t)}{\mathrm{d}\epsilon}.$$

For this deformation, the matrices A *and* B*, and the position vector of the surface take the following form*

$$A = \delta U, \qquad B = \delta V, \qquad F = \Phi^{-1}\delta\Phi. \tag{44}$$

d) *The deformation of parameters for solution of integrable equation is the fourth deformation. This is introduced in* [44]. *In this case,* A*,* B*, and* F *are obtained as*

$$A_i = \frac{\partial U}{\partial \xi_i}, \qquad B_i = \frac{\partial V}{\partial \xi_i}, \qquad F_i = \Phi^{-1}\frac{\partial \Phi}{\partial \xi_i}, \qquad i = 1, 2, \ldots, N.$$

Here ξ_i *are parameters of solution* $u(x, t, \xi_i)$ *of integrable nonlinear equations, where* $i = 1, 2, \ldots, N$*. Here* N *is the number of parameters of the solution of integrable equation.*

The Proposition 7 establishes the relationship between differential equations which has Lax representation and related surfaces. Now we give two proposition about the surface of sphere.

Proposition 8. *For any differential equation, if the determinant of the matrix M, given in* Proposition 7 b)*, is constant, i.e.,* $\det M = R^2 = \mathrm{const}$*, the corresponding surface is a sphere with radius R.*

To prove it is enough take the determinant of both side of the equation given in equation (43). Some of the differential equations have transitional symmetry in either x direction or t direction (or in both direction). In this case the deformation operator can be considered as $\delta = \partial_x$ or $\delta = \partial_t$. Here we consider the transition in both directions such as $\delta = a\partial_x + b\partial_t$, where a and b are arbitrary constants.

Proposition 9. *If $\delta = a\partial_x + b\partial_t$ is the symmetry operator such that a and b are free parameters and* $\det(aU + bV) = \mathrm{const} = R^2$*, the corresponding surface is a surface of sphere with radius R.*

If $\delta = a\partial_x + b\partial_t$, using the third equation in (44), Proposition 7 and equation (38), we obtain F in the following form

$$F = \Phi^{-1}(aU + bV)\Phi$$

which yields

$$\det F = \det(aU + bV) = R^2.$$

Both of the above results are independent of the integrable equations. First one says the surface is a sphere if the gauge transformation is a special one and the second proposition says that the surface is again a sphere if the Lax representation is special.

6. Surfaces From a Variational Principle

In 1833, Poisson considered the free energy of a solid shell as

$$\mathcal{F} = \oiint_S H^2 \mathrm{d}A. \tag{45}$$

Here S is a smooth closed surface, A and H denote the surface area and the mean curvature of the surface S. In 1982, Willmore obtained the equation of the surface as a result of variational derivative of $\mathcal{F}$. We give this in the following proposition.

Proposition 10. *Let S be a smooth closed surface, K and H be the Gaussian and the mean curvatures of the surface, respectively. Variation of the functional $\mathcal{F}$ gives the following Euler-Lagrange equation* [51]

$$\nabla^2 H + 2(H^2 - K)H = 0. \tag{46}$$

Here ∇^2 is the Laplace-Beltrami operator defined as

$$\nabla^2 = \frac{1}{\sqrt{\tilde{g}}} \frac{\partial}{\partial x^i} \left(\sqrt{\tilde{g}} g^{ij} \frac{\partial}{\partial x^j} \right) \tag{47}$$

where $\tilde{g} = \det(g_{ij})$, g^{ij} is the inverse components of the first fundamental form, and $i, j = 1, 2$, where $x^1 = x$, $x^2 = t$. Solutions of the equation (46) *are called Willmore surfaces.*

Helfrich [21] obtained the curvature energy per unit area of the bilayer as

$$\mathcal{E}_{lb} = (k_c/2)\,(2H + c_0)^2 + \bar{k}K \tag{48}$$

where k_c and $\bar{k}$ are elastic constants, and c_0 is spontaneous curvature of the lipid bilayer. Using the Helfrich curvature energy given by equation (48), the free energy functional of the lipid vesicle is written as

$$\mathcal{F} = \oiint_S (\mathcal{E}_{lb} + \omega) \mathrm{d}A + p \iiint_V \mathrm{d}V. \tag{49}$$

Ou-Yang and Helfrich [34] obtained the shape equation of the bilayer by taking the first variation of free energy $\mathcal{F}$ given in equation (49). We give this result in the following proposition.

Proposition 11. *Let S be a smooth surface of lipid vesicle, V be the volume enclosed by the surface and p and ω be the osmotic pressure and surface tension of the vesicle, respectively. First order variation of the functional in equation* (49) *yields the following Euler-Lagrange equation* [34]

$$k_c \nabla^2 (2H) + k_c (2H + c_0)(2H^2 - c_0 H - 2K) + p - 2\omega H = 0. \tag{50}$$

Later Ou-Yang *et al* considered the more general energy functional

$$\mathcal{F} = \oiint_S \mathcal{E}(H, K) \mathrm{d}A + p \iiint_V \mathrm{d}V \tag{51}$$

which arises both in red blood cells and liquid crystals [33], [47–50]. Here $\mathcal{E}$ is function of mean and Gaussian curvatures H and K, respectively, p is a constant, and V is the volume enclosed within the surface S.

Proposition 12. *Let S be a closed smooth surface. The first variation of $\mathcal{F}$ given in equation* (51) *results a highly nonlinear Euler-Lagrange equation* (*see* [33], [47–49])

$$(\nabla^2 + 4H^2 - 2K) \frac{\partial \mathcal{F}}{\partial H} + 2(\nabla \cdot \bar{\nabla} + 2KH) \frac{\partial \mathcal{E}}{\partial K} - 4H\mathcal{E} + 2p = 0 \tag{52}$$

where ∇^2 is the Laplace-Beltrami operator given in equation (47) *and $\nabla \cdot \bar{\nabla}$ is defined by the formula*

$$\nabla \cdot \bar{\nabla} = \frac{1}{\sqrt{\tilde{g}}} \frac{\partial}{\partial x^i} \left(\sqrt{\tilde{g}} K h^{ij} \frac{\partial}{\partial x^j} \right). \tag{53}$$

For open surfaces, we let $p = 0$.

Some of the surfaces can be obtained from a variational principle for a suitable choice of $\mathcal{E}$ are given as

a) Minimal surfaces: $\mathcal{E} = 1,\ p = 0$
b) Surfaces with constant mean curvature: $\mathcal{E} = 1$
c) Linear Weingarten surfaces: $\mathcal{E} = aH + b$, where a and b are some constants, $aK + 2bH - p = 0$
d) Willmore surfaces: $\mathcal{E} = H^2$, [51], [52]
e) Surfaces that solve the shape equation of lipid membrane: $\mathcal{E} = (H - c)^2$, where c is a constant [29], [33–35], [47–50]
f) Shape equation of closed lipid bilayer: $\mathcal{E} = (k_c/2)\,(2H + c_0)^2 + \bar{k}K$, where k_c and $\bar{k}$ are elastic constants, and c_0 is the spontaneous curvature of the lipid bilayer [34].

Definition 13. *Surfaces that solve the following equation*

$$\nabla^2 H + aH^3 + bH\,K = 0 \tag{54}$$

are called Willmore-like surfaces, where a *and* b *are arbitrary constants.*

Remark 14. *When* $a = 2$ *and* $b = -2$*, the surface becomes Willmore surface which arise from a variational problem.*

7. Soliton Surfaces in $\mathbb{R}^3$

In this section, we obtain surfaces in $\mathbb{R}^3$ using soliton surface technique and variational principle. Consider the immersion F of $\mathcal{U} \in \mathbb{R}^2$ into $\mathbb{R}^3$. Let $N(x,t)$ denotes the vector field at every point of the surface. Let us denote the tangent space by $T_{(x,t)}S$ of the surface S. A basis for the $T_{(x,t)}S$ can be defined as $\{F_x, F_t, N\}$. Here S is a surface parameterized by $F(x,t)$. Let us denote the first and second fundamental forms, respectively, as

$$\mathrm{d}s_I^2 \equiv g_{ij}\mathrm{d}x^i\mathrm{d}x^j, \qquad \mathrm{d}s_{II}^2 \equiv h_{ij}\mathrm{d}x^i\mathrm{d}x^j, \quad i,j = 1,2, \quad x^1 = x, \quad x^2 = t.$$

As we discussed in previous sections, in order to develop surfaces using integrable equations we use Lie group and its Lie algebra. To study the immersions in $\mathbb{R}^3$, we use $\mathrm{SU}(2)$ as a Lie group and $\mathfrak{su}(2)$ as its corresponding Lie algebra. Consider $\mathrm{e}_k = -\mathrm{i}\sigma_k$, $k = 1,2,3$ as a basis for the Lie algebra $\mathfrak{su}(2)$. Here σ_k denotes the standard Pauli sigma matrices (cf. equation (11)).

Consider the following inner product defined on $\mathfrak{su}(2)$ Lie algebra

$$\langle X, Y\rangle = -\frac{1}{2}\mathrm{trace}(XY)$$

where $X, Y \in \mathfrak{su}(2)$ and $[.,.]$ denotes the usual commutator.

We follow Fokas and Gelfand's approach introduced in Section 5 to construct surfaces using integrable equations such as mKdV, SG, and NLS equations. We start with $\mathfrak{su}(2)$ valued Lax pairs U and V of these integrable equations. We use the deformations that we introduced in Section 5 to find the matrices A and B that satisfy equation (37). Using the matrices U, V, A and B we find the first and second fundamental forms of the surfaces corresponding to mKdV, SG, and NLS equations. We also find the Gaussian and the mean curvatures of these surfaces using first and second fundamental forms. Finding K and H allows us to classify some of these surfaces. Furthermore, in order to find the position vector F explicitly, we solve the Lax equations of the integrable equation using the Lax pairs U and V, and a solution of the integrable equation that we consider. Using the solution of Lax equations, and the matrices A and B we find the $\mathfrak{su}(2)$ valued position vector F of the surface. Considering some special values of the parameters in position vector, we plot some of these surfaces that we obtained using integrable equations. We also obtain some new Willmore-like surfaces and surfaces that satisfy generalized shape equation.

7.1. mKdV Surfaces From Spectral Parameter Deformations

In soliton surface technique, finding the matrices A and B that satisfy the equation (37) is crucial. There are four methods to determine them as we mentioned in Section 5. In this section we use the spectral parameter deformation of the Lax pairs of mKdV equation. In this section we closely follow the references [5] and [43].

Let u satisfy the mKdV equation

$$u_t = u_{xxx} + \frac{3}{2}u^2 u_x. \tag{55}$$

When we use the travelling wave ansatz $u_t - \alpha u_x = 0$ in mKdV equation (55) and integrating that equation, we obtain the simpler form of the mKdV equation as

$$u_{xx} = \alpha u - \frac{u^3}{2}. \tag{56}$$

Here α is an arbitrary real constant and the integration constant is set to zero. The Lax pairs for the mKdV equation in equation (56) are given as

$$
\begin{aligned}
U &= \frac{\mathrm{i}}{2}\begin{pmatrix} \lambda & -u \\ -u & -\lambda \end{pmatrix} \\
V &= -\frac{\mathrm{i}}{2}\begin{pmatrix} \frac{1}{2}u^2 - (\alpha + \alpha\lambda + \lambda^2) & (\alpha+\lambda)u - \mathrm{i}u_x \\ (\alpha+\lambda)u + \mathrm{i}u_x & -\frac{1}{2}u^2 + (\alpha + \alpha\lambda + \lambda^2) \end{pmatrix}
\end{aligned} \tag{57}
$$

and λ is the spectral parameter.

In the following proposition, using the Lax pairs of mKdV equation and their spectral parameter deformation we obtain the surfaces for mKdV equation.

Proposition 15. *Let u be a travelling wave solution of the* mKdV *equation given in equation* (56) *and $\mathfrak{su}(2)$ valued Lax pairs U and V are defined by equations* (57). *The matrices A and B, defined as spectral parameter deformations of the Lax pairs U and V, respectively, are given by the following equations*

$$
\begin{aligned}
A &= \mu\frac{\partial U}{\partial \lambda} = \frac{\mathrm{i}}{2}\begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix} \\
B &= \mu\frac{\partial V}{\partial \lambda} = -\frac{\mathrm{i}}{2}\begin{pmatrix} -(\alpha\,\mu + 2\,\mu\,\lambda) & \mu\,u \\ \mu\,u & \alpha\,\mu + 2\,\mu\,\lambda \end{pmatrix}
\end{aligned} \tag{58}
$$

where μ is a constant and λ is the spectral parameter. First and second fundamental forms of the surface S are given as

$$
\begin{aligned}
\mathrm{d}s_I^2 &\equiv g_{jk}\mathrm{d}x^j\mathrm{d}x^k = \frac{\mu^2}{4}\Big((\mathrm{d}x + (\alpha + 2\,\lambda)\mathrm{d}t)^2 + u^2\mathrm{d}t^2\Big) \\
\mathrm{d}s_{II}^2 &\equiv h_{jk}\mathrm{d}x^j\mathrm{d}x^k = \frac{\mu\,u}{2}\big(\mathrm{d}x + (\alpha+\lambda)\mathrm{d}t\big)^2 + \frac{\mu\,u}{4}(u^2 - 2\,\alpha)\mathrm{d}t^2
\end{aligned}
$$

and the other two important geometric invariants of the surface, namely Gaussian and mean curvatures are obtained as

$$
K = \frac{2}{\mu^2}\big(u^2 - 2\,\alpha\big), \qquad H = \frac{1}{2\mu\,u}\big(3\,u^2 + 2\,(\lambda^2 - \alpha)\big)
$$

where $x^1 = x$, $x^2 = t$. Repeated indices are summed up.

Proposition 15 gives us the invariants of the surfaces developed using mKdV equations. In the following three propositions, we will classify some of these surfaces. The following proposition gives surfaces belongs to Weingarten surfaces [5], [43].

Proposition 16. *Let u be a travelling wave solution of the* mKdV *equation given in equation* (56) *and S be the surface obtained using spectral parameter deformation in* Proposition 15. *Then the surface S is a Weingarten surface that has the following algebraic relation between Gaussian and mean curvatures of the surface*

$$
8\mu^2H^2(4\alpha + \mu^2K) = (8\alpha + 4\lambda^2 + 3\mu^2K)^2.
$$

When $\alpha = \lambda^2$ in Proposition 15*, the surface reduces to a quadratic Weingarten surface which has the following relation*

$$16\mu^2 H^2 = 18(\mu^2 K + 4\lambda^2).$$

Integrating the equation given in equation (56) and taking the integration constant zero, we obtain the following equation

$$u_x^2 = \alpha\, u^2 - \frac{u^4}{4}. \tag{59}$$

The following proposition gives another class of mKdV surfaces, namely Willmore-like surfaces [43]. In this case, Gaussian and mean curvatures satisfy some partial differential equation.

Proposition 17. *Let u satisfy equation* (59) *and S be the surface obtained using spectral parameter deformation in* Proposition 15*. Then the surface S is called a Willmore-like surface. This means that the Gaussian and mean curvatures of the surface S satisfy the partial differential equation given in equation* (54)*, where*

$$a = \frac{4}{9}, \qquad b = 1, \qquad \alpha = \lambda^2$$

and λ is an arbitrary constant.

In the following proposition we investigate mKdV surfaces which arise from a variational principle. It gives solutions to the Euler-Lagrange equation (52).

Proposition 18. *Let u satisfy equation* (59) *and S be the surface obtained using spectral parameter deformation in* Proposition 15*. Then there are* mKdV *surfaces satisfying the generalized shape equation* (52) *with Lagrange functions which are polynomials of Gaussian and mean curvatures of the surface S.*

Let us now give some examples of polynomial Lagrange functions of H and K that solve the equation given in equation (52) and provide the constraints [43]. Now we give following examples where the mKdV surfaces mentioned in the previous Proposition 18 are the critical points of the functionals with $\deg(\mathcal{E}) = N$.

Example 3.

i) For $N = 3$, the Lagrange function is in the following form

$$\mathcal{E} = a_1\, H^3 + a_2\, H^2 + a_3\, H + a_4 + a_5\, K + a_6\, K\, H$$

where

$$\alpha = \lambda^2, \quad a_1 = -\frac{p\,\mu^4}{72\,\lambda^4}, \quad a_2 = a_3 = a_4 = 0, \quad a_6 = \frac{p\,\mu^4}{32\,\lambda^4}.$$

Here $\lambda \neq 0$, and μ, p, and a_5 are arbitrary constants.

ii) For $N=4$, the Lagrange function is in the following form

$$\begin{aligned}\mathcal{E} &= a_1\,H^4+a_2\,H^3+a_3\,H^2+a_4\,H+a_5+a_6\,K\\ &+a_7\,K\,H+a_8\,K^2+a_9\,K\,H^2\end{aligned}$$

where

$$\alpha=\lambda^2,\quad a_2=-\frac{p\,\mu^4}{72\,\lambda^4},\quad a_3=-\frac{8\,\lambda^2}{15\,\mu^2}\,(27\,a_1-8\,a_8),\quad a_4=0$$
$$a_5=\frac{\lambda^4}{5\,\mu^4}\,(81\,a_1+16\,a_8),\;\; a_7=\frac{p\,\mu^4}{32\,\lambda^4},\;\; a_9=-\frac{1}{120}\,(189\,a_1+64\,a_8).$$

Here $\lambda\neq 0$, $\mu\neq 0$, and p, a_1, a_6, and a_8 are arbitrary constants.

iii) For $N=5$, the Lagrange function is in the following form

$$\begin{aligned}\mathcal{E} &= a_1\,H^5+a_2\,H^4+a_3\,H^3+a_4\,H^2+a_5\,H+a_6+a_7K\\ &+a_8\,K\,H+a_9\,K^2+a_{10}\,K\,H^2+a_{11}\,K^2\,H+a_{12}\,K\,H^3\end{aligned}$$

where

$$\begin{aligned}&\alpha=\lambda^2,\quad a_3=-\frac{1}{504\,\mu^2\,\lambda^4}\left(\lambda^6(4212\,a_1+256\,a_{11})+7\,p\,\mu^6\right)\\ &a_4=-\frac{8\,\lambda^2}{15\,\mu^2}\,(27\,a_2-8\,a_9)\,,\quad a_5=\frac{6\,\lambda^4}{7\,\mu^4}\,(135\,a_1-88\,a_{11})\\ &a_6=\frac{\lambda^4}{5\,\mu^4}\,(81\,a_2+16\,a_9)\\ &a_8=\frac{1}{32\,\mu^2\,\lambda^4}\left(\lambda^6(-324\,a_1+512\,a_{11})+p\,\mu^6\right)\\ &a_{10}=-\frac{1}{120}\,(189\,a_2+64\,a_9)\,,\quad a_{12}=-\frac{1}{756}\,(1053\,a_1+512\,a_{11}).\end{aligned}$$

Here $\lambda\neq 0$, $\mu\neq 0$, and p, a_1, a_2, a_7, a_9, and a_{11} are arbitrary constants.

iv) For $N=6$, the Lagrange function is in the following form

$$\begin{aligned}\mathcal{E} &= a_1\,H^6+a_2\,H^5+a_3\,H^4+a_4\,H^3+a_5\,H^2+a_6\,H\\ &+a_7+a_8\,K+a_9\,K\,H+a_{10}\,K^2+a_{11}\,K\,H^2+a_{12}\,K^2\,H\\ &+a_{13}\,K\,H^3+a_{14}\,K^3+a_{15}\,K^2\,H^2+a_{16}\,K\,H^4\end{aligned}$$

where

$$\begin{aligned}&\alpha=\lambda^2,\qquad a_4=-\frac{1}{504\,\mu^2\,\lambda^4}\left(\lambda^6(4212\,a_2+256\,a_{12})+7\,p\,\mu^6\right)\\ &a_5=-\frac{\lambda^4}{900\,\mu^4}\,(-359397\,a_1+191488\,a_{14}-203472\,a_{16})\\ &\quad-\frac{8\lambda^2}{15\mu^2}\,(27a_3-8a_{10})\\ &a_6=\frac{6\,\lambda^4}{7\,\mu^4}\,(135\,a_2-88\,a_{12})\end{aligned}$$

$$
\begin{aligned}
a_7 &= \frac{\lambda^6}{25\,\mu^6}\,(29889\,a_1 - 9856\,a_{14} + 11664\,a_{16}) + \frac{\lambda^4}{5\mu^4}\,(81a_3 + 16a_{10}) \\
a_9 &= \frac{1}{32\,\mu^2\,\lambda^4}\,\left(\lambda^6[-324\,a_2 + 512\,a_{12}] + p\,\mu^6\right) \\
a_{11} &= -\frac{\lambda^2}{1800\,\mu^2}\,(59778\,a_1 - 13312\,a_{14} + 23328\,a_{16}) \\
&\quad - \frac{1}{120}\,(189a_3 + 64a_{10}) \\
a_{13} &= -\frac{1}{756}\,(1053\,a_2 + 512\,a_{12}) \\
a_{15} &= -\frac{1}{2880}\,(5103\,a_1 + 2048\,a_{14} + 3888\,a_{16})\,.
\end{aligned}
$$

Here $\lambda \neq 0$, $\mu \neq 0$, and $p, a_1, a_2, a_3, a_8, a_{10}, a_{12}, a_{14}$, and a_{16} are arbitrary constants.

For general $N \geq 3$, from the above examples, the polynomial function $\mathcal{E}$ takes the following form

$$
\mathcal{E} = \sum_{n=0}^{N} H^n \sum_{l=0}^{\lfloor\frac{(N-n)}{2}\rfloor} a_{nl} K^l
$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, and a_{nl} are constants.

7.1.1. Position Vector of mKdV Surfaces

In the previous section, we obtained local invariants of the mKdV surfaces. We also classified some of these surfaces such as Weingarten surfaces, Willmore-like surfaces and surfaces that solves generalized shape equation. It is also important to determine the position vector of the mKdV surfaces.

We start with one soliton solution of mKdV equation given in equation (56). Consider the following one soliton solution

$$
u = k_1\,\mathrm{sech}\xi \tag{60}
$$

where $\alpha = k_1^2/4$ in equation (56) and $\xi = k_1\,\left(k_1^2\,t + 4\,x\right)/8$. Using this one soliton solution and corresponding matrix Lax pairs U and V given by equations (57) of mKdV equation, we solve the Lax equations given in equation (38). The solution of Lax equation is a 2×2 matrix Φ

$$
\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}.
$$

We find the components of Φ as

$$
\begin{aligned}
\Phi_{11} &= -\frac{\mathrm{i}}{k_1} A_1\, \mathrm{e}^{\mathrm{i}\left(k_1^2+4\lambda^2\right)t/8} \left(2\,\lambda + \mathrm{i}\, k_1\, \tanh\xi\right) \\
&\quad \times \left(\tanh\xi + 1\right)^{\mathrm{i}\lambda/2k_1} \left(\tanh\xi - 1\right)^{-\mathrm{i}\lambda/2k_1} \\
&\quad + \mathrm{i}\, k_1\, B_1\, \mathrm{e}^{-\mathrm{i}\left(k_1^2+4\lambda^2\right)t/8} \left(\tanh\xi - 1\right)^{\mathrm{i}\lambda/2k_1} \left(\tanh\xi + 1\right)^{-\mathrm{i}\lambda/2k_1} \operatorname{sech}\xi \\
\Phi_{12} &= -\frac{\mathrm{i}}{k_1} A_2\, \mathrm{e}^{\mathrm{i}\left(k_1^2+4\lambda^2\right)t/8} \left(2\,\lambda + \mathrm{i}\, k_1\, \tanh\xi\right) \\
&\quad \times \left(\tanh\xi + 1\right)^{\mathrm{i}\lambda/2k_1} \left(\tanh\xi - 1\right)^{-\mathrm{i}\lambda/2k_1} \\
&\quad + \mathrm{i}\, k_1\, B_2\, \mathrm{e}^{-\mathrm{i}\left(k_1^2+4\lambda^2\right)t/8} \left(\tanh\xi - 1\right)^{\mathrm{i}\lambda/2k_1} \left(\tanh\xi + 1\right)^{-\mathrm{i}\lambda/2k_1} \operatorname{sech}\xi \\
\Phi_{21} &= \mathrm{i}\, A_1\, \mathrm{e}^{\mathrm{i}\left(k_1^2+4\lambda^2\right)t/8} \left(\tanh\xi + 1\right)^{\mathrm{i}\lambda/2k_1} \left(\tanh\xi - 1\right)^{-\mathrm{i}\lambda/2k_1} \operatorname{sech}\,\xi \\
&\quad + B_1\, \mathrm{e}^{-\mathrm{i}\left(k_1^2+4\lambda^2\right)t/8} \left(k_1 \tanh\xi + 2\mathrm{i}\lambda\right) \\
&\quad \times \left(\tanh\xi - 1\right)^{\mathrm{i}\lambda/2k_1} \left(\tanh\xi + 1\right)^{-\mathrm{i}\lambda/2k_1} \\
\Phi_{22} &= \mathrm{i}\, A_2\, \mathrm{e}^{\mathrm{i}\left(k_1^2+4\lambda^2\right)t/8} \left(\tanh\xi + 1\right)^{\mathrm{i}\lambda/2k_1} \left(\tanh\xi - 1\right)^{-\mathrm{i}\lambda/2k_1} \operatorname{sech}\,\xi \\
&\quad + B_2\, \mathrm{e}^{-\mathrm{i}\left(k_1^2+4\lambda^2\right)t/8} \left(k_1 \tanh\xi + 2\mathrm{i}\lambda\right) \\
&\quad \times \left(\tanh\xi - 1\right)^{\mathrm{i}\lambda/2k_1} \left(\tanh\xi + 1\right)^{-\mathrm{i}\lambda/2k_1}.
\end{aligned} \tag{61}
$$

Here A_1, A_2, B_1, and B_2 are arbitrary constants. The determinant of the matrix Φ is constant which is obtained in terms of k_1, λ, A_1, A_2, B_1, and B_2 as

$$\det(\Phi) = ((k_1^2 + 4\lambda^2)/k_1)\,(A_1 B_2 - A_2 B_1) \neq 0.$$

In order to find the immersion function F explicitly, we first find F_x and F_t from equation (39). For this purpose we substitute the $\mathfrak{su}(2)$ valued matrices A and B given by equations (58), and the the matrix Φ given by equations (61) into the equations for F_x and F_t given by equation (39). We solve the resultant equation by letting $A_1 = A_2$, $B_1 = (A_1/k_1)\mathrm{e}^{\pi\lambda/k_1}$, $B_2 = -B_1$ and obtain the function F as

$$F = \mathrm{e}_1 y_1 + \mathrm{e}_2 y_2 + \mathrm{e}_3 y_3$$

where y_1, y_2, and y_3 are given as

$$
\begin{aligned}
y_1 &= \frac{1}{4\,k_1\,(\mathrm{e}^{2\xi}+1)} W_1 \left(\Omega_1\,(\mathrm{e}^{2\xi}+1) + 32 k_1\right) \\
y_2 &= -4\,W_1\, \cos\Omega_2 \operatorname{sech}\xi \\
y_3 &= 4\,W_1\, \sin\Omega_2 \operatorname{sech}\xi.
\end{aligned} \tag{62}
$$

Here e_1, e_2, e_3 form a basis for $\mathfrak{su}(2)$, Ω_1, Ω_2, and W_1 are given as

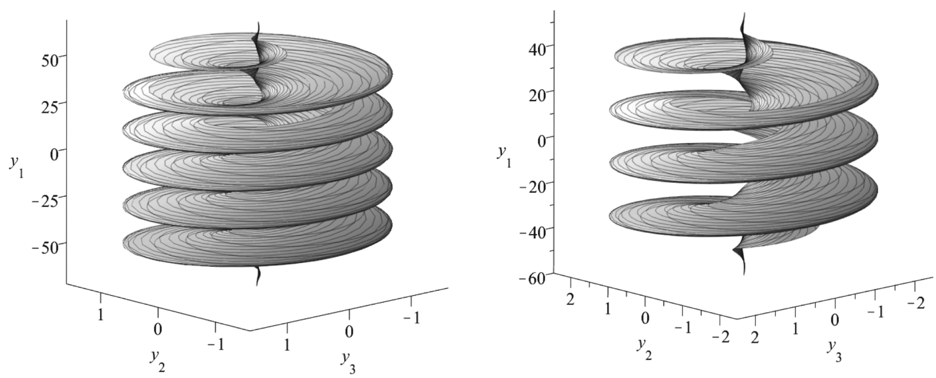
$$W_1 = -\frac{\mu\, k_1}{2\,(k_1^2 + 4\,\lambda^2)}, \qquad \Omega_1 = \big(t\,(8\,\lambda + k_1^2) + 4\,x\big)\,\big(k_1^2 + 4\,\lambda^2\big)$$

$$\Omega_2 = t\left(\lambda^2 + \frac{1}{4}\,k_1^2(1+\lambda)\right) + x\,\lambda, \qquad \xi = \frac{k_1^3}{8}(t + \frac{4x}{k_1^2}).$$

7.1.2. Plotting mKdV Surfaces From Spectral Parameter Deformation

Position vector $\mathbf{Y} = (y_1, y_2, y_3)$ of the mKdV surfaces corresponds to the spectral parameter deformation is given by equations (62) that we obtained using one soliton solution. We plot some of these mKdV surfaces for some special values of the constants μ, λ, and k_1.

Example 4. *Taking* $\mu = 5,\ k_1 = 1.5$, *and changing* λ *as* $a)\ \lambda = 1,\ \ b)\ \lambda = 2$, *in the equations provided by equations* (62), *we get the surfaces given in* Fig. 1.



(a) **(b)**

Figure 1. $(x, t) \in [-3, 3] \times [-3, 3]$

Example 5. *Taking* $\mu = 2$, $\lambda = 0$, *and* $k_1 = 1.25$ *in the equations* (62), *we get the surface given in* Fig. 2.

Example 6. *Taking* $\mu = 3$, $k_1 = -2$, *and changing* λ *as* $a)\ \lambda = 0.08,\ \ b)\ \lambda = 0.2,\ \ c)\ \lambda = 0.5,\ \ d)\ \lambda = 0.8$ *in the equations* (62), *we get the surfaces given in* Fig. 3.

Even though for small values of x and t these surfaces given in Examples 4 - 6 have different behaviors, asymptotically they are similar to each other. As ξ tends to $\pm\infty$, y_1 approaches $\pm\infty$, and y_2 and y_3 become zero.

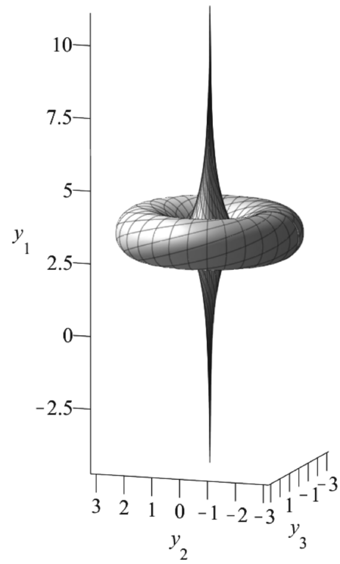


Figure 2. $(x,t) \in [-8,8] \times [-8,8]$

7.2. mKdV Surfaces From the Spectral-Gauge Deformations

When we consider a combination of the spectral parameter and gauge deformations of Lax pairs U and V given by equations (57), for mKdV equation, local invariants of the surface are more complicated than the just spectral deformations case. In this case the matrices A and B are obtained as

$$A = \mu \frac{\partial U}{\partial \lambda} + \nu [\sigma_2, U], \qquad B = \mu \frac{\partial V}{\partial \lambda} + \nu [\sigma_2, U].$$

Here we just give the Gaussian and mean curvatures of the surfaces of the surface as

$$K = \frac{2\,u\,(u^2 - 2\,\alpha)}{\nu\Big(2\,\nu u[u^2 - 2\,\alpha] - 3\,\mu u^2 - 2\mu(\lambda^2 - \alpha)\Big) + \mu^2 u}$$

$$H = \frac{\mu(3\,u^2 + 2(\lambda^2 - \alpha)) - 4\,u\,\nu(u^2 - 2\,\alpha)}{2\,\nu\Big(2\,\nu\,u[u^2 - 2\,\alpha] - 3\,\mu\,u^2 - 2\,\mu(\lambda^2 - \alpha)\Big) + 2\,\mu^2\,u}.$$

The mKdV surfaces obtained from spectral-gauge deformation do not belong to Willmore-like surfaces and surfaces that solve the generalized shape equation.

In order to find the position vector of the surfaces we use the same method that we used for the spectral deformation. We use one soliton solution, given in equation

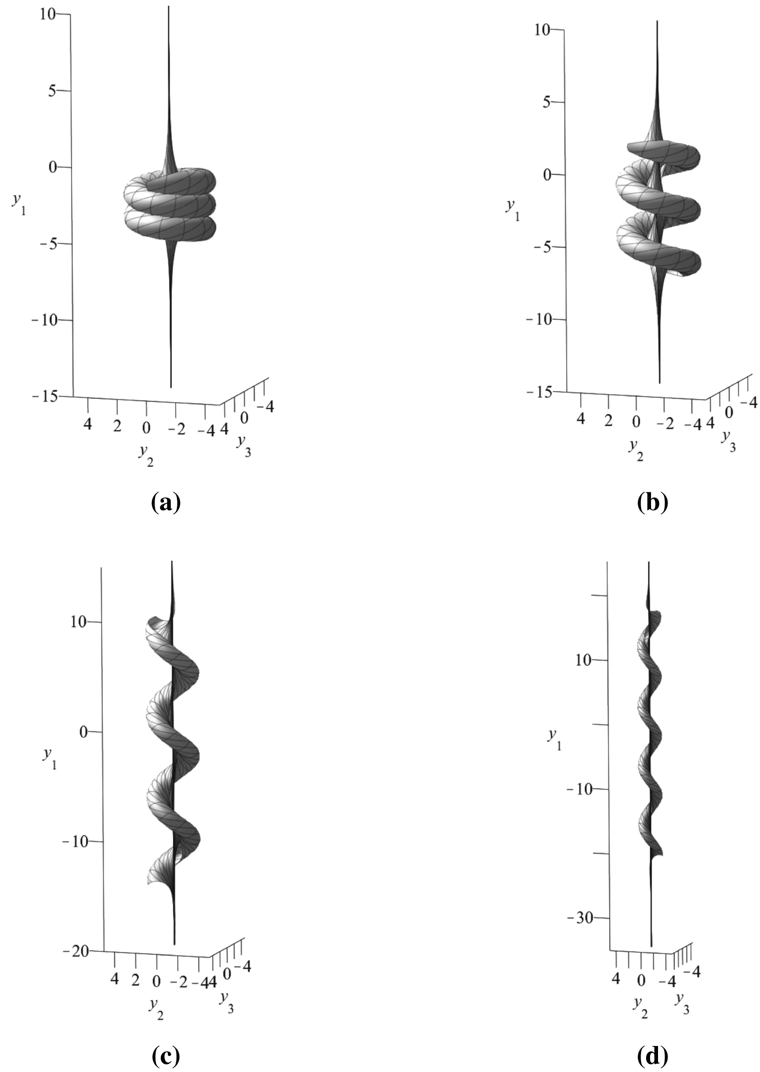


Figure 3. $(x, t) \in [-8, 8] \times [-8, 8]$

(60), of the mKdV equation. Lax pairs U and V, and solution, Φ, of the Lax equation are same as the spectral deformation given by equations (57), and equations (61). The components of the position vector $\mathbf{Y} = (y_1, y_2, y_3)$ for mKdV surfaces correspond to spectral-gauge deformation are given as

$$
\begin{aligned}
y_1 &= -W_2\,\frac{\mathrm{e}^{2\xi}-1}{(\mathrm{e}^{2\xi}+1)}\,\mathrm{sech}\,\xi - W_3\,\Omega_3 - W_4\frac{1}{\mathrm{e}^{2\xi}+1}\\
y_2 &= \Big(\frac{1}{2}\,W_4\,\mathrm{sech}\,\xi + W_5\,\frac{\big(\mathrm{e}^{4\xi}+1\big)}{(\mathrm{e}^{2\xi}+1)^2} - W_6\,\mathrm{sech}^2\,\xi\Big)\cos\Omega_2\\
&\quad +W_7\,\frac{\big(\mathrm{e}^{2\xi}-1\big)}{\big(\mathrm{e}^{2\xi}+1\big)}\sin\Omega_2 \qquad\qquad (63)\\
y_3 &= \Big(\frac{1}{2}\,W_4\,\mathrm{sech}\,\xi + W_5\,\frac{\big(\mathrm{e}^{4\xi}+1\big)}{(\mathrm{e}^{2\xi}+1)^2} - W_6\,\mathrm{sech}^2\,\xi\Big)\sin\Omega_2\\
&\quad -W_7\,\frac{\big(\mathrm{e}^{2\xi}-1\big)}{\big(\mathrm{e}^{2\xi}+1\big)}\cos\Omega_2
\end{aligned}
$$

where

$$
\begin{aligned}
W_2 &= \frac{2\,k_1^2\,\nu}{k_1^2+4\lambda^2}, & W_3 &= \frac{\mu}{8}, \qquad W_4 = \frac{4\,\mu\,k_1^2}{k_1^2+4\lambda^2}\\
W_5 &= \frac{\nu\,(k_1^2-4\lambda^2)}{k_1^2+4\lambda^2}, & W_6 &= \frac{\nu\,(4\,\lambda^2+3\,k_1^2)}{2(k_1^2+4\lambda^2)}\\
W_7 &= \frac{4\,\lambda\,k_1^2\,\nu}{k_1^2+4\lambda^2}, & \xi &= \frac{k_1^3}{8}(t+\frac{4\,x}{k_1^2})\\
\Omega_2 &= t\left(\lambda^2+k_1^2\,(1+\lambda)/4\right)+x\lambda, & \Omega_3 &= \left(t\,(8\,\lambda+k_1^2)+4\,x\right).
\end{aligned}
$$

7.2.1. Plotting mKdV Surfaces From Spectral-Gauge Deformation

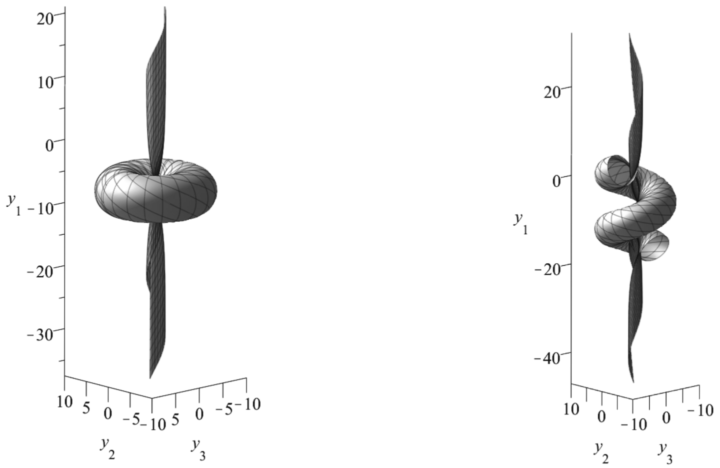
In this section, we plot some of mKdV surfaces that we obtained using spectral-gauge deformation where the position vector is given by equations (63) for some special values of the constants μ, ν, λ, and k_1.

Example 7. *Taking* $\mu=-6$*,* $\nu=1.5$*, and* $k_1=1.5$*, and changing* λ *as* $a)$ $\lambda=0$*,* $b)$ $\lambda=0.2$ *in the equations provided by equations* (63)*, we get the surface given in* Fig. 4.

Example 8. *Taking* $\mu=3$*,* $\nu=-1$*, and* $k_1=1$*, and changing* λ *as* $a)$ $\lambda=1$*,* $b)$ $\lambda=-4$ *in the equations* (63)*, we get the surface given in* Fig. 5.

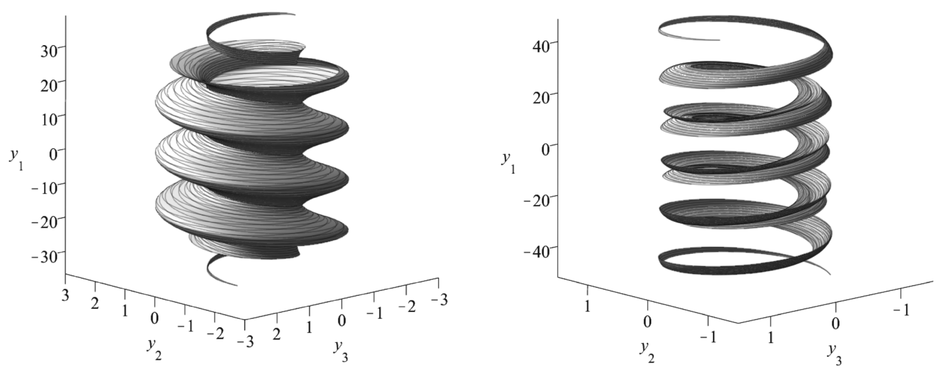
Example 9. *Taking* $\mu=1.5$*,* $\nu=0.1$*,* $k_1=1.7$*, and* $\lambda=0.1$ *in the equations* (63)*, we get the surface given in* Fig. 6.

Example 10. *Taking* $\mu=-3$*,* $\nu=-1$*,* $k_1=1$*, and* $\lambda=-0.2$ *in the equations* (63)*, we get the surface given in* Fig. 7.



(a) **(b)**

Figure 4. $(x, t) \in [-8, 8] \times [-8, 8]$



(a) **(b)**

Figure 5. $(x, t) \in [-8, 8] \times [-8, 8]$

Asymptotic behavior of these surfaces in Examples 7 - 10 are as follows, ξ tends to $\pm\infty$, y_2 approaches $W_5 \cos\Omega_2 \pm W_7 \sin\Omega_2$, y_3 approaches $-W_5 \sin\Omega_2 \pm W_7 \cos\Omega_2$, and y_1 goes to $\pm\infty$.

7.3. SG Surfaces

In this section, we obtain surfaces corresponding the SG equation [5], [44].

Let $u(x,t)$ satisfy the following SG equation

$$u_{xt} = \sin u. \tag{64}$$

The Lax pairs U and V of the SG equation (64) are given as

$$U = \frac{\mathrm{i}}{2}\begin{pmatrix} \lambda & -u_x \\ -u_x & -\lambda \end{pmatrix}, \qquad V = \frac{1}{2\,\lambda}\begin{pmatrix} -\mathrm{i}\,\cos u & \sin u \\ -\sin u & \mathrm{i}\,\cos u \end{pmatrix} \tag{65}$$

were λ is a spectral constant.

In the following proposition, we obtain SG surfaces using spectral parameter deformation of U and V.

Proposition 19. *Let u satisfy the* SG *equation* (64) *and $\mathfrak{su}(2)$ valued Lax pairs U and V are defined by equations* (65). *The matrices A and B, defined as spectral parameter deformations of the Lax pairs U and V, respectively, are given by the following equations*

$$A = \mu\frac{\partial U}{\partial \lambda} = \frac{\mathrm{i}\,\mu}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \mu\frac{\partial V}{\partial \lambda} = \frac{\mu}{2\,\lambda}\begin{pmatrix} \mathrm{i}\,\cos u & -\sin u \\ \sin u & -\mathrm{i}\,\cos u \end{pmatrix}$$

where μ is a constant and λ is a spectral parameter. Then the first and the second fundamental forms of the surface S are given as

$$\mathrm{d}s_I^2 \equiv g_{jk}\mathrm{d}x^j\mathrm{d}x^k = \frac{\mu^2}{4}\Big(\mathrm{d}x^2 + \frac{2}{\lambda^2}\,\cos u\,\mathrm{d}x\mathrm{d}t + \frac{1}{\lambda^4}\mathrm{d}t^2\Big)$$

$$\mathrm{d}s_{II}^2 \equiv h_{jk}\mathrm{d}x^j\mathrm{d}x^k = -\frac{\mu}{\lambda}\sin u\,\mathrm{d}x\mathrm{d}t$$

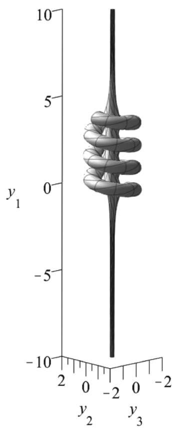


Figure 6. $(x,t) \in [-15,15]\times[-15,15]$

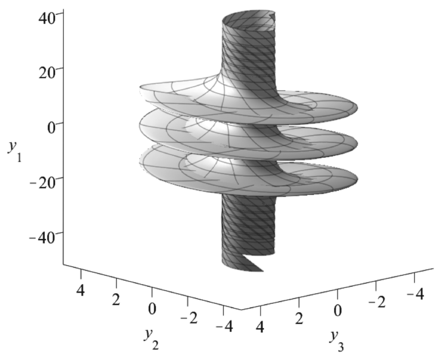


Figure 7. $(x,t) \in [-30,30]\times[-30,30]$

while the Gaussian and mean curvatures are

$$K = -\frac{4\,\lambda^2}{\mu^2}, \qquad H = \frac{2\,\lambda}{\mu}\cot u$$

where $x^1 = x$, $x^2 = t$.

In the following proposition, we use spectral and Gauge deformation to obtain SG surfaces.

Proposition 20. *Let u satisfy the SG equation* (64) *and $\mathfrak{su}(2)$ valued Lax pairs U and V are defined by equations* (65)*. The matrices A and B defined as*

$$\begin{aligned}
A &= \mu\frac{\partial U}{\partial\lambda} + \frac{\mathrm{i}\nu}{2}[\sigma_1, U] = \frac{1}{2}\begin{pmatrix} \mathrm{i}\mu & \nu\lambda \\ -\nu\lambda & -\mathrm{i}\mu \end{pmatrix}\\
B &= \mu\frac{\partial V}{\partial\lambda} + \frac{\mathrm{i}\nu}{2}[\sigma_1, V]\\
&= \frac{1}{2\,\lambda^2}\begin{pmatrix} \mathrm{i}(\mu\cos u - \lambda\nu\sin u) & -\mu\sin u - \lambda\nu\cos u \\ \mu\sin u + \lambda\nu\cos u & -\mathrm{i}(\mu\cos u - \lambda\nu\sin u) \end{pmatrix}
\end{aligned}$$

where μ is a constant and λ is a spectral parameter. Then the first and second fundamental forms of the surface S are given as

$$\mathrm{d}s_I^2 \equiv g_{jk}\,\mathrm{d}x^j\,\mathrm{d}x^k, \qquad \mathrm{d}s_{II}^2 \equiv h_{jk}\,\mathrm{d}x^j\,\mathrm{d}x^k$$

where

$$\begin{aligned}
g_{11} &= \frac{1}{4}(\mu^2 + \lambda^2\,\nu^2)\\
g_{12} &= g_{21} = \frac{1}{4\lambda^2}\Big((\mu^2 - \lambda^2\,\nu^2)\cos u - 2\mu\,\nu\lambda\sin u\Big)\\
g_{22} &= \frac{1}{4\lambda^2}(\mu^2 + \lambda^2\,\nu^2), \qquad h_{11} = \frac{1}{2}\lambda^2\nu\\
h_{12} &= h_{21} = -\frac{1}{2\lambda}(\mu\sin u + \lambda\,\nu\cos u), \qquad h_{22} = \frac{\nu}{2\lambda^2}.
\end{aligned}$$

The Gaussian and mean curvatures are obtained as

$$\begin{aligned}
K &= \frac{L_1\cos^2 u + L_2\sin u\cos u - L_1}{L_3\cos^2 u + L_4\sin u\cos u + L_5}\\
H &= \frac{L_6\cos^2 u + L_7\sin u\cos u + L_8}{L_3\cos^2 u + L_4\sin u\cos u + L_5}
\end{aligned}$$

where

$$
\begin{aligned}
L_1 &= 4\lambda^2(\lambda^2\,\nu^2-\mu^2), & L_2 &= 8\,\mu\,\nu\lambda^3 \\
L_3 &= \mu^4+\lambda^2\,\nu^2(\lambda^2\,\nu^2-6\mu^2), & L_4 &= 4\mu\,\lambda\,\nu(\lambda^2\,\nu^2-\mu^2) \\
L_5 &= -\mu^4-\lambda^2\,\nu^2(\lambda^2\,\nu^2-2\mu^2), & L_6 &= 2\nu\,\lambda^2(\lambda^2\,\nu^2-3\,\mu^2) \\
L_7 &= 2\,\mu\,\lambda(3\lambda^2\nu^2-\mu^2), & L_8 &= 2\nu\,\lambda^2(\mu^2-\lambda^2\,\nu^2).
\end{aligned}
$$

The following proposition gives SG surfaces belongs to Weingarten surfaces [5].

Proposition 21. *Let u satisfy the* SG *equation given in equation* (64) *and S be the surface obtained using spectral parameter deformation in* Proposition 20. *Then the surface S is a Weingarten surface that has the following algebraic relation between Gaussian and mean curvatures of the surface*

$$
(\mu^2+\lambda^2\nu^2)K-4\nu\lambda^2\,H+4\lambda^2=0. \tag{66}
$$

Proposition 22. *Let u satisfy the* SG *equation given in equation* (64). *The $\mathfrak{su}(2)$ valued Lax pairs U and V of the* SG *equation are given by equations* (65). *Respectively, the $\mathfrak{su}(2)$ valued matrices A and B are defined as*

$$
A=U'\varphi=-\frac{\mathrm{i}}{2}\varphi_x\sigma_1, \qquad B=V'\varphi=\frac{\mathrm{i}}{2\lambda}\,\varphi(\cos u\sigma_2+\sin u\sigma_3)
$$

where λ is constant and $\sigma_1,\sigma_2,\sigma_3$ are the Pauli sigma matrices. Here primes denote Frèchet differentiation and φ is a symmetry of (64)*, i.e., φ is a solution of*

$$
\varphi_{xt}=\varphi\,\cos u \tag{67}
$$

Then the surface S has the following first and second fundamental forms

$$
\begin{aligned}
\mathrm{d}s_I^2 &\equiv g_{jk}\,\mathrm{d}x^j\,\mathrm{d}x^k=\frac{1}{4}\Big(\varphi_x^2\,\mathrm{d}x^2+\frac{1}{\lambda^2}\,\varphi^2\mathrm{d}t^2\Big)\\
\mathrm{d}s_{II}^2 &\equiv g_{jk}\,\mathrm{d}x^j\,\mathrm{d}x^k=\frac{1}{2}\Big(\lambda\,\varphi_x\,\sin u\,\mathrm{d}x^2+\frac{1}{\lambda}\,\varphi\,u_t\,\mathrm{d}t^2\Big)
\end{aligned} \tag{68}
$$

and the Gaussian and mean curvatures are given as

$$
K=\frac{4\lambda^2u_t\sin u}{\varphi\varphi_x}, \qquad H=\frac{\lambda(\varphi_xu_t+\varphi\sin u)}{\varphi\varphi_x}.
$$

Indeed, the equation given in equation (67) has infinitely many explicit solutions in terms of u and its derivatives. The following corollary gives the surfaces corresponding to $\varphi=u_x$ which is special case of Proposition 22.

Corollary 23. *Let* $\varphi = u_x$ *in* Proposition 22*, then the surface turns out to be a sphere with the following first and second fundamental forms*

$$\begin{aligned} \mathrm{d}s_I^2 &= \frac{1}{4}\Big(\sin^2 u \, \mathrm{d}x^2 + \frac{1}{\lambda^2}\, u_t^2 \mathrm{d}t^2 \Big) \\ \mathrm{d}s_{II}^2 &= \frac{1}{2}\Big(\lambda \, \sin^2 u \, \mathrm{d}x^2 + \frac{1}{\lambda}\, u_t^2 \, \mathrm{d}t^2 \Big) \end{aligned} \tag{69}$$

and the corresponding Gaussian and mean curvatures are constants given by

$$K = 4\,\lambda^2, \qquad H = 2\,\lambda. \tag{70}$$

For the following solutions of the equation given in equation (67)

$$\varphi = u_x, \;\; \varphi = u_{3x} + \frac{u_x^3}{2}, \;\; \varphi = u_{3t} + \frac{u_t^3}{2}, \;\; \varphi = u_{5x} + \frac{5}{2}\, u_x^2\, u_{3x} + \frac{5}{2}\, u_x\, u_{2x}^2 + \frac{3}{8}\, u_x^5$$

the Gaussian and mean curvatures of the surfaces which are constructed in Proposition 22 are not constant directly as we get in Corollary 23. But they are constant when we use one soliton solution of the SG equation. One soliton solution of SG equation is given by

$$u = 4 \, \arctan\,(\mathrm{e}^{\xi}), \quad \xi = \Big(k_1\,(t+x) + (k_1^2 - 1)^{1/2}\,(t-x) + k_2 \Big) \tag{71}$$

7.3.1. SG Surfaces From Deformation of Parameters

In this section, we use the deformation of parameters (k_1 and k_2) of one soliton solution of SG equation given in equation (71) to develop SG surfaces.

Proposition 24. *Let* u*, provided by equation* (71)*, satisfy the* SG *equation given in equation* (64) *and* $\mathfrak{su}(2)$ *valued Lax pairs* U *and* V *of the SG equation are given by equations* (65)*. The matrices* A *and* B *are defined as*

$$A = \mu \frac{\partial U}{\partial k_2} = \frac{\mathrm{i}\,\mu}{2} \begin{pmatrix} 0 & -(\phi_1)_x \\ -(\phi_1)_x & 0 \end{pmatrix}$$

$$B = \mu \frac{\partial V}{\partial k_2} = \frac{\mu}{2\,\lambda} \begin{pmatrix} \mathrm{i}\,\sin(u)\,\phi_1 & \cos(u)\,\phi_1 \\ -\cos(u)\,\phi_1 & -\mathrm{i}\,\sin(u)\,\phi_1 \end{pmatrix}$$

where $\phi_1 = \partial\, u/\partial k_2$*,* k_2 *is a parameter of the one soliton solution* u*, and* μ *is a constant. Then the surface* S *has the following first and second fundamental forms*

$$\begin{aligned} \mathrm{d}s_I^2 &\equiv g_{jk}\, \mathrm{d}x^j\, \mathrm{d}x^k = \mu^2\, \mathrm{sech}^2\xi \Big(\tanh^2\xi \Big((k_1^2-1)^{1/2} - k_1 \Big)^2 \mathrm{d}x^2 + \frac{1}{\lambda^2}\, \mathrm{d}t^2 \Big) \\ \mathrm{d}s_{II}^2 &\equiv g_{jk} \mathrm{d}x^j\, \mathrm{d}x^k = 2\,\mu\, \mathrm{sech}^2\xi \Big(\lambda\, \tanh^2\xi\, \Big(k_1 - (k_1^2-1)^{1/2} \Big)\, \mathrm{d}x^2 \\ &\qquad + \frac{1}{\lambda}\, \Big(k_1 + (k_1^2-1)^{1/2} \Big)\, \mathrm{d}t^2 \Big). \end{aligned}$$

Gaussian and mean curvatures of S are obtained as

$$K=\frac{4\,\lambda^2\left(k_1+[k_1^2-1]^{1/2}\right)^2}{\mu^2},\qquad H=\frac{2\,\lambda\left(k_1+[k_1^2-1]^{1/2}\right)}{\mu}$$

where $x^1=x$, $x^2=t$.

These surfaces given Proposition 24 are also sphere in $\mathbb{R}^3$.

In the following proposition, we obtain SG surfaces using deformation of the other parameter k_1.

Proposition 25. *Let u, provided by equation* (71)*, satisfy the* SG *equation given in equation* (64) *and $\mathfrak{su}(2)$ valued Lax pairs U and V of the* SG *equation are given by equations* (65)*. The matrices A and B are defined as*

$$\begin{aligned} A &= \mu\frac{\partial U}{\partial k_1}=\frac{\mathrm{i}\,\mu}{2}\begin{pmatrix}0 & -(\phi_2)_x\\ -(\phi_2)_x & 0\end{pmatrix}\\ B &= \mu\frac{\partial V}{\partial k_1}=\frac{\mu}{2\,\lambda}\begin{pmatrix}\mathrm{i}\,\sin(u)\,\phi_2 & \cos(u)\,\phi_2\\ -\cos(u)\,\phi_2 & -\mathrm{i}\,\sin(u)\,\phi_2\end{pmatrix}\end{aligned}$$

where $\phi_2=\partial\,u/\partial k_1$, k_1 is a parameter of the one soliton solution u,and μ is a constant. Then the surface S has the following first and second fundamental forms

$$\begin{aligned} \mathrm{d}s_I^2\equiv g_{jk}\,\mathrm{d}x^j\,\mathrm{d}x^k &= L_9\,\mathrm{sech}^4\xi\,(\xi_2\,\sinh\xi+\cosh\xi)^2\,\mathrm{d}x^2\\ &\quad +L_{10}\,\xi_2^2\,\mathrm{sech}^2\xi\,\mathrm{d}t^2\\ \mathrm{d}s_{II}^2\equiv g_{jk}\,\mathrm{d}x^j\,\mathrm{d}x^k &= L_{11}\tanh\xi\,\mathrm{sech}^3\xi\,(\xi_2\,\sinh\xi+\cosh\xi)\,\mathrm{d}x^2\\ &\quad +L_{12}\,\xi_2\,\mathrm{sech}^2\xi\,\mathrm{d}t^2.\end{aligned}$$

Gaussian and mean curvatures of S have the following form

$$K=L_{13}\frac{\sinh\xi}{\xi_2\,(\xi_2\,\sinh\xi+\cosh\xi)},\qquad H=L_{14}\frac{(2\,\xi_2\,\sinh\xi+\cosh\xi)}{\xi_2\,(\xi_2\,\sinh\xi+\cosh\xi)}$$

where

$$\xi = \Big(k_1\,(t+x) + (k_1^2-1)^{1/2}\,(t-x) + k_2\Big)$$
$$\xi_2 = (t+x)(k_1^2-1)^{1/2} + k_1(t-x)$$
$$L_9 = \frac{\mu^2\left(k_1-(k_1^2-1)^{1/2}\right)^2}{k_1^2-1}, \qquad L_{10} = \frac{\mu^2}{\lambda^2\left(k_1^2-1\right)}$$
$$L_{11} = \frac{2\,\mu\,\lambda\left(k_1-(k_1^2-1)^{1/2}\right)}{(k_1^2-1)^{1/2}}, \qquad L_{12} = \frac{2\,\mu\left((k_1^2-1)^{1/2}+k_1\right)}{\lambda\,(k_1^2-1)^{1/2}}$$
$$L_{13} = \frac{4\,\lambda^2\left((k_1^2-1)^{1/2}+k_1\right)^2\left(k_1^2-1\right)}{\mu^2}$$
$$L_{14} = \frac{\lambda}{\mu}\Big(k_1+({k_1}^2-1)^{1/2}\Big)\,({k_1}^2-1)^{1/2}.$$

7.4. NLS Surfaces

In this section we obtain surfaces in $\mathbb{R}^3$ corresponding NLS equation [5], [46]. Let complex function $u(x,t) = r(x,t) + \mathrm{i}s(x,t)$ satisfy the NLS equation

$$r_t = s_{xx} + 2s(r^2+s^2), \qquad s_t = -r_{xx} - 2r(r^2+s^2) \tag{72}$$

where r, s are real functions.
By changing the variables r and s as

$$r = q\cos\phi, \qquad s = q\sin\phi \tag{73}$$

and NLS given in equations (72) take the following form

$$q\phi_t = -q_{xx} - 2q^3 + q\phi_x^2, \qquad q_t = q\phi_{xx} + 2q_x\phi_x. \tag{74}$$

The Lax pairs U and V of these equations are given as

$$\begin{aligned} U &= \frac{\mathrm{i}}{2}\begin{pmatrix} -2\lambda & 2\,q\,(\sin\phi - \mathrm{i}\,\cos\phi) \\ 2\,q\,(\sin\phi + \mathrm{i}\,\cos\phi) & 2\lambda \end{pmatrix} \\ V &= -\frac{\mathrm{i}}{2}\begin{pmatrix} -2\left(2\,\lambda^2-q^2\right) & z_1+\mathrm{i}z_2 \\ z_1-\mathrm{i}z_2 & 2\left(2\,\lambda^2-q^2\right) \end{pmatrix} \end{aligned} \tag{75}$$

where

$$z_1 = 2\,(q_x + 2\,\lambda\,q)\cos\phi - 2\,q\,\phi_x\sin\phi$$
$$z_2 = 2\,(q_x + 2\,\lambda\,q)\sin\phi - 2\,q\,\phi_x\cos\phi$$

and λ is a constant.
In the following proposition we obtain the NLS surfaces using spectral deformation.

Proposition 26. *Let q and ϕ satisfy* NLS *equation given in equations* (74)*. The Lax pairs U and V of the* NLS *equation are given by equations* (75)*. The $\mathfrak{su}(2)$ valued matrices A and B are defined as*

$$A = \mu \frac{\partial U}{\partial \lambda} = \frac{\mathrm{i}}{2} \begin{pmatrix} -2\mu & 0 \\ 0 & 2\mu \end{pmatrix}$$

$$B = \mu \frac{\partial V}{\partial \lambda} = -\frac{\mathrm{i}}{2} \begin{pmatrix} -8\lambda\mu & 4\mu\, q(\cos\phi - \mathrm{i}\,\sin\phi) \\ 4\mu\, q(\cos\phi + \mathrm{i}\,\sin\phi) & 8\lambda\mu \end{pmatrix}$$

where λ is spectral parameter and μ is a constant. Then the surface S has the following first and second fundamental forms $(j,k=1,2)$

$$\mathrm{d}s_I^2 \equiv g_{jk}\, \mathrm{d}x^j\, \mathrm{d}x^k = \mu^2 \Big((\mathrm{d}x - 4\lambda\, \mathrm{d}t)^2 + 4\, q^2\, \mathrm{d}t^2 \Big)$$

$$\mathrm{d}s_{II}^2 \equiv h_{jk}\, \mathrm{d}x^j\, \mathrm{d}x^k = -2\,\mu\, q \left(\mathrm{d}x - (2\,\lambda - \phi_x)\mathrm{d}t \right)^2 + 2\,\mu\, q_{2x}\, \mathrm{d}t^2.$$

The Gaussian and mean curvatures of S are obtained as

$$K = -\frac{q_{xx}}{\mu^2\, q}, \qquad H = \frac{q_{xx} - q\,(\phi_x + 2\,\lambda)^2 - 4\, q^3}{4\,\mu\, q^2} \tag{76}$$

where $x^1 = x$, $x^2 = t$.

Let $\phi = \alpha\, t$ and $q = q(x)$ satisfies the following equation

$$q_{xx} = -2q^3 - \alpha\, q. \tag{77}$$

When we multiply the equation given in equation (77) by q_x and integrate the resultant equation, $q(x)$ satisfy the following equation

$$q_x^2 = -q^4 - \alpha\, q^2. \tag{78}$$

The following proposition gives a class of NLS surfaces which are Willmore-like.

Proposition 27. *Let $\phi = \alpha\, t$ and $q = q(x)$ satisfy the equation given in equation* (78) *and S be the surface obtained in* Proposition 26*. Then the surface S is called a Willmore-like surface. This means that the Gaussian and mean curvatures of the surface S satisfy the partial differential equation given in equation* (54)*, where a, b, and α have the following form*

$$a = \frac{4}{3}, \qquad b = 0, \qquad \alpha = -2\,\lambda^2$$

and λ is an arbitrary constant.

The following proposition contains the Weingarten surfaces.

Proposition 28. *Let S be the surface obtained in* Proposition 26, $\theta = \alpha\, t$ *and* $q = q(x)$ *satisfy equation* (78). *Then the surface S is a Weingarten surface that has the following algebraic relation between Gaussian and mean curvatures of the surface*

$$8\,\mu^2 H^2 \left(K\mu^2 - \alpha\right) = \left(3\,K\mu^2 - 2\,\alpha + 4\,\lambda^2\right)^2$$

where α, μ, and λ are constants. This surface S is a Weingarten surface. When $\alpha = -4\lambda^2$, the surface S reduces to a quadratic Weingarten surface

$$K - \frac{8}{9}\,H^2 + 4\,\frac{\lambda^2}{\mu^2} = 0.$$

Proposition 29. *Let $\theta = \alpha\, t$ and $q = q(x)$ satisfy the equation given in equation* (78) *and S be the surface in* Proposition 26. *Then there are* NLS *surfaces satisfying the generalized shape equation given in equation* (52) *where the Lagrange function $\mathcal{E}$ is a polynomial of Gaussian and mean curvatures of the surface S.*

We now give some examples of $\mathcal{E}$ with deg($\mathcal{E}$) $= N$, for the NLS surfaces that solve the Euler-Lagrange equation given in equation (52) and provide the constraints [46].

Example 11.

i) For $N = 3$, the Lagrange function is in the following form

$$\mathcal{E} = a_1\,H^3 + a_2\,H^2 + a_3\,H + a_4 + a_5\,K + a_6\,K\,H$$

where

$$\alpha = -2\,\lambda^2,\ \ a_1 = -\frac{p\,\mu^4}{18\,\lambda^4},\ \ a_2 = a_4 = 0,\ \ a_3 = \frac{p\,\mu^2}{16\,\lambda^2},\ \ a_6 = \frac{p\,\mu^4}{8\,\lambda^4}.$$

Here $\lambda \neq 0$, and μ, p, and a_5 are arbitrary constants.

ii) For $N = 4$, the Lagrange function is in the following form

$$\begin{aligned}\mathcal{E} &= a_1\,H^4 + a_2\,H^3 + a_3\,H^2 + a_4\,H + a_5 + a_6\,K \\ &\quad + a_7\,K\,H + a_8\,K^2 + a_9\,K\,H^2\end{aligned}$$

where

$$\alpha = -2\,\lambda^2, \qquad a_1 = -\frac{8}{189}\,(8\,a_8 + 15\,a_9), \qquad a_2 = -\frac{p\,\mu^4}{18\,\lambda^4}$$

$$a_3 = \frac{2\,\lambda^2}{7\,\mu^2}\,(32\,a_8 + 25\,a_9), \qquad a_4 = \frac{p\,\mu^2}{16\,\lambda^2}$$

$$a_5 = -\frac{2\,\lambda^4}{21\,\mu^4}\,(38\,a_8 + 45\,a_9), \qquad a_7 = \frac{p\,\mu^4}{8\,\lambda^4}.$$

Here $\lambda \neq 0$, $\mu \neq 0$, and p, a_6, a_8, and a_9 are arbitrary constants.

iii) For $N = 5$, the Lagrange function is in the following form

$$\mathcal{E} = a_1 H^5 + a_2 H^4 + a_3 H^3 + a_4 H^2 + a_5 H + a_6 \\ +a_7 K + a_8 K H + a_9 K^2 + a_{10} K H^2 + a_{11} K^2 H + a_{12} K H^3$$

where

$$\alpha = -2\lambda^2, \qquad a_1 = -\frac{4}{1053}(128\, a_{11} + 189\, a_{12})$$

$$a_2 = -\frac{8}{169}(8\, a_9 + 15\, a_{10})$$

$$a_3 = \frac{1}{936\,\mu^2\,\lambda^4}\left(\lambda^6\,(784\, a_{11} + 2313\, a_{12}) - 52\, p\, \mu^6\right)$$

$$a_4 = \frac{2\,\lambda^2}{7\,\mu^2}(32\, a_9 + 25\, a_{10})$$

$$a_5 = \frac{1}{832\,\mu^4\,\lambda^2}\left(52\, p\, \mu^6 - \lambda^6\,(111248\, a_{11} + 6449\, a_{12})\right)$$

$$a_6 = -\frac{2\,\lambda^4}{21\,\mu^4}(38\, a_9 + 45\, a_{10})$$

$$a_8 = \frac{1}{416\,\mu^2\,\lambda^4}\left(\lambda^6\,[8048\, a_{11} + 3591\, a_{12}] + 52\, p\, \mu^6\right).$$

Here $\lambda \neq 0$, $\mu \neq 0$, and $p, a_7, a_9, a_{10}, a_{11}$, and a_{12} are arbitrary constants.

For general $N \geq 3$, from the above examples, the polynomial function $\mathcal{E}$ takes the form

$$\mathcal{E} = \sum_{n=0}^{N} H^n \sum_{l=0}^{\lfloor\frac{(N-n)}{2}\rfloor} a_{nl} K^l$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and a_{nl} are constants.

7.4.1. Position Vector of NLS Surfaces

In this section, we find the position vector of the NLS surfaces that we obtained using spectral parameter deformation in Proposition 26.

Let $q = 2\eta\,\mathrm{sech}\xi$ and $\theta(t) = \rho$ be solution of NLS equation, where $\xi = 2\eta x - \kappa$ and $\rho = -4\eta^2 t$.

In order to find the position vector first we solve the Lax equation given in equation (38). The solution of Lax equation is a 2×2 matrix Φ

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}$$

where $\Phi_{11},\ \Phi_{12},\ \Phi_{21},\ \Phi_{22}$ are given as

$$
\begin{aligned}
\Phi_{11} &= \frac{\mathrm{i}}{\eta(\sin\rho+\mathrm{i}\cos\rho)}\Big(C_1\mathrm{e}^{2\,\mathrm{i}\left(\lambda^2+2\,\eta^2\right)t}(\eta\tanh\xi+\mathrm{i}\lambda) \\
&\quad\times(\tanh\xi+1)^{-\mathrm{i}\lambda/4\eta}(\tanh\xi-1)^{\mathrm{i}\lambda/4\eta} \\
&\quad-D_1\mathrm{e}^{-2\,\mathrm{i}\lambda^2t}\eta^2\mathrm{sech}\,\xi(\tanh\xi+1)^{\mathrm{i}\lambda/4\eta}(\tanh\xi-1)^{-\mathrm{i}\lambda/4\eta}\Big) \\
\Phi_{12} &= \frac{\mathrm{i}}{\eta(\sin\rho+\mathrm{i}\cos\rho)}\Big(C_2\mathrm{e}^{2\,\mathrm{i}\left(\lambda^2+2\,\eta^2\right)t}(\eta\tanh\xi+\mathrm{i}\lambda) \\
&\quad\times(\tanh\xi+1)^{-\mathrm{i}\lambda/4\eta}(\tanh\xi-1)^{\mathrm{i}\lambda/4\eta} \\
&\quad-D_2\mathrm{e}^{-2\,\mathrm{i}\lambda^2t}\eta^2\mathrm{sech}\,\xi(\tanh\xi+1)^{\mathrm{i}\lambda/4\eta}(\tanh\xi-1)^{-\mathrm{i}\lambda/4\eta}\Big) \\
\Phi_{21} &= C_1\mathrm{e}^{2\,\mathrm{i}\left(\lambda^2+2\,\eta^2\right)t}\mathrm{sech}\,\xi\,(\tanh\xi+1)^{-\mathrm{i}\lambda/4\eta}(\tanh\xi-1)^{i\lambda/4\eta} \\
&\quad+D_1\mathrm{e}^{-2\,\mathrm{i}\lambda^2t}\,(\eta\,\tanh\xi-\mathrm{i}\lambda)(\tanh\xi+1)^{\mathrm{i}\lambda/4\eta}(\tanh\xi-1)^{-\mathrm{i}\lambda/4\eta} \\
\Phi_{22} &= C_2\mathrm{e}^{2\,\mathrm{i}\left(\lambda^2+2\,\eta^2\right)t}\mathrm{sech}\,\xi\,(\tanh\xi+1)^{-\mathrm{i}\lambda/4\eta}(\tanh\xi-1)^{\mathrm{i}\lambda/4\eta} \\
&\quad+D_2\mathrm{e}^{-2\,\mathrm{i}\lambda^2t}\,(\eta\,\tanh\xi-\mathrm{i}\lambda)(\tanh\xi+1)^{\mathrm{i}\lambda/4\eta}(\tanh\xi-1)^{-\mathrm{i}\lambda/4\eta}.
\end{aligned}
\tag{79}
$$

Here the determinant of the solution of the Lax equation Φ is constant and it has the following form

$$
\det(\Phi) = \frac{\left(\eta^2+\lambda^2\right)\left(C_1D_2-C_2D_1\right)}{\eta} \neq 0.
$$

We use the equation given in equation (39) in order to find the immersion function F. We obtain F in the following form

$$
F = y_1\mathrm{e}_1 + y_2\mathrm{e}_2 + y_3\mathrm{e}_3
$$

where y_1, y_2, and y_3 are given as

$$
\begin{aligned}
y_1 &= -\frac{1}{\eta\left(\mathrm{e}^{2\xi}+1\right)}W_8\left(\Omega_4\left(\mathrm{e}^{2\xi}+1\right)-2\eta\right) \\
y_2 &= -W_8\,\mathrm{sech}(\xi)\sin(\Omega_5) \\
y_3 &= W_8\,\mathrm{sech}(\xi)\cos(\Omega_5)
\end{aligned}
\tag{80}
$$

where

$$
\begin{aligned}
W_8 &= \frac{\mu\eta}{\eta^2+\lambda^2}, \qquad \Omega_4 = (4\lambda t - x)(\eta^2+\lambda^2) \\
\Omega_5 &= \frac{1}{\eta}\Big(4\eta(\eta^2+\lambda^2)t - \lambda(2\eta x-\kappa)\Big), \qquad \xi = 2\eta x - \kappa.
\end{aligned}
$$

7.4.2. Plotting NLS Surfaces

In this section, we plot some of the NLS surfaces where the position vector is provided by the equations in equations (80) for some special values of the constants μ, η, and κ.

Example 12. *If we take* $\lambda = 2$*,* $\mu = 3$*,* $\kappa = 10$ *and changing* η *as* $a)\,\eta = 0.5$*,* $b)\,\eta = 0.75$*, and* $c)\,\eta = 1$ *in the equations* (80)*, we get the surface given in* Fig. 8.

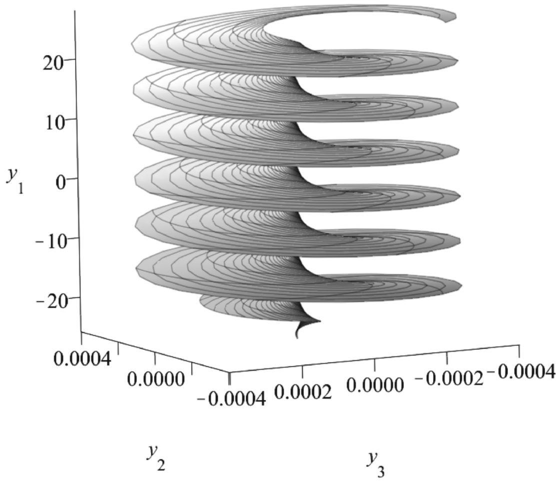


Figure 8. $(x,t) \in [-1,1] \times [-1,1]$

Example 13. *If we take* $\lambda = 0$*,* $\mu = 0.2$*,* $\eta = 0.3$ *and* $\kappa = 4$ *in the equations* (80)*, we get the surface given in* Fig. 9.

Example 14. *If we take* $\lambda = 0.5$*,* $\mu = 1$*,* $\eta = 2$ *and* $\kappa = 0$ *in the equations* (80)*, we get the surface given in* Fig. 10.

8. Soliton Surfaces in $\mathbb{M}_3$

In this section, we develop surfaces in three dimensional Minkowski space using the similar techniques that we used in Section 7. Consider the isometric immersion $F : \mathcal{U} \to \mathbb{M}_3$. Here $\mathcal{U} \in \mathbb{M}_2$ is the domain of the immersion, $\mathbb{M}_2$ and $\mathbb{M}_3$ are two and three dimensional Minkowski spaces. To investigate the surfaces in $\mathbb{M}_3$, the Lie group G that we use is $\mathrm{SL}(2,\mathbb{R})$, the corresponding Lie algebra $\mathfrak{g}$ is $\mathfrak{sl}(2,\mathbb{R})$. The 2×2 base matrices of $\mathfrak{sl}(2,\mathbb{R})$ are provided by

$$\mathrm{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathrm{e}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathrm{e}_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

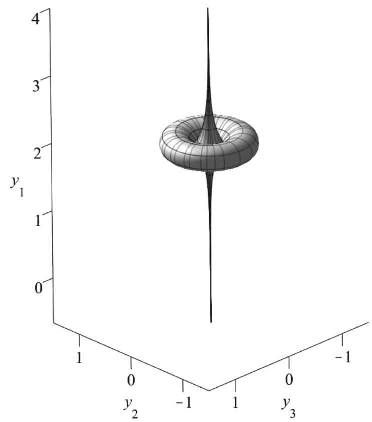


Figure 9. $(x,t) \in [-15,15] \times [-15,15]$

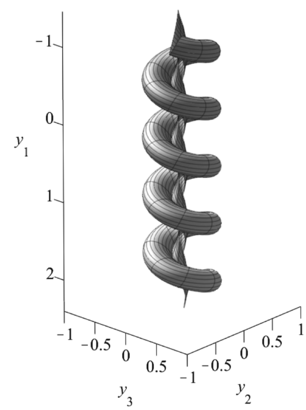


Figure 10. $(x,t) \in [-0.8,0.8] \times [-0.8,0.8]$

The inner product defined on $\mathfrak{sl}(2,\mathbb{R})$ is given as

$$\langle X, Y\rangle = \frac{1}{2}\ \mathrm{trace}(XY)$$

for X, $Y \in \mathfrak{sl}(2,\mathbb{R})$.

8.1. KdV Surfaces from Spectral Parameter Deformations

In this section, we obtain surfaces corresponding KdV equation using spectral parameter deformation [15].

Let $u(x,t)$ satisfy the following KdV equation

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x. \tag{81}$$

The Lax pairs U and V of the KdV equation given in equation (81) have the following forms

$$U = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} -\frac{1}{4}u_x & \frac{1}{2}u + \lambda \\ -\frac{1}{4}u_{xxx} + \frac{1}{2}\,(2\,\lambda + u)\,(\lambda - u) & \frac{1}{4}u_x \end{pmatrix} \tag{82}$$

where λ is the spectral parameter.

In the following proposition, we obtain KdV surfaces using spectral parameter deformation of U and V.

Proposition 30. *Let u satisfy the* KdV *equation given in equation* (81) *and* $\mathfrak{sl}(2,\mathbb{R})$ *valued Lax pairs* U *and* V *are provided by equations* (82). *The matrices* A *and* B *defined as spectral parameter deformations of the Lax pairs* U *and* V, *respectively*

$$A = \mu \frac{\partial U}{\partial \lambda} = \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix}, \qquad B = \mu \frac{\partial V}{\partial \lambda} = \begin{pmatrix} 0 & \mu \\ \frac{\mu}{2}(4\lambda - u) & 0 \end{pmatrix}$$

where λ *is spectral parameter, and* μ *is a constant. Then the first and second fundamental forms of the surface* S *are given as*

$$\mathrm{d}s_I^2 \equiv g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j = \mu^2\, \mathrm{d}x\, \mathrm{d}t + \frac{\mu^2}{2}(4\lambda - u)\mathrm{d}t^2$$

$$\mathrm{d}s_{II}^2 \equiv h_{ij}\, \mathrm{d}x^i \mathrm{d}x^j = -\mu\, \mathrm{d}x^2 - \mu(2\,\lambda + u)\mathrm{d}x\, \mathrm{d}t - \frac{\mu}{4}\Big(u_{xx} + (u + 2\,\lambda)^2\Big)\mathrm{d}t^2$$

and the Gaussian and mean curvatures are obtained as

$$K = -\frac{u_{xx}}{\mu^2}, \qquad H = \frac{2(\lambda - u)}{\mu}$$

where $x^1 = x$, $x^2 = t$.

When we use traveling wave ansatz $u_t + u_t/c = 0$ in KdV equation given in equation (81) and integrate the resultant equation, we obtain the following form of the KdV equation

$$u_{xx} = -3u^2 - \frac{4}{c}u + 4\beta \tag{83}$$

where c and β are constants.

We obtained the invariants such as K, H, first and second fundamental forms of the KdV surfaces in Proposition 30. In the following proposition, we give quadratic Weingarten surfaces.

Proposition 31. *Let u be a traveling wave solution of the* KdV *equation given in equation* (83) *and S be the surface obtained using spectral parameter deformation in* Proposition 30. *Then the surface S is a Weingarten surface that has the following algebraic relation between Gaussian and mean curvatures of the surface*

$$4\,c\,\mu^2\,K + 4\,\mu\,(2 + 3\,c\,\lambda)\,H - 3\,c\,\mu^2\,H^2 - 4\,(3\,c\,\lambda^2 + 4\,\lambda - 4\,\beta\,c) = 0$$

where c and β *are constants,* $\mu \neq 0$ *and* $c \neq 0$.

When we multiply the KdV equation in equation (83) by u_x and integrate the resultant equation, we obtain the following form of the KdV equation

$$u_x^2 = -2u^3 + 4\alpha u^2 + 8\beta u + 2\gamma \tag{84}$$

where $\alpha = -1/c$, $c \neq 0$.

The following proposition contains Willmore-like surfaces.

Proposition 32. *Let u satisfy the* KdV *equation given in equation* (84) *and S be the surface S obtained in* Proposition 30. *Then the surface S is called a Willmore-like surface. This means that the Gaussian and mean curvatures of the surface S satisfy the partial differential equation given in equation* (54)*, where a, b, β, and γ have the following form*

$$a = \frac{7}{4}, \qquad b = 1, \qquad \beta = \frac{1}{20}\big(28\lambda\alpha - 16\alpha^2 - 21\lambda^2\big)$$
$$\gamma = \frac{1}{5}\big(16\alpha^3 - 56\lambda\alpha^2 + 70\alpha\lambda^2 - 28\lambda^3\big).$$

Here $\alpha = -1/c$ ($c \neq 0$), λ and c are arbitrary constants.

In the following proposition we give KdV surfaces that solve the Euler-Lagrange equation given in equation (52).

Proposition 33. *Let u satisfy equation* (84) *and S be the surface in* Proposition 30. *Then there are* KdV *surfaces satisfying the generalized shape equation* (52) *with Lagrange functions which are polynomials of Gaussian and mean curvatures of the surface S.*

Let us now give some examples of polynomial Lagrange functions $\mathcal{E}$ of H and K with deg$(\mathcal{E}) = N$ that solve the Euler-Lagrange equation given in equation (52) and provide the constraints [15].

Example 15.

i) For $N = 3$, the Lagrange function is in the following form

$$\mathcal{E} = a_1 H^3 + a_2 H^2 + a_3 H + a_4 + a_5 K + a_6 K H$$

where

$$a_1 = -\frac{11\,p\,\mu^4}{64\,\Xi_1}, \qquad a_2 = -\frac{15}{32\,\Xi_1}\,p\,\mu^3\big(2\,\alpha - 3\,\lambda\big)$$
$$a_3 = -\frac{p\,\mu^2}{16\,\Xi_1}\big(33\,\lambda^2 - 44\,\alpha\,\lambda + 8\,\alpha^2 - 20\,\beta\big)$$
$$a_4 = \frac{p\,\mu}{8\,\Xi_1}\,\big(47\,\lambda^3 - 94\,\alpha\,\lambda^2 + 4\,(10\,\alpha^2 - 17\,\beta)\,\lambda + 40\,\alpha\,\beta - 2\,\gamma\big)$$
$$a_6 = \frac{7\,p\,\mu^4}{16\,\Xi_1}.$$

Here $\Xi_1 = 12\,\lambda^4 - 32\,\alpha\,\lambda^3 + (20\,\alpha^2 - 36\,\beta)\lambda^2 + (40\,\alpha\,\beta - 3\,\gamma)\lambda + 2\,\alpha\,\gamma + 16\,\beta^2$, $\mu \neq 0$, $p \neq 0$, λ, α, β, γ and a_5 are arbitrary constants, but λ, α, β and γ cannot be zero at the same time.

ii) For $N = 4$, the Lagrange function is in the following form

$$\mathcal{E} = a_1 H^4 + a_2 H^3 + a_3 H^2 + a_4 H + a_5 + a_6 K + a_7 K H + a_8 K^2 + a_9 K H^2$$

where

$$
\begin{aligned}
a_1 &= -\frac{1}{64}\,(34\,a_9 + 15\,a_8)\\
a_2 &= \frac{1}{56\,\mu}\,\Big((210\,\lambda - 140\,\alpha)\,a_9 + (195\,\lambda - 130\,\alpha)\,a_8 - 22\,\mu\,a_7\Big)\\
a_3 &= \frac{1}{56\,\mu^2}\,\Big(\Big(1512\,\alpha\,\lambda - 308\,\alpha^2 - 1134\,\lambda^2 + 588\,\beta\Big)a_9 + \Big(546\,\beta - 718\,\alpha^2\\
&\quad -2025\,\lambda^2 + 2700\,\alpha\,\lambda\Big)a_8 + 60\,\mu\,\Big(3\,\lambda - 2\,\alpha\Big)a_7\Big)\\
a_4 &= \frac{1}{14\,\mu^3}\,\Big(\Big(1414\,\lambda^3 - 2828\,\lambda^2\alpha + \big(1652\,\alpha^2 - 700\,\beta\big)\,\lambda + 392\,\beta\,\alpha\\
&\quad -28\,\gamma - 280\,\alpha^3\Big)a_9 + \Big(2265\,\lambda^3 - 4530\,\lambda^2\alpha + \big(2702\,\alpha^2 - 954\,\beta\big)\,\lambda\\
&\quad -42\,\gamma + 524\,\beta\,\alpha - 484\,\alpha^3\Big)a_8 - 2\,\Big(33\lambda^2 - 20\,\beta + 8\,\alpha^2 - 44\,\alpha\,\lambda\Big)\mu\,a_7\Big)\\
a_5 &= \frac{1}{28\,\mu^3}\,\Big(\Big(19960\,\lambda^3\alpha - 7485\,\lambda^4 + \big(2844\,\beta - 19012\,\alpha^2\big)\,\lambda^2\\
&\quad + \big(96\,\gamma + 7664\,\alpha^3 - 3536\,\beta\,\alpha\big)\,\lambda + 784\,\beta^2 - 1008\,\alpha^4 + 1616\,\alpha^2\beta\\
&\quad -64\,\alpha\,\gamma\Big)a_8 + \Big(9744\,\lambda^3\alpha - 3654\,\lambda^4 + \big(168\,\beta - 9688\,\alpha^2\big)\,\lambda^2\\
&\quad + \big(4256\,\alpha^3 - 224\,\beta\,\alpha\big)\,\lambda + 224\,\beta^2 + 224\,\alpha^2\beta - 672\,\alpha^4\Big)a_9\\
&\quad +8\,\mu\,a_7\Big(47\,\lambda^3 - 94\,\lambda^2\alpha + \big(-68\,\beta + 40\,\alpha^2\big)\,\lambda - 2\,\gamma + 40\,\beta\,\alpha\Big)\Big)\\
a_7 &= \frac{1}{16\,\mu\,\Xi_2}\,\Big(-672\,\Big(4\,\alpha\,\lambda - \alpha^2 + \beta - 3\,\lambda^2\Big)\Big(7\,\lambda^3/6 - 7\,\lambda^2\alpha/3\\
&\quad + \big(\alpha^2 - 5\,\beta/3\big)\,\lambda + \beta\,\alpha - \gamma/24\Big)a_9 + \Big(4680\,\lambda^5 - 15600\,\lambda^4\alpha\\
&\quad + \big(17576\,\alpha^2 - 9672\,\beta\big)\lambda^3 - \big(7664\,\alpha^3 + 414\,\gamma - 18240\,\beta\,\alpha\big)\lambda^2\\
&\quad + \big(552\,\alpha\,\gamma + 1008\,\alpha^4 + 3216\,\beta^2 - 9280\,\alpha^2\beta\big)\,\lambda - 170\,\alpha^2\gamma\\
&\quad +42\,\gamma\,\beta - 2032\,\alpha\,\beta^2 + 1008\,\beta\,\alpha^3\Big)a_8 + 7\,p\,\mu^5\Big).
\end{aligned}
$$

Here $\Xi_2 = 4\,\lambda^3\,(3\,\lambda - 8\,\alpha) + 4\,\big(5\,\alpha^2 - 9\,\beta\big)\,\lambda^2 + (-3\,\gamma + 40\,\beta\,\alpha)\,\lambda + 2\,\alpha\,\gamma + 16\,\beta^2$, and $\mu \neq 0, p \neq 0$, λ, α, β, γ, a_6, a_8 and a_9 are arbitrary constants, but λ, α, β and γ cannot be zero at the same time.

iii) For $N = 3$, the Lagrange function is in the following form

$$
\begin{aligned}
\mathcal{E} \,=\, & a_1\,H^5 + a_2\,H^4 + a_3\,H^3 + a_4\,H^2 + a_5\,H + a_6 + a_7\,K\\
& +a_8\,K\,H + a_9\,K^2 + a_{10}\,K\,H^2 + a_{11}\,K^2\,H + a_{12}\,K\,H^3
\end{aligned}
$$

where $a_1, a_2, a_3, a_4, a_5, a_6, a_8$ can be written in terms of $a_9, a_{10}, a_{11}, a_{12}$, α, β, γ, μ, p and λ.

For general $N \geq 3$, from the above examples, the polynomial function $\mathcal{E}$ takes the form

$$\mathcal{E} = \sum_{n=0}^{N} H^n \sum_{l=0}^{\lfloor \frac{(N-n)}{2} \rfloor} a_{nl} K^l$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and a_{nl} are constants.

8.1.1. Position Vector of KdV Surfaces

In this section, we find the position vector of the KdV surfaces using the solution of KdV equation and its the Lax pairs. We will consider two different solutions of the KdV equation.

Example 16. *Consider the constant solution*

$$u = u_0 = \frac{2}{3}(\alpha \pm \sqrt{\alpha^2 + 3\beta}) \tag{85}$$

of the integrated form of the KdV equation (84)*, where* $\alpha = -1/c$, $c \neq 0$.
Using this solution and corresponding matrix Lax pairs U *and* V *given by equations* (82) *of* KdV *equation, we solve the Lax equations given in equation* (38)*. The solution of Lax equation is a* 2×2 *matrix* Φ

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}.$$

We find these components of Φ *as*

$$\begin{aligned} \Phi_{11} &= C_1 \,\mathrm{e}^{m(nt+x)} + D_1 \,\mathrm{e}^{-m(nt+x)} \\ \Phi_{12} &= C_2 \,\mathrm{e}^{m(nt+x)} + D_2 \,\mathrm{e}^{-m(nt+x)} \\ \Phi_{21} &= m\Big(C_1 \,\mathrm{e}^{m(nt+x)} - D_1 \,\mathrm{e}^{-m(nt+x)}\Big) \\ \Phi_{22} &= m\Big(C_2 \,\mathrm{e}^{m(nt+x)} - D_2 \,\mathrm{e}^{-m(nt+x)}\Big) \end{aligned} \tag{86}$$

where $\lambda - u_0 = m^2$, $(2\lambda + u_0)/2 = n$, C_1, C_2, D_1 *and* D_2 *are arbitrary constants. Here we find that* $\det(\Phi) = 2m(C_2 D_1 - C_1 D_2) \neq 0$.

By using A, B, and Φ, we solve the equation given in equation (39) and write the immersion function F in the following form

$$F = \Phi^{-1} \frac{\partial \Phi}{\partial \lambda} = y_1 \mathrm{e}_1 + y_2 \mathrm{e}_2 + y_3 \mathrm{e}_3$$

where $\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3$ are basis elements of $\mathfrak{sl}(2,\mathbb{R})$ and

$$
\begin{aligned}
y_1 &= -\left(\frac{D_1C_2 + C_1D_2}{D_1C_2 - C_1D_2}\right) \frac{(4\lambda - u_0)t + x}{2\sqrt{\lambda - u_0}} \\
y_2 &= \left(\frac{D_1C_1 - D_2C_2}{D_1C_2 - D_2C_1}\right) \frac{(4\lambda - u_0)t + x}{2\sqrt{\lambda - u_0}} \\
y_3 &= -\left(\frac{D_1C_1 + D_2C_2}{D_1C_2 - D_2C_1}\right) \frac{(4\lambda - u_0)t + x}{2\sqrt{\lambda - u_0}}.
\end{aligned} \tag{87}
$$

Hence we find the position vector $\mathbf{Y} = (y_1(x,t), y_2(x,t), y_3(x,t))$ of KdV surfaces in $\mathbb{M}_3$ using the constant solution given in equation (85). The components y_1, y_2 and y_3 of the position vector the KdV surfaces are given by equations (87), respectively. This surface is plane in $\mathbb{M}_3$.

Example 17. *In this example, we consider nonconstant solution. Consider the following one soliton solution of the* KdV *equation*

$$u = 2\,k^2\,c^2 \mathrm{sech}^2 k(t - cx) \tag{88}$$

where $k^2 = -1/c^3$.

We solve the Lax equations given in equation (38) *using one soliton solution and corresponding matrix Lax pairs U and V given by equations* (82) *of* KdV *equation. Here we denote* $k(t - cx) = \xi$ *and let* $\lambda = k^2c^2$. *The solution of Lax equation is a* 2×2 *matrix* Φ

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}$$

where Φ_{11}, Φ_{12}, Φ_{21}, Φ_{122} *are given as*

$$
\begin{aligned}
\Phi_{11} &= B_1\left(2kt\,\mathrm{sech}\,\xi + \sinh\xi + \xi \mathrm{sech}\,\xi\right) + C_1 \mathrm{sech}\,\xi \\
\Phi_{12} &= B_2\left(2kt\,\mathrm{sech}\,\xi + \sinh\xi + \xi \mathrm{sech}\,\xi\right) + C_2 \mathrm{sech}\,\xi \\
\Phi_{21} &= kc\Bigg(B_1\Big(2kt\,\mathrm{sech}\,\xi\tanh\xi - \cosh\xi - \mathrm{sech}\,\xi \\
&\quad + \xi \mathrm{sech}\,\xi\tanh\xi\Big) + C_1 \mathrm{sech}\,\xi\tanh\xi\Bigg) \\
\Phi_{22} &= kc\Bigg(B_2\Big(2kt\,\mathrm{sech}\,\xi\tanh\xi - \cosh\xi - \mathrm{sech}\,\xi \\
&\quad + \xi \mathrm{sech}\,\xi\tanh\xi\Big) + C_2 \mathrm{sech}\,\xi\tanh\xi\Bigg)
\end{aligned} \tag{89}
$$

where B_1, B_2, C_1 *and* C_2 *are arbitrary constants. The determinant of the matrix* Φ *is a constant, we find it as*

$$\det(\Phi) = 2\,k\,c(C_2B_1 - C_1B_2) \neq 0.$$

In order to find the immersion function F explicitly, we insert the matrices A and B provided by equations (82), and the the matrix Φ given by equations (89) into the equations for F_x and F_t given in equation (39). When we solve the consequent equations, we acquire the immersion function F as

$$F = y_1 \mathrm{e}_1 + y_2 \mathrm{e}_2 + y_3 \mathrm{e}_3$$

where

$$\begin{aligned} y_1 &= \frac{2W_9}{\zeta_1}\Big(\Big(\Omega_6\zeta_2 + \Omega_7\zeta_1 + \zeta_3\Big)W_{10} + \Omega_8\zeta_2 W_{11} + W_{12}\Big) \\ y_2 &= \frac{W_9}{\zeta_1}\Big(\Big(\Omega_9\zeta_2 + \Omega_7\zeta_1 + \zeta_4\Big)W_{13} + \Big(\Omega_{10}\zeta_2 + \Omega_{11}\Big)W_{14} + W_{15}\Big) \\ y_3 &= \frac{W_9}{\zeta_1}\Big(\Big(\Omega_9\zeta_2 + \Omega_7\zeta_1 + \zeta_4\Big)W_{16} + \Big(\Omega_{10}\zeta_2 + \Omega_{11}\Big)W_{17} + W_{18}\Big) \end{aligned} \tag{90}$$

and $\mathrm{e}_1, \mathrm{e}_2,\ e_3$ are basis elements of $\mathfrak{sl}(2,\mathbb{R})$. Here $\zeta_i,\ i = 1,2,3,4,\ \Omega_j,\ j = 6,7,...,10$, and $W_l,\ l = 9,10,...,18$ are given as

$$\begin{aligned} \zeta_1 &= 1 + \mathrm{e}^{-2\xi}, \qquad \zeta_2 = \mathrm{e}^{-2\xi} - 1, \qquad \zeta_3 = c^3(\mathrm{e}^{-4\xi} - 1 - 2\sinh(2\xi)) \\ \zeta_4 &= \zeta_3 + 288t^2, \qquad \Omega_6 = -8(cx+3t)^2, \qquad \Omega_7 = 4kc^3(9t - cx) \\ \Omega_8 &= 8kc^3(3t - cx), \qquad \Omega_9 = -8(c^2x^2 - 6tcx - 9t^2) \\ \Omega_{10} &= -16kc^3(cx+3t), \qquad \Omega_{11} = -192kc^3 t \\ W_9 &= \mu/32c^2\,(B_1C_2 - B_2C_1), \qquad W_{10} = B_1B_2, \qquad W_{11} = C_1B_2 + C_2B_1 \\ W_{12} &= -16c^3C_1C_2, \qquad W_{13} = B_2^2 - B_1^2, \qquad W_{14} = B_2C_2 - B_1C_1 \\ W_{15} &= 16c^3\left(C_1^2 - C_2^2\right), \qquad W_{16} = B_1^2 + B_2^2, \qquad W_{17} = B_1C_1 + B_2C_2 \\ W_{18} &= -16c^3(C_1^2 + C_2^2) \end{aligned}$$

where $\zeta_i,\ i = 1,2,3,4$ and $\Omega_j,\ j = 6,7,...,10$ are functions of x and t, and $W_l,\ l = 9,10,...,18$ are constants given in terms of arbitrary constants B_1, B_2, C_1, and C_2.

Hence we obtain the position vector $\mathbf{Y} = (y_1(x,t), y_2(x,t), y_3(x,t))$ of the KdV surfaces in $\mathbb{M}_3$ using one soliton solution of KdV equation given in equation (88). The components y_1, y_2 and y_3 of the position vector the KdV surfaces are provided by equations (90) respectively. Here y_3 is the time like and y_1 and y_2 are space like coordinates in $\mathbb{M}_3$.

8.1.2. Plotting KdV Surfaces From Spectral Parameter Deformation

Position vector $\mathbf{Y} = (y_1, y_2, y_3)$ of the KdV surfaces corresponds to the spectral parameter deformation is given by equations (90) that we obtained in Example 17. We plot some of these KdV surfaces for some special values of the constants μ, k, c, B_1, B_2, C_1, and $C_2 = 1$.

Example 18. *Taking* $\mu = 1$, $k = 1$, $c = 1$, $B_1 = -1$, $B_2 = 1$, $C_1 = 1$, *and* $C_2 = 1$ *in the equations provided by equations* (90)*, we get the surface given in* Fig. 11.

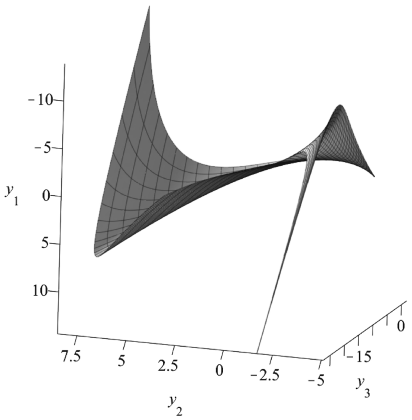


Figure 11. $(x,t) \in [-1.7, 1.7] \times [-1.7, 1.7]$

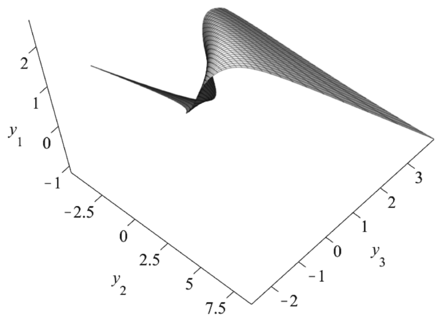


Figure 12. $(x,t) \in [-2, 2] \times [-2, 2]$

Example 19. *Taking* $\mu = 1$, $k = 1$, $c = 3$, $B_1 = -1$, $B_2 = 1$, $C_1 = 1$, *and* $C_2 = 1$ *in the equations* (90)*, we get the surface given in* Fig. 12.

8.2. KdV Surfaces From the Spectral-Gauge Deformations

In this section, we develop KdV surfaces using spectral-Gauge deformation.

Proposition 34. *Let u satisfy the* KdV *equation given in equation* (81) *and* $\mathfrak{sl}(2,\mathbb{R})$ *valued Lax pairs* U *and* V *are defined by equations* (82)*, respectively.* $\mathfrak{sl}(2,\mathbb{R})$ *valued matrices* A *and* B *are defined as*

$$
\begin{aligned}
A &= \mu_1 \frac{\partial U}{\partial \lambda} + \mu_2 [\mathrm{e}_1, U] = \begin{pmatrix} 0 & 2\mu_2 \\ 2\mu_2(u-\lambda) + \mu_1 & 0 \end{pmatrix} \\
B &= \mu_1 \frac{\partial V}{\partial \lambda} + \mu_2 [\mathrm{e}_1, U] \\
&= \begin{pmatrix} 0 & \mu_2(2\lambda + u) + \mu_1 \\ \frac{\mu_2}{2}\left(u_{xx} - 2(2\lambda - u)(u+\lambda)\right) + \frac{\mu_1}{2}(4\lambda - 4) & 0 \end{pmatrix}
\end{aligned}
$$

where μ_1 and μ_2 are arbitrary constants. First and second fundamental forms of the surface S are given as

$$\begin{aligned}
\mathrm{d}s_I^2 \equiv g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j &= 2\,\mu_2\Big(2\mu_2(u-\lambda)+\mu_1\Big)\mathrm{d}x^2 \\
&+\Big(\mu_2\Big(\mu_2\,u_{2x}-2(u+2\,\lambda)(\mu_1-2\,\mu_2\,(\lambda-u))\Big)+\mu_1^2\Big)\mathrm{d}x\mathrm{d}t \\
&-\frac{1}{2}\Big(2(u+2\lambda)+\mu_1\Big)\Big(2\mu_2(u+2\lambda)(\lambda-u)-\mu_1(4\lambda-u)-\mu_2 u_{xx}\Big)\mathrm{d}t^2 \\
\mathrm{d}s_{II}^2 \equiv h_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j &= \Big(4\mu_2(\lambda-u)-\mu_1\Big)\mathrm{d}x^2 \\
&-\Big(\mu_2\,u_{2x}+\big(\mu_1-4\,\mu_2\,(\lambda-u)\big)\big(2\,\lambda+u\big)\Big)\mathrm{d}x\,\mathrm{d}t \\
&-\frac{1}{4}\Big(\big(\mu_1+2\,\mu_2(2\,\lambda+u)\big)u_{2x}+\big(\mu_1-4\,\mu_2(\lambda-u)\big)\big(u+2\,\lambda\big)^2\Big)\mathrm{d}t^2
\end{aligned}$$

and the corresponding Gaussian and mean curvatures have the following form

$$K_1=\frac{u_{xx}}{\mu_2^2 u_{xx}+\mu_1\,(4\,\mu_2(\lambda-u)-\mu_1)},\qquad H_1=\frac{2\mu_1(\lambda-u)+\mu_2 u_{xx}}{\mu_2^2 u_{xx}+\mu_1\,(4\,\mu_2(\lambda-u)-\mu_1)}$$

where $x^1=x$, $x^2=t$.

8.3. HD Surfaces

In this section we obtain surfaces in M_3 corresponding Harry Dym (HD) equation [44], [45]. Let $u(x,t)$ satisfy the following HD equation

$$u_t=-u^3\,u_{xxx}. \tag{91}$$

The Lax pairs U and V of the HD equation in equation (91) are given as

$$U=\begin{pmatrix} 0 & 1 \\ \dfrac{\lambda^2}{u^2} & 0\end{pmatrix},\qquad V=2\,\lambda^2\begin{pmatrix} u_x & -2\,u \\ u_{xx}-\dfrac{2\,\lambda^2}{u} & -u_x\end{pmatrix} \tag{92}$$

were λ is a spectral parameter.

In the following proposition, we develop HD surfaces using spectral deformation of the Lax pairs U and V.

Proposition 35. *Let u satisfy the HD equation given in equation* (91) *and $\mathfrak{sl}(2,\mathbb{R})$ valued Lax pairs U and V are defined by equation* (92). *The matrices A and B are obtained as*

$$A=\mu\,\frac{\partial U}{\partial\lambda}=2\,\mu\,\lambda\begin{pmatrix} 0 & 0 \\ \dfrac{1}{u^2} & 0\end{pmatrix},\qquad B=\mu\frac{\partial V}{\partial\lambda}=4\,\mu\,\lambda\begin{pmatrix} u_x & -2\,u \\ u_{xx}-\dfrac{4\,\lambda}{u} & -u_x\end{pmatrix}$$

where μ is a constant and λ is a spectral parameter. The first and second fundamental forms of the surface S are given as

$$\begin{aligned} \mathrm{d}s_I^2 \equiv g_{jk}\,\mathrm{d}x^j\,\mathrm{d}x^k &= -16\,\mu^2\,\lambda^2\Big(\frac{1}{u}\mathrm{d}x\,\mathrm{d}t + (u_x^2 - 2\,u\,u_{xx} + 8\,\lambda^2)\mathrm{d}t^2\Big) \\ \mathrm{d}s_{II}^2 \equiv h_{jk}\,\mathrm{d}x^j\,\mathrm{d}x^k &= -\frac{2\,\mu\,\lambda}{u^2}\Big(\mathrm{d}x^2 - 8\,\lambda^2\,u\,\mathrm{d}x\,\mathrm{d}t \\ &\quad +2\,u^2\left(2\,u^2\,u_x\,u_{xxx} + u^3\,u_{4x} + 8\,\lambda^4\right)\mathrm{d}t^2\Big). \end{aligned}$$

The Gaussian and mean curvatures of the surface S are obtained as

$$K = -\frac{u^2}{8\,\mu^2\,\lambda^2}\left(2\,u_x\,u_{xxx} + u\,u_{xxxx}\right), \qquad H = \frac{1}{4\,\mu\,\lambda}\left(u_x^2 - 2\,u\,u_{xx} + 4\,\lambda^2\right)$$

where $x^1 = x$, $x^2 = t$.

When we use traveling wave ansatz $u_t - \alpha\,u_x = 0$ in HD equation given by equation (91), we obtain the following form of the HD equation

$$u_{xx} = \frac{\alpha}{2}\frac{1}{u} - C_1 \tag{93}$$

where α and C_1 are arbitrary constants.
When we multiply the HD equation in equation (93) by u_x and integrate the resultant equation, we obtain the following equation

$$u_x^2 = -\alpha\frac{1}{u} - 2\,C_1\,u + 2\,C_2. \tag{94}$$

In the following proposition we give HD surfaces belong to Willmore-like surfaces.

Proposition 36. *Let u satisfy the equation given in equation* (94) *and S be the surface obtained in* Proposition 35. *Then the surface S is called a Willmore-like surface. This means that the Gaussian and mean curvatures of the surface S satisfy the partial differential equation given in equation* (54)*, where a, b, C_1, and C_2 have the following form*

$$a = -2, \qquad b = 6, \qquad C_1 = \frac{16\,\lambda^4}{\alpha}, \qquad C_2 = -6\,\lambda^2$$

and λ is an arbitrary constant.

The following proposition gives HD surfaces belongs to Weingarten surfaces.

Proposition 37. *Let u be a travelling wave solution of the* HD *equation given in equation* (94) *and S be the surface obtained using spectral parameter deformation in* Proposition 35. *Then the surface S is a Weingarten surface that has the following algebraic relation between Gaussian and mean curvatures of the surface*

$$4\mu^2\lambda^2(4K - 3H^2) + (24\mu\lambda^3 + 4\mu\lambda C_2)H + C_3 = 0$$

where $C_3 = -4\lambda^2(3\lambda^2 - C_2) - 2\alpha C_1 + C_2^2$.

In the following proposition we obtain HD surfaces arise from a variational principle in another words solve the Euler-Lagrange equation (52).

Proposition 38. *Let* u *satisfy the equation given in equation* (94) *and* S *be the surface in* Proposition 35. *Then there are* HD *surfaces satisfying the generalized shape equation given in equation* (52) *where the Lagrange function* $\mathcal{E}$ *is a polynomial of Gaussian and mean curvatures of the surface* S.

We now give some examples of $\mathcal{E}$ with deg $(\mathcal{E}) = N$ for the HD surfaces that solve the Euler-Lagrange equation given in equation (52) and provide the constraints [45].

Example 20.

i) For $N = 3$, the Lagrange function is in the following form

$$\mathcal{E} = a_1\,H^3 + a_2\,H^2 + a_3\,H + a_4 + a_5\,K + a_6\,K\,H$$

where

$$a_1 = -\frac{11\,\mu\,a_2}{30\,\lambda}, \qquad a_3 = -\frac{4\,\lambda\,a_2}{15\,\mu}, \qquad a_6 = \frac{14\,\mu\,a_2}{15\,\lambda}$$

$$a_4 = 0, \qquad C_1 = p = 0, \qquad C_2 = 2\,\lambda.$$

Here $\lambda \neq 0$, μ, and a_5 are arbitrary constants.

ii) For $N = 4$, the Lagrange function is in the following form

$$\mathcal{E} = a_1\,H^4 + a_2\,H^3 + a_3\,H^2 + a_4\,H + a_5 + a_6\,K + a_7\,K\,H + a_8\,K^2 + a_9\,K\,H^2$$

where

$$C_1 = p = 0, \quad C_2 = 2\,\lambda, \qquad a_1 = -\frac{1}{64}\,(15\,a_8 + 34\,a_9)$$

$$a_2 = \frac{1}{480\,\mu\,\lambda}\,\big(\lambda^2\,(358\,a_9 - 7\,a_8) - 176\,\mu^2\,a_3\big)$$

$$a_4 = \frac{4\,\lambda}{15\,\mu^3}\,\big(\lambda^2\,(13\,a_8 + 8\,a_9) - \mu^2\,a_3\big), \qquad a_5 = -\frac{3\,\lambda^4}{4\,\mu^4}\,(3\,a_8 + 2\,a_9)$$

$$a_7 = \frac{1}{120\,\mu\,\lambda}\,\big(\lambda^2\,(359\,a_8 + 154\,a_9) + 112\,\mu^2\,a_3\big).$$

Here $\lambda \neq 0$, $\mu \neq 0$, and a_6 are arbitrary constants.

iii) For $N = 5$, the Lagrange function is in the following form

$$\begin{aligned}\mathcal{E} \;=\; & a_1\,H^5 + a_2\,H^4 + a_3\,H^3 + a_4\,H^2 + a_5\,H + a_6 + a_7\,K \\ & + a_8\,K\,H + a_9\,K^2 + a_{10}\,K\,H^2 + a_{11}\,K^2\,H + a_{12}\,K\,H^3\end{aligned}$$

where

$$C_1 = p = 0, \qquad C_2 = 2\,\lambda, \qquad a_1 = -\frac{3}{464}\,(51\,a_{11} + 92\,a_{12})$$

$$a_2 = -\frac{1}{1856\,\mu}\,(\mu\,(435\,a_9 986 + \,a_{10}) - \lambda\,(2590\,a_{11} + 2268\,a_{12}))$$

$$a_3 = -\frac{1}{13920\,\mu^2\,\lambda}\Big(\mu\,\lambda^2\,(203\,a_9 - 10382\,a_{10}) \\ -\lambda^3\,(14486\,a_{11} + 14220\,a_{12}) + 5104\,\mu^3\,a_4\Big)$$

$$a_5 = -\frac{4\,\lambda}{435\,\mu^4}\Big(- \mu\,\lambda^2\,(377\,a_9 + 232\,a_{10}) \\ +\lambda^3\,(1544\,a_{11} + 720\,a_{12}) + 29\,\mu^3\,a_4\Big)$$

$$a_6 = -\frac{3\,\lambda}{116\,\mu^5}\Big(\mu\,(87\,a_9 + 58\,a_{10}) - \lambda\,(494\,a_{11} + 252\,a_{12})\Big)$$

$$a_8 = \frac{1}{3480\,\mu^2\,\lambda}\Big(\mu\,\lambda^2\,(10411\,a_9 + 4466\,a_{10}) \\ -\lambda^3\,(34582\,a_{11} + 17100\,a_{12}) + 3248\,\mu^3\,a_4\Big).$$

Here $\lambda \neq 0$, $\mu \neq 0$, a_7 are arbitrary constants.

iv) For $N = 6$, the Lagrange function is in the following form

$$\begin{aligned} \mathcal{E} \;=\; & a_1\,H^6 + a_2\,H^5 + a_3\,H^4 + a_4\,H^3 + a_5\,H^2 + a_6\,H + a_7 \\ & + a_8\,K + a_9\,K\,H + a_{10}\,K^2 + a_{11}\,K\,H^2 + a_{13}\,K\,H^3 \\ & + a_{12}\,K^2\,H + a_{14}\,K^3 + a_{15}\,K^2\,H^2 + a_{16}\,K\,H^4 \end{aligned}$$

where

$a_1, a_2, a_3, a_4, a_6, a_7, a_9$ can be written in terms of a_5, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16} and $C_1 = p = 0, C_2 = 2\,\lambda$.

Here $\lambda \neq 0$, $\mu \neq 0$, are arbitrary constants.

For general $N \geq 3$, from the above examples, the polynomial function $\mathcal{E}$ takes the form

$$\mathcal{E} = \sum_{n=0}^{N} H^n \sum_{l=0}^{\lfloor \frac{(N-n)}{2} \rfloor} a_{nl} K^l$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and a_{nl} are constants.

8.3.1. Position Vector of HD Surfaces

In this section, we find the position vector of the HD surfaces that we obtained using spectral parameter deformation in Proposition 35.

Consider the solution

$$u = -(\alpha/2)\, 18^{1/3}\, \xi^{2/3} \tag{95}$$

of the HD equation, where $\xi = t + x/\alpha$ and $\alpha \neq 0$ is a constant.

In order to find the position vector first we need to solve the Lax equation given in equation (38). We insert solution of the HD equation given in equation (95) and Lax pairs U and V given by equations (92) into the Lax equations provided by equation (38). We solve the resulting equation and obtain the solution of Lax equation 2×2 matrix Φ

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}$$

where $\Phi_{11},\, \Phi_{12},\, \Phi_{21},\, \Phi_{22}$ are given as

$$\begin{aligned}
\Phi_{11} &= \frac{1}{\lambda^{3/2}}\Big(A_1\,(18^{1/3} - 6\,\lambda\,\xi^{1/3})\,\mathrm{Exp}\,\Big(4\,\lambda^3\,t + \lambda\,18^{2/3}\,\xi^{1/3}/3\Big) \\
&\quad + B_1\,(18^{1/3} + 6\,\lambda\,\xi^{1/3})\,\mathrm{Exp}\,\Big(-4\,\lambda^3\,t - \lambda\,18^{2/3}\,\xi^{1/3}/3\Big)\Big) \\
\Phi_{21} &= -\frac{2\,\sqrt{\lambda}\,18^{2/3}}{3\,\alpha\,\xi^{1/3}}\Big(A_1\ \mathrm{Exp}\,\Big(4\,\lambda^3\,t + \lambda\,18^{2/3}\,\xi^{1/3}/3\Big) \\
&\quad + B_1\ \mathrm{Exp}\,\Big(-4\,\lambda^3\,t - \lambda\,18^{2/3}\,\xi^{1/3}/3\Big)\Big) \\
\Phi_{12} &= \frac{1}{\lambda^{3/2}}\Big(A_2\,(18^{1/3} - 6\,\lambda\,\xi^{1/3})\,\mathrm{Exp}\,\Big(4\,\lambda^3\,t + \lambda\,18^{2/3}\,\xi^{1/3}/3\Big) \\
&\quad + B_2\,(18^{1/3} + 6\,\lambda\,\xi^{1/3})\,\mathrm{Exp}\,\Big(-4\,\lambda^3\,t - \lambda\,18^{2/3}\,\xi^{1/3}/3\Big)\Big) \\
\Phi_{22} &= -\frac{2\,\sqrt{\lambda}\,18^{2/3}}{3\,\alpha\,\xi^{1/3}}\Big(A_2\ \mathrm{Exp}\,\Big(4\,\lambda^3\,t + \lambda\,18^{2/3}\,\xi^{1/3}/3\Big) \\
&\quad + B_2\ \mathrm{Exp}\,\Big(-4\,\lambda^3\,t - \lambda\,18^{2/3}\,\xi^{1/3}/3\Big)\Big)
\end{aligned} \tag{96}$$

where $\xi = t + x/\alpha$, and A_1, A_2, B_1, B_2, and $\alpha \neq 0$ are constants.

Here the determinant of the solution,Φ, of the Lax equation is constant and it has the following form

$$\det(\Phi) = \frac{8 \cdot 18^{2/3}}{\alpha}\,(A_1 B_2 - A_2 B_1) \neq 0.$$

We use the equation

$$F = \mu\,\Phi^{-1}\,\frac{\partial \Phi}{\partial \lambda}$$

in order to find the immersion function F. We obtain F as

$$F = y_1 \mathrm{e}_1 + y_2 \mathrm{e}_2 + y_3 \mathrm{e}_3$$

where y_1, y_2, and y_3 are given as

$$\begin{aligned}
y_1 &= \Omega_{12}\Big(\Omega_{13}\, W_{19} + \Omega_{14}\, W_{20} + \Omega_{15}\, W_{21}\Big) \\
y_2 &= \frac{\Omega_{12}}{2}\Big(\Omega_{13}\, W_{22} + \Omega_{14}\, W_{23} + \Omega_{15}\, W_{24}\Big) \\
y_3 &= \frac{\Omega_{12}}{2}\Big(\Omega_{13}\, W_{25} + \Omega_{14}\, W_{26} + \Omega_{15}\, W_{27}\Big)
\end{aligned} \tag{97}$$

and

$$\begin{aligned}
\Omega_{12} &= \frac{\mu}{3\,\alpha^{2/3}\,\lambda^2\,(A_1\,B_2 - A_2\,B_1)\,(\alpha\,t + x)^{1/3}} \\
\Omega_{13} &= \frac{1}{2}\Big(3\,\lambda\,\alpha^{2/3}(\alpha\,t + x)^{1/3} + \alpha\,18^{1/3}\Big)\ \mathrm{Exp}\Big\{-2\,\lambda\Big(12\,\lambda^2\,\alpha^{1/3} t \\
&\quad + 18^{2/3}\,(\alpha\,t + x)^{1/3}\Big)/(3\,\alpha^{1/3})\Big\} \\
\Omega_{14} &= \frac{1}{2}\Big(-3\,\lambda\,\alpha^2/3(\alpha\,t + x)^{1/3} + \alpha\,18^{1/3}\Big)\ \mathrm{Exp}\Big\{2\,\lambda\Big(12\,\lambda^2\,\alpha^{1/3} t \\
&\quad + 18^{2/3}\,(\alpha\,t + x)^{1/3}\Big)/(3\,\alpha^{1/3})\Big\} \\
\Omega_{15} &= 2\,\lambda^2\,18^{2/3}\,\alpha^{1/3}\,(\alpha\,t + x)^{2/3} + 72\,\lambda^4\,\alpha^{2/3}\,t\,(\alpha\,t + x)^{1/3} + 18^{1/3}\,\alpha \\
W_{19} &= B_1\,B_2, \qquad W_{20} = A_1\,A_2, \qquad W_{21} = \frac{1}{2}\,(A_1\,B_2 + A_2\,B_1) \\
W_{22} &= B_2^2 - B_1^2, \qquad W_{23} = A_2^2 - A_1^2, \qquad W_{24} = A_2\,B_2 - A_1\,B_1 \\
W_{25} &= B_2^2 + B_1^2, \qquad W_{26} = A_2^2 + A_1^2, \qquad W_{27} = A_2\,B_2 + A_1\,B_1.
\end{aligned}$$

8.3.2. Plotting HD Surfaces

In this section, we plot some of the HD surfaces given by equations (97) for some special values of the constants μ, k, c, B_1, B_2, C_1, and $C_2 = 1$.

Example 21. *Taking* $\mu = 1$, $\alpha = 1$, $\lambda = 1$, $A_1 = 1$, $A_2 = -1$, $B_1 = -1$, $B_2 = -1$ *in the equations provided by equations* (97)*, we get the surface given in* Fig. 13.

Example 22. *Taking* $\mu = 1$, $\alpha = 0.2$, $\lambda = 0.7$, $A_1 = 1$, $A_2 = -1$, $B_1 = -1$, $B_2 = -1$ *in the equations provided by equations* (97)*, we get the surface given in* Fig. 14.

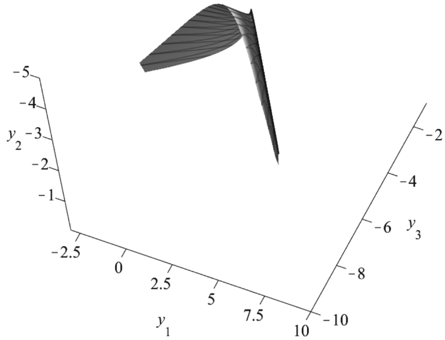


Figure 13. $(x,t) \in [-0.2, 0.2] \times [-0.2, 0.2]$

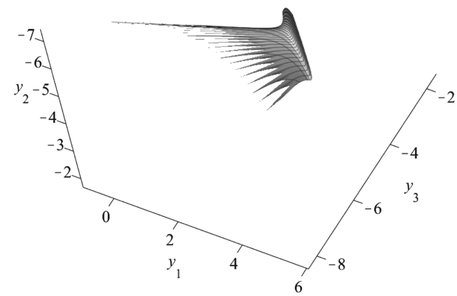


Figure 14. $(x,t) \in [-0.2, 0.2] \times [-0.2, 0.2]$

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References

[1] Ablowitz M. and Segur H., *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge 1991.

[2] Blaszak M., *Multi-Hamilton Theory of Dynamical Systems*, Springer, Berlin 1998.

[3] Bobenko A., *Surfaces in Terms of 2 by 2 Matrices, Old and New Integrable Cases* in Harmonic Maps and Intgerable Systems, A. Fordy and J. Wood (Eds), Aspects of Mathematics, Friedr. Vieweg, Braunscweig 1994, pp.83-127.

[4] Bobenko A., *Integrable Surfaces*, Funct. Anal. Prilozh. **24** (1990) 68-69.

[5] Ceyhan Ö., Fokas A. and Gürses M., *Deformations of Surfaces Associated with Integrable Gauss-Minardi-Codazzi Equations*, J. Math. Phys. **41** (2000) 2251-2270.

[6] Cieśliński J., Goldstein P. and Sym A., *Isothermic Surfaces in E^3 as Soliton Surfaces*, Phys. Lett. A **205** (1995) 37-43.

[7] Cieśliński J., *A Generalized Formula for Integrable Classes of Surfaces in Lie Algebraas*, J. Math. Phys. **38** (1997) 4255-4272.

[8] Do Carmo M., *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Englewood Cliffs 1976.

[9] Drazin P., *Solitons*: *An Introduction*, Cambridge Univ. Press, New York 1989.

[10] Eisenhart L., *A Treatise on the Differential Geometry of Curves and Surfaces*, Dover, New York 1909.

[11] Eisenhart L., *Riemannian Geometry*, Princeton Univ. Press, Princeton 1964.

[12] Fokas A. and Gelfand I., *Surfaces on Lie Groups, on Lie Algebras, and Their Integrability*, Commun. Math. Phys. **177** (1996) 203-220.

[13] Fokas A., Gelfand I., Finkel F. and Liu Q.-M., *A Formula for Constructing Infinitely Many Surfaces on Lie Algebras and Integrable Equations*, Selecta Math. New Ser. **6** (2000) 347-375.

[14] Gürses M., *Some Special Integrable Surfaces*, J. Nonlinear Math. Phys. **9** (2002) 59-66.

[15] Gürses M. and Tek S., *Korteweg-de Vries Surfaces*, Nonlinear Analysis: Theory, Method and Applications **95** (2014) 11-22.

[16] Gürses M. and Tek S., *Surfaces From Deformation of Parameters*, Geom. Integrability & Quantization **17** (2016) xx-xx.

[17] Gürses M. and Nutku Y., *New Nonlinear Evolution Equations From Surface Theory,* J. Math. Phys. **22** (1981) 1393-1398.

[18] Gürses M., *Motion of Curves on Two Dimensional Curves Surfaces and Soliton Equations*, Phys. Lett. A **241** (1998) 329-332.

[19] Gürses M., *Inverse Scattering, Differential Geometry, Einstein-Maxwell 2N Solitons and Gravitational One Soliton Backlund Transformations*, In: Solutions of Einstein's Equations: Techniques and Results, C. Hoensalears and W. Ditez (Eds), Lecture Notes in Physics **205**, Springer, Berlin 1984, pp.199-234.

[20] Hasimoto H., *A Soliton on a Vortex Filament*, J. Fluid. Mech. **51** (1972) 477-485.

[21] Helfrich W., *Elastic Properties of Lipid Bilayers–Theory and Possible Experiments*, Z. Naturforsch. C **28** (1973) 693-703.

[22] Konopelchenko B., *Nets in $\mathbb{R}^3$, Their Integrable Evolutions and the DS Hierarchy*, Phys. Lett. A **183** (1993) 153-159.

[23] Konopelchenko B. and Taimanov I., *Generalized Weierstrass Formulae, Soliton Equations and Willmore Surfaces. I. Tori of Revolution and the mKdV Equation*, arXiv:dg-ga/9506011.

[24] Konopelchenko B. and Landolfi G., *Induced Surfaces and Their Integrable Dynamics II. Generalized Weierstrass Representation in 4-D Spaces and Deformations via DS Hierarchy*, Studies in Applied Mathematics **104** (2000) 129-169.

[25] Konopelchenko B., *Surfaces of Revolution and Their Integrable Dynamics via the Schrödinger and KdV Equations*, Inverse Problems **12** (1996) L13–L18.

[26] Lamb G., *Solitons on Moving Space Curves*, J. Math. Phys. **18** (1977) 1654.

[27] McLachlan R. and Segur H., *A Note on the Motion of Surfaces*, Phys. Lett. A **194** (1994) 165-172.

[28] Melko M. and Sterling I., *Integrable Systems, Harmonic Maps and the Classical Theory of Surfaces*, In: Harmonic Maps and Integrable Systems, A. Fordy and J. Wood (Eds), Friedr. Vieweg, Braunscweig 1994, pp.129-144.

[29] Mladenov I., *New Solutions of the Shape Equation*, Eur. Phys. J. B **29** (2002) 327-330.

[30] Nakayama K., Segur H., and Wadati M., *Integrability and the Motion of Curves*, Phys. Rev. Lett. **69** (1992) 2603.

[31] Nakayama K., *Motion of Curves in Hyperbolid in the Minkowski Space*, J. Phys. Soc. Japan **67** (1998) 3031-3037.

[32] Nakayama K., *Motion of Curves in Hyperbolid in the Minkowski Space II*, J. Phys. Soc. Japan **68** (1999) 3214-3218.

[33] Ou-Yang Z.-C., Liu J. and Xie Y., *Geometric Methods in the Elastic Theory of Membranes in Liquid Crystal Phases*, World Scientific, Singapore 1999.
[34] Ou-Yang Z.-C. and Helfrich W., *Instability and Deformation of a Spherical Vesicle by Pressure*, Phys. Rev. Lett. **59** (1987) 2486-2488.
[35] Ou-Yang Z.-C. and Helfrich W., *Bending Energy of Vesicle Membranes*: *General Expansion for the First, Second, and Third Variation of the Shape Energy and Applications to Sphere and Cylinders*, Phys. Rev. A **39** (1989) 5280-5288.
[36] Osserman R., *A Survey of Minimal Surfaces*, Dover, New York 1986.
[37] Olver P., *Applications of Lie Groups to Differential Equations*, Springer, Berlin 1991.
[38] Parthasarathy R. and Viswanathan K., *Geometric Properties of QCD String From Willmore Functional*, J. Geom. Phys. **38** (2001) 207-216.
[39] Pinkall U., *Hamiltonian Flows on the Space of Star-Shaped Curves*, Results Math. **27** (1995) 328-332.
[40] Sym A., *Soliton Surfaces*, Lett. Nuovo Cimento **33** (1982) 394-400.
[41] Sym A., *Soliton Surfaces II*, Lett. Nuovo Cimento **36** (1983) 307-312.
[42] Sym A., *Soliton Surfaces and Their Applications*, In: Geometrical Aspects of the Einstein Equations and Integrable Systems, Lecture Notes in Physics **239**, R. Martini (Ed), Springer, Berlin 1985, pp.154-231.
[43] Tek S., *Modified Korteweg-de Vries Surfaces*, J. Math. Phys. **48** (2007) 013505.
[44] Tek S., *Soliton Surfaces and Surfaces From a Variational Principle*, PhD Thesis. Bilkent University, 2007.
[45] Tek S., *Some Classes of Surfaces in* $\mathbb{R}^3$ *and* $\mathbb{M}_3$ *Arising from Soliton Theory and a Variational Principle*, Discrete and Continuous Dynamical System Supplement (2009) 761-770.
[46] Tek S., *Using Nonlinear Schrödinger Equation to Obtain Some New Surfaces in* $\mathbb{R}^3$, *under preparation*.
[47] Tu Z.-C. and Ou-Yang Z.-C., *A Geometric Theory on the Elasticity of Bio-Membranes*, J. Phys. A: Math. & Gen. **37** (2004) 11407-11429.
[48] Tu Z.-C. and Ou-Yang Z.-C., *Lipid Membranes with Free Edges*, Phys. Rev. E **68** (2003) 061915.
[49] Tu Z.-C. and Ou-Yang Z.-C., *Variational Problems in Elastic Theory of Biomembranes, Smectic-A Liquid Crystals, and Carbon Related Structures*, Geom. Integrability & Quantization **7** (2006) 23-248.
[50] Tu Z.-C., *Elastic Theory of Biomembranes*, Thin Solid Films **393** (2001) 19-23.
[51] Willmore T., *Total Curvature in Riemannian Geometry*, Wiley, New York 1982.
[52] Willmore T., *Surfaces in Conformal Geometry*, Annals of Global Analysis and Geometry **18** (2000) 255-264.